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EXPONENTIALLY LONG TIME STABILITY FOR NON-LINEARIZABLE ANALYTIC GERMS OF $(\mathbb{C}^n, 0)$

by Timoteo CARLETTI

1. Introduction.

In this paper we consider the *Siegel–Schröder center problem* [He], [CM], [Ca] in some class of ultradifferentiable germs of $(\mathbb{C}^n, 0)$, $n \geqslant 1$; let us consider two classes of formal power series $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathbb{C}^n\,[[z_1, \dots, z_n]]$, closed w.r.t. to derivation and composition, let $F \in \mathcal{A}_1$ and call $DF(0) = A \in GL(n, \mathbb{C})$, we say that F is *linearizable in* $\mathcal{A}_2$ if there exists $H \in \mathcal{A}_2$, normalized with $DH(0) = \mathbb{I}$, which solves [1]:

$$F \circ H(z) = H \circ R_A(z), \tag{1.1}$$

where $R_A(z) = Az$. In the following we will assume A to be diagonal with eigenvalues of unit modulus $\lambda_1, \dots, \lambda_n$, thus $A = \operatorname{diag} \lambda_1, \dots, \lambda_n$.

If both $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide with the ring of formal power series then generically the formal linearization holds if and only if A is *non–resonant*, namely for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| = \sum_{1 \leqslant i \leqslant n} \alpha_i \geqslant 2$, and for all $j \in \{1, \dots, n\}$ then $\lambda^\alpha - \lambda_j \neq 0$ (where we used the standard notation $\lambda^\alpha = \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n}$).

When F is a germ of analytic diffeomorphisms defined in a neighborhood of the origin and we want to solve (1.1) in the same class of analytic

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[1] Here $F \circ H$ means the composition of F and H; in the following we will denote the composition of F n-times with itself, by F^n instead of $F^{\circ n}$.

germs, we have to consider several cases. If A is the *Poincaré domain*, namely $\sup_{1\leqslant j\leqslant n}|\lambda_j| < 1$ or $\sup_{1\leqslant j\leqslant n}|\lambda_j^{-1}| < 1$, then Koenigs [Ko] and Poincaré [Po] proved that every analytic germ $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that $F(0) = 0$ and $DF(0) = A$, is analytically linearizable. When A is not in the Poincaré domain, we say that it is in the *Siegel domain*; the question is harder and some additional arithmetical conditions on $(\lambda_j)_j$ are needed (see [He] §17, 158).

Let $p \in \mathbb{N}$, $p \geqslant 2$ and let us define for non–resonant $\lambda_1, \dots, \lambda_n$:

$$\Omega(p) = \min_{1\leqslant j\leqslant n} \inf_{\substack{\alpha\in\mathbb{N}^n \\ 0<|\alpha|<p}} |\lambda^\alpha - \lambda_j|; \tag{1.2}$$

we say that A verifies a *Diophantine condition* of type (γ, τ) if there exist $\gamma > 0$ and $\tau \geqslant n-1$ such that for all $\beta \in \mathbb{N}^n \setminus \{0\}$ we have $\Omega(|\beta|) \geqslant \gamma|\beta|^{-\tau}$. Siegel [Si] in 1942 for the $n = 1$ case and then Sternberg [St] and Gray [Gr] in the general case proved that if A verifies a Diophantine condition then the linearization problem has an analytic solution. Bruno [Br] weakened the arithmetical condition by asking the convergence of the series $\sum_k \frac{\log \Omega^{-1}(2^{k+1})}{2^k}$. We remark that in the one dimensional case the Bruno condition [2] is optimal, as proved by Yoccoz [Yo].

In [CM] authors studied the Siegel–Schröder center problem in the case of general algebras of ultradifferentiable germs of $(\mathbb{C}, 0)$, including the Gevrey case. In [Ca] the multidimensional case is considered: if $\mathcal{A}_1 = \mathcal{A}_2$ and A verifies a Bruno condition, then every $F \in \mathcal{A}_1$ with $F(0) = 0$ and $DF(0) = A$ is linearizable in $\mathcal{A}_2$, whereas if $\mathcal{A}_1$ is properly contained in $\mathcal{A}_2$ new conditions weaker than Bruno are sufficient to ensure linearizability in $\mathcal{A}_2$.

In this paper we consider in detail the case where $\mathcal{A}_1$ is the ring of germs of analytic diffeomorphisms at the origin of $n \geqslant 1$ complex variables, and $\mathcal{A}_2$ is the algebra of *Gevrey–s*, $s > 0$, formal power series: the *Gevrey–s linearization of analytic germs.*

Let $\hat{F} = \sum f_\alpha z^\alpha$, $(f_\alpha)_{\alpha\in\mathbb{N}^n} \subset \mathbb{C}^n$ be a formal power series, then we say that it is *Gevrey–s* [Ba], [Ra], $s > 0$, if there exist two positive constants C_1, C_2 such that:

$$|f_\alpha| \leqslant C_1 C_2^{-s|\alpha|}(|\alpha|!)^s \quad \forall \alpha \in \mathbb{N}^n. \tag{1.3}$$

[2] In this case let $\omega \in (0,1) \setminus \mathbb{Q}$ such that $\lambda = e^{2\pi i\omega}$ and let $(q_n)_n$ be the denominators of the convergents [HW] to ω, then the Bruno condition is equivalent to the convergence of the series $\sum_{k\geqslant 0} \frac{\log q_{k+1}}{q_k}$.

