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Thomas ECKL

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NUMERICALLY TRIVIAL FOLIATIONS

by **Thomas ECKL**

1. Introduction.

In the last few years several fibrations related to a nef or even pseudoeffective line bundle L on a projective complex manifold were constructed whose fibers satisfy certain numerical properties with respect to a sometimes modified intersection theory:

For a nef line bundle L, the usual intersection theory is taken by [BCE$^+$00] to define (and construct) the so called *nef fibration* whose fibers contain only curves C with $L.C = 0$. The base dimension of this fibration is called the nef dimension of L , and it can be proven that it is never smaller than the numerical dimension $\nu(L)$ of L. Note however that already for surfaces there are explicit counter examples to equality, cf. Section 4.2.

Even earlier, Tsuji [Tsu00] associated an intersection theory to positive singular hermitian metrics h on pseudoeffective line bundles L by defining

$$(L, h).C := \limsup_{m\to\infty} \frac{1}{m} h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(m\pi^*L) \otimes \mathcal{I}((\pi^*h)^m)).$$

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Here, $\pi : \tilde{C} \to C$ is the normalization of an irreducible curve C not contained in the singular locus of h, and $\mathcal{I}((\pi^*h)^m))$ denotes the multiplier ideal sheaf of the pulled back metric $(\pi^*h)^m$ on $\tilde{C}$. A projective complex manifold is called numerically trivial in Tsuji's sense iff $(L,h).C = 0$ for all such curves C. In [Eck02] other possible definitions of these intersection numbers are discussed, their relations are studied, and the fibration map with numerically trivial fibers is constructed, according to the suggestions of Tsuji.

Finally, Takayama [Tak02] defined intersection numbers reflecting properties of the linear sytems $|mL|$ by using the asymptotic multiplier ideal sheaf $\mathcal{J}(||L||)$:

$$||L, C|| := \lim_{m \to \infty} m^{-1} \deg_C mL \otimes \mathcal{J}(||mL||),$$

where C is an irreducible curve not contained in the stable base locus $\bigcap_{m\in\mathbb{N}} \mathrm{Bs}|mL|$ of L. The resulting fibration turns out to be the well known Iitaka fibration.

The motivation for this work is to give a more unified treatment of all these fibrations and to give geometric reasons for the deviation of nef, numerical, and Kodaira-Iitaka dimension of a nef line bundle on a projective manifold. Three surface examples will illustrate the ideas developed to this purpose.

The first example is due to Mumford and has the property that the nef dimension is bigger than the numerical dimension: Start with a smooth projective curve C of genus $\geqslant 2$ with the unit circle Δ as universal covering and an irreducible unitary representation $\rho : \pi_1(C) \to GL(2,\mathbb{C})$ of the fundamental group of C. This defines a rank 2 vector bundle $E = (\Delta \times \mathbb{C}^2)/\pi_1(C)$ on C of degree 0 where the action of $\pi_1(C)$ is given by covering transformations on Δ and the representation ρ on $\mathbb{C}^2$.

Mumford proved that the nef line bundle $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ on the projectivized bundle $\mathbb{P}(E)$ is stable hence the restriction of L to all curves $D \subset \mathbb{P}(E)$ is positive. On the other hand $\deg E = 0$ hence $L.L = 0$. Hence the numerical dimension $\nu(L)$ is 1, while the nef reduction map is the identity, and the nef dimension is 2.

It seems quite obvious how to explain this deviation: the ruled surface $\mathbb{P}(E)$ carries a foliation induced by the images of the $\Delta \times l$ in $\mathbb{P}(E)$ (where l is a line through the origin in $\mathbb{C}^2$). Furthermore, locally the leaves of this foliation are mapped to points by the morphism induced by $|L|$, which is a kind of numerical triviality.

This motivates the construction of numerically trivial foliations w.r.t. some positive closed current (which may be the curvature current of some hemitian metric on a nef line bundle) on a complex manifold X. The starting point is an interesting criterion for numerical triviality in Tsuji's sense:

THEOREM 1.1. — *Let X be a smooth projective complex manifold, let L be a pseudo-effective line bundle on X with positive singular hermitian metric h such that X is $(L,h)-$ numerically trivial. Then the curvature current Θ_h may be decomposed as*

$$\Theta_h = \sum_i a_i[D_i]$$

where the D_i form a countable set of prime divisors on X and the a_i are > 0.

This is proven in [Eck02], and by trivial arguments the converse of this theorem is also true. It shows that numerical triviality is a local property of currents and does not depend on projectivity. Hence it is possible to localize the notion of numerically trivial fibrations to the notion of a foliation with numerically trivial leaves (details in Sections 2.1, 2.2). The main result is the following:

THEOREM 1.2. — *On a (not necessarily compact) complex manifold X with a positive closed $(1,1)-$ current T there exists a* maximal *foliation with numerically trivial leaves w.r.t. T, that is the leaves of every foliation with numerically trivial leaves are contained in leaves of this foliation.*

It is called the numerically trivial foliation w.r.t. T. The construction rests essentially on the Local Key Lemma which allows to unite different foliations with numerically trivial leaves, and the proof of this lemma is an easy consequence of another interpretation of numerically trivial fibrations $f : X \to Y$ w.r.t. to some closed positive $(1,1)-$ current T: The residue current R of the Siu decomposition $T = \sum a_i[D_i] + R$ must be the pull back of a (positive) current on Y (details in 2.2).

If X is projective and T the curvature current of a positive singular hermitian metric on a line bundle, Tsuji's numerically trivial fibration will be the fibration maximal among those whose fibers are contained in the leaves of the numerically trivial foliation w.r.t. T; details in Section 2.3. The same construction gives the Iitaka fibration of a line bundle L with Kodaira-Iitaka dimension $\kappa(L) \geqslant 0$ provided one uses the positive singular hermitian metric h on L defined as

$$h = \limsup_{m \to \infty} (h_{|mL|})^{\frac{1}{m}}$$

where $h_{|mL|}$ is the (singular) hermitian metric on $|mL|$ defined by the global sections of mL (see [Tsu99]). In this case even more is true: The numerical trivial foliation w.r.t. h is already the Iitaka fibration (section 2.4).

It is not possible to find a positive singular hermitian metric which defines the nef fibration in a similar way, as shown by an example constructed in [DPS94] which is quite similar to Mumford's example (section 4.1): Start with an elliptic curve C and take as the rank 2 vector bundle E the unique nontrivial extension of the structure sheaf $\mathcal{O}_C$. As in Mumford's example the numerical dimension of the nef line bundle $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ is 1, while the nef dimension is 2.

The remarkable feature of this example is the fact that the only positive singular hermitian metric on L is given by the unique section of L, the "section at infinity" of $\mathbb{P}(E)$ (proof in [DPS94]), and $\mathbb{P}(E)$ is numerically trivial w.r.t. this metric. Hence the obvious foliation on $\mathbb{P}(E)$ induced by the universal cover $\mathbb{C}$ of C cannot be interpreted as the numerically trivial foliation w.r.t. some positive metric on $L = \mathcal{O}_{\mathbb{P}(E)}(1)$.

Ideas how to deal with this situation may be found in Boucksom's construction of a divisorial Zariski decomposition and his definition of "moving" intersection numbers on pseudoeffective line bundles [Bou02] on compact Kähler manifolds. Both notions show that it is extremely useful to loosen the restriction on positivity and to consider sequences of almost positive $(1,1)-$currents in a fixed cohomology class α whose negative parts tend to 0.

For nef line bundles the moving intersection numbers coincide with the usual ones. In particular, if $C \subset X$ is a smooth compact curve on a compact Kähler manifold X with Kähler form ω, and L is a nef line bundle with first Chern class $\alpha := c_1(L) \in H^{1,1}(X, \mathbb{R})$,

$$L.C = \lim_{\epsilon \downarrow 0} \sup_{T} \int_{X - \mathrm{Sing}T} (T + \epsilon\omega) \wedge [C] = \lim_{\epsilon \downarrow 0} \sup_{T} \int_{C - \mathrm{Sing}T} (T + \epsilon\omega),$$

where the T's run through all closed currents representing α such that $T \geqslant -\epsilon\omega$, and $[C]$ is the integration current of the submanifold C of bidegree $(n-1, n-1)$. (For further details see Section 3.1.)

It is obvious that in this case $L.C = 0$ iff $\lim_{\epsilon\downarrow 0}\sup_T \int_{\Delta-\mathrm{Sing}T}(T + \epsilon\omega) = 0$ for all disks $\Delta \subset C$. Thus it is justified to interpret numerical triviality w.r.t. a pseudo-effective class as a local property: An immersed submanifold $Y \subset X$ (closed or not) is called numerically trivial w.r.t. a pseudo-effective class α iff

$$\lim_{\epsilon\downarrow 0}\sup_T \int_{\Delta-\mathrm{Sing}T}(T + \epsilon\omega) = 0$$

(where the T's run through all closed currents on X contained in $\alpha[-\epsilon\omega]$) for all holomorphically immersed disks $\Delta \subset Y$. And a foliation will be called numerically trivial w.r.t. α iff (locally) almost every leaf is numerically trivial w.r.t. α.

It is possible to prove an analog to the Local Key Lemma, hence there is a maximal numerically trivial foliation w.r.t. α. It is contained in every numerically trivial foliation w.r.t. a positive current representing α. If α is the first Chern class of some nef line bundle L on a projective manifold X, the nef fibration of L is the maximal fibration contained in the foliation (which will be called the *nef foliation* in that case). Furthermore, the Kodaira-Iitaka fibration contains the nef foliation, and one gets a nice geometric reason for deviations of the Kodaira-Iitaka and the nef dimension of nef line bundles on projective manifolds: $\kappa(L) < n(L)$ if the nef foliation is not a fibration. It is a very interesting open question whether the converse of this statement is also true. More generally: Is the fibration with the smallest fiber dimension which contains the nef foliation the Kodaira-Iitaka fibration ?

Finally, it is shown that the codimension of the leafs is an upper bound for the numerical dimension of α, if the singularities of the foliation are isolated points. It is not clear to the author how to weaken this assumption or if there are counter examples. To get better answers it seems necessary to have a closer look at the structure of numerical trivial foliations around the singularities.

The last section of the paper constructs nef foliations of nef line bundles on surfaces. The first two examples are those due to Mumford and Demailly-Peternell-Schneider. In Mumford's example it is easy to construct a smooth closed positive $(1,1)-$ current on $L = \mathcal{O}_{\mathbb{P}^1}(1)$ such that the associated nef foliation is the obvious one: Take a measure ω invariant w.r.t. the representation of $\pi(C)$ in PGL(2). This gives a measure on $(\Delta\times\mathbb{P}^1)/\pi(C)$ transversal to the foliation induced by the images of $\Delta\times\{p\}$. Averaging out the integration currents of the leaves with this transverse measure gives an (even smooth) closed positive $(1,1)-$ current in the first Chern class of $L=\mathcal{O}_{\mathbb{P}(E)}(1)$ which vanishes on the leaves but not in any transverse direction.

The Demailly-Peternell-Schneider example is more difficult: A complicated glueing argument leads to almost positive currents which determine the obvious foliation.

The last example deals with $\mathbb{P}^2$ blown up in 9 points and is interesting in many ways. In particular, if one fixes 8 points in sufficiently general position, varying the last point will give a nef fibration in the torsion points, but there is no nef foliation on the whole family with 1- dimensional leaves. Hence, the nef fibrations in varieties over torsion points do not converge against a foliation in varieties over (general) non-torsion points. This somehow answers a question asked in [DPS96].

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2. Numerically trivial foliations.

2.1. Numerical triviality.

As proposed in the introduction numerical triviality of (not necessarily compact) complex manifolds is defined via the criterion of Theorem 1.1:

DEFINITION 2.1. — *Let X be a complex manifold and T a positive closed $(1,1)-$ current on X. Then X is called numerically trivial w.r.t. T iff*

$$T=\sum a_i[D_i],$$

for countably many prime divisors D_i in X and real numbers $a_i\geqslant 0$.

To compare later on Tsuji's numerically trivial fibration with the numerically trivial foliation it is useful to define numerical triviality for any irreducible analytic subsets (not only for submanifolds):

DEFINITION 2.2. — *Let X be compact complex manifold and Θ a positive closed $(1,1)-$ current on X. Let $Y \subset X$ be a positive dimensional analytic subset of X such that Θ may be restricted to Y_{reg}, the smooth part of Y. Then Y is called numerically trivial with respect to Θ iff for all holomorphic maps $f : \Delta^k \to Y$ such that $f^*\Theta$ exists the complex manifold Δ^k is numerically trivial with respect to $f^*\Theta$.*

This definition is consistent with the definition of numerically trivial manifolds:

PROPOSITION 2.3. — *Let X be a complex manifold and Θ a positive closed $(1,1)-$ current. X is numerically trivial w.r.t. Θ iff for all holomorphic maps $f : \Delta^k \to X$ such that $f^*\Theta$ exists Δ^k is numerically trivial w.r.t. $f^*\Theta$.*

Proof. — The "only if" part is a trivial consequence of the equality $f^*([D_i]) = [f^*(D_i)]$ for (integration currents of) divisors. The other direction follows from the Siu-decomposition [Dem00, (2.18)]

$$\Theta = \sum_i \nu_i [D_i] + R,$$

where R is a positive closed $(1,1)-$ current such that the Lelong number level sets $E_c(R)$ have no codim 1 components. If $R \not\equiv 0$ there will exist an open set $U \cong \Delta^n$ in X such that $R_{|U} \not\equiv 0$, hence U is not numerically trivial w.r.t. $\Theta_{|U}$. □

The definition of numerical triviality can be further simplified by means of the following proposition:

PROPOSITION 2.4. — *Let X be a complex manifold and Θ a positive closed $(1,1)-$ current. X is numerically trivial w.r.t. Θ iff for all holomorphic maps $f : \Delta \to X$ such that $f^*\Theta$ exists Δ is numerically trivial w.r.t. $f^*\Theta$.*

Proof. — The "only if" part follows by definition. For the other direction start again with the Siu decomposition $\Theta = \sum_i \nu_i [D_i] + R$. Let $\Delta^n \cong U \subset X$ be an open subset and let $q : \Delta^n \to \Delta^{n-1}$ be the projection onto the first $n-1$ factors. Since the Lelong number level sets $E_c(R)$ contain no codim 1 component, very general fibers F of q do not intersect any of the $E_c(R)$. By the results of [ME00] there is a pluripolar set $N \subset \Delta^{n-1}$ such that the level sets $E_c(R_{|F}) = \emptyset$ for the restriction of R to all fibers F over points outside of N. By assumption $R_{|F} \cong 0$.

By the following lemma there exists a positive closed $(1,1)-$ current S on Δ^{n-1} such that $R = q^*S$. Let $D = \Delta^{n-1} \times \{p\}$ be a section of q such that $R_{|D}$ is well defined. By induction $R_{|D} \equiv 0$. Since the projection $q : D \to \Delta^{n-1}$ is an isomorphism $S \equiv 0$ hence $R \equiv 0$. □

LEMMA 2.5. — *Let T be a positive closed $(1,1)-$ current on Δ^n and let $q : \Delta^n \to \Delta^{n-1}$ be the projection onto all factors but the last one. If $T_{|q^{-1}(x)} \equiv 0$ for all x outside a pluripolar set $N \subset \Delta^{n-1}$ then there will be a positive closed $(1,1)-$ current S on Δ^{n-1} such that $T = q^*S$.*

Proof. — The positive current T may be written as

$$T = i \sum_{i,j} \Theta_{ij} dz_i \wedge d\overline{z}_j$$

where the Θ_{ij} are complex measures on Δ^n ([Dem00, (1.15)]). That T is a real current implies $\Theta_{ij} = \overline{\Theta}_{ji}$. Since T is positive, $\sum \lambda_i \overline{\lambda_j} \Theta_{ij}$ is a positive measure for all vectors $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. Hence

$$\lambda_i \overline{\lambda_i} \Theta_{ii} + \lambda_i \overline{\lambda_n} \Theta_{in} + \lambda_n \overline{\lambda_i} \Theta_{ni} + \lambda_n \overline{\lambda_n} \Theta_{nn} \geqslant 0 \quad \forall (\lambda_i, \lambda_n) \in \mathbb{C}^2.$$

Claim. — As a $(1,1)-$ current $i\Theta_{nn} dz_n \wedge d\overline{z}_n = 0$.

