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A SPECTRAL ANALYSIS OF AUTOMORPHIC DISTRIBUTIONS AND POISSON SUMMATION FORMULAS

by André UNTERBERGER

1. Introduction.

The standard Poisson formula is the equation

$$(1.1) \quad \sum_{m \in \mathbb{Z}^d} (\mathcal{F}h)(m) = \sum_{m \in \mathbb{Z}^d} h(m),$$

valid for every function h in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of C^∞ functions on \mathbb{R}^d , rapidly decreasing at infinity: the Fourier transformation \mathcal{F} is normalized as

$$(1.2) \quad (\mathcal{F}h)(x) = \int_{\mathbb{R}^d} h(y) e^{-2i\pi\langle x, y \rangle} dy.$$

A special case is the so-called identity of theta functions, in which $t > 0$ is otherwise arbitrary,

$$(1.3) \quad \sum_{m \in \mathbb{Z}^d} e^{-\pi t |m|^2} = t^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}^d} e^{-\pi t^{-1} |m|^2}.$$

In this paper, we shall suggest a variety of identities very similar in appearance to these two formulas. One of the simplest such generalizations of (1.3) is the following: let $\Phi_{d,1}$ be the function on \mathbb{R}^d characterized by the equation

$$(1.4) \quad (\mathcal{F}\Phi_{d,1})(\xi) = |\xi|^{-d-1} \exp\left(-\frac{\pi}{4|\xi|^2}\right).$$

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For every integer $n \geq 1$, let $\text{Sq}_{2d}(n)$ be the number of ways n can be decomposed as the sum of squares of $2d$ integers (of any sign; the order is taken into account). Setting, for $m \in \mathbb{Z}^d \setminus \{0\}$,

$$(1.5) \quad c_m = \sum_{n|(m_1, \dots, m_d)} \text{Sq}_{2d}(n)$$

and $c_0 = -1$, one has for every $t > 0$

$$(1.6) \quad \sum_{m \in \mathbb{Z}^d} c_m \Phi_{d,1}(tm) = t^{-d} \sum_{m \in \mathbb{Z}^d} c_m \Phi_{d,1}\left(\frac{m}{t}\right).$$

One may generalize (1.1) instead. Let $\Phi \in \mathcal{S}(\mathbb{R}^d)$ (less stringent conditions, making the consideration of the function $\Phi_{d,1}$ possible, suffice) satisfy $\int \Phi(x) dx = 0$, and set, for $\xi \neq 0$,

$$(1.7) \quad \Psi(\xi) = \pi \int_0^\infty s^{\frac{d-3}{2}} J_{d-1}\left(\frac{2\pi}{\sqrt{s}}\right) (\mathcal{F}\Phi)(s\xi) ds,$$

where J_{d-1} is the Bessel function so denoted. The function Ψ extends as a continuous function on the whole of \mathbb{R}^d , and the identity

$$(1.8) \quad \sum_{m \in \mathbb{Z}^d} c_m \Psi(m) = \sum_{m \in \mathbb{Z}^d} c_m \Phi(m)$$

holds.

The main features of the generalizations of Poisson's formula we have in mind are already apparent on this example. First, the coefficients c_m are no longer trivial: in the examples we shall discuss, they will always be borrowed from the consideration of modular forms. Next, the transformation $\Phi \mapsto \Psi$ will not be the Fourier transformation any more: instead, it will be the composition of the Fourier transformation by some function — a product of Gamma factors — of the Euler operator

$$(1.9) \quad 2i\pi\mathcal{E} = \sum x_j \frac{\partial}{\partial x_j} + \frac{d}{2}$$

on \mathbb{R}^d (the extra constant makes \mathcal{E} a formally self-adjoint operator on $L^2(\mathbb{R}^d)$). One has for instance, in the preceding example,

$$(1.10) \quad \Psi = \pi^{4i\pi\mathcal{E}} \frac{\Gamma(\frac{d}{2} - 2i\pi\mathcal{E})}{\Gamma(\frac{d}{2} + 2i\pi\mathcal{E})} \mathcal{F}\Phi.$$

One of the main tools of the theory of modular forms is Hecke's theory, which starts with the consideration, in association with such a form f , of a certain Dirichlet series $\mathcal{D}(s) = \sum_{r \geq 1} a_r r^{-s}$ (the L -function of f), the coefficients of which are taken from the Fourier expansion of f .