We denote the class of all formal vector valued power series Gevrey–s by $\mathcal{C}_s$. It is closed w.r.t. derivation and composition.

In the Gevrey–s case the arithmetical condition introduced in [CM], [Ca] will be called *Bruno–s* condition, $s > 0$: for short $A \in \mathcal{B}_s$ if there exists a strictly increasing sequence of positive integer $(p_k)_k$ such that:

$$\limsup_{|\alpha|\to+\infty} \left(2\sum_{m=0}^{\kappa(\alpha)} \frac{\log \Omega^{-1}(p_{m+1})}{p_m} - s\log|\alpha| \right) < +\infty, \tag{1.4}$$

where $\kappa(\alpha)$ is defined by $p_{\kappa(\alpha)} \leqslant |\alpha| < p_{\kappa(\alpha)+1}$.

Remark 1.1. — This definition recall the classical one of Bruno [Br], where first one suppose the existence of a strictly increasing sequence of positive integer such that (1.4) holds, then one can prove (see [Br] § IV p. 222) that one can take an exponentially growing sequence, e.g. $p_k = 2^k$. This holds also in our case, in fact we can prove that (1.4)is equivalent to:

$$\limsup_{N\to+\infty} \left(\sum_{l=0}^{N} \frac{\log \Omega^{-1}(2^{l+1})}{2^l} - sN2\log 2 \right) < +\infty.$$

Let us give a sketch of the proof of this claim. Take a sequence (p_k) for which (1.4) holds, then we can find sequences of positive integer $(m_k)_k$ and $(l_k)_k$ such that: $m_0 = 0$, $l_k \geqslant 1$, $m_{k+1} = m_k + l_k$ and

$$2^{m_k} \leqslant p_k < 2^{m_k+1} < \ldots < 2^{m_k+l_k} \leqslant p_{k+1} < 2^{m_k+l_k+1}.$$

The function $\Omega^{-1}(p)$ is increasing, hence: $\Omega^{-1}(2^{q+1}) < \Omega^{-1}(p_{k+1})$ for all $q = m_k, \ldots, m_k + l_k - 1$, and so:

$$\sum_{q=m_k}^{m_k+l_k-1} \frac{\log \Omega^{-1}(2^{q+1})}{2^q} < 4\frac{\log \Omega^{-1}(p_{k+1})}{p_k}. \tag{1.5}$$

Take any $|\alpha| > 1$, the corresponding $\kappa(\alpha)$ and fix $N(\alpha) = m_{\kappa(\alpha)} + l_{\kappa(\alpha)} - 2$. Then, dividing the sum of (1.4) into pieces from m_h to $m_h + l_h - 1$, for $h = 0, \ldots, \kappa(\alpha)$, and using estimate (1.5) for each piece, we get:

$$2\left(2\sum_{m=0}^{\kappa(\alpha)} \frac{\log \Omega^{-1}(p_{m+1})}{p_m} - s\log|\alpha| \right) > \sum_{l=0}^{N(\alpha)} \frac{\log \Omega^{-1}(2^{l+1})}{2^l} - 2s\log|\alpha|.$$

Now the claim follows remarking that $-2s\log|\alpha| \geqslant -sN(\alpha)2\log 2 + c$.

For real number, the Bruno–s condition can be slightly weakened (see [CM]); let $\omega \in (0,1) \setminus \mathbb{Q}$, then the Bruno–s, $s > 0$, condition reads:

$$\limsup_{k\to+\infty} \left(\sum_{j=0}^{k} \frac{\log q_{j+1}}{q_j} - s\log q_k \right) < +\infty, \tag{1.6}$$

where $(q_k)_k$ are the denominators of the convergents to ω. We remark that in both cases the new conditions are weaker than Bruno condition, which is recovered when $s = 0$. In dimension one we prove that the set $\bigcup_s \mathcal{B}_s$ is $PSL(2,\mathbb{Z})$–invariant (see remark 3.1). The main result of [Ca] in the case of Gevrey–s classes reads:

THEOREM 1.2 (Gevrey–s linearization). — *Let $\lambda_1, \dots, \lambda_n$ be complex numbers of unit modulus and $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$; let*

$$D_1 = \{z \in \mathbb{C}^n : |z_i| < 1\,, 1 \leqslant i \leqslant n\}$$

be the isotropic polydisk of radius 1 *and let $F : D_1 \to \mathbb{C}^n$ be an analytic function, such that $F(z) = Az + f(z)$, with $f(0) = Df(0) = 0$. If A is non–resonant and verifies a Bruno–s, $s > 0$, condition (1.4) (or (1.6) when $n = 1$), then there exists a formal Gevrey–s linearization $\widehat{H}$ which solves (1.1).*

The aim of this paper is to show that the Gevrey character of the formal linearization can give information concerning the dynamics of the analytic germ. Let $F(z) = Az + f(z)$ be a germ of analytic diffeomorphism verifying the hypothesis of Theorem 1.2, assume moreover F not to be analytically linearizable. We will show that even if there is not *Siegel disk*, where the dynamics of F is conjugate to the dynamics of its linear part, we have an open neighborhood of the origin which "behaves as a Siegel disk" under the iterates of F for finite but long time, which results exponentially long: the *effective stability* [GDFGS] of the fixed point.