Proof. — By definition one has to show that

$$\int_{\Delta^n} i\Theta_{nn} dz_n \wedge d\overline{z}_n \wedge \alpha i dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i dz_{n-1} \wedge d\overline{z}_{n-1} = 0$$

for all complex valued functions $\alpha \in \mathcal{C}^\infty_c(\Delta^n)$. Since $T_{|q^{-1}(x)} = i\Theta_{nn} dz_n \wedge d\overline{z}_n$

$$\int_{\Delta^n} i\Theta_{nn} dz_n \wedge d\overline{z}_n \wedge \alpha i dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i dz_{n-1} \wedge d\overline{z}_{n-1} =$$
$$\int_{\Delta^n} T \wedge \alpha i dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge i dz_{n-1} \wedge d\overline{z}_{n-1},$$

and the slicing formula [Dem00, (1.22)] implies that this is equal to

$$\int_{\Delta^{n-1}} \left(\int_{q^{-1}(x')} T_{|q^{-1}(x')} \wedge \alpha_{|q^{-1}(x')} \right) idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_{n-1} \wedge d\overline{z}_{n-1}.$$

This is 0 because $T_{|q^{-1}(x)} \equiv 0$ for all x outside a pluripolar set $N \subset \Delta^{n-1}$. □

Consequently,

$$\Theta_{\imath\imath} + \overline{\lambda}_n \Theta_{\imath n} + \lambda_n \Theta_{n\imath} = \Theta_{ii} + \overline{\lambda}_n \overline{\Theta}_{ni} + \lambda_n \Theta_{ni} \geqslant 0$$

for all $\lambda_n \in \mathbb{C}$. Now suppose that $\Theta_{ni} \neq 0$, i.e. there is a smooth real valued function $\alpha \geqslant 0$ with compact support such that $\Theta_{ni}(\alpha) \neq 0$. Then there is a $\lambda_n \in \mathbb{C}$ such that

$$\Theta_{ii}(\alpha) + \overline{\lambda}_n \overline{\Theta_{ni}(\alpha)} + \lambda_n \Theta_{ni}(\alpha) < 0.$$

This is a contradiction. Hence $\Theta_{in} = \Theta_{ni} = 0$ for all $i \leqslant n-1$.

Next, the closedness of T implies

$$\frac{\partial}{\partial z_n} \Theta_{\imath\jmath} = \frac{\partial}{\partial \overline{z_n}} \Theta_{\imath\jmath} = 0 \quad \forall i, j \leqslant n-1.$$

Hence the $\Theta_{\imath\jmath}$ only depend on $z_1, \ldots, z_{n-1}$. One finally gets

$$T = q^* S = i \sum_{i,j \leqslant n-1} \Theta_{ij} dz_i \wedge d\overline{z_j}$$

and S is a closed positive $(1,1)-$ current on Δ^{n-1}. □

Proposition 2.4 has an easy

Corollary 2.6. — *Let X be a complex manifold and Θ a positive closed $(1,1)-$ current. Let $Y \subset X$ be an irreducible analytic subset such that Θ may be restricted to Y_{reg}, the smooth part of Y. Then Y is numerically trivial w.r.t. Θ iff for all holomorphic maps $f : \Delta \to Y \subset X$ such that $f^*\Theta$ exists the complex manifold Δ is numerically trivial w.r.t. $f^*\Theta$.*

As a consequence one can give an alternative definition of numerically trivial irreducible analytic subsets using embedded resolutions of singularities by blowing up smooth centers. Such resolutions exist for arbitrary complex manifolds, at least on relatively compact open subsets ([Hir64],[BM97]).

PROPOSITION 2.7. — *Let X be a complex manifold and Θ a positive closed $(1,1)-$ current. Let $Y \subset X$ be an irreducible analytic subset such that Θ may be restricted to Y_{reg}, the smooth part of Y. Then Y is numerically trivial w.r.t. Θ iff for an embedded resolution $f : \widetilde{Y} \to Y$ the complex manifold $\widetilde{Y}$ is numerically trivial w.r.t. the pulled back current $f^*\Theta$.*

Proof. — By the universal property of the blowup a map $f : \Delta \to Y$ will factorize through the blow up $\pi : \widetilde{Y} \to Y$ of a smooth center if its image is not contained in the center. Furthermore the exceptional divisor is a projectivized bundle hence locally trivial. So at least locally there will be a map $\widetilde{f} : \Delta \to \widetilde{Y}$ such that $\pi \circ \widetilde{f} = f$ if the image is contained in the center. □

Finally there is a useful criterion for numerical triviality:

PROPOSITION 2.8. — *Let X be a complex manifold and let Θ be an almost positive $(1,1)-$ current. Then X is numerically trivial w.r.t. Θ iff there is an analytic subset $A \subset X$ such that $X - A$ is numerically trivial w.r.t. Θ.*

Proof. — This is a direct consequence of the following standard arguments. First a closed $(1,1)-$current is 0 iff it is already 0 outside a set of real codimension 4. [Dem00, (1.21)]. Second, for complete pluripolar sets E (as are analytic subsets) $\Theta = 1_{X-E}\Theta + 1_E\Theta$: This is true for the closed positive current $\Theta + C\omega$ by [Dem00, (1.19)], hence also for Θ. But for E a codimension 1 analytic subset $1_E\Theta = m_E[E]$ where m_E is the generic Lelong number on E [Dem00, (2.17)]. □

2.2. Existence of maximal numerically trivial foliations.

Consider singular foliations as described in the Appendix:

DEFINITION 2.9. — *Let X be an $n-$ dimensional compact complex manifold with hermitian metric ω and $\Theta \geqslant 0$ a positive closed $(1,1)-$ current on X. A singular foliation $\{\mathcal{F}, (U_i, p_i)\}$ is said to induce a (singular) numerically trivial foliation w.r.t. Θ iff almost every fiber of p_i is numerically trivial w.r.t. Θ.*

Note that the condition about the fibers of the p_i is much stronger than in the case of numerical trivial fibrations: Here it was only necessary to assume that the union of numerically trivial fibers is not a Lebesgue zero set, and the numerical triviality of all fibers over points lying in the complement of a pluripolar set followed. In the foliation case there exist counter examples to this conclusion: On Δ^2 consider the plurisubharmonic function

$$\phi(z_1, z_2) = \max(\log(1 + |z_1 z_2|^2) - \log\frac{5}{4}, 0).$$

For every z_2- fiber F with $|z_2| < \frac{1}{2}$ the restriction $\phi_{|F}$ is $\equiv 0$. But for $|z_1 z_2| > \frac{1}{2}$ one sees that $\phi \equiv \log(1 + |z_1 z_2|^2) - \log\frac{5}{4}$.

Theorem 1.2 states the existence of a maximal numerically trivial foliation with respect to the inclusion relation "$\sqsubset$" of singular foliations, see the Appendix. The strategy to prove the existence of this maximal foliation is essentially the same as for the existence proof of numerical trivial fibrations: one proves that the common refinement $\{\mathcal{H}, (W_k, r_k : W_k \to \Delta^{n-m})\}$ of two numerically trivial foliations $\{\mathcal{F}, (U_i, p_i : U_i \to \Delta^{n-k})\}$, $\{\mathcal{G}, (V_j, q_j : V_j \to \Delta^{n-l})\}$, (see the Appendix) is again a numerically trivial foliation.

The main step is to establish a local analog to the Key Lemma in [Eck02]. It is stated for the following configuration: Let $W \cong \Delta^n$ be a complex manifold with two projections $p_1 : W \to \Delta^{n-k}$, $p_2 : W \to \Delta^{n-l}$ such that a smallest projection $p : W \to \Delta^{n-m}$ as constructed in the Appendix exists.

Local key lemma 2.10. — *If the fibrations induced by p_1 and p_2 are numerically trivial w.r.t. a positive closed $(1,1)-$current Θ on W, then the foliation induced by p will also be numerically trivial w.r.t. Θ.*

Since any two points on Δ^{n-k} may be connected by a sequence of images of p_2-fibers this is a consequence of

Lemma2.11. — *Let Θ be a positive closed $(1,1)-$current on Δ^n, let $q : \Delta^n \to \Delta^k$ be the projection onto the last k factors and let*

$$V = \{x \in \Delta^n | x_1 = \ldots = x_l = 0\} \subset \Delta^n$$

be an analytic subset mapping surjectively on Δ^k (that is, $l < k$). If almost every $q-$ fiber and V are numerically trivial w.r.t. Θ then Δ^n will be numerically trivial w.r.t. Θ.

Proof. — Let $\Theta = \sum_i a_i[D_i] + R$ be the Siu decomposition. Lemma 2.5 shows that $R = q^*S$ for some closed positive $(1,1)-$current S on Δ^k. Now, at least locally each map $f : \Delta \to \Delta^k$ is liftable to a map $\widetilde{f} : \Delta \to V$, that is $f = q \circ \widetilde{f}$ (this is obvious for projections). Hence S is numerically trivial by the criterion in Proposition 2.8. But divisorial components in the Siu-decomposition of S would give divisorial components of the Lelong number level sets of $R = q^*S$. Therefore $S \equiv 0$ hence $R \equiv 0$. □

Note again that the Local Key Lemma needs stronger assumptions on the fibers than the Key Lemma. This is shown by the same counter example as above: The horizontal sections $\{z_2 = a\}$ are also numerically trivial as long as $|a| < \frac{1}{2}$.

Now it is an easy consequence of the Local Key Lemma to show that common refinement $\{\mathcal{H}, (W_k, r_k : W_k \to \Delta^{n-m})\}$ of two numerically trivial foliations $\{\mathcal{F}, (U_i, p_i : U_i \to \Delta^{n-k})\}$, $\{\mathcal{G}, (V_j, q_j : V_j \to \Delta^{n-l})\}$ is again numerically trivial.

This ends the proof of Theorem 1.2.

2.3. Tsuji's numerically trivial fibrations.

Now let X be a smooth projective complex manifold and L a pseudoeffective holomorphic line bundle on X with positive singular hermitian metric h. As already mentioned in the beginning, the notion of numerical triviality used to construct Tsuji's numerically trivial fibrations is derived from an intersection number $(L,h).C$ of an irreducible curve $C \subset X$ not contained in the singular locus of h with the pair (L,h). A subvariety $Y \subset X$ is numerically trivial iff $(L,h).C = 0$ for all irreducible curves $C \subset X$. The analysis of these intersection numbers in [Eck02] shows that

$$(L,h).C = (\pi^*L, \pi^*h).\overline{C} = \pi^*L.\overline{C} - \sum_{x \in \overline{C}} \nu(\pi^*h, x),$$

where $\pi : \overline{C} \to C$ is the normalization. In particular, if $(L,h).C = 0$, the curvature current of π^*h on $\overline{C}$ may be written as $\sum_{x \in \overline{C}} \nu(\pi^*h, x)[x]$. Hence proposition 2.4 shows that numerically trivial subvarieties in the sense of definition 2.2 are also numerically trivial in the sense just described.

The converse is also true: By the birational invariance of numerical triviality [Eck02, 2.6] the normalization and desingularization $\overline{Y}$ of a numerically trivial subvariety Y is also numerically trivial (in Tsuji's sense).

Hence the curvature current of the pulled back metric is of the form $\sum \nu_i[D_i]$, by Theorem 1.1. But this implies certainly numerical triviality of $\overline{Y}$ in the sense of Definition 2.2, and since every holomorphic map $f : \Delta \to Y$ may be lifted to a holomorphic map $f : \Delta \to \overline{Y}$, the numerical triviality of Y follows.

Now remind the construction of Tsuji's numerically trivial fibration: It is the (up to birational equivalence unique) element with maximal fiber dimension in the set of families $\widetilde{f} : \mathfrak{X} \to \mathcal{N}$ with the following properties ([Eck02, 3.3]):

(i) $\mathfrak{X} \subset X \times \mathcal{N}$, $\mathfrak{X}, \mathcal{N}$ quasi-projective, irreducible, general fibres are subvarieties of X;

(ii) the projection $p : \mathfrak{X} \to X$ is generically finite;

(iii) (L, h) is defined and not numerically trivial on sufficiently general fibres of $\widetilde{f}$, i.e. on a set of fibres $\mathcal{M} \subset \mathcal{N}$ which has not Lebesgue measure 0;

(iv) the fibres are generically unique, i.e. if $U \subset \mathcal{N}$ is an open subset such that $\widetilde{f}_{|U}$ is flat then the induced map $U \to \mathrm{Hilb}(X)$ will be generically bijective.

It is shown that for the maximal element, the projection $p : \mathfrak{X} \to X$ is really birational, and that all fibers where $h_{|F} \not\equiv \infty$ are numerically trivial. But by the observation above, such a fibration can be interpreted as a numerically trivial foliation $\{\mathcal{F}, (U_i, p_i)\}$: Take $\mathcal{F}$ as $p_* T_{\mathfrak{X}/\mathcal{N}}$, and let $Z \subset X$ be an algebraic subset of points where p is an isomorphism and $\widetilde{f}$ is smooth. Then $X - Z$ may be covered by (analytically) open sets U_i such that there exist maps $p_i : U_i \to \Delta^{n-k}$ with $T_{U_i/\Delta^{n-k}} = \mathcal{F}_{|U_i}$. This implies that the p_i- fibers are numerically trivial. Consequently, it is possible to characterize Tsuji's numerically trivial fibration in the following way:

Proposition 2.12. — *Let X be a smooth projective complex manifold and L a pseudoeffective holomorphic line bundle on X with positive singular hermitian metric h. Then the birational fibration with maximal fiber dimension contained in the numerically trivial foliation w.r.t. the curvature current Θ_h is Tsuji's numerically trivial fibration w.r.t. (L, h).*

2.4. The Iitaka fibration

Let X be a projective complex manifold and L a line bundle with non-negative Kodaira-Iitaka dimension $\kappa(X,L) \geqslant 0$. Consider the set $N(L)$ of all $m \in \mathbb{N}$ such that the linear systems $|mL| \neq \emptyset$. Let m_0 be the greatest common divisor of the numbers in $N(L)$. Then there is a positive integer $m(L)$ such that $|mm_0L| \neq \emptyset$ for all positive integers $m \geqslant m(L)$. Choose generating sets $f_1, \ldots, f_{k_m}$ for the linear systems $|mm_0L| \neq \emptyset$ and let h_m be the (possibly singular) hermitian metric on L with plurisubharmonic weight (on the base $\Omega \subset \mathbb{C}^n$ of a local trivialization $L \cong \Omega \times \mathbb{C}$)

$$\phi_m = \frac{1}{2mm_0} \log \left(\sum_{i=1}^{k_m} |f_i|^2 \right)$$

and curvature current $\Theta_m = i\partial\overline{\partial}\phi_m$ (on Ω). Let h_L be a smooth hermitian metric on L with weight ϕ_L on Ω and smooth curvature form Θ_L. Write $\Theta_m = \Theta_L + i\partial\overline{\partial}\phi'_m$ and normalize the ϕ'_m by subtracting (if necessary) a positive constant C_m such that $\sup \phi'_m \leqslant 0$ (this is possible because ϕ'_m is defined on the compact manifold X hence bounded from above). Then take the upper semicontinuous upper envelope ϕ' of the ϕ'_m and call h the (singular) hermitian metric on L given by the plurisubharmonic weight $\phi = \phi_L + \phi'$.