In the case when f is a modular form for the full modular group $SL(2, \mathbb{Z})$, the function \mathcal{D} satisfies a *functional equation*: the product $\mathcal{D}^*(s)$ of the function $\mathcal{D}(s)$ by $(2\pi)^{-s}\Gamma(s)$ is invariant, or changes to its negative, under the symmetry $s \mapsto k - s$, where the integer k is the weight of f . This is a characterization, which extends, if one substitutes for the Dirichlet series a finite set of “twisted” versions thereof, to the case when the full modular group is replaced by some congruence subgroup (cf. [1, p. 60] for Weil’s so-called converse theorem). Similar species of functional equations hold for the Dirichlet series associated with non-holomorphic modular forms: however, in this case, the extra factor from \mathcal{D} to \mathcal{D}^* (referred to, in general, as the Archimedean factor) involves the product of two Gamma functions.

In the present paper, we interpret the function \mathcal{D} in some specific spectral-theoretic sense, substituting for the argument s the Euler operator (1.9). The class of functions, or rather distributions, on which these Dirichlet series operate, consists of *combs*, which are measures on \mathbb{R}^d supported in \mathbb{Z}^d , automorphic in the sense that they are invariant under the linear action of the group $SL(d, \mathbb{Z})$: the best-known such example is of course the Dirac comb, the sum of unit masses at the points of $\mathbb{Z}^d \setminus \{0\}$. Considering the image \mathfrak{T} of the Dirac comb under some operator immediately related to $\mathcal{D}^*(2i\pi\mathcal{E})$, one can reinterpret the functional equation of the Dirichlet series \mathcal{D} as being the invariance of \mathfrak{T} under the Fourier transformation.

It is convenient, since Dirichlet series in the Euler operator as an argument preserve the class of combs, to set aside the Archimedean factor of the function \mathcal{D}^* . We then end up with Poisson-like formulas such as (1.8), in which the Archimedean factor appears, as an operator on the test-function Φ , in transformations such as (1.10). Some specially nice formulas, closer to a generalization of the identity of theta functions (1.3), involve the same Φ on both sides of the equation: Section 5 of this paper is entirely devoted to the construction of such functions.

Up to some point, the machinery in this paper could be considered as being one step further down Hecke’s path from modular forms to Dirichlet series: here, Dirichlet series with functional equations are transformed into Poisson-like formulas. Let us stress that the second step is independent from the first one, *i.e.* does not demand any understanding of the origin of the Dirichlet series under consideration. As a consequence, the proper general frame for these developments would probably be close to Selberg’s class [12] of L -functions. Space and time limitations, as well as the point of view that examples are usually more exciting than axiomatics, led us

to our present choice of working, instead, with Dirichlet series built with the help of (holomorphic or non-holomorphic) modular form theory; in the non-holomorphic case (Prop. 4.2 and Theorem 4.7), we shall also have to consider at some point an identity related to the so-called theory of convolution L -functions.

In a short conclusion, at the end of this paper, we shall refer to some other types of formulas known as non-Euclidean Poisson formulas, which bear no relation to the ones discussed here. On the other hand, a special case (*cf.* Proposition 3.2) of the present Poisson formulas has been known for a century as the Voronoï identity: it is the starting point of most investigations on Gauss's circle problem. Finally, but we shall leave this to the conclusion too, we want to call the reader's attention to the general concept of automorphic distributions: this is an approach to modular form theory with some advantages, which made possible, in particular, the development [18] of automorphic pseudodifferential analysis.

2. Combs and Euler's operator.

DEFINITION 2.1. — *Let $d = 1, 2, \dots$. A distribution on \mathbb{R}^d will be said to be automorphic if it is invariant under the linear action on \mathbb{R}^d of the group $SL(d, \mathbb{Z})$. A comb on \mathbb{R}^d is any measure \mathfrak{S} supported in $\mathbb{Z}^d \setminus \{0\}$, at the same time an automorphic distribution: in the case when $d = 1$, it is also assumed that it is even, i.e. that it vanishes when applied to odd functions.*

LEMMA 2.2. — *If $d \geq 2$, $SL(d, \mathbb{Z})$ acts transitively on the set of (column) vectors $(m_1 \cdots m_d)^\top \in \mathbb{Z}^d \setminus \{0\}$ with a fixed (m_1, \dots, m_d) (the positive g.c.d. of all coordinates m_1, \dots, m_d).*