In the case of analytic linearization, $|H_i^{-1}(z)|$, $i = 1, \dots, n$, (which is well defined sufficiently close to the origin because H is tangent to the identity) is *constant along the orbits*, namely it is a *first integral* and $|F^m(z_0)|$ is bounded for all m and sufficiently small $|z_0|$.

We will prove that any non–zero z_0 belonging to a polydisk of sufficiently small radius $r > 0$, can be iterate a number of times $K = \mathcal{O}(\exp\{r^{-1/s}\})$, being $s > 0$ the Gevrey exponent of the formal linearization, and we can find an *almost first integral*: a function which varies by a quantity of order r under $m \leqslant K$ iterations, which implies that $F^m(z_0)$ is well defined and bounded for $m \leqslant K$. More precisely we prove the following.

THEOREM 1.3. — *Let $\lambda_1, \dots, \lambda_n$ be complex numbers of unit modulus and $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$; let $F : D_1 \to \mathbb{C}^n$ be an analytic and univalent function, such that $F(z) = Az + f(z)$, with $f(0) = Df(0) = 0$. If*

A is non–resonant and verifies a Bruno–s, $s>0$, condition (1.4) (or (1.6) when $n=1$), then for all sufficiently small $0<r_{**}<1$, there exist positive constants A_{**}, B_{**}, C_{**} such that for all $0<|z_0|<r_{**}/2$, the m–th iterate of z_0 by F is well defined and verifies $|z_m| = |F^m(z_0)| \leqslant C_{**}r_{**}$, for all $m \leqslant K_* = \left\lfloor A_{**}^{-1} \exp\left\{ B_{**} \left(r_{**}/|z_0|\right)^{1/s} \right\}\right\rfloor$.

The hypothesis on the domain for F is a natural normalization condition being the whole problem invariant by homothety.

In Section 3 we compare our stability result with the stronger results which can be proved using Yoccoz's renormalization method [PM2] in the case $n=1$. Moreover we discuss the relation between our Bruno–s condition and the arithmetical condition of Pérez–Marco [PM1], [PM2] ensuring that in the non–linearizable case the fixed point is accumulate by periodic orbits.

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2. Proof of the main Theorem.

In this part we will prove our main result, Theorem 1.3. The proof will be divided into three steps: first we use the Gevrey–s character of the formal linearization $\widehat{H}$, given by Theorem 1.2, to find an approximate solution of the conjugacy equation (1.1) up to a (exponentially) small correction (paragraph 2.1); then we prove an iterative Lemma allowing us to control how the small error introduced in the solution propagates (paragraph 2.2). Finally we collect all the informations to conclude the proof (paragraph 2.3).

2.1. Determination of an approximate solution.

We apply Theorem 1.2: The formal power series solution $\widehat{H}$ belongs to $\mathcal{C}_s$, as well as its inverse $\widehat{H}^{-1}$ which solves (formally):

$$\widehat{H}^{-1} \circ F(z) = R_A \circ \widehat{H}^{-1}(z). \tag{2.1}$$

Since $\widehat{H}^{-1} = \sum h_\alpha z^\alpha \in \mathcal{C}_s$, there exist positive constants A_1 and B_1 such that

$$|h_\alpha| \leqslant A_1 B_1^{-s|\alpha|}(|\alpha|!)^s \quad \forall\, |\alpha| \geqslant 1. \tag{2.2}$$

For any positive integer N we consider the *vectorial polynomial*, sum of homogeneous vector monomials of degree $1 \leqslant l \leqslant N$, defined by: $\mathcal{H}_N(z) = \sum_{l=1}^N \sum_{|\alpha|=l} h_\alpha z^\alpha$ and the *Remainder Function*:

$$\mathcal{R}_N(z) = \mathcal{H}_N \circ F(z) - R_A \circ \mathcal{H}_N(z). \tag{2.3}$$

The following Proposition collects some useful properties of the remainder function.

PROPOSITION 2.1. — *Let $\mathcal{R}_N(z)$ be the remainder function defined in (2.3) and let $\alpha \in \mathbb{N}^n$, then:*

1) $\partial_z^\alpha \mathcal{R}_N(0) = 0$ *if* $|\alpha| \leqslant N$.