It is useful to construct the ϕ_m in such a way that ϕ' has the singularities exactly at the stable base locus

$$\mathrm{SBs}(L) := \bigcap_{m \in \mathbb{N}} \mathrm{Bs}(|mL|)$$

of L. This is possible by defining ϕ_{m+1} from ϕ_m as follows: multiply the generators of $|mm_0L|$ by a section in $|m_0L|$ and complete this set to a generating set of $|(m+1)m_0L|$. By multiplying the completing sections with small positive constants one can reach around points $x \in \mathrm{Bs}(|(m+1)m_0L|)$ that

$$\phi_{m+1} \leqslant (1 - \epsilon_m)\phi_m$$

for arbitrarily small ϵ. Hence given a positive integer M, for appropriately chosen ϵ_m, there is a constant $C_x > 0$ such that $\sup \phi_m \leqslant C_x\phi_M$. This implies that ϕ' has also a singularity in x.

The aim is to prove that the Iitaka fibration is (up to birational equivalence) the same as the numerically trivial foliation with respect to the current $i\partial\overline{\partial}\phi$. Since the Iitaka fibration is a fibration this implies in particular that in this case the numerically trivial foliation is the same as Tsuji's numerically trivial fibration.

To prove these assertions, first compare Tsuji's and Takayama's intersection numbers:

LEMMA 2.13. — *With L, h as above,*

$$(L,h).C \leqslant ||L,C||$$

for smooth irreducible curves C not contained in a Lebesgue zero set.

Proof. — To begin with, one has to relate the multiplier ideals $\mathcal{J}(c\cdot|mm_0L|)$ of the linear system $|mm_0L|$ and the positive rational number c with the (analytic) multiplier ideals $\mathcal{J}(\phi_m)$. The ideal $\mathcal{J}(c\cdot|mm_0L|)$ is defined via a log resolution, but since ϕ_m is a plurisubharmonic function with analytic singularities defined by generating elements of $|mm_0L|$, it follows that

$$\mathcal{J}(c\cdot mm_0\phi_m) = \mathcal{J}(c\cdot|mm_0L|)$$

by [Dem00, (5.9)].

As already mentioned in the introduction,

$$||L,C|| := \lim_{m\to\infty} m^{-1}\deg_C mL\otimes\mathcal{J}(||mL||)$$

is defined by using the asymptotic multiplier ideal $\mathcal{J}(||mL||)$. This ideal is defined to be the unique maximal element among all multiplier ideals $\mathcal{J}(\frac{1}{pm_0}\cdot|pm_0mL|)$ ([Kaw99],[Laz00]). Consequently,

$$\begin{aligned}
(2.1)\qquad ||L,C|| &= L.C + \lim_{m\to\infty} m^{-1}\max_{p\in\mathbb{N}}\deg_C\mathcal{J}(\frac{1}{m_0p}|m_0pmL|)\\
&= L.C + \lim_{m\to\infty} m^{-1}\lim_{p\to\infty}\deg_C\mathcal{J}(m\phi_{mp})\\
&= L.C + \lim_{m\to\infty} m^{-1}\lim_{n\to\infty}\deg_C\mathcal{J}(m\phi_n).
\end{aligned}$$

The last equality is true because $\mathcal{J}(m\phi_n)\subset\mathcal{J}(m\phi_{n+1})$ for all n: The multiplier ideals do not depend on the generating set used to define ϕ_n. By multiplying the generators defining ϕ_n with a section in $H^0(X,m_0L)$ and completing this set to a generating set of $H^0(X,(n+1)m_0L)$ it is possible to choose $\phi_n\leqslant\phi_{n+1}$ (as above), hence the inclusion.

Next, Tsuji's intersection number may be expressed as

$$(L, h_n).C = L.C + \limsup_{m\to\infty} m^{-1} \deg_C \mathcal{J}(m\phi_n),$$

by [Eck02, (2.3.1)] and the fact that h_n is a metric with analytic singularities, hence restriction to C and taking the multiplier ideal in the lim sup above may be interchanged on smooth curves ([Eck02, Prop. 2.11]). An easy analysis shows that

$$\begin{aligned}\lim_{n\to\infty}(L, h_n).C &= L.C + \lim_{n\to\infty}\limsup_{m\to\infty} m^{-1} \deg_C \mathcal{J}(m\phi_n)\\ &\leqslant L.C + \lim_{m\to\infty} m^{-1} \lim_{n\to\infty} \deg_C \mathcal{J}(m\phi_n) = ||L;C||.\end{aligned}$$

On the other hand, $(L, h_n).C = L.C - \sum_{x\in C}\nu(h_{n|C}, x)$ by [Eck02, Prop. 1.2]. Since the upper semicontinuous upper envelope ϕ' of the ϕ'_m equals $\sup_m \phi'_m$ outside a set of Lebesgue measure zero ([Lel68]), the envelope of the restrictions $\phi'_{m|C}$ equals almost everywhere the restriction $(\phi'_m)_{|C}$ on all curves outside a Lebesgue zero set. For these curves the lemma follows from the next statement, using the definition of Lelong numbers via integrals ([Dem00, (2.7)]). □

Lemma 2.14. — *Let $C \subset X$ be a smooth curve not contained in $\{x \in X : \sup_m \phi'_m(x) < \phi'(x)\}$. Then for all $x \in C$*

$$\lim_{n\to\infty} \nu(h_{n|C}, x) \xrightarrow{\geqslant} \nu(h_{|C}, x).$$

Proof. — By definition of Lelong numbers, $\nu(\phi, x) \geqslant \nu(\psi, x)$ if $\phi \leqslant \psi$. Consequently, by the same construction as for the inclusion $\mathcal{J}(m\phi_n) \subset \mathcal{J}(m\phi_{n+1})$, the Lelong numbers $\nu(h_{n|C}, x)$ of the ϕ_n form a decreasing sequence of non-negative numbers in every point $x \in C$ whose limit is $\geqslant \nu(h_{|C}, x)$. It remains to show the equality:

If z is a local parameter of C centered in x, the restriction ϕ'_n may locally be written as

$$\begin{aligned}\phi'_n(z) &= \phi_n(z) - \phi_L(z) - C_n\\ &= \nu(h_{n|C}, 0)\log|z| + d_n \log(1 + \sum_{i=0}^{\infty} a_i|z|^i) - \phi_L(z) - C_n.\end{aligned}$$

For every $\epsilon > 0$ and a sufficiently small neighborhood of 0 it is true that

$$d_n \log(1 + \sum_{i=0}^{\infty} a_i |z|^i) - \phi_L(z) - C_n \leqslant -\epsilon \log |z|,$$

hence $\phi'_n(z) \leqslant (\nu(h_{n|C}, 0) - \epsilon) \log |z|$, which implies

$$\phi'(z) \leqslant (\lim_{n\to\infty} \nu(h_{n|C}, 0) - \epsilon) \log |z|$$

for almost all z around 0. Consequently, $\nu(\phi', 0) \geqslant \lim_{n\to\infty} \nu(h_{n|C}, 0) - \epsilon$ for all $\epsilon > 0$, and the equality follows. □

This already implies that the Iitaka fibration is contained in Tsuji's numerically trivial fibration: Take a birational morphism $\mu : X' \to X$ from a smooth projective variety X' such that the Iitaka fibration induced by the linear system $|m\mu^* L|$ is a morphism $f : X \to Y$ on another smooth variety Y. The general fiber of this fibration is smooth. Smooth varieties are numerically trivial w.r.t. some pair (L, h) iff $(L, h).C = 0$ for all sufficiently general smooth curves in this variety ([Eck02, 3.1]). Hence by the above inequality the numerically trivial fibration w.r.t. $(\mu^* L, \mu^* h)$ contains the Iitaka fibration. By birational equivalence of intersection numbers ([Eck02, 2.6]), the numerically trivial fibration w.r.t. $(\mu^* L, \mu^* h)$ is birationally equivalent to that on X w.r.t. (L, h).

Next note that there is a positive integer m such that the Iitaka fibration of L is induced by the linear system $|mL|$ [Ii82, 10.3].

LEMMA 2.15. — *Let L be a holomorphic line bundle on a projective complex manifold X such that $|mL|$ is a non-empty linear system which induces a rational map $\phi_{|mL|} : X \longrightarrow Y$. Then $\phi_{|mL|}$ is the numerically trivial foliation w.r.t. $h_{|mL|}$.*

Proof. — By corollary 2.6 it is enough to show that for every holomorphic map $f : \Delta \to X$ such that Δ is not mapped to a point and does not intersect the base locus of $|mL|$, the unit disk Δ is *not* numerically trivial w.r.t. $f^* h_{|mL|}$. But when $|mL|$ has no base points in the image of Δ, the metric $f^* h_{|mL|}$ is a smooth metric with smooth positive curvature form different from 0. □

LEMMA 2.16. — *For X, L and h as above, let $m > 0$ be an integer such that $|mL|$ is a non-empty linear system, and $f : \Delta \to X$ a holomorphic* *map such that* $f^*h \neq \infty$, $f^*h_{|mL|} \neq \infty$. *Then*

$$\Delta \text{ numerically trivial w.r.t. } f^*h \Longrightarrow \Delta \text{ numerically trivial w.r.t. } f^*h_{|mL|}.$$

Proof. — This is a trivial consequence of the Lelong number inequality $\nu(f^*h, x) \leqslant \nu(f^*h_m, x)$, see above. □

The last lemma implies that the numerically trivial foliation w.r.t. h is contained in the numerically trivial foliation w.r.t. $h_{|mL|}$, and the lemma before shows that this foliation is the Iitaka fibration which in turn is contained in Tsuji's numerically trivial fibration by the arguments above.

Remark. — This also shows that the Iitaka fibration is the numerically trivial foliation w.r.t. $h_{|mL|}$ for an appropriate m.

3. Bounds for the numerical dimension.

In this section the ideas of Boucksom and Demailly are used to construct a numerically trivial foliation for pseudo-effective $(1,1)-$ classes on compact Kähler manifolds. The next paragraph tries to collect the scattered and mostly unpublished definitions and properties of volumes and moving intersection products of pseudo-effective classes without claiming any originality or completeness and mostly without proof (in many cases they may be found in [Bou02a]). The main result about the numerically trivial foliations will be that the codimension of their leaves determines an upper bound for the numerical dimension of the pseudo-effective class, if the singularities of the foliation are isolated points.

3.1. Moving intersection numbers of pseudo-effective classes.

Starting with Fujita's approximate Zariski decomposition ([Fuj94], [DEL00]) Boucksom developped a notion of volume for arbitrary pseudo-effective classes ([Bou01]) on compact Kähler manifolds. This was generalized (with small modifications) by Demailly to a "moving intersection product" of pseudo-effective classes ([Dem02]). This in turn allows the definition of a numerical dimension for pseudo-effective classes.

Logically one has to start with defining the "moving intersection numbers":

DEFINITION 3.1. — *Let X be a compact Kähler manifold with Kähler form ω. Let $\alpha_1, \dots, \alpha_p \in H^{1,1}(X, \mathbb{R})$ be pseudo-effective classes and let Θ be a closed positive current of bidimension (p,p). Then the moving intersection number $(\alpha_1 \cdot \ldots \cdot \alpha_p \cdot \Theta)_{\geqslant 0}$ of the α_i and Θ is defined to be the limit when $\epsilon > 0$ goes to 0 of*

$$\sup \int_{X-F} (T_1 + \epsilon\omega) \wedge \ldots \wedge (T_p + \epsilon\omega) \wedge \Theta$$

where the T_i's run through all currents with analytic singularities in $\alpha_i[-\epsilon\omega]$, and F is the union of the $Sing(T_i)$.

It is not difficult to justify the existence of the limit above: First, on $X-F$ the currents $T_i+\epsilon\omega$ may locally be written as $T_i+\epsilon\omega = dd^c u_i$ for some bounded plurisubharmonic function u_i. By results of Bedford-Taylor [BT76] this implies the existence of the integral. In addition Boucksom [Bou01] showed that these integrals are bounded by a constant only depending on the cohomological classes $\{T_i\}$ and $\{\Theta\}$ (this is where the Kähler assumption comes in). Hence the supremum always exists, and is increasing with increasing ϵ. This implies the existence of the limit. Finally it is easy to see that this limit does not depend on the choice of the Kähler form ω.

The $(\alpha_1 \cdot \ldots \cdot \alpha_p \cdot \Theta)_{\geqslant 0}$ are symmetric in the α_i and concave and homogeneous in every variable separately. For nef classes $\alpha_i \in H^{1,1}(X, \mathbb{R})$ the moving intersection number equals the normal cohomological intersection number $(\alpha_1 \cdot \ldots \cdot \alpha_p \cdot \{\Theta\})$ [Bou02a]. If some of the pseudo-effective classes coincide one has

LEMMA 3.2. — *For pseudo-effective classes $\alpha, \alpha_{p+1}, \dots, \alpha_n$ the moving intersection number $(\alpha^p \cdot \alpha_{p+1} \cdot \ldots \cdot \alpha_n)_{\geqslant 0}$ is the limit for $\epsilon \to 0$ of*

$$\sup \int_{X-F} (T + \epsilon\omega)^p \wedge (T_{p+1} + \epsilon\omega) \wedge \ldots \wedge (T_n + \epsilon\omega)$$

where $T \in \alpha[-\epsilon\omega]$ and $T_i \in \alpha_i[-\epsilon\omega]$ have analytic singularities.

Proof. — See Lemma 3.2.7 in [Bou02a]. □

DEFINITION 3.3. — *Let X be a compact Kähler manifold. Then the numerical dimension $\nu(\alpha)$ of a pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$ is defined as*

$$\max\{k \in \{0, \dots, n\} : (\alpha^k \cdot \omega^{n-k})_{\geqslant 0} > 0\}$$

for some (and hence all) Kähler classes ω.

Now the volume of a pseudo-effective class $\alpha \in H^{1,1}(X,\mathbb{R})$ on a compact Kähler manifold may be defined as a special case of the moving intersection product:

$$\mathrm{vol}(\alpha) = (\alpha^n)_{\geqslant 0}.$$

But there are other useful possibilities to define it: First remember that Fujita considered projective $n-$ dimensional algebraic varieties X and line bundles L over X, and defined the volume of L by

$$\mathrm{vol}(L) := \limsup_{k\to+\infty} \frac{n!}{k^n} h^0(X, kL).$$

If L is nef the volume of L is the self-intersection L^n, by Riemann-Roch and $h^q(X,kL) \sim O(k^{n-q})$ ([Dem00, (6.7)]). For arbitrary pseudo-effective classes $\alpha \in H^{1,1}(X,\mathbb{R})$ on compact Kähler manifolds X Boucksom generalized this volume by defining

$$\mathrm{vol}(\alpha) = \sup \int_X T_{ac}^n$$

where the supremum is taken over all closed positive $(1,1)-$ currents T with $\{T\} = \alpha$ and T_{ac} is the absolute continuous part of the Lebesgue decomposition $T = T_{ac} + T_{sg}$. Again, the Kähler assumption is necessary to guarantee that T_{ac}^n is locally integrable. By using singular Morse inequalities and the Calabi-Yau theorem Boucksom proved that $\mathrm{vol}(L) = \mathrm{vol}(c_1(L))$ and that $\mathrm{vol}(L) > 0$ iff L is a big line bundle, i.e. iff there is a closed strictly positive current representing $c_1(L)$.

Note that it is not necessary to look at all closed positive $(1,1)-$currents for taking the supremum. This is a consequence of an approximation theorem of Demailly:

Theorem 3.4 ([Dem92]). — *Let $T = \theta + dd^c\phi$ be a closed almost positive $(1,1)-$ current on a complex manifold X with hermitian metric ω such that θ is a smooth form. Suppose that $T \geqslant \gamma$ for some real $\mathcal{C}^\infty-$ form γ. Then there exists a decreasing sequence ϕ_k of almost plurisubharmonic functions with analytic singularities such that the $T_k := \theta + dd^c\phi_k$ verify*

(i) The T_k converge pointwise and L^1_{loc} against ϕ, hence the T_k converge weakly against T.

(ii) $T_k \geqslant \gamma - \epsilon_k\omega$ for some sequence of positive numbers $\epsilon_k \to 0$.