Proof. — Let $r = (m_1, \dots, m_d)$: in the case when $d = 2$, let a and $c \in \mathbb{Z}$ satisfy $am_2 - cm_1 = r$: then

$$(2.1) \quad (m_1 \ m_2)^\top = \begin{pmatrix} a & \frac{m_1}{r} - a \\ c & \frac{m_2}{r} - c \end{pmatrix} (r \ r)^\top.$$

If $d \geq 3$, set $q = (m_1, \dots, m_{d-1})$, so that $(q, m_d) = r$. By induction, we may assume that there exists some matrix $B \in SL(d-1, \mathbb{Z})$ such that

$$(2.2) \quad (m_1 \cdots m_d)^\top = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} (q \cdots q \ m_d)^\top;$$

then, with $C \in SL(2, \mathbb{Z})$ such that $(q m_d)^\top = C(r r)^\top$, one has

$$(2.3) \quad (q \cdots q m_d)^\top = \begin{pmatrix} I_{d-2} & 0 \\ 0 & C \end{pmatrix} (q \cdots q r r)^\top$$

and, with $D_{d-2,2} = \begin{pmatrix} 0 & \frac{q-1}{r} \\ \vdots & \vdots \\ 0 & \frac{q-1}{r} \end{pmatrix}$,

$$(2.4) \quad (q \cdots q r r)^\top = \begin{pmatrix} I_{d-2} & D_{d-2,2} \\ 0 & I_2 \end{pmatrix} (r \cdots r)^\top.$$

□

Notation. — If $m \in \mathbb{Z}^d \setminus \{0\}$, $r(m)$ will always denote the g.c.d. of the coordinates of m .

Thus combs in \mathbb{R}^d are just measures of the kind

$$(2.5) \quad \mathfrak{S}_{\mathbf{a}}(x) = 2\pi \sum_{\substack{m \in \mathbb{Z}^d \\ m \neq 0}} a_{r(m)} \delta(x - m),$$

where $\mathbf{a} = (a_1, \dots, a_r, \dots)$ is any sequence of complex numbers: in most cases, we shall have to add to such a comb a constant plus a multiple of the unit mass at 0, which will of course not destroy the invariance under the action of $SL(d, \mathbb{Z})$. In the case when $a_r = 1$ for all r , we get the *Dirac comb*

$$(2.6) \quad \mathfrak{D}(x) = 2\pi \sum_{\substack{m \in \mathbb{Z}^d \\ m \neq 0}} \delta(x - m),$$

and in the case when $a_1 = 1$ but $a_r = 0$ for $r \geq 2$, we get the comb

$$(2.7) \quad \mathfrak{D}^{\text{prime}}(x) = 2\pi \sum_{\substack{m \in \mathbb{Z}^d \\ (m_1, \dots, m_d) = 1}} \delta(x - m).$$

The distribution $\mathfrak{S}_{\mathbf{a}}$ is tempered if and only if $|a_r|$ is bounded by some power of r for $r \geq 2$. In order to decompose tempered combs into their homogeneous components, we introduce the *Eisenstein distributions*. First, define the Euler operator \mathcal{E} on \mathbb{R}^d by the equation (1.9). If a function h lies in $\mathcal{S}(\mathbb{R}^d)$, one sets

$$(2.8) \quad (t^{2i\pi\mathcal{E}} h)(x) = t^{\frac{d}{2}} h(tx)$$

for every $t > 0$ and, for every tempered distribution \mathfrak{S} ,

$$(2.9) \quad \langle t^{2i\pi\mathcal{E}} \mathfrak{S}, h \rangle = \langle \mathfrak{S}, t^{-2i\pi\mathcal{E}} h \rangle$$

for every $h \in \mathcal{S}(\mathbb{R}^d)$. A tempered distribution \mathfrak{E} is homogeneous of degree $-\frac{d}{2} - \nu$ if and only if $t^{2i\pi\mathcal{E}} \mathfrak{E} = t^{-\nu} \mathfrak{E}$. Since the operator \mathcal{E} is formally self-adjoint on $L^2(\mathbb{R}^d)$, we are specially interested in distributions homogeneous of degree $-\frac{d}{2} - i\lambda$, $\lambda \in \mathbb{R}$.