2) *For all $0 < r < 1$ there exists a positive constants A_2 and B_2 such that if $|\alpha| \geqslant N+1$, then:*

$$\left|\frac{1}{\alpha!}\partial_z^\alpha \mathcal{R}_N(0)\right| \leqslant A_2 r^{-|\alpha|} B_2^{-sN}(N!)^s.$$

3) *For all $0 < r < 1$ and $|z| < r/2$ there exist positive constants A_3, B_3 such that:*

$$|\mathcal{R}_N(z)| \leqslant A_3 B_3^{-sN}(N!)^s \left(\frac{|z|}{r}\right)^{N+1}. \tag{2.4}$$

Where we used the compact notation $\frac{1}{\alpha!}\partial_z^\alpha = \frac{1}{\alpha_1!\dots\alpha_n!}\frac{\partial^{|\alpha|}}{\partial_{z_1}^{\alpha_1}\dots\partial_{z_n}^{\alpha_n}}$.

Proof. — Statement 1) is an immediate consequence of the definition of $\mathcal{R}_N$.

To prove 2) we observe that $\mathcal{R}_N(z)$ is an analytic function on D_1, then one gets by Cauchy's estimates for all $0 < r < 1$ and for all $|\alpha| \geqslant N+1$:

$$\left|\frac{1}{\alpha!}\partial_z^\alpha \mathcal{R}_N(0)\right| \leqslant \frac{1}{(2\pi)^n}\frac{1}{r^{|\alpha|+1}}\max_{|z|=r}|\mathcal{H}_N \circ F(z)|. \tag{2.5}$$

Recalling the Gevrey estimate (2.2) for $\mathcal{H}_N$ and the analyticity of F we obtain:

$$\left|\frac{1}{\alpha!}\partial_z^\alpha \mathcal{R}_N(0)\right| \leqslant A_2 B_2^{-sN}(N!)^s r^{-|\alpha|}, \tag{2.6}$$

for some positive constants A_2 and B_2 depending on the previous constants, on the dimension n and on F.

To prove 3) let us write the Taylor series

$$\mathcal{R}_N(z) = \sum_{|\alpha|\geqslant N+1}\frac{1}{\alpha!}\partial_z^\alpha \mathcal{R}_N(0) z^\alpha,$$

then the bound on derivatives (2.6) implies the estimate (2.4) for all $|z| < r/2$ and for some positive constants A_3 and B_3. □

The bound (2.4) on $\mathcal{R}_N(z)$ depends on the positive integer N, so we can determine the value of N for which the right hand side of (2.4) attains its minimum, that's Poincaré's idea of *summation at the smallest term.*

LEMMA 2.2 (Summation at the smallest term). — *Let $\mathcal{R}_N(z)$ defined as before and let $0 < r_* < 1/2$ then there exist positive constants A_4, B_4 such that for all $0 < |z| < r_*$ we have:*

$$|\mathcal{R}_{\bar{N}}(z)| \leqslant A_4 \exp\left\{ - B_4 \left(\frac{r_*}{|z|}\right)^{1/s} \right\}, \tag{2.7}$$

where $\bar{N} = \lfloor B_4 \left(r_*/|z|\right)^{1/s} \rfloor$ and $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$.

Proof. — Let us fix $0 < r_* < 1/2$, then for $0 < |z| < r_*$ by Stirling formula we obtain:

$$|\mathcal{R}_N(z)| \leqslant A_4 \left(N B_3^{-1} \left(|z|/r_*\right)^{1/s}\right)^{Ns} e^{-sN}, \tag{2.8}$$

for some positive constant A_4. The right hand side of (2.8) attains its minimum at $\bar{N} = B_3 \left(r_*/|z|\right)^{1/s}$, evaluating the value of this minimum we get (2.7) with $B_4 = B_3$. □

2.2. Control of the "errors".

Let us define $\mathcal{H}(z) = \mathcal{H}_{\bar{N}}(z)$ and $\mathcal{R}(z) = \mathcal{R}_{\bar{N}}(z)$, being $\bar{N}$ the "optimal value" obtained in Lemma 2.2. We remark that $\mathcal{H}(z)$ doesn't solve (2.1) but the "error", $\mathcal{R}(z)$, is very small: exponentially small. We will prove that for initial conditions in a sufficiently small disk, one can iterate an exponentially large number of times without leaving a disk, say, of double size.

LEMMA 2.3 (Iteration lemma). — *Let a, b, α and R be positive real numbers. Let us consider the sequence of positive number $(\mu_j)_{j\geqslant 0}$ defined by:*

$$\mu_0 = R \quad \textit{and} \quad \mu_{j+1} = \mu_j + a\exp\{-b/\mu_j^{\alpha}\}.$$

Let $K = \lfloor Ra^{-1} \exp\{b/(2R)^{\alpha}\} \rfloor$, then $\mu_j \leqslant 2R$ for all $j \leqslant K$.