(iii) The Lelong numbers $\nu(T_k, x)$ converge uniformly against $\nu(T, x)$ w.r.t. $x \in X$.

Using another approximation theorem ([Dem82]) Boucksom slightly modified this statement ([Bou01]):

THEOREM 3.5. — *Let the assumptions and notations be the same as in the theorem before. Then there exists a decreasing sequence ϕ_k of plurisubharmonic functions with analytic singularities such that the $T_k := \theta + dd^c\phi_k$ verify*

(i) The T_k converge weakly against T, and $T_{k,ac} \to T_{ac}$ almost everywhere.

(ii) $T_k \geqslant \gamma - \epsilon_k\omega$ for some sequence of positive numbers $\epsilon_k \to 0$.

(iii) The Lelong numbers $\nu(T_k, x)$ converge uniformly against $\nu(T, x)$ w.r.t. $x \in X$.

So one may define instead

$$\mathrm{vol}(\alpha) = \lim_{\epsilon \to 0^+} \sup \int_X T_{ac}^n$$

where the T's run through all closed $(1,1)-$ currents with analytic singularities in $\alpha[-\epsilon\omega]$, that is $\{T\} = \alpha$ and $T \geqslant -\epsilon\omega$ for some hermitian metric ω on X.

Here, closed $(1,1)-$ currents with analytic singularities are currents whose almost plurisubharmonic potentials locally look like

$$\frac{\alpha}{2}\log(|f_1|^2 + \ldots + |f_p|^2)$$

with $f_1, \ldots, f_n$ holomorphic, up to a bounded $\mathcal{C}^\infty-$function. Such currents T are particularly useful because their absolut continuous part is the same as the residual part R in the Siu-decomposition $T = \sum_i a_i[D_i] + R$. Consequently, one may compute $\int_X T_{ac}^n$ by blowing up the (integral closure) of the ideal of singularities locally generated by the f_i and integrating the smooth form given by the pull back of T minus the integration currents of the exceptional divisors as they occur in the inverse image of the singularity ideal. In Fujita's setting this corresponds to blowing up the base locus of the multiples mL and decomposing the pull back of L into an effective part E_m and a free part D_m, and Fujita's theorem [Dem00, (14.6)] tells us that

$$\mathrm{vol}(L) = \lim_{m \to \infty} D_m^n.$$

Finally, the last definition of $\mathrm{vol}(\alpha)$ is equivalent to the first one, with moving intersection numbers, by Lemma 3.2.

3.2. Numerical triviality for pseudo-effective classes

First repeat and codify the informal definitions of numerical triviality and numerically trivial foliations w.r.t. a pseudo-effective class from the introduction:

Definition 3.6. — *Let X be a compact Kähler manifold with Kähler form ω and pseudo-effective class $\alpha \in H^{1,1}(X,\mathbb{R})$. A submanifold $Y \subset X$ (closed or not) is numerically trivial w.r.t. α iff for every immersed disk $\Delta \subset Y$,*

$$\limsup_{\epsilon \downarrow 0} \sup_{T} \int_{\Delta' - \mathrm{Sing}\ T} (T + \epsilon\omega) = 0,$$

where the T's run through all currents with analytic singularities in $\alpha[-\epsilon\omega]$ and $\Delta' = \{t : |t| < 1-\delta\}$ is any smaller disk contained in $\Delta = \{t : |t| < 1\}$.

As a convention set $\int_{\Delta - \mathrm{Sing}\ T}(T + \epsilon\omega) = 0$ if $\Delta - \mathrm{Sing}\ T = \emptyset$. Furthermore note that the restriction to disks Δ' may be replaced by the assumption that it is possible to continue the immersion $\Delta \subset Y$ holomorphically.

Definition 3.7. — *Let X be a compact Kähler manifold with a pseudo-effective class $\alpha \in H^{1,1}(X,\mathbb{R})$. A foliation $\{\mathcal{F}, (U_\imath, p_\imath)\}$ is numerically trivial w.r.t. α iff*

(i) every fiber of $p_\imath$ is numerically trivial w.r.t. α,

(ii) and if $\Delta^2 \hookrightarrow U_i$ is an immersion such that the projection onto the first coordinate coincides with the projection $p_i : U_i \to \Delta^{n-k}$, then for any $\Delta' \subset\subset \Delta$ and any sequence of currents $T_k \in \alpha[-\epsilon_k\omega]$, $\epsilon_k \to 0$, the integrals $\int_{(\{z_1=a\}\cap\Delta') - \mathrm{Sing}\ T_k}(T_k + \epsilon_k\omega)$ are uniformly (in a) bounded from above.

Note that no exceptional fibers are allowed: if the fibers are completely contained in the common singularity locus of the $T \in \alpha[-\epsilon\omega]$, then they are numerically trivial by the convention above, otherwise the limit in Definition 3.6 is supposed to be 0. The uniform boundedness is essential for the proof of the Local Key Lemma below.

To construct a maximal numerical trivial foliation w.r.t. this notion, it is enough to prove an analog for the Local Key Lemma 2.10:

LEMMA 3.8 (Local Key Lemma for pseudo-effective classes). — *Let X be a compact Kähler manifold with a pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$. Let $W \cong \Delta^n$ be an open subset of X with a projection $p : W \to \Delta^k$ onto the last k factors, and let $V = \{z_1 = \ldots = z_{n-k} = 0\}$ be a complex submanifold of W. If every fiber of p and also Y are numerically trivial w.r.t. α, then W will also be numerically trivial w.r.t. α.*

Then the maximal numerically trivial foliation w.r.t. α may be constructed in the same way as in Section 2.2.

The proof of this Local Key Lemma for pseudo-effective classes imitates the proof of the Local Key Lemma for closed positive $(1,1)-$currents: There, the numerical triviality of the fibers of the projection implies that the residue current of the Siu decomposition is a pull back of a current on the base (see the Pullback Lemma 2.5). Of course, the Pullback Lemma is not true for pseudo-effective classes. But it is enough to prove that the restriction onto different horizontal sections are quite the same, hence the numerical triviality of V implies the numerical triviality of all horizontal sections, hence that of W. This argument is made exact by

PROPOSITION 3.9. — *Let X be a compact Kähler manifold with Kähler form ω, and let $T_k = T'_k + \epsilon_k\omega$, $\epsilon_k \to 0$, be a sequence of closed positive $(1,1)$-currents on X such that the T'_k represent the same cohomology class. Let $\Delta^2 \hookrightarrow X$ be an immersion (with coordinates z_1, z_2). Let $\Delta' \subset\subset \Delta$ be a disk, and consider the functions $f_k : \Delta' \to \mathbb{R}^+, a \mapsto \int_{(\{z_1=a\}\cap\Delta')-\mathrm{Sing}\ T_k} T_k$ and $g_k : \Delta' \to \mathbb{R}^+, b \mapsto \int_{(\{z_2=b\}\cap\Delta')-\mathrm{Sing}\ T_k} T_k$. Suppose that $\lim_{k\to\infty} f_k(a) = 0$ for all $a \in \Delta$, and that the f_k are uniformly (in a) bounded from above. Suppose furthermore that $\lim_{k\to\infty} g_k(0) = 0$. Then $\lim_{k\to\infty} g_k(b) = 0$ for all $b \in \Delta'$, and the g_k are uniformly (in b) bounded from above.*

Proof. — Since the integrals are always evaluated outside the singularities of T_k, and since the mass of the integration current of a divisor is always concentrated in the divisor, one can assume without loss of generality that the Siu decomposition of T_k does not contain any integration currents of divisors. Consequently, T_k has only finitely many isolated singularities on any compact subset of $p_1^{-1}(\Delta')$ where $\Delta' \subset\subset \Delta$ is any disk and $p_1 : \Delta^2 \to \Delta$ is the projection onto the first coordinate, and T_k may be written on $p_1^{-1}(\Delta')$ as

$$T_k = \theta_{11}^k i dz_1 \wedge d\bar{z}_1 + \theta_{12}^k i dz_1 \wedge d\bar{z}_2 + \theta_{21}^k i dz_2 \wedge d\bar{z}_1 + \theta_{22}^k i dz_2 \wedge d\bar{z}_2,$$

where the θ_{ij}^k are smooth functions outside these singularities, and integrable on Δ^2. That T_k is a real current implies $\theta_{ij}^k = \overline{\theta_{ji}^k}$.

To prove the proposition it is enough to show that

$$\lim_{k\to\infty} \left| \int_{\Delta'_b - \mathrm{Sing}\ T_k} T_k - \int_{\Delta'_0 - \mathrm{Sing}\ T_k} T_k \right| = 0.$$

for a sequence of disks $\Delta' \subset\subset \Delta$ exhausting Δ (where $\Delta'_b = \{z_2 = b\} \cap \Delta'$). Now, choose a path $\gamma \in \Delta$ from 0 to b. Then,

$$\left| \int_{\Delta'_b - \mathrm{Sing}\ T_k} T_k - \int_{\Delta'_0 - \mathrm{Sing}\ T_k} T_k \right| = \left| \int_{\Delta'} (\theta_{11}^k(z_1, b) - \theta_{11}^k(z_1, 0)) i dz_1 \wedge d\bar{z}_1 \right|$$

equals (by Stokes and Fubini)

$$\left| \int_{\Delta'} \left(\int_\gamma d\theta_{11}^k \right) i dz_1 \wedge d\bar{z}_1 \right| = \left| \int_{\Delta' \times \gamma} d(\theta_{11}^k i dz_1 \wedge d\bar{z}_1) \right|.$$

Since the closedness of T implies

$$d(\theta_{11}^k i dz_1 \wedge d\bar{z}_1) = -d(\theta_{12}^k i dz_1 \wedge d\bar{z}_2 + \theta_{21}^k i dz_2 \wedge d\bar{z}_1 + \theta_{22}^k i dz_2 \wedge d\bar{z}_2),$$

this integral equals by Stokes

$$\left| \int_{\partial(\Delta' \times \gamma)} (\theta_{12}^k i dz_1 \wedge d\bar{z}_2 + \theta_{21}^k i dz_2 \wedge d\bar{z}_1 + \theta_{22}^k i dz_2 \wedge d\bar{z}_2) \right|,$$

and since z_2 is constant on $\Delta' \times \partial\gamma$, this simplifies to

$$\left| \int_{(\partial\Delta') \times \gamma} (\theta_{12}^k i dz_1 \wedge d\bar{z}_2 + \theta_{21}^k i dz_2 \wedge d\bar{z}_1 + \theta_{22}^k i dz_2 \wedge d\bar{z}_2) \right|.$$

Observe that these integrals do not depend on the chosen path γ. Consequently, cover the disk $\Delta_{0,b}$ with center in $b/2$ and radius $|b/2|$ with a family of paths γ_a from 0 to b. Then to prove $\lim_{k\to\infty} |\int_{\Delta'_b} T_k - \int_{\Delta'_0} T_k| = 0$ it is enough to show that

$$\lim_{k\to\infty} \int_a \left| \int_{(\partial\Delta')\times\gamma_a} (\theta^k_{12} i dz_1 \wedge d\bar{z}_2 + \theta^k_{21} i dz_2 \wedge d\bar{z}_1 + \theta^k_{22} i dz_2 \wedge d\bar{z}_2)) \right| da = 0.$$

The term with θ^k_{22} vanishes since $idz_2 \wedge d\bar{z}_2$ is pulled back to 0 in any chart of $(\partial\Delta')\times\gamma_a$. Since $\theta^k_{12} = \overline{\theta^k_{21}}$ the remaining integral may be bounded from above by

$$C \cdot \int_{\partial\Delta'\times\Delta_{0,b}} |\theta^k_{12}| dV,$$

where C is independent of b and k, and dV is a volume element on $\partial\Delta' \times \Delta_{0,b}$.

Now interpret T_k as a semipositive hermitian form $\langle .\ ,.\rangle$ on every tangent space $T_{X,x}$ (where T has no singularities). Then the Schwarz inequality implies that

$$|\theta^k_{12}| = |\langle \frac{\partial}{\partial\bar{z}_1}, \frac{\partial}{\partial\bar{z}_2}\rangle| \leqslant |\langle \frac{\partial}{\partial\bar{z}_1}, \frac{\partial}{\partial\bar{z}_1}\rangle|^{\frac{1}{2}} \cdot |\langle \frac{\partial}{\partial\bar{z}_2}, \frac{\partial}{\partial\bar{z}_2}\rangle|^{\frac{1}{2}} = |\theta^k_{11}|^{\frac{1}{2}} \cdot |\theta^k_{22}|^{\frac{1}{2}}.$$

Hence the integral above is $\leqslant$ the square root of the product

$$\int_{\partial\Delta'\times\Delta_{0,b}} |\theta^k_{11}| dV \cdot \int_{\partial\Delta'\times\Delta_{0,b}} |\theta^k_{22}| dV,$$

again by the Schwarz inequality.

Claim. — There exists a bound $M' > 0$ such that for all k there is a disk $\Delta'_k \subset\subset \Delta$ containing Δ' with

$$\int_{\partial\Delta'_k\times\Delta'} |\theta^k_{11}| dV < M'.$$

Proof. — Suppose that $\Delta' \subset\subset \Delta'' \subset\subset \Delta$, and look at the $(1,1)$-form $\eta = idz_2 \wedge d\bar{z}_2$. There exists a $C > 0$, such that $\eta \leqslant C\cdot\omega$ on $\Delta''\times\Delta'$. Hence,

$$\int_{(\Delta''-\Delta')\times\Delta'} |\theta^k_{11}| dV = \int_{(\Delta''-\Delta')\times\Delta'} (T'_k + \epsilon_k\omega)\wedge\eta \leqslant C\cdot \int_X (T'_k + \epsilon_k\omega)\wedge\omega,$$

and the last integral only depends on the cohomology class of T'_k (and ω). By Fubini one gets a disk Δ'_k as above. $\square$

For the second term note that the assumptions on the functions f_k imply $\lim_{k\to\infty}\int_{\Delta'} f_k i da\wedge d\bar{a} = 0$, by Lebesgue's dominated convergence, and the measure of the sets $\{a : f_k(a) > \delta\}$ tends to 0, too, for $k \to \infty$.

Hence, as above, for a given $\epsilon > 0$ it is possible to bound the measure of $\{a : f_k(a) > \delta\}$ small enough such that for all k big enough there is a disk $\Delta''_k \subset\subset \Delta$ containing Δ' with

$$\int_{\partial\Delta''_k\times\Delta'} |\theta^k_{22}|dV < \epsilon.$$

Choosing δ small enough and M' big enough (but both independent of k!) one can assume that the two disks Δ'_k and Δ''_k coincide (at least for k big enough). Since M' is independent of ϵ, the difference $\int_{\Delta_{k,b}} T_k - \int_{\Delta_{k,0}} T_k$ tends to 0 for $k \to \infty$, and uniformly in b. Since $\int_{\Delta_{k,0}} T_k \overset{k\to\infty}{\longrightarrow} 0$, this is also true for $\int_{\Delta'_b} T_k - \int_{\Delta'_0} T_k$. Consequently, $\lim_{k\to\infty} g_k(a) = 0$, and the uniformity in b implies the uniform boundedness of the g_k. □

Proof of the Local Key Lemma for pseudo-effective classes. — If Δ is a disk immersed in W such that p projects it on a point in Δ^k, there is nothing to prove.

If Δ is a disk immersed in W not intersecting Y which is projected biholomorphically onto Δ^k, then a coordinate change and further cutting down leads to the configuration described in the proposition. Note that it is sufficient to check on any disk $\Delta' \subset\subset \Delta$ that

$$\lim_{k\to\infty}\int_{\Delta'-\mathrm{Sing}T_k} T_k + \frac{1}{k}\omega = 0$$

for arbitrary sequences T_k of currents with analytic singularities in $\alpha[-\frac{1}{k}\omega]$. The assumptions of the Local Key Lemma imply that

$$\lim_{k\to\infty}\int_{\{z_1=a\}-\mathrm{Sing}T_k} T_k + \frac{1}{k}\omega = \lim_{k\to\infty} f_k(a) = 0$$

for all a and $\lim_{k\to\infty}\int_{\{z_2=0\}-\mathrm{Sing}T_k} T_k + \frac{1}{k}\omega = 0$. The definition of a numerically trivial foliation implies the uniform boundedness of the f_k, so it is possible to apply the proposition.