DEFINITION 2.3. — *If $\nu \in \mathbb{C}$, $\text{Re } \nu < -\frac{d}{2}$, we define the Eisenstein distribution \mathfrak{E}_ν^d by the equation, valid for every $h \in \mathcal{S}(\mathbb{R}^d)$,*

$$(2.10) \quad \langle \mathfrak{E}_\nu^d, h \rangle = \sum_{\substack{m \in \mathbb{Z}^d \\ m \neq 0}} \int_0^\infty t^{-\nu + \frac{d}{2}} h(tm) \frac{dt}{t}.$$

It is immediate (but we shall prove more in a moment) that the integral converges if $\text{Re } \nu < -\frac{d}{2}$, so that \mathfrak{E}_ν^d is well defined as a tempered distribution (not a comb, of course) as soon as $\text{Re } \nu < -\frac{d}{2}$. Obviously, it is $SL(d, \mathbb{Z})$ -invariant as a distribution, also an even distribution (i.e. it vanishes on odd functions), finally it is homogeneous of degree $-\nu - \frac{d}{2}$. In the two-dimensional case, the relation between this notion and the classical one of non-holomorphic Eisenstein series can be found in [18, p. 18–20].

Before we analyze the convergence in more detail, it is necessary to introduce a class of functions with a very specific behaviour, as $\mathcal{S}(\mathbb{R}^d)$ will not do for our purpose.

DEFINITION 2.4. — *Let $\varepsilon > 0$ be given: we shall say that a function h on \mathbb{R}^d lies in the (Banach) space \mathcal{C}_ε if it is continuous and satisfies for all large $|x|$ the estimate $h(x) = O(|x|^{-d-\varepsilon})$ and if, moreover, the same condition holds for $\mathcal{F}h$ in place of h . We shall say that h lies in the space $\mathcal{C}_\varepsilon^{[\infty]}$ if, besides, the function $\mathcal{E}^j h$, associated with the operation of taking the derivative in the radial direction only, lies in \mathcal{C}_ε for every $j = 0, 1, \dots$*

THEOREM 2.5. — *As a tempered distribution, \mathfrak{E}_ν^d extends as a meromorphic function of $\nu \in \mathbb{C}$, whose only poles are at $\nu = \pm \frac{d}{2}$: these poles are simple, and the residues of \mathfrak{E}_ν^d there are given as follows:*

$$(2.11) \quad \text{Res}_{\nu = -\frac{d}{2}} \mathfrak{E}_\nu^d = -1 \quad \text{and} \quad \text{Res}_{\nu = \frac{d}{2}} \mathfrak{E}_\nu^d = \delta,$$

the unit mass at the origin of \mathbb{R}^d . Let \mathcal{F} be the Fourier transformation on $\mathcal{S}'(\mathbb{R}^d)$, where the Fourier transformation on $\mathcal{S}(\mathbb{R}^d)$ is normalized as

$$(2.12) \quad (\mathcal{F}h)(x) = \int_{\mathbb{R}^d} f(y) e^{-2i\pi \langle x, y \rangle} dy :$$

then one has

$$(2.13) \quad \mathcal{F} \mathfrak{E}_\nu^d = \mathfrak{E}_{-\nu}^d \quad \text{for } \nu \neq \pm \frac{d}{2}.$$

In the case when $-\frac{d}{2} - \varepsilon < \operatorname{Re} \nu < \frac{d}{2} + \varepsilon$, $\nu \neq \pm \frac{d}{2}$, \mathfrak{E}_ν^d extends as a continuous linear functional on the space \mathcal{C}_ε .

Proof. — We concentrate on the proof of the second part, which will entail that of the first one as well. Since

$$(2.14) \quad \sum_{\substack{m \in \mathbb{Z}^d \\ m \neq 0}} \int_0^\infty t^{-\operatorname{Re} \nu + \frac{d}{2}} (1 + t|m|)^{-d-\varepsilon} \frac{dt}{t} = C \sum_{m \neq 0} |m|^{\operatorname{Re} \nu - \frac{d}{2}}$$