Proof. — Let us prove by induction on j that for all $0 \leqslant j \leqslant K$ we have

$$\mu_j \leqslant R + ja\exp\{-b/(2R)^\alpha\}, \tag{2.9}$$

then the claim will follow from (2.9) and the definition of K, in fact for all $j \leqslant K$:

$$\mu_j \leqslant R+ja\exp\{-b/(2R)^\alpha\} \leqslant R+Ra^{-1}\exp\{b/(2R)^\alpha\}a\exp\{-b/(2R)^\alpha\} \leqslant 2R.$$

The basis of induction is easily verified; assume (2.9) for all $j \leqslant K-1$, we will prove it for $j = K$. By definition of $(\mu_j)_j$ and the induction hypothesis we have:

$$\begin{aligned}\mu_K &= \mu_{K-1} + a\exp\{-b/\mu_{K-1}^\alpha\}\\ &\leqslant R + (K-1)a\exp\{-b/(2R)^\alpha\} + a\exp\{-b/\mu_{K-1}^\alpha\},\end{aligned}$$

we remark that from (2.9) with $j = K-1$, using $K-1 < Ra^{-1}\exp\{b/(2R)^\alpha\}$, we get $\mu_{K-1} \leqslant 2R$ and $\exp\{-b/\mu_{K-1}^\alpha\} \leqslant \exp\{-b/(2R)^\alpha\}$. Then we conclude:

$$\mu_K \leqslant R + (K-1)a\exp\{-b/(2R)^\alpha\} + a\exp\{-b/(2R)^\alpha\},$$

which ends the induction. □

Let r_* as in Lemma 2.2, define $\rho(z) = |\mathcal{H}(z)|$ for all $0 < |z| < r_*$, then Lemma 2.2 admits the following Corollary, which allows us to control the function $\rho(z)$ on consecutive points of an orbit of $F(z)$.

COROLLARY 2.4. — *Let $0 < r_* < 1/2$, let r_1 be the radius of the maximal polydisk where $\mathcal{H}(z)$ is invertible and let $r_{**} = \min(r_*, r_1)$. Then there exist positive constants A_*, B_* such that for all $0 < |z| < r_{**}$ we have:*

$$\Big|\rho(F(z)) - \rho(z)\Big| \leqslant A_*\exp\Big\{-B_*\left(\frac{r_*}{\rho(z)}\right)^{1/s}\Big\}. \tag{2.10}$$

Proof. — By definition $\rho(F(z)) = |\mathcal{H}\circ F(z)|$ and $\rho(z) = |R_A\circ\mathcal{H}(z)|$, since $|\lambda_j| = 1$ for $1 \leqslant j \leqslant n$, and $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, therefore:

$$\Big|\rho(F(z)) - \rho(z)\Big| \leqslant \Big|\mathcal{H}\circ F(z) - R_A\circ\mathcal{H}(z)\Big| = |\mathcal{R}(z)|,$$

and from Lemma 2.2 we get:

$$\Big|\rho(F(z)) - \rho(z)\Big| \leqslant A_4\exp\Big\{-B_4\left(\frac{r_*}{|z|}\right)^{1/s}\Big\}. \tag{2.11}$$

We want to express this condition in terms of $\rho(z)$ instead of $|z|$, to do this we have to consider the distortion properties of $\mathcal{H}(z)$ and of its inverse. Let $J(z) = \partial_z \mathcal{H}(z)$ be the Jacobian of $\mathcal{H}(z)$ and let $J_1 = \max_{|z|\leqslant r_1} |J(z)|$, where r_1 has been defined previously. Let $0 < |z| < r_1$ and let us call $z' = \mathcal{H}(z)$, clearly $|z'| \leqslant J_1 r_1 = r_2$. Let us call $J_2 = \max_{|z'|<r_2} |\partial_{z'} \mathcal{H}^{-1}(z')|$, then for any $0 < |z'| < r_2$ there exists only one z such that $z = \mathcal{H}^{-1}(z')$, which satisfies $|z| \leqslant J_2 |z'| = J_2 |\mathcal{H}(z)|$.

Let $r_{**} = \min(r_*, r_1)$ then from (2.11) for any $0 < |z| < r_{**}$ we get:

$$\Big|\rho(F(z)) - \rho(z)\Big| \leqslant A_* \exp\Big\{ - B_* \left(\frac{r_*}{\rho(z)}\right)^{1/s} \Big\},$$

where $A_* = A_4$ and $B_* = B_4 J_2^{-1/s}$. □

2.3. End of the proof.

We are now able to conclude the proof of the main Theorem 1.3. Take any $0 < |z_0| < r_{**}/2$ and let us define $\rho_0 = |z_0|$, $\rho_m = \rho(F^m(z_0))$ for all positive integer m for which $F^m(z_0)$ is well defined, by Corollary 2.4 we have:

$$\rho_m \leqslant \rho_{m-1} + A_* \exp\Big\{ - B_* \left(r_*/\rho_{m-1}\right)^{1/s} \Big\}. \tag{2.12}$$

Let us call $R = |z_0|$, $a = A_*$, $b = B_* r_*^{1/s}$ and $\alpha = 1/s$, then we can apply Lemma 2.3, being $\mu_m \geqslant \rho_m$, to conclude that:

$$\rho_m \leqslant r_{**} \quad \forall m \leqslant K_* = \Big\lfloor |z_0| A_*^{-1} \exp\Big\{ B_* \left(\frac{r_*}{2|z_0|}\right)^{1/s} \Big\} \Big\rfloor. \tag{2.13}$$

This implies that $\mathcal{H}(z_m)$ is well defined in this range of values of m, it is not constant and it evolves only by $\big||\mathcal{H}(z_m)| - |\mathcal{H}(z_0)|\big| \leqslant r_{**}$. Recalling that $z_m = F^m(z_0)$ we also have $|F^m(z_0)| \leqslant J_2 r_{**}$ and $\big||z_m| - |z_0|\big| \leqslant J_2 r_{**}$ for all $0 < |z_0| < r_{**}/2$ and all $m \leqslant K_*$.