If Δ is a disk immersed in W not satisfying one of the two conditions above, then for any $\Delta' \subset\subset \Delta$ there are disks $\Delta''_i \subset\subset \Delta'_i \subset \Delta$ such that

$\bigcup \Delta_i'' \supset \Delta'$ (hence it is enough to consider finitely many of these disks), and there are projections $p_i : W \to \Delta^{n-k}$ (possibly different from p) such that the restriction onto Δ_i' is a submersion. Since the fibers and sections of these p_i are composed of disks already shown to be numerically trivial, it is possible to apply again the proposition on $\Delta_i'' \subset\subset \Delta_i'$ (by possibly further cutting down and a coordinate change). Since there are only finitely many i's, Δ' is also numerically trivial.

Finally, the uniform boundedness property of the foliation follows directly from the uniform boundedness shown in the proposition. □

3.3. The Iitaka fibration and the nef fibration.

A remarkable fact about the construction of a numerically trivial foliation w.r.t. a pseudo-effective class α is that it works also if one restricts to non-empty subsets of currents with analytic singularities in $\alpha[-\epsilon\omega]$.

LEMMA 3.10. — *Let X be a compact Kähler manifold with Kähler form ω and Θ a closed positive $(1,1)-$current representing the cohomology class $\alpha \in H^{1,1}(X,\mathbb{R})$. Then the foliation constructed w.r.t. the subsets $\{\Theta\} \subset \alpha[-\epsilon\omega]$ is the numerically trivial foliation w.r.t. Θ.*

Proof. — Comparing Definitions 2.2 and 3.6, and taking into account the Criterion 2.4 for numerical triviality one immediately gets that the numerically trivial foliation w.r.t. Θ is contained in that constructed w.r.t. the subsets $\{\Theta\} \subset \alpha[-\epsilon\omega]$. But the other inclusion is also not difficult to prove: Every holomorphic map $\Delta \to X$ maps Δ onto a $1-$ dimensional analytic subset, and the integrals in Definition 3.6 may be taken outside the singularities of this set. □

It is also clear that this foliation contains the numerically trivial foliation w.r.t. α. In particular:

PROPOSITION 3.11. — *Let X be a projective manifold and L a nef line bundle on X such that the Kodaira-Iitaka dimension $\kappa(L) \geqslant 0$. Then the nef foliation of L is contained in the Iitaka fibration.*

In analogy to Tsuji's numerically trivial fibration one can define the pseudo-effective fibration of a pseudo-effective line bundle L as the maximal fibration contained in the numerically trivial foliation w.r.t. $c_1(L)$.

Proposition 3.12. — *Let X be a projective manifold and L a nef line bundle on X. Then the nef fibration of [BCE+00] is equal to the pseudo-effective fibration.*

Proof. — This is just a consequence of the definitions: A curve C which is numerically trivial w.r.t. $c_1(L)$ satisfies $(L.C)_{\geqslant 0} = L.C = 0$, and vice versa. □

To summarize, all this gives a (sufficient) geometric reason that the fibers of the nef fibration are strictly contained in the fibers of the Iitaka fibration: this happens if the nef foliation is not a fibration. It would be interesting to decide if the converse is also true.

3.4. Currents with minimal singularities.

To state and to prove the results about upper bounds for the numerical dimension of a pseudo-effective class, a further notion is still missing: that of currents with minimal singularities.

Definition 3.13. — *Let ϕ_1 and ϕ_2 be two almost plurisubharmonic functions on a complex manifold X. Then ϕ_1 is said to be less singular than ϕ_2 in $x \in X$ iff*

$$\phi_2 \leqslant \phi_1 + O(1)$$

in a neighborhood of X. The fact that ϕ_1 is less singular than ϕ_2 in every point is denoted by $\phi_1 \preceq \phi_2$.

Now let X be compact Kähler and $\alpha \in H^{1,1}(X, \mathbb{R})$. Let θ be a smooth $(1,1)-$ form representing α. Then every current in α may be written as $T = \theta + dd^c\phi$ for some almost plurisubharmonic function ϕ and

$$T_1 \preceq T_2$$

shall denote the fact that $\phi_1 \preceq \phi_2$.

Proposition 3.14. — *Let γ be a smooth $(1,1)-$ form on X. Every non-empty subset of $\alpha[\gamma]$ admits a lower bound in $\alpha[\gamma]$ w.r.t. $\preceq$.*

Proof. — The proof is almost trivial and of course contained in [DPS01] but is repeated for emphasizing a certain uniqueness property.

Let $(T_i)_{i \in I}$ be the given subset of $\alpha[\gamma]$. Write $T_i = \theta + dd^c\phi_i$ where ϕ_i is almost plurisubharmonic and $dd^c\phi_i \geqslant \gamma - \theta$. Since X is compact, all almost plurisubharmonic functions are bounded from above hence one may suppose that $\phi_i \leqslant 0$ by subtracting a constant. If one choose this constant such that $\sup_{x \in X} \phi_i(x) = 0$ the ϕ_i will be unique: An almost plurisubharmonic function ϕ with $dd^c\phi = 0$ is a holomorphic function.

The ϕ_i have an almost plurisubharmonic upper envelope ϕ such that $\theta + dd^c\phi \in \alpha[\gamma]$. The current $T = \theta + dd^c\phi$ is obviously a lower bound for the $(T_i)_{i \in I}$, with the following property: If $S \preceq T_i$ for all I, then $S \preceq T$. □

Remark. — The construction above shows that this lower bound $T = T_{min}$ is unique only up to L^∞. On the other hand, given the smooth $(1,1)-$ form θ in α, the construction leads to a well defined current $T_{min} = \theta + dd^c\phi_{min}$ via the upper envelope. Here, the almost plurisubharmonic function ϕ_{min} satisfies $\phi_i \leqslant \phi_{min}$ where the ϕ_i are chosen as above.

This current will be used in the following.

The currents with minimal singularities may be used to define minimal multiplicities of pseudo-effective classes, having a look at Boucksom's construction of higher dimensional Zariski decompositions [Bou02b]. In this paper, he interpreted the Lelong numbers of a current $T_{min,\epsilon}$ with minimal singularities in $\alpha[-\epsilon\omega]$ as the obstructions to reach smooth currents in $\alpha[-\epsilon\omega]$. This led him to

DEFINITION 3.15. — *The minimal multiplicity of a pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$ in $x \in X$ is defined as*

$$\nu(\alpha, x) := \sup_{\epsilon > 0} \nu(T_{min,\epsilon}, x).$$

The generic minimal multiplicity on a prime divisor $D \subset X$ is defined as

$$\nu(\alpha, D) := \inf_{x \in D} \nu(\alpha, x).$$

Denoting by T_{min} a current with minimal singularities in $\alpha[0]$ one has always

$$\nu(\alpha, x) \leqslant \nu(T_{min}, x), \ \nu(\alpha, D) \leqslant \nu(T_{min}, D).$$

There are examples where $\nu(\alpha, D) < \nu(T_{min}, D)$, see Section 4.1

The following approximation of T_{min} which will be useful later on:

THEOREM 3.16. — *Let X be a compact Kähler manifold with Kähler form ω, let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a pseudo-effective class. Then there exists a sequence of closed $(1,1)-$ currents T_k with analytic singularities in $\alpha[-\epsilon_k\omega]$ for some sequence $(\epsilon_k) \to 0$ of positive real numbers such that*

(i) the T_k converge weakly against a closed positive $(1,1)-$ current T which has minimal singularities in $\alpha[0]$,

(ii) $\nu(T_k, x) \to \nu(\alpha, x)$ for every point $x \in X$,

(iii) for all i

$$\int_{X-Sing(T_k)} (T_k + \epsilon_k\omega)^p \wedge \omega^{n-p} \to (\alpha^p.\omega^{n-p})_{\geqslant 0}.$$

Proof. — To compute $(\alpha^p.\omega^{n-p})_{\geqslant 0}$ it is enough to determine the limit of the

$$s_\epsilon := \sup_T \int_{X-Sing(T)} (T + \epsilon\omega)^p \wedge \omega^{n-p}$$

where $T \in \alpha[-\epsilon\omega]$ has analytic singularities, by Lemma 3.2. Consequently, for each p there is a sequence of closed $(1,1)-$ currents $(T_k^{(p)})_{k\in\mathbb{N}}$ with analytic singularities such that $T_k^{(p)} \in \alpha[-\epsilon_k\omega]$ for some sequence $\epsilon_k \to 0$ of positive real numbers and

$$\int_{X-Sing(T_k^{(p)})} (T_k^{(i)} + \epsilon_k\omega)^p \wedge \omega^{n-p} \to (\alpha^p.\omega^{n-p})_{\geqslant 0}.$$

Now let θ be a smooth $(1,1)-$ form on X representing α. Let $T_{min,k} = \theta + dd^c\phi_{min,k}$ be the current with minimal singularities in $\alpha[-\epsilon_k\omega]$ associated to θ, as described in the remark above. Since $T_k^{(p)} = \theta + dd^c\phi_k^{(p)} \in \alpha[-\epsilon_k\omega]$ this implies $\phi_k^{(p)} \leqslant \phi_{min,k} \leqslant 0$. Furthermore the $T_{min,k}$ converge weakly against a current T_{min} with minimal singularities in $\alpha[0]$.

By Demailly's Approximation Theorem 3.4 there exists a decreasing sequence of almost plurisubharmonic functions $\phi_{k,l}$ with analytic singularities converging pointwise and L^1_{loc} against $\phi_{min,k}$ such that $T_{k,l} = \theta + dd^c\phi_{k,l} \in \alpha[-\epsilon_{k,l}\omega]$ for some sequence $(\epsilon_{k,l})_{l\in\mathbb{N}} \overset{>}{\to} \epsilon_k$ of positive real numbers. Furthermore $\nu(T_{k,l}, x) \overset{<}{\to} \nu(T_{min,k}, x)$ for every point $x \in X$.

Let $\mu : Y \to X$ be a common resolution of the singularities of $T_{k,l}$ and the $T_k^{(p)}$. Then

$$\mu^* T_k^{(p)} = R_k^{(p)} + [D_k^{(p)}], \mu^* T_{k,l} = R_{k,l} + [D_{k,l}]$$

where $R_k^{(p)}, R_{k,l}$ are smooth and $D_k^{(p)}, D_{k,l}$ are effective $\mathbb{R}-$ divisors. Since the $\phi_{k,l}$ form a decreasing sequence, $\phi_k^{(p)} \leqslant \phi_{k,l}$ and $T_{k,l}$ is less singular than $T_k^{(p)}$. In particular $D_{k,l} \leqslant D_k^{(p)}$, hence the *class* $\{R_{k,l} - R_k^{(i)}\} = \{D_k^{(i)} - D_{k,l}\}$ is pseudo-effective. Consequently,

$$\int_Y (R_{k,l} + \epsilon_{k,l}\mu^*\omega) \wedge (R_k^{(p)} + \epsilon_{k,l}\mu^*\omega)^{p-1} \wedge \mu^*\omega^{n-p} \geqslant \int_Y (R_k^{(p)} + \epsilon_{k,l}\mu^*\omega)^p \wedge \mu^*\omega^{n-p},$$

since the integrals over the compact manifold Y only depend on the cohomology classes, and all factors besides $R_{k,l} + \epsilon_{k,l}\mu^*\omega$ and $R_k^{(p)} + \epsilon_{k,l}\mu^*\omega$ are smooth. Iterating gives

$$\int_Y (R_{k,l} + \epsilon_{k,l}\mu^*\omega)^p \wedge \mu^*\omega^{n-p} \geqslant \int_Y (R_k^{(p)} + \epsilon_{k,l}\mu^*\omega)^p \wedge \mu^*\omega^{n-p}.$$

Noting that

$$\int_Y (R_{k,l} + \epsilon_{k,l}\mu^*\omega)^p \wedge \mu^*\omega^{n-p} = \int_{X-Sing(T_{k,l})} (T_{k,l} + \epsilon_{k,l}\omega)^p \wedge \omega^{n-p}$$

and similarly for $R_k^{(p)}$ and $T_k^{(p)}$ one finally gets

$$\int_{X-Sing(T_k^{(p)})} (T_k^{(p)} + \epsilon_{k,l}\omega)^p \wedge \omega^{n-p} \leqslant \int_{X-Sing(T_{k,l})} (T_{k,l} + \epsilon_{k,l}\omega)^p \wedge \omega^{n-p}.$$

Since $\epsilon_{k,l} \longrightarrow \epsilon_k$ the same line of arguments shows

$$\int_{X-Sing(T_k^{(p)})} (T_k^{(p)} + \epsilon_{k,l}\omega)^p \wedge \omega^{n-p} \longrightarrow \int_{X-Sing(T_k^{(p)})} (T_k^{(p)} + \epsilon_k\omega)^p \wedge \omega^{n-p}.$$

For l big enough (depending on k) this gives

$$s_k - \delta_k \leqslant \int_{X-Sing(T_{k,l})} (T_{k,l} + \epsilon_{k,l}\omega)^p \wedge \omega^{n-p} \leqslant s_{k+1}.$$

Combining all these facts one gets a sequence of closed positive $(1,1)-$ currents $T_k = T_{k,l(k)}$ with analytic singularities in $\alpha[-\epsilon_{k+1}\omega]$ such that the T_k converge weakly against T_{min}, and conditions (ii) and (iii) of the theorem are also satisfied. □

Remark. — As long as $T_{k,min} \to T_{min}$ weakly for $k \to \infty$, in the construction above it is not necessary that the $T_{k,min}$ are computed w.r.t. the same smooth $(1,1)-$ form on α.

The approximation may be used e.g. to prove

LEMMA 3.17. — *Let X be a compact Kähler manifold and $\alpha \in H^{1,1}(X,\mathbb{R})$ a pseudo-effective class. Let $\Delta^n \cong U \subset X$ be an open subset, and let $p : \Delta^n \to \Delta^{n-1}$ be the projection onto the last $n-1$ coordinates. Then there is a pluripolar set $E \subset \Delta^{n-1}$ such that for all fibers Δ over points in $\Delta^{n-1} \setminus E$*

$$\lim_{\epsilon \downarrow 0} \inf_{T} \nu(T_{|\Delta}, x) = \nu(\alpha, x) \quad \text{for all } x \in \Delta,$$

where the T's run through all currents in $\alpha[-\epsilon\omega]$ with analytic singularities, for which the restriction to Δ is well-defined.

Proof. — The proof is an application of the theory of (L,h)- resp. T-general curves. If T is an almost psoitive $(1,1)$- current on X, a smooth curve C (compact or not) will be called T- general iff the restriction of T on C is well-defined and

(i) C intersects no codim-2-component in any of the Lelong number level sets $E_c(T)$,

(ii) C intersects every prime divisor $D \subset E_c(T)$ in the regular locus D_{reg} of this divisor, C does not intersect the intersection of two such prime divisors, and every intersection point x has the minimal Lelong number $\nu(T,x) = \nu(T,D) := \min_{z\in D} \nu(T,z)$,

(iii) for all $x \in C$, the Lelong numbers

$$\nu(T_{|\Delta}, x) = \nu(T, x).$$

Then theorem 2.1. in [Eck02] states that in a family of curves over a smooth base there is a pluripolar subset in the base such that every curve over points outside this pluripolar set is T-general. In particular, this is true for currents T_k approximating T_{min} as in the theorem above. Since the union of countably many pluripolar sets is again pluripolar, this proves the lemma. □

3.5. Upper bound for the numerical dimension.

The numerically trivial foliation w.r.t. a pseudo-effective class may be also used to bound its numerical dimension, provided that the singularities of the foliation are nice enough:

THEOREM 3.18. — *Let X be a compact Kähler manifold with Kähler form ω and $\alpha \in H^{1,1}(X,\mathbb{R})$ a pseudo-effective class. Let $\mathcal{F}$ be the numerically trivial foliation w.r.t. α and suppose that the singularities of $\mathcal{F}$ are isolated points. Then the numerical dimension $\nu(\alpha)$ is less or equal to the codimension of the leaves of $\mathcal{F}$.*

Proof. — Applying Theorem 3.16, one gets a sequence of closed $(1,1)$-currents T_k with analytic singularities in $\alpha[-\epsilon_k\omega]$ such that

$$\lim_{k\to\infty}\int_{X-\mathrm{Sing}\ T_k}(T_k+\epsilon_k\omega)^p\wedge\omega^{n-p}=(\alpha^p.\omega^{n-p})_{\geqslant 0}$$

for all $p=1,\ldots,n$. In these integrals, the T_k's may be replaced by the residue currents

$$R_k=T_k-\sum\nu(T_k,D)[D]$$

of the Siu decomposition of the T_k.