with $C = \int_0^\infty t^{-\operatorname{Re} \nu + \frac{d}{2}} (1 + t)^{-d-\varepsilon} \frac{dt}{t}$, a convergent integral if $\frac{d}{2} > \operatorname{Re} \nu > -\frac{d}{2} - \varepsilon$, the distribution \mathfrak{E}_ν^d extends as a continuous linear form on \mathcal{C}_ε provided that $-\frac{d}{2} - \varepsilon < \operatorname{Re} \nu < -\frac{d}{2}$. Introducing the decomposition

$$(2.15) \quad \mathfrak{E}_\nu^d = (\mathfrak{E}_\nu^d)_{\text{princ}} + (\mathfrak{E}_\nu^d)_{\text{res}}$$

with

$$\langle (\mathfrak{E}_\nu^d)_{\text{princ}}, h \rangle = \sum_{m \neq 0} \int_0^1 t^{-\nu + \frac{d}{2}} h(tm) \frac{dt}{t}$$

and

$$(2.16) \quad \langle (\mathfrak{E}_\nu^d)_{\text{res}}, h \rangle = \sum_{m \neq 0} \int_1^\infty t^{-\nu + \frac{d}{2}} h(tm) \frac{dt}{t},$$

it is immediate that the second of these two expressions extends as a holomorphic function of ν in the half-plane $\operatorname{Re} \nu > -\frac{d}{2} - \varepsilon$ if h lies in \mathcal{C}_ε . On the other hand, if $-\frac{d}{2} - \varepsilon < \operatorname{Re} \nu < -\frac{d}{2}$, one has

$$(2.17) \quad \begin{aligned} \langle (\mathfrak{E}_{-\nu}^d)_{\text{res}}, \mathcal{F}h \rangle &= \sum_{\substack{m \in \mathbb{Z}^d \\ m \neq 0}} \int_1^\infty t^{\nu + \frac{d}{2}} (\mathcal{F}h)(tm) \frac{dt}{t} \\ &= \int_1^\infty t^{\nu + \frac{d}{2}} \left[\sum_{m \in \mathbb{Z}^d} (\mathcal{F}h)(tm) - (\mathcal{F}h)(0) \right] \frac{dt}{t} \\ &= \int_1^\infty t^{\nu + \frac{d}{2}} \left[\sum_{m \in \mathbb{Z}^d} t^{-d} h(t^{-1}m) - (\mathcal{F}h)(0) \right] \frac{dt}{t} \\ &= \int_0^1 t^{-\nu + \frac{d}{2}} \left[\sum_{m \in \mathbb{Z}^d} h(tm) - t^{-d} (\mathcal{F}h)(0) \right] \frac{dt}{t} \\ &= \int_0^1 t^{-\nu + \frac{d}{2}} \left[\sum_{m \neq 0} h(tm) + h(0) - t^{-d} (\mathcal{F}h)(0) \right] \frac{dt}{t} \\ &= \langle (\mathfrak{E}_\nu^d)_{\text{princ}}, h \rangle + \frac{h(0)}{\frac{d}{2} - \nu} + \frac{(\mathcal{F}h)(0)}{\frac{d}{2} + \nu}. \end{aligned}$$

This equation yields the extension, as a linear form on \mathcal{C}_ε , of $(\mathfrak{E}_\nu^d)_{\text{princ}}$ in the half-plane $\text{Re } \nu < \frac{d}{2} + \varepsilon$, thus completing the proof: note the use of Poisson's formula in the middle. \square

Combs often decompose as integral superpositions of Eisenstein distributions, for instance:

$$(2.18) \quad \begin{aligned} \mathfrak{D} &= 2\pi + \int_{-\infty}^{\infty} \mathfrak{E}_{i\lambda}^d d\lambda, \\ \mathfrak{D}^{\text{prime}} &= \frac{2\pi}{\zeta(d)} + \int_{-\infty}^{\infty} \left(\zeta\left(\frac{d}{2} - i\lambda\right)\right)^{-1} \mathfrak{E}_{i\lambda}^d d\lambda, \end{aligned}$$

the proof being the same as the one given in [17, (16.2), (16.57)] in the case when $d = 2$: these are also particular cases of Proposition 2.6 to follow. Note that $\mathfrak{E}_\nu^1(x) = \zeta(\frac{1}{2} - \nu)|x|^{-\frac{1}{2} - \nu}$ if $\nu \neq \pm\frac{1}{2}$, so that (2.18) reduces when $d = 1$ to

$$(2.19) \quad 2\pi(\delta(x - 1) + \delta(x + 1)) = \int_{-\infty}^{\infty} |x|^{-\frac{1}{2} - i\lambda} d\lambda;$$

of course, when $d \geq 2$, $\zeta(\frac{d}{2} - i\lambda)$ does not vanish on the real line.