This conclude the proof by setting $A_{**} = 2A_* r_{**}^{-1}$, $B_{**} = B_*(r_*/(2r_{**}))^{1/s}$ and $C_{**} = J_2$.

3. One dimensional case.

In this paper we proved that any analytic germs of diffeomorphisms of $(\mathbb{C}^n, 0)$ with diagonal, non–resonant linear part has an *effective stability* domain, i.e. stable up to finite but "long times", close to the fixed point, provided the linear part verifies a new arithmetical Bruno–like condition (1.4) depending on a parameter $s > 0$.

Remark 3.1 (Invariance of $\cup_{s>0}\mathcal{B}_s$, $n = 1$ under the action of $PSL(2,\mathbb{Z})$). — The continued fraction development [HW], [MMY] of an irrational number ω gives us the sequences: $(a_k)_{k\geqslant 0}$ and $(\omega_k)_{k\geqslant 0}$. Then we introduce $(\beta_k)_{k\geqslant -1}$ defined by $\beta_{-1} = 1$ and for all integer $k \geqslant 0$: $\beta_k = \prod_{j=0}^{k}\omega_k$, which verifies : $1/2 < \beta_k q_{k+1} < 1$ and $q_n\beta_{n-1} + q_{n-1}\beta_n = 1$, where q_k's are the denominators of the continued fraction development of ω. We claim that condition Bruno–s (1.6) is equivalent to the following one:

$$\limsup_{k\to+\infty}\left(\sum_{j=0}^{k}\beta_{j-1}\log\omega_j^{-1} + s\log\beta_{k-1}\right) < +\infty. \tag{3.1}$$

This can be proved by using the relations between β_l and q_l, to obtain the bound, for all integer $k > 0$:

$$\left|\sum_{l=0}^{k}\left(\beta_{l-1}\log\omega_l + \frac{\log q_{l+1}}{q_l}\right)\right| \leqslant \sum_{l=0}^{k}\left|\beta_{l-1}\log\beta_l q_{l+1}\right| + \left|\beta_{l-1}\log\beta_{l-1}\right| + \left|\frac{q_{l-1}}{q_l}\beta_l\log q_{l+1}\right| \leqslant 18,$$

where we used the convergence of series $\sum q_l^{-1}$ and $\sum q_l^{-1}\log q_l$ (see [MMY] p. 272).

To prove the invariance of $\cup_s\mathcal{B}_s$ under the action of $PSL(2,\mathbb{Z})$, is enough to consider its generators: $T\omega = \omega + 1$ and $S\omega = 1/\omega$. For any irrational ω, T acts trivially being $\beta_k(T\omega) = \beta_k(\omega)$ for all k, whereas for S we have $\beta_k(\omega) = \omega\beta_{k-1}(S\omega)$ for all $k \geqslant 1$. Let ω be an irrational and let $\omega' = \omega^{-1}$, let us also denote with a $'$ quantities given by the continued fraction algorithm applied to ω', then using (3.1) one can prove:

$$\omega_0\left(\sum_{j=0}^{k}\beta'_{j-1}\log{\omega'_j}^{-1} + s\omega_0^{-1}\log\beta'_{k-1}\right) = C(\omega,s) + \sum_{j=0}^{k+1}\beta_{j-1}\log\omega_j^{-1} + s\log\beta_k,$$

where $C(\omega,s) = \omega_0\left(\log\omega_1^{-1} - s\log\omega_0\right) + \sum_{l=0}^{1}\beta_{l-1}\log\omega_l^{-1}$, from which the claim follows.

Let us consider a slightly stronger version of the Bruno–s condition: $\omega\in\mathbb{R}\setminus\mathbb{Q}$ belongs to $\tilde{\mathcal{B}}_s$ if:

$$\lim_{k\to+\infty}\left(\sum_{l=0}^{k}\frac{\log q_{l+1}}{q_l} - s\log q_k\right) < +\infty, \tag{3.2}$$

where $(q_n)_n$ are the convergents to ω.

Remark 3.2. — This new condition is stronger than Bruno–s, because the existence of the limit is required. One can construct numbers ω which verify $\mathcal{B}_s$ but not $\tilde{\mathcal{B}}_s$, as follows.