Now the proof consists of two steps: first, let $\Delta^n\cong U\subset X$ be an open set such that the projection $q:U\cong\Delta^n\to\Delta^l$ on the last l coordinates describes the numerical trivial foliation w.r.t. α locally in U. Then use as in Proposition 3.9 that the R_k's get close to pulled back currents from the base Δ^l to show

Claim 1. — For $l<p\leqslant n$ and an open subset $U'\subset\subset U$,

$$\int_{U'}(R_k+\epsilon_k\omega)^p\wedge\omega^{n-p}\to 0.$$

Proof. — Every $R_k+\epsilon_k\omega$ may be written as a sum $\sum_{i,j}\theta^k_{ij}dz_i\wedge d\overline{z}_j$. Then every coefficient of $(R_k+\epsilon_k\omega)^p$ w.r.t. the base $dz_I\wedge d\overline{z}_J$ (I, J multi-indices of length $|I|=|J|=p$) is a product of p of these θ^k_{ij}. If $p>l$, then one of these θ^k_{ij} has index $i\leqslant n-l$ or $j\leqslant n-l$.

As in proposition 3.9 one can argue with the Schwarz inequality that

$$|\theta^k_{ij}|\leqslant|\theta^k_{ii}|^{\frac{1}{2}}\cdot|\theta^k_{jj}|^{\frac{1}{2}}.$$

Furthermore, let F_i be a sufficiently general fiber of the projection $\Delta^n \to \Delta^{n-1}$ onto all but the ith coordinate, $i = 1, \dots, n-l$. Since R_k is a current with analytic singularities only in codimension 2, a sufficiently general F_i does not hit the singularities of R_k. Then $\theta^k_{ii|F_i \cap U'}$ is smooth and positive, and numerical triviality applied on the 1-dimensional fibers F_i which are leaves of q implies that

$$\int_{F_i \cap U'} |\theta^k_{ii}| dz_i \wedge d\bar{z}_i = \int_{F_i \cap U'} \theta^k_{ii} dz_i \wedge d\bar{z}_i \overset{k \to \infty}{\longrightarrow} 0.$$

This leads to the following chain of inequalities: Let $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_p)$ be two multi-indices of length p such that (without loss of generality) $i_1 \leqslant n - l$. Then

$$\begin{aligned}\int_{U'} |\theta^k_{i_1 j_1} \cdots \theta^k_{i_p j_p}| dV_\omega &\leqslant \int_{U'} |\theta^k_{i_1 i_1} \cdots \theta^k_{i_p i_p}|^{\frac{1}{2}} |\theta^k_{j_1 j_1} \cdots \theta^k_{j_p j_p}|^{\frac{1}{2}} dV_\omega \\ &\leqslant \left(\int_{U'} |\theta^k_{i_1 i_1}| dV_\omega \right)^{\frac{1}{2}} \cdot \left(\int_{U'} |\theta^k_{i_2 i_2} \cdots \theta^k_{i_p i_p}| |\theta^k_{j_1 j_1} \cdots \theta^k_{j_p j_p}| dV_\omega \right)^{\frac{1}{2}}.\end{aligned}$$

The second integral of the last term remains bounded for $k \to \infty$ because the $R_k + \epsilon_k \omega$ (weakly) converge against some current according to Theorem 3.16. The first integral may be computed via Fubini as

$$\int_{U'} |\theta^k_{i_1 i_1}| dV_\omega = \int_{\Delta^{n-1}} \left(\int_{F_{i_1}} |\theta^k_{i_1 i_1}| dz_{i_1} \wedge d\bar{z}_{i_1} \right) dV_{\Delta^{n-1}},$$

hence tends to 0 for $k \to \infty$ since the integrals $\int_{F_{i_1}} |\theta^k_{i_1 i_1}| dz_{i_1} \wedge d\bar{z}_{i_1}$ are uniformly bounded from above by definition of numerically trivial foliations. Consequently, $\int_{U'} |\theta^k_{i_1 j_1} \cdots \theta^k_{i_p j_p}| dV_\omega \overset{k \to \infty}{\longrightarrow} 0$ and the claim follows. □

The second step is to give an estimate of the considered integrals around the isolated singularities of the foliation by using the uniform boundedness of the Lelong numbers of (almost) positive currents in the same cohomology class.

Claim 2.— There is a sequence of compact sets $K_i \subset X$ exhausting $X - \operatorname{Sing} \mathcal{F}$ and a constant $C > 0$ such that for all $1 \leqslant p \leqslant n$

$$\int_{X - K_i} (R_k + \epsilon_k \omega)^p \wedge \omega^{n-p} \leqslant \delta_i,$$

and $\lim_{i \to \infty} \delta_i = 0$.

Proof. — This is just an expanded version of Boucksom's argument in [Bou02a, Lem 3.1.11]. Choose a finite covering of X by open charts U_i isomorphic to the unity ball $B \subset \mathbb{C}^n$, such that the balls with half of the diameter still cover X. If $z^{(i)}$ denote coordinates on U_i one may find two constants $C_1, C_2 > 0$ such that

$$C_1\omega \leqslant \frac{i}{2}\partial\overline{\partial}|z^{(i)}|^2 \leqslant C_2\omega$$

in U_i, for all i.

If $x \in X$ lies in U_i, the Lelong number $\nu((R_k+\epsilon\omega)^p, x)$ is by definition the decreasing limit for $r \to 0$ of

$$\nu((R_k+\epsilon\omega)^p, x, r) := \frac{1}{(\pi r^2)^{n-p}} \int_{|z^{(i)}-x|<r} (R_k+\epsilon\omega)^p \wedge (\frac{i}{2}\partial\overline{\partial}|z^{(i)}|^2)^p.$$

On the one hand, for $r \leqslant r_0$ one has

$$\nu((R_k+\epsilon\omega)^p, x, r) \leqslant \nu((R_k+\epsilon\omega)^p, x, r_0) \leqslant \frac{C_2}{(\pi r_0^2)^{n-p}} \int_X (R_k+\epsilon\omega)^p \wedge \omega^{n-p}.$$

But $\int_X (R_k+\epsilon\omega)^p \wedge \omega^{n-p} \leqslant \int_X (T_k+\epsilon\omega)^p \wedge \omega^{n-p}$, and the last integral depends only on the cohomology class of T_k, since ω is closed.

On the other hand,

$$\begin{aligned}(\pi r^2)^{n-p}\nu((T_k+\epsilon_k\omega)^p, x, r) &\geqslant C_1 \int_{|z^{(i)}-x|<r} (T_k+\epsilon_k\omega)^p \wedge \omega^{n-p} \\ &\geqslant C_1 \int_{|z^{(i)}-x|<r} (R_k+\epsilon_k\omega)^p \wedge \omega^{n-p}.\end{aligned}$$

For $p < n$ the claim follows since Sing $\mathcal{F}$ is compact, hence consists of only finitely many points. For $p < n$ there is nothing to argue, since $\nu(\alpha) = n$ implies that α is big ([Bou02a, Thm. 3.1.31]). Hence the numerically trivial foliation coincides with the Iitaka fibration w.r.t. α, because it is the identity map. □

Both claims together show the theorem. □

4. Surface Examples.

If one constructs the numerical trivial foliation w.r.t. an incomplete system of currents with analytic singularities in $\alpha[-\epsilon_k\omega]$, $\epsilon_k \to 0$, then the leaf dimension is greater or equal than that of the numerical trivial foliation w.r.t. α. Unfortunately, the author could not prove any criterion when the leaf dimension remains the same (hence the two foliations are equal). In general, it seems quite difficult to decide whether a given foliation is numerically trivial w.r.t. some pseudo-effective class α. In the first two surface examples which follow, some ad-hoc arguments are used to show the identity of the constructed foliations and the numerically trivial foliation w.r.t. the given pseudo-effective classes.

4.1. A nef line bundle without smooth positive curvature form.

This example was already discussed in [DPS94]: Let $\Gamma = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, $\mathrm{Im}\tau > 0$, be an elliptic curve and let E be the rank 2 vector bundle over Γ defined by

$$E = \mathbb{C} \times \mathbb{C}^2/(\mathbb{Z}+\mathbb{Z}\tau)$$

where the action is given by the two automorphisms

$$\begin{aligned} g_1(x, z_1, z_2) &= (x + 1, z_1, z_2) \\ g_\tau(x, z_1, z_2) &= (x + \tau, z_1 + z_2, z_2), \end{aligned}$$

and where the projection $E \to \Gamma$ is induced by the first projection $(x, z_1, z_2) \mapsto x$. Then $\mathbb{C} \times \mathbb{C} \times \{0\}/(\mathbb{Z}+\mathbb{Z}\tau)$ is a trivial line subbundle $\mathcal{O} \hookrightarrow E$, and the quotient $E/\mathcal{O} \cong \Gamma \times \{0\} \times \mathbb{C}$ is also trivial. Let L be the line bundle $L = \mathcal{O}_E(1)$ over the ruled surface $X = \mathbb{P}(E)$. The exact sequence

$$0 \to \mathcal{O} \to E \to \mathcal{O} \to 0$$

shows that L is nef over X.

Now, in [DPS94] all hermitian metrics h (including singular metrics) are determined such that the curvature current $\Theta_h(L)$ is semipositive (in the sense of currents): These metrics have all the same curvature current

$$\Theta_h(L) = [C],$$

where C is the curve on X induced by $\{z_2 = 0\}$. (This implies in particular that there exists no *smooth* positive hermitian metric on L.) To exclude the possibility that there exist positive currents in $c_1(L)$ which are not the curvature current of a metric on L one proves the following

LEMMA 4.1. — *Let X be a projective complex manifold and L a holomorphic line bundle on X. Then for every closed positive current in $c_1(L)$ there is a possibly singular hermitian metric h on L such that the curvature current*

$$\Theta_h(L) = T.$$

Proof. — Let T be any positive current in $c_1(L)$. By [Bon95] there exists a line bundle L' on X with a possibly singular hermitian metric h' such that $\Theta_{h'}(L') = T$. (This is just the usual construction of a cycle in $H^1(X, \mathcal{O}^*)$). The line bundle $N = (L')^{-1} \otimes L$ is numerically trivial, hence nef. Consequently there exists a positive singular hemitian metric h_N on N such that the class of the curvature current

$$\{\Theta_{h_N}(N)\} = 0 \in H^{1,1}(X, \mathbb{R}).$$

Now, all closed positive currents in $0 \in H^{1,1}(X, \mathbb{R})$ have the form $dd^c\phi$ for some plurisubharmonic function on X. Since ϕ is upper semi-continuous it attains its supremum. But then the maximum principle implies that ϕ is a constant function. Therefore the only closed positive current in $0 \in H^{1,1}(X, \mathbb{R})$ is the zero form. This implies $\Theta_{h_N}(N) = 0$ (as a current).

Furthermore this gives the hermitian metric $h = h_N \otimes h'$ on $L = N \otimes L'$ with $\Theta_h(L) = T$. □

So $[C]$ really is a positive current with minimal singularities in $c_1(L)$. But then X is numerically trivial w.r.t. $[C]$, and the associated numerical trivial foliation has only one leaf X with codimension 0.

On the other hand, L is certainly not numerically trivial since it intersects a fiber of $X = \mathbb{P}(E)$ with intersection number 1. Consequently, the moving intersection number $(c_1(L))_{\geqslant 0} = c_1(L)$ is strictly positive, and $(X, c_1(L))$ is a counter example to equality of the numerically trivial foliation w.r.t. the positive closed $(1,1)$- current with minimal singularities and that w.r.t. the associated pseudo-effective cohomology class.

Now there is an obvious candidate for a numerically trivial foliation w.r.t. $c_1(L)$: its leaves are the projection of the curves $\mathbb{C} \times \{p\}$ in $\mathbb{P}_C(E)$. The strategy to show this has two parts: first, one constructs a sequence of currents $T_k \in c_1(L)[-\epsilon_k\omega]$ for some Kähler form ω on X and a sequence ϵ_k of positive real numbers tending to 0 such that the foliation mentioned above is the numerically trivial foliation w.r.t. this sequence of T_k's. Second, one uses that the restriction of the T_k's to any $\mathbb{P}^1$-fiber of $\mathbb{P}_C(E)$ is $\geqslant c\cdot\omega$, for some fixed number $c > 0$.

The construction of the T_k requires a careful study of almost positive (singular) hermitian metrics h on L: As the total space of L^{-1} is equal to E^* blown up along the zero section, the function

$$\phi(\zeta) = \log \parallel \zeta \parallel^2_{h^{-1}},\ \zeta \in L^{-1}$$

associated to any hermitian metric h on L can also be seen as a function on E^* satisfying the log-homogeneity condition

$$\phi(\lambda\zeta) = \log|\lambda| + \phi(\zeta) \text{ for every } \lambda \in \mathbb{C}.$$

One has

$$\frac{i}{2\pi}\partial\overline{\partial}\phi(\zeta) = \pi^*_{L^{-1}}\Theta_h(L),\ \pi_{L^{-1}} : L^{-1} \to X.$$

Thus $\Theta_h(L)$ is almost positive iff ϕ is almost plurisubharmonic on E^*.

The total space of E^* is the quotient $E^* = \mathbb{C}\times\mathbb{C}^2/(\mathbb{Z}+\mathbb{Z}\tau)$ by the dual action

$$\begin{aligned} g_1^*(x, w_1, w_2) &= (x+1, w_1, w_2)\\ g_\tau^*(x, w_1, w_2) &= (x+\tau, w_1, w_1+w_2). \end{aligned}$$

The function ϕ gives rise to a function $\widetilde{\phi}$ on $\mathbb{C}\times\mathbb{C}^2$ which is invariant by g_1^*, g_τ^* and log-homogeneous w.r.t. (w_1, w_2), and $\widetilde{\phi}$ is almost plurisubharmonic iff ϕ is almost psh. Even more is true: Interpret X as the zero section of the total space of L^{-1} and let $\omega_X, \omega_{L^{-1}}$ be positive $(1,1)$-forms on X, L^{-1}. Then there are constants $C_1, C_2 > 0$ such that

$$0 \leqslant \pi^*_{L^{-1}}\omega_X \leqslant C_1\omega_{L^{-1}},\ 0 \leqslant \omega_{L^{-1}|X} \leqslant C_2\omega_X.$$

Hence $\pi^*_{L^{-1}}\Theta_h \geqslant -\epsilon\omega_{L^{-1}}$ implies $\Theta_h \geqslant -\epsilon C_2\omega_X$, and $\Theta_h \geqslant -\epsilon\omega_X$ implies $\pi^*_{L^{-1}}\Theta_h \geqslant -\epsilon C_1\omega_{L^{-1}}$. Consequently, instead of constructing currents $T_k \geqslant -\epsilon_k\omega_X$, $\epsilon_k \to 0$ on X, it suffices to construct currents $\Theta_k \geqslant -\epsilon'_k\omega_{L^{-1}}$, $\epsilon'_k \to 0$, and functions $\widetilde{\phi}_k$ on $\mathbb{C}\times\mathbb{C}^2$ such that $i\partial\overline{\partial}\widetilde{\phi}_k = \Theta_k$ and the $\widetilde{\phi}_k$ are invariant by g_1^*, g_τ^* and log-homogeneous w.r.t. (w_1, w_2).