Remark. — Let $\|\cdot\|$ and $\|\!\|\!\|$ be two Euclidean norms on \mathbb{R}^d defined by quadratic forms of discriminant 1, dual of each other with respect to the canonical bilinear form. Setting, when $\text{Re } s > d$,

$$(2.20) \quad Z^d(s; \|\cdot\|) = \frac{1}{2} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \|m\|^{-s},$$

one remarks that, for $\text{Re } \nu < -\frac{d}{2}$, the distribution \mathfrak{E}_ν^d coincides when tested against functions depending only on the norm $\|x\|$ of the variable x with the function $\pi^{-\frac{d}{2}} \Gamma(\frac{d}{2}) Z^d(\frac{d}{2} - \nu; \|\cdot\|) \|x\|^{-\frac{d}{2} - \nu}$. As a consequence of Theorem 2.5, the function $Z^d(\cdot; \|\cdot\|)$ extends as a meromorphic function in the complex plane, with a simple pole at $s = d$ with residue $\pi^{\frac{d}{2}} / \Gamma(\frac{d}{2})$. Next, in view of the equation (5.5) expressing the Fourier transform of the distribution $\|x\|^{-\frac{d}{2} - i\lambda}$, the equation $\mathcal{F}\mathfrak{E}_{i\lambda}^d = \mathfrak{E}_{-i\lambda}^d$ shows that the function $(Z^d)^*(s; \|\cdot\|) := \pi^{-\frac{d}{2}} \Gamma(\frac{s}{2}) Z^d(s; \|\cdot\|)$ coincides with $(Z^d)^*(d - s; \|\!\|\!\|)$. This may be considered as a proof of the functional equation of the Epstein (in particular, Riemann!) zeta function *slightly* different from the more usual ones but of course, like all such proofs, it relies on the use of the Poisson formula, which occurs in the middle of the sequence (2.17).

The two integral decompositions (2.18) are meant in the weak sense in $\mathcal{S}'(\mathbb{R}^d)$. However, it is well-known [2, 14] that, for $\sigma \geq 1$, the function $(\zeta(\sigma - i\lambda))^{\pm 1}$ is bounded, for large $|\lambda|$, by some power of $1 + |\lambda|$: then, these

decompositions can also be applied when tested against any function h in some space $C_\varepsilon^{[\infty]}$ as introduced in Definition 2.4.

More generally, the decomposition of any tempered comb will involve a Dirichlet series: we set, with the notation introduced just after (2.5),

$$(2.21) \quad \mathcal{D}_a(s) = \sum_{r \geq 1} \frac{a_r}{r^s},$$

a convergent series if $\text{Re } s$ is large. For the applications we have in mind, we shall always be dealing with functions \mathcal{D}_a with the following properties: such a function will extend as a meromorphic function in the whole complex plane, with only a finite number of poles; next, for every compact interval $[\alpha, \beta]$, there will exist some $c > 0$ and $C > 0$, finally some $t_0 > 1$ such that $|\mathcal{D}_a(s)| \leq C |\text{Im } s|^c$ whenever $\alpha \leq \text{Re } s \leq \beta$ and $|\text{Im } s| \geq t_0$. We shall say that \mathcal{D}_a has at most a polynomial increase on vertical strips when we need to refer to this latter property: it will hold in all cases to be considered here, where the Dirichlet series to be dealt with originate from modular form theory.

Besides the function \mathcal{D}_a , we shall have to consider also the function \mathcal{D}_b linked to it by the equation $\mathcal{D}_b(s) = \frac{\mathcal{D}_a(s)}{\zeta(s)}$. Recall [3, p. 236] that it is indeed (when $\text{Re } s$ is large) the sum of a Dirichlet series, namely that associated with the sequence b linked to a by the pair of (equivalent) formulas

$$(2.22) \quad a_r = \sum_{1 \leq n|r} b_n, \quad b_n = \sum_{1 \leq r|n} \mu\left(\frac{n}{r}\right) a_r,$$

in which μ is the Möbius indicator function ($\mu(m) = (-1)^j$ if m is the product of j distinct