Let us call for short $s_k = \sum_{l=0}^{k} \frac{\log q_{l+1}}{q_l} - s \log q_k$. Fix $\alpha > \beta > 0$, $\delta > 0$ and to simplify take $s > (2+\delta)/\log 2$. We claim that one can choose large enough positive integers k_1 and $a_1, \dots, a_{k_1+1}$ such that:

$$s_k - s_{k-1} \geqslant \delta \quad \text{and} \quad s_{k_1} \in (\alpha, 2\alpha),$$

for instance take $k_1 > (\alpha - \log q_1)\delta^{-1}$ and inductively

$$a_{l+1} \geqslant \left[e^{\delta q_l} \left(\frac{q_{l-1}}{q_{l-2}} \right)^{2q_l} - q_{l-1} \right] q_l^{-1}, \quad \text{for all } \ l = 1, \dots, k_1.$$

Then one can take sufficiently many a_l's equal to one, $a_{k_1+1+l} = 1$ for $l = 1, \dots, 2k_2 + 1$, and verify that:

$$s_{k_1+1+2j} - s_{k_1+1+2j-2} < -\delta, \ \forall j = 0, \dots, k_2 \quad \text{and} \quad s_{k_1+1+2k_2} < \beta.$$

Then one iterate taking a sufficiently large block of large enough a_k's, followed by a sufficiently large block of $a_k = 1$. The real number whose continued fraction development is given by $x = [0, a_1, \dots, a_n, \dots]$, verifies by construction Bruno–s, with $s > (2+\delta)/\log 2$, being $(s_k)_k$ bounded by 2α, but it doesn't verify $\tilde{\mathcal{B}}_s$, in fact $(s_k)_k$ oscillates from values larger than α to values smaller than β, without reaching any limit.

Let us introduce two other arithmetical conditions. Let us denote by $\mathcal{B}'_s$ the set of irrational numbers whose convergents verify:

$$\lim_{k \to +\infty} \frac{\log q_{k+1}}{q_k \log q_k} = s. \tag{3.3}$$

And a second condition as follows, let $(\gamma_m)_{m \geqslant 1}$ and $(s_m)_{m \geqslant 1}$ be two positive sequences of real numbers such that: $\sum_1^{+\infty} \gamma_m = \gamma < +\infty$ and $\sum_1^{+\infty} s_m = \sigma < +\infty$, then we define condition $\mathcal{B}_{\gamma,\sigma}$ by:

$$\frac{\log q_{m+1}}{q_m} \leqslant s_m \log q_m + \gamma_m \quad \forall m \geqslant 1. \tag{3.4}$$

PROPOSITION 3.3. — *Let $\omega \in (0,1) \setminus \mathbb{Q}$ and let $s > 0$ then we have the following inclusions:*

1) *let $\omega \in \tilde{\mathcal{B}}_s$, if ω is not a Bruno number then $\omega \in \mathcal{B}'_s$, otherwise $\omega \in \mathcal{B}'_0$.*

2) *Let* $\sigma \leqslant s$ *and* $\omega \in \mathcal{B}_{\gamma,\sigma}$ *then* $\omega \in \tilde{\mathcal{B}}_s$*;*

Proof. — To prove the first statement let us write the following identity:

$$(3.5)\quad \sum_{l=0}^{k} \frac{\log q_{l+1}}{q_l} - s\log q_k = C + \sum_{l=2}^{k}\left[\frac{\log q_{l+1}}{q_l} - s\left(\log q_l - \log q_{l-1}\right)\right],$$

where $C = (1-s)\log q_1 + \frac{\log q_2}{q_1}$. By condition $\tilde{\mathcal{B}}_s$, this series converges and then its generic term goes to zero, from which we get:

$$(3.6)\qquad \lim_{k\to+\infty} \frac{\log q_{k+1}}{q_k \log q_k} = s\left(1 - \lim_{k\to+\infty}\frac{\log q_{k-1}}{\log q_k}\right).$$

Let us denote by s' be value of the right hand side of (3.6), then clearly $s' \in [0,s]$. Let us suppose $s' > 0$, but then we have for all sufficiently large k:

$$\frac{C_1}{q_k} \leqslant \frac{\log q_k}{\log q_{k+1}} \leqslant \frac{C_2}{q_k},$$

for some positive constants C_1, C_2, from which we get $\frac{\log q_k}{\log q_{k+1}} \to 0$, and from (3.6) we conclude that $s' = s$.

If $s' = 0$, namely $\frac{\log q_k}{\log q_{k+1}} \to 1$, then it is easy to check that ω is a Bruno number.

Let us prove the second statement. For any positive integer k, using the definition of $\mathcal{B}_{\gamma,\sigma}$ we can write:

$$(3.7)\qquad \sum_{l=0}^{k} \frac{\log q_{l+1}}{q_l} - s\log q_k \leqslant \sum_{l=0}^{k} s_l \log q_l + \sum_{l=0}^{k}\gamma_l - s\log q_k,$$

for all $0 \leqslant l \leqslant k$ we have $\log q_l \leqslant \log q_k$ then the right hand side of (3.7) is bounded by: $-\log q_k\left(s - \sum_{l=0}^{k} s_l\right) + \sum_{l=0}^{k}\gamma_l$. By hypothesis $\sum_{l=0}^{k} s_l \leqslant s$, for all k, then using $-\log q_k \leqslant -\log q_1$ we obtain:

$$\sum_{l=0}^{k} \frac{\log q_{l+1}}{q_l} - s\log q_k \leqslant -\log q_1\left(s - \sum_{l=0}^{k} s_l\right) + \sum_{l=0}^{k}\gamma_l,$$

then passing to the limit on k we have:

$$\lim_{k\to+\infty}\sum_{l=0}^{k} \frac{\log q_{l+1}}{q_l} - s\log q_k \leqslant -\log q_1\left(s-\sigma\right) + \gamma < +\infty.$$