This can be done by using a gluing procedure developed in [Dem92]: Choosing an appropriate partition of unity which is g_1^* and g_τ^* invariant and only depends on the imaginary part of x one gets the desired almost plurisubharmonic functions $\widetilde{\phi}_k$ from plurisubharmonic functions

$$k\widetilde{\psi}_j = \frac{k}{2}\log(|w_1|^2 + |jw_1 + w_2|^2),\ k \in \mathbb{N},\ j \in \mathbb{Z},$$

defined on stripes of type

$$\{(x, w_1, w_2) : (j-a)\mathrm{Im}\tau < \mathrm{Im}x < (j-a+1)\mathrm{Im}\tau\},\ 0 \leqslant a \leqslant 1,$$

and the associated currents T_k have arbitrary small negative part for $k \to \infty$.

On the other hand, it follows from the construction that the restriction of the induced currents T_k to the $\mathbb{P}^1$-fibers of $X = \mathbb{P}(E)$ remain $> \epsilon\omega$ for some $\epsilon > 0$.

Let $T'_k \in \alpha[-\epsilon_k\omega]$ be another sequence of currents representing α. If $\Delta^2 \cong U \subset X$ is an open subset with coordinates z_1, z_2 such that the lines $\{z_1 = a\}$ belong to $\mathbb{P}^1$-fibers and $\{z_2 = b\}$ are subsets of the leaves of the foliation one can write

$$T_k + \epsilon_k\omega = \sum_{i,j=1}^{2} \theta_{ij}^{(k)} idz_i \wedge d\overline{z}_j,\ T'_k + \epsilon_k\omega = \sum_{i,j=1}^{2} \theta_{ij}^{\prime(k)} idz_i \wedge d\overline{z}_j.$$

By the remark above,

$$(\theta_{22}^{(k)})_{|\{z_1=a\}} idz_2 \wedge d\overline{z}_2 > \epsilon\omega$$

for all a, and

$$\widetilde{\theta}^{(k)} := \theta_{11}^{(k)} idz_1 \wedge d\overline{z}_1 + \theta_{12}^{(k)} idz_1 \wedge d\overline{z}_2 + \theta_{21}^{(k)} idz_2 \wedge d\overline{z}_1 \overset{k\to\infty}{\longrightarrow} 0$$

by the numerical triviality (use as before the Schwarz inequality for the terms with $\theta_{12}^{(k)}, \theta_{21}^{(k)}$).

Since the numerical dimension of L is 1, one knows furthermore that

$$\lim_{k\to\infty} \int_{X-\mathrm{Sing}\ T'_k} (T_k + \epsilon_k\omega) \wedge (T'_k + \epsilon_k\omega) = 0.$$

But

$$(T_k + \epsilon_k\omega) \wedge (T'_k + \epsilon_k\omega) = \widetilde{\theta}^{(k)} \wedge (T'_k + \epsilon_k\omega) + \theta_{22}^{(k)} idz_2 \wedge d\overline{z}_2 \wedge \theta_{11}^{\prime(k)} idz_1 \wedge d\overline{z}_1,$$

hence the vanishing of the limits above implies

$$\int_{(\Delta')^2-\mathrm{Sing}\ T'_k} \theta'^{(k)}_{11} i dz_1 \wedge d\bar{z}_1 \wedge i dz_2 \wedge d\bar{z}_2 \overset{k\to\infty}{\longrightarrow} 0,$$

where $\Delta' \subset\subset \Delta$ is any open disk such that $(\Delta')^2 \subset U \cong \Delta^2$.

Consequently, $\int_{\Delta'_b-\mathrm{Sing}\ T'_k}(T'_k + \epsilon_k\omega) \overset{k\to\infty}{\longrightarrow} 0$ for almost all $b \in \Delta'$ (where $\Delta'_b = \{b\} \times \Delta'$). The definition of the numerically trivial foliation requires that $\int_{\Delta'_b-\mathrm{Sing}\ T'_k}(T'_k + \epsilon_k\omega) \overset{k\to\infty}{\longrightarrow} 0$ for *all* $b \in \Delta'$. To prove this one can use the same line of arguments as in the proof of the Local Key Lemma for pseudo-effective classes: One tries to show that

$$\lim_{k\to\infty} \left| \int_{\Delta'_b-\mathrm{Sing}\ T'_k} (T'_k + \epsilon_k\omega) - \int_{\Delta'_0-\mathrm{Sing}\ T'_k} (T'_k + \epsilon_k\omega) \right| = 0.$$

Following the proof of Proposition 3.9 one sees that it is enough to show that

$$\lim_{k\to\infty} \int_{\partial\Delta'\times\Delta_{0,b}} |\theta'^{(k)}_{11}| dV \cdot \int_{\partial\Delta'\times\Delta_{0,b}} |\theta'^{(k)}_{22}| dV = 0,$$

where $\Delta_{0,b}$ is the disk with center in $b/2$ and radius $|b/2|$, and dV is a volume element of $\partial\Delta' \times \Delta_{0,b}$.

As in the proof of Proposition 3.9 there is a bound $M > 0$ such that for all k there is a disk $\Delta'_k \subset\subset \Delta$ containing Δ' with

$$\int_{\partial\Delta'_k\times\Delta'} |\theta'^{(k)}_{22}| dV < M.$$

For the first term, look at the $(1,1)$-form $\eta = i dz_2 \wedge d\bar{z}_2$ and take a disk $\Delta' \subset\subset \Delta'' \subset\subset \Delta$. Then by the arguments above,

$$\int_{(\Delta''-\Delta')\times\Delta'} |\theta^k_{11}| dV = \int_{(\Delta''-\Delta')\times\Delta'} (T'_k + \epsilon_k\omega) \wedge \eta \overset{k\to\infty}{\longrightarrow} 0.$$

By Fubini, one gets a disk Δ'_k such that

$$\int_{\partial\Delta'_k\times\Delta'} |\theta^k_{11}| dV \overset{k\to\infty}{\longrightarrow} 0,$$

and one concludes that the limit above is indeed 0.

4.2. Mumford's example

Back to our counter example at the beginning: it is easy to construct a closed positive $(1,1)-$ current on $L = \mathcal{O}_{\mathbb{P}^1}(1)$ such that the leaves of the associated numerically trivial foliation are 1-dimensional. Take a measure ω invariant w.r.t. the representation of $\pi(C)$ in PGL(2). This gives a measure on $(\Delta \times \mathbb{P}^1)/\pi(C)$ transversal to the foliation induced by the images of $\Delta \times \{p\}$. Averaging out the integration currents of the leaves with this transverse measure gives an (even smooth) closed positive $(1,1)-$ current in the first Chern class of $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ which vanishes on the leaves but not in any transverse direction.

Most of this example is explained in the introduction; the only assertions not already discussed are the existence of a measure ω in $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ invariant w.r.t. the unitary representation of $\pi(C)$ in GL(2) and the smoothness of the metric which results from averaging out the integration currents of the leaves. But this is easy, too: Take the Haar measure ω on the Lie group $U(2)$ which is absolutely continuous ([Die70], [Ch.14]). Since $U(2)$ operates transitively on $\mathbb{P}^1$ this measure induces a $U(2)-$ invariant measure on the homogeneous quotient space $\mathbb{P}^1$. Since $U(2)$ is compact it is possible to normalize ω such that $\mathbb{P}^1$ has measure 1. Hence averaging over the integration currents of the leaves w.r.t. ω gives a smooth positive $(1,1)-$ form which is still in the first Chern class of $L = \mathcal{O}_{\mathbb{P}(E)}(1)$. Since it is smooth it is a current with minimal singularities on L, and obviously, this current is numerically trivial on the leaves.

On the other hand it is strictly positive on the $\mathbb{P}^1$-fibers, hence the foliation is numerically trivial w.r.t. the cohomology class by the same argument as in the first example.

Remark. — The difference to the previous example is that the unitary group is compact and consequently its Haar measure is finite. This is not the case for the group of linear automorphisms generated by $(z_1, z_2) \mapsto (z_1 + z_2, z_2)$.

4.3. $\mathbb{P}^2$ blown up in 9 points.

Consider the following situation: Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve and let $p_1, \ldots, p_8 \in C$ be sufficiently general points. The aim is to study the numerically trivial foliation w.r.t. the anticanonical bundle $-K_X$ on varieties $X_p = \mathbb{P}^2(p_1, \ldots, p_8, p)$ blown up in points $p \in C$.

Let $E_i = \pi^{-1}(p_i)$ be the exceptional divisor on X over p_i. First of all, $-K_X = \mathcal{O}_{\mathbb{P}^2}(3) + \sum E_i$ is nef and $-K_X^2 = 0$. Next, the pencil of elliptic curves on $\mathbb{P}^2$ through $p_1, \ldots, p_8$ has a base point q. So $X_q = \mathbb{P}^2(p_1, \ldots, p_8, q)$ is an elliptic fibration $\pi_q : X_q \to \mathbb{P}^1$. The pull back of a smooth positive metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ gives a smooth semipositive hermitian metric on $-K_{X_q}$ which is strictly positive in directions transverse to the fibers. Hence by the same arguments as in the two examples above, the fibration is the numerically trivial foliation w.r.t. $-K_{X_q}$.

For points $p \neq q$ in C there is only one section in $-K_{X_p}$, the strict transform C' of C. But if one considers torsion points (w.r.t. to q) of order m on C then a calculation in [DPS96] shows that $-mK_{X_p}$ defines again an elliptic fibration over $\mathbb{P}^1$. This fibration yields a smooth semipositive hermitian metric on $-mK_{X_p}$, hence on $-K_{X_p}$, and again the fibration is the numerically trivial foliation w.r.t. $-mK_{X_p}$.

The question is: What happens if non-torsion points $p \in C$ are blown up ? In particular: Is there always a smooth semipositive hermitian metric on $-K_{X_p}$ inducing a holomorphic foliation on X_p, which may be seen as the limit of the fibrations of X_{p_k} where the p_k are torsion points ? (The last question was asked in [DPS96].) A strategy to answer it is to use the theory of holomorphic foliations on surfaces, as developed e.g. in [Bru00].

Definition 4.2. — *A (holomorphic)* foliation $\mathcal{F}$ *on a compact complex surface X is a coherent analytic rank 1 subsheaf $T_{\mathcal{F}}$ of the tangent bundle T_X (the tangent bundle of the foliation) fitting into an exact sequence*

$$0 \to T_{\mathcal{F}} \to T_X \to \mathcal{J}_Z \otimes N_{\mathcal{F}} \to 0$$

for a suitable invertible sheaf $N_{\mathcal{F}}$ (the normal bundle of the foliation) and an ideal sheaf $\mathcal{J}_Z$ whose zero locus consists of isolated points called the singularities $\mathrm{Sing}(\mathcal{F})$ *of $\mathcal{F}$.*

Furthermore, one can easily show that $T_{\mathcal{F}}^* \otimes N_{\mathcal{F}}^* = K_X$.

Numerically trivial foliations $\{\mathcal{F}, (U_i, p_i)\}$ on surfaces X with $\mathcal{F}$ of rank 1 are such foliations: If $\mathcal{F}$ is not a line bundle then replace it by $\mathcal{F}^{**}$. As a reflexive sheaf on a surface this is a line bundle [OSS80, 1.1.10], and dualizing the inclusion $\mathcal{F} \subset T_X$ twice shows that it is still a subsheaf of T_X. Furthermore, $\mathcal{F}$ is locally integrable because it has rank 1, hence the maps p_i exist trivially.

Let $\mathcal{X}$ be $\mathbb{P}^2(p_1,\ldots,p_8)\times C$ blown up in the diagonal

$$\Delta_{C\times C}\subset C\times C\subset \mathbb{P}^2(p_1,\ldots,p_8)\times C.$$

The fibers of $\mathcal{X}$ over $p\in C$ are just the X_p for all p. If there is an algebraic family of foliations on the X_p such that over torsion points, the foliation coincides with the fibration described above, then (at least generically) the conormal line bundles $N^*_{F_p}$ should also fit into a family. But this is impossible, as the following computation shows:

LEMMA 4.3. — *Let C, q, X_p be as above, and let p be a torsion point w.r.t. q of order m. Let N_{F_p} be the normal bundle of the foliation induced by the fibration $\pi_p : X_p \to \mathbb{P}^1$. Then*

$$N^*_{F_p}\cong (m+1)K_{X_p}.$$

Proof. — Let D be an irreducible component of a fiber of $\pi=\pi_p$ with multiplicity l_D. If η is a local non-vanishing 1- form on $\mathbb{P}^1$ then $\pi^*(\eta)$ is a local section of $\pi^*(K_{\mathbb{P}^1})$ vanishing of order l_D-1 on D. Hence,

$$N^*_{F_p}=\pi^*(K_{\mathbb{P}^1})\otimes \mathcal{O}_{X_p}(\textstyle\sum(l_D-1)D).$$

The relative canonical bundle formula (for elliptic fibrations, see [Fri98]) tells that

$$K_{X_p}=\pi^*(K_{\mathbb{P}^1}\otimes (R^1\pi_*\mathcal{O}_{X_p})^*)\otimes \mathcal{O}_{X_p}(\textstyle\sum(l_F-1)F),$$

where the sum is taken over all fibers F occuring with multiplicity l_F in the fibration.

There are two differences between the two formulas: First, in the relative canonical bundle formula occurs the term

$$L:=(R^1\pi_*\mathcal{O}_{X_p})^*.$$

Now, $\deg L\geqslant 0$, and $\deg L=0$ would imply that L is a torsion bundle on $\mathbb{P}^1$, hence it is trivial, and $X_p=C\times\mathbb{P}^1$ – a contradiction. If L is nontrivial, a short calculation with spectral sequences shows that

$$0=p_g=\deg L-g(\mathbb{P}^1)+1,$$

hence $\deg L = 1$, and $L = \mathcal{O}_{\mathbb{P}^1}(1)$ (see again [Fri98, Ch.VII]). This shows

$$\pi^*(K_{\mathbb{P}^1} \otimes L) = \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) = mK_{X_p},$$

and together with the relative canonical bundle formula this shows that mC is the only multiple fiber.

The second difference is that some fibers may contain multiple components, but are not multiple themselves. By the classification of singular fibers of elliptic fibrations this is only possible if there are -2-curves ([Fri98]). But on $\mathbb{P}^2$ blown up in 9 points in general position, there are no -2-curves. Hence

$$\mathcal{O}_{X_p}(\sum(l_D - 1)D) = \mathcal{O}_{X_p}(\sum(l_F - 1)F),$$

and the claim of the lemma follows. □

The threefold $\mathcal{X}$ is also a counter example to equality of numerical dimension and codimension of the leaves of the numerically trivial foliation w.r.t. some pseudo-effective class: Set

$$L := -\pi^*(p_1^* K_{\mathbb{P}^2(p_1,\ldots,p_8)}) - E_\Delta + p_2^*\mathcal{O}(nr),$$

where p_1 is the projection of $\mathbb{P}^2(p_1,\ldots,p_8) \times C$ onto $\mathbb{P}^2(p_1,\ldots,p_8)$, p_2 is the projection of $\mathcal{X}$ onto C, r is any point on C and $n > 0$ an integer. The restriction of L to any fiber over $p \in C$ is the anticanonical bundle $K_{X_p}^*$.

For n sufficiently big, L is nef: L is effective, since $D = C \times C + nX_r$ is contained in $|L|$. Consequently, to prove the nefness of L it suffices to show that all curves $E \subset C \times C$ have non-negative intersection number with L. To this purpose first get an overview over all curves on $C \times C$: According to the general theory of abelian surfaces the Picard number of $C \times C$ is 4 or 3 depending on whether C has complex multiplication or not ([BL99, 2.7]. Hence it suffices to look at the fibers of the two projections of $C \times C$ onto C, the diagonal, and if necessary, on some other curve constructed as the graph of complex multiplication in $C \times C$. Since it is a graph of an isomorphism, such a curve maps isomorphically to C under both projections.

Now, one has to compute the degree of the restriction of L to E. This restriction may also be seen as the restriction of the divisor $D_{|D}$ to such an E. Let C' be a sufficiently general curve in the pencil $|-K_{\mathbb{P}^2(p_1,\ldots,p_8)}|$. Then the strict transform of $C' \times C$ is an element of $-\pi^*(p_1^* K_{\mathbb{P}^2(p_1,\ldots,p_8)})$

and intersects $C \times C$ in $\{q\} \times C$. Furthermore, E_Δ intersects $C \times C$ in the diagonal $\Delta_{C\times C}$. Therefore,

$$D_{|D} \sim \{q\} \times C + n(C' \times \{r\}) - \Delta_{C\times C} + n(C \times \{r\}),$$

where E_r is the exceptional divisor over r in X_r. And L is nef if n is $\geqslant$ the maximum of 1 (this is the intersection number of fibers $C \times \{p\}$ with the diagonal) and the intersection number of the curve coming from complex multiplication (if existing) with the diagonal. (The self intersection number of the diagonal is 0 since the tangent bundles on $C \cong \Delta_{C\times C}$ and $C \times C$ are trivial.)

PROPOSITION 4.4. — *Let $\mathcal{X}, L$ be as above. Then the numerical dimension $\nu(L)$ of L is 2, but the numerically trivial foliation w.r.t. $c_1(L)$ is the identy map.*

Proof. — To prove $L^2 \neq 0$, observe that L^2 is represented by the cycles in the expression above for $D_{|D}$. This is not $\equiv 0$, since the intersection number with $\{q\} \times C$ is positive for $n \geqslant 1$.

The numerically trivial foliation w.r.t. $c_1(L)$ cannot be the trivial map onto a point, because in fibers X_p over torsion points p there are curves which are not numerically trivial. Since immersed disks which do not lie in a fiber of the projection onto C are not numerically trivial, the only possible numerically trivial foliation w.r.t. $c_1(L)$ with 2-dimensional leaves is the fibration onto C. But this is impossible by the same reason as above. To exclude the possibility that the numerically trivial foliation has 1-dimensional leaves, one notes first that over torsion points p, the fibers of $\pi_p : X_p \to \mathbb{P}^1$ are numerically trivial: This is clear since these fibers F are projective, hence $\int_F T_k$ only depends on the cohomology class of the T_k, and $\int_F c_1(L)$ is certainly 0.

This can be used to show that the 1-dimensional leaves of a numerically trivial foliation must lie in the fibers X_p of $\mathcal{X}$: Otherwise, let $\Delta^3 \cong U \subset \mathcal{X}$ be any open subset with coordinates x, z_1, z_2 such that the projection onto C is given by the projection onto the first coordinate, and the foliation is described by the projection onto the two last coordinates. Choose x such that $x = 0$ corresponds to a torsion point p_0. Shrinking U if necessary, one can suppose that the fibers of π_{p_0} are smooth in U. But then the Local Key Lemma for pseudo-effective classes implies that there are 2-dimensional numerically trivial leaves, contradiction.

Next one shows that the 1-dimensional leaves in fibers X_p, where p is a torsion point, must be the fibers of $\pi_p : X_p \to \mathbb{P}^1$: Take an ample line bundle A on $\mathcal{X}$. Since L is nef, $L^k \otimes A$ is also ample, and some multiple is very ample. The global sections of this very ample line bundle generate a smooth metric on $L^k \otimes A$ whose strictly positive curvature form may be written as $k(T_k + \frac{1}{k}\omega_A)$, for some form $T_k \in c_1(L)[-\frac{1}{k}\omega_A]$.

Let $p \in C$ be any torsion point of order m and $\pi_p : X_p \to \mathbb{P}^1$ the induced fibration. Let $T = i\partial\overline{\partial}\log(|z_1|^2 + |z_2|^2)$ be a strictly positive curvature form in $c_1(\mathcal{O}^1_{\mathbb{P}}(1))$. Then

$$(T_k + \frac{1}{k}\omega_A)_{|X_p} \geqslant \frac{1}{m}\pi_p^* T.$$

But this means in particular that for any disk $\Delta \subset X_p$ not immersed into a fiber of π_p,

$$\int_\Delta T_k + \frac{1}{k}\omega_A \geqslant \frac{1}{m}\int_\Delta \pi_p^* T > 0.$$

Hence the leaves of the numerically trivial foliation w.r.t. $c_1(L)$ coincide with the fibers of π_p in X_p.

But this is impossible, as shown above. □

Remark. — This proposition does not exclude the possibility that (some of) the X_p over non-torsion points p have a numerically trivial foliation with 1-dimensional leaves.

Another result dealing with this type of foliations is

PROPOSITION 4.5 [Brunella]. — *Let $\mathcal{F}$ be a foliation on a compact algebraic surface X and suppose that $\mathcal{F}$ is tangent to a smooth elliptic curve E, free of singularities of $\mathcal{F}$. Then either E is a (multiple) fiber of an elliptic fibration or, up to ramified coverings and birational maps, $\mathcal{F}$ is the suspension of a representation $\rho : \pi_1(\widehat{E}) \to \mathrm{Aut}(\mathbb{CP}^1)$, $\widehat{E}$ an elliptic curve.*

Appendix A. Singular foliations

One can define foliations on complex manifolds as involutive subbundles of the tangent bundle. Then the classical theorem of Frobenius asserts that through any point there is a unique integral complex submanifold [Miy86]. Singular foliations may be defined as involutive coherent subsheaves of the tangent bundle, which are furthermore saturated, that is, their quotient with the tangent bundle is torsion free. In the points where the rank is maximal, one may use again the Frobenius theorem to get leaves. Since in this paper the reasoning is always explicitely using the leaves their existence is directly incorporated in the definition of a singular foliation:

DEFINITION A.1. — *Let X be an $n-$ dimensional compact complex manifold. Let $\mathcal{F} \subset T_X$ be a saturated subsheaf of the tangent bundle with maximal rank k and $Z \subset X$ be the analytic subset where $\mathcal{F}/m_{X,x}\mathcal{F} \to T_{X,x}$ is not injective.*

$\mathcal{F}$ induces a singular foliation described by the following data: $X - Z$ is covered by open sets $U_i \cong \Delta^n$ such that for the smooth holomorphic map $p_i : U_i \to \Delta^{n-k}$ coming from the projection $\Delta^n \to \Delta^{n-k}$,

$$\mathcal{F}_{|U_i} = T_{U_i/\Delta^{n-k}}.$$

Such a foliation will be denoted by $\{\mathcal{F}, (U_i, p_i)\}$.

Next, one defines the inclusion relation for numerically trivial foliations as above

DEFINITION A.2. — *A numerically trivial foliation is contained in another one,*

$$\{\mathcal{F}, (U_i, p_i)\} \sqsubset \{\mathcal{G}, (V_j, q_j)\},$$

iff there is a Zariski open set $U \in X$ such that $\mathcal{F}_{|U} \subset \mathcal{G}_{|U}$.

In particular this means that the leaves of $\{\mathcal{F}, (U_i, p_i)\}$ are contained in those of $\{\mathcal{G}, (V_j, q_j)\}$.

The next aim is to construct a common refinement $\{\mathcal{H}, (W_k, r_k : W_k \to \Delta^{n-m})\}$ of two singular foliations $\{\mathcal{F}, (U_i, p_i : U_i \to \Delta^{n-k})\}$, $\{\mathcal{G}, (V_j, q_j : V_j \to \Delta^{n-l})\}$, that is

$$\{\mathcal{F}, (U_i, p_i)\} \sqsubset \{\mathcal{H}, (W_k, r_k)\}, \ \{\mathcal{G}, (V_j, q_j)\} \sqsubset \{\mathcal{H}, (W_k, r_k)\}.$$

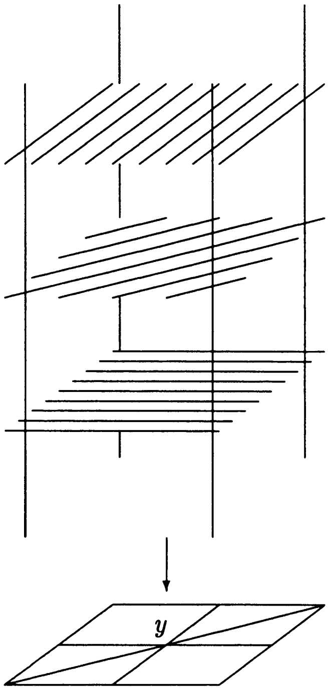
To this purpose one has first to analyze the local picture when two foliations meet transversally everywhere: Let W be a complex manifold with two isomorphisms $q_1 : W \to \Delta^n$, $q_2 : W \to \Delta^n$. Let $p_1 : W \to \Delta^{n-k}$, $p_2 : W \to \Delta^{n-l}$ be the composition of q_1, q_2 with the projections of Δ^n onto the last $n-k$ resp. $n-l$ factors.

$$\begin{array}{ccc} W & \xrightarrow{p_1} & \Delta^{n-k} \\ {\scriptstyle p_2}\downarrow & & \\ \Delta^{n-l} & & \end{array}$$

Suppose that the p_1- and p_2- fibers intersect transversally everywhere.

If $n \leqslant k+l$, by choosing appropriate coordinates p_1 will be the projection on the first $n-k$ coordinates while p_2 is the projection on the last $n-l$ coordinates. In particular, the smallest fibration p whose fibers contain all fibers of p_1 and p_2 is the trivial fibration onto a point.

So suppose from now on that $n > k+l$. Again by choosing appropriate coordinates via the implicit function theorem and possibly further restricting W one can describe the configuration of the two sets of fibers in the following way (look at the next figure): The horizontal sections of p_1 consist of p_2-fibers which are parallel hyperplanes, and over each point $y \in \Delta^{n-k}$ the p_2-fibers through the points in $q^{-1}(y)$ project into a pencil of hyperplanes through a common central hyperplane $\Delta^{k'} \subset \Delta^{n-k}$ containing y.



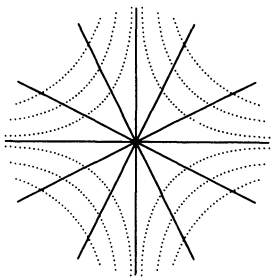
This central hyperplane is isomorphic to $\Delta^{k'}$ for all $y \in \Delta^{n-k}$, and different central hyperplanes are parallel in Δ^k. Let $r : \Delta^{n-k} \to \Delta^{n-k-k'}$ be the projection with the central hyperplanes as fibers. Consequently the new projection

$$p : W \xrightarrow{p_1} \Delta^{n-k} \xrightarrow{r} \Delta^{n-k-k'}$$

is the smallest fibration whose fibers contain both the fibers of p_1 and p_2.

So outside the singularities of $\mathcal{F}$ and $\mathcal{G}$ and the locus where the leaves of the foliation do not intersect transversally or even coincide, it is clear how to define the common refinement.

To get a better feeling for the locus of the other points, look at the following two-dimensional toy example, where the two foliations are marked with dotted and dashed lines.



At least, the exceptional points form an analytic subset of X: A point $x \in U_\imath \cap V_\jmath$ is contained in this set iff the differential of $p_i \times q_\jmath : U_i \cap V_j \rightarrow \Delta^{n-k} \times \Delta^{n-l}$ in x has not full rank, that is iff all maximal minors of this differential vanish in x. But it still remains the task to define a saturated subsheaf $\mathcal{H} \subset T_X$ which locally coincides with the relative tangential sheaf of the projections $r_k : W_k \rightarrow \Delta^{n-m}$.

To do this one goes back to the purely algebraic definition of (singular) foliations: The subsheaf $\mathcal{F} \subset T_X$ induces such a foliation iff it is involutive, that is, closed under the Lie bracket, which means $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$. Then there is a natural candidate for a subsheaf defining the union of the foliations given by $\mathcal{F}$ and $\mathcal{G}$: the smallest involutive subsheaf $\mathcal{H} \subset T_X$ containing both $\mathcal{F}$ and $\mathcal{G}$. It exists because it may be constructed as the subsheaf generated by $\mathcal{F}$, $\mathcal{G}$ $[\mathcal{F}, \mathcal{G}]$, $[[\mathcal{F}, \mathcal{G}], \mathcal{F}]$, $[[\mathcal{F}, \mathcal{G}], \mathcal{G}]$, $[[\mathcal{F}, \mathcal{G}], [\mathcal{F}, \mathcal{G}]]$ and so on.

LEMMA A.3. — *Let* $\{\mathcal{F},(U_\imath, p_\imath : U_\imath \rightarrow \Delta^{n-k})\}$, $\{\mathcal{G},(V_j, q_\jmath : V_\jmath \rightarrow \Delta^{n-l})\}$ *be two singular foliations on an n-dimensional complex manifold X. Let x be a point not in the analytic subset $Z \subset X$ consisting of the singular locus of $\mathcal{F}$, $\mathcal{G}$ and the points where the leaves of $\mathcal{F}$ and $\mathcal{G}$ do not intersect* *transversally. Then on the common refinement $r_k : W_k \to \Delta^{n-m}$ around x constructed as above, the smallest involutive subsheaf $\mathcal{H}$ containing both $\mathcal{F}$ and $\mathcal{G}$ coincides with the relative tangential sheaf of r_k.*

Proof. — Since the two foliations intersect transversally around x, it is obvious that the smallest saturated involutive subsheaf in $T_{X|W_k}$ containing both $\mathcal{F}_{|W_k}$ and $\mathcal{G}_{|W_k}$ is the relative tangent sheaf of the projection r_k. Glueing together one gets a saturated involutive subsheaf $\mathcal{H}_{X-Z} \subset T_{X-Z}$ on the open set $X - Z$.

Let $U \cong \Delta^n$ be a neighborhood of some point $z \in Z$, and let $H = \{f = 0\}$, $f \in \mathcal{O}(U)$, be an analytic hyperplane in U containing the analytic subset $Z \cap U$. Now, $\mathcal{O}(U - H) = \mathcal{O}(U)_f$, and one can define the sections of $\mathcal{H}$ on U as the intersection

$$\mathcal{H}(U) = T_X(U) \cap \mathcal{H}_{X-Z}(U - H)$$

in $T_X(U - H) \cong \mathcal{O}(U)^n$. Since $T_X(U)$ and $\mathcal{H}_{X-Z}(U - H)$ are involutive, $\mathcal{H}(U) \subset T_X(U)$ is closed under the Lie bracket, too. Furthermore, $\mathcal{H}(U)$ is the smallest *saturated* submodule of $T_X(U)$ such that $\mathcal{H}(U)_f = \mathcal{H}_{X-Z}(U - H)$, as the following algebraic lemma shows.

Finally, since the same is true for $\mathcal{F}(U)$ and $\mathcal{G}(U)$, they are both contained in $\mathcal{H}(U)$. □

LEMMA A.4. — *Let R be a commutaive integral ring, $f \in R$, and $M^f \subset R_f^k$ a submodule such that R_f^k/M^f is torsion free. Then $M = M^f \cap R^k$ is the smallest submodule of R^k such that $M_f = M^f$ and R^k/M is torsion free.*

Proof. — If $N \subset M^f \subset R$ such that there exists $m \in M - N$, but still $N_f = M^f$, then $m \in N_f$. Hence there is an $n \in N$ and $l \in \mathbb{N}$ such that $m = \frac{n}{f^l}$ or $m \cdot f^l = n$. But then f is a torsion element of R^k/N. □

This shows that $\{\mathcal{H}, (W_k, r_k : W_k \to \Delta^{n-m})\}$ is really a singular foliation.

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Thomas ECKL,
Universität Bayreuth
Institut für Mathematik
95440 Bayreuth, (Germany).

thomas.eckl@uni-bayreuth.de