□

Remark 3.4. — These new arithmetical conditions are weaker than the Bruno one, for instance condition $\mathcal{B}'_s$ is verified by numbers ω whose denominators $(q_k)_k$ satisfy a growth condition like $q_{k+1} \sim (q_k!)^s$. Condition $\mathcal{B}_{\gamma,\sigma}$ implies convergence of the series: $\sum_{k\geqslant 0} \frac{\log q_{k+1}}{q_k \log q_k}$.

Let us conclude recalling a stability result of Pérez–Marco [PM1], [PM2] and compare it with our result. In [PM2] author proved (Theorem V.2.1, Annexe 2 § f) using a *geometric renormalization* scheme "à la Yoccoz" valid in the one dimensional case, a stability result that can be stated as follows:

THEOREM 3.5 (Pérez–Marco, Contrôle de la diffusion). — *Let $\omega \in (0,1) \setminus \mathbb{Q}$ and let $(q_k)_k$ be the denominators of its convergents. Let F be an analytic and univalent function defined in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ such that $F(z) = \lambda z + \mathcal{O}(|z|^2)$, where $\lambda = e^{2\pi i\omega}$. There exist two positive constants C_1, C_2 such that if:*

$$|z| \leqslant C_1 e^{-\sum_{j=0}^{k-1} \frac{\log q_{j+1}}{q_j}}, \tag{3.8}$$

then for all integer $0 \leqslant m \leqslant q_k$ we have:

$$|F^m(z)| \leqslant C_2 e^{-\sum_{j=0}^{k-1} \frac{\log q_{j+1}}{q_j}}. \tag{3.9}$$

The meaning of the Theorem is clear: if we start inside a disk of radius $r = C_1 e^{-\sum_{j=0}^{k-1} \frac{\log q_{j+1}}{q_j}}$ then we can apply F, up to q_k times, without leaving a disk of radius rC_2/C_1. To compare this result with our effective stability result we have to make explicit the relation between r and q_k, which give the time of "stability". Using our Bruno–s condition (3.2) we can say that $C \leqslant r q_{k-1}^s \leqslant C'$ for some positive constants C, C'. But from (3.3) we get $\log q_k \leqslant C_3 q_{k-1} \log q_{k-1}$ for some positive constant C_3, namely there exist positive constants C_3', C_4 such that:

$$q_k \leqslant \exp\left\{ \frac{C_3'}{r^{1/s}} \log \frac{C_4}{r^{1/s}} \right\}.$$

We can then restate Theorem 3.5 as follows: if $|z| \leqslant r$, then $|F^m(z)| \leqslant rC_2/C_1$ for all integer $0 \leqslant m \leqslant \exp\left\{ \frac{C_3'}{r^{1/s}} \log \frac{C_4}{r^{1/s}} \right\}$, obtaining a better estimate on the time of effective stability.

This improvement has been obtained thanks to a good understanding of the geometry of the dynamics, if one would like to obtain these better estimates also for germs in higher dimensions, one should extend the Pérez–Marco ideas to understand the geometry of dynamics of germs in higher dimension. This could be very difficult whereas our results are easy to adapt to any dimension.

We end with a last remark related again to the work of Pérez–Marco.

Remark 3.6. — Pérez–Marco proved in [PM1], [PM2] that any non–analytically linearizable analytic germ, univalent in the unit disk, whose multiplier at the fixed point, verifies the following arithmetical condition:

$$\sum_{k\geqslant 0} \frac{\log\log q_{k+1}}{q_k} < +\infty, \tag{3.10}$$

has a sequence of periodic orbits accumulating the fixed point, whose periods, $(q_{n_k})_k$, make the Bruno series diverging.

Our Bruno–s condition implies (3.10), in fact from (3.2) we get:

$$\sum_{k=0}^{N} \frac{\log\log q_{k+1}}{q_k} \leqslant \sum_{k=0}^{N} \left(\frac{\log C_3}{q_k} + \frac{\log q_k}{q_k} + \frac{\log\log q_k}{q_k} \right),$$

we can let N grow and using standard number theory results concerning the convergents, we obtain the Pérez–Marco condition. Then we can suppose these periodic orbits accumulating the fixed point to "produce the effective stability: preventing the orbits from a too fast escape", a situation similar to the one holding in the Nekoroshev Theorem for Hamiltonian systems [Ne], where the resonant web confines the flow for exponentially long times. It would be very interesting to know whether a similar phenomenon takes place in higher dimension.

We conclude by pointing out that our method gives us a stability exponent depending on the Gevrey exponent and independent of the dimension: the bigger is the exponent, longer is the time interval of stability, we can always take s small enough to have a very long time of stability.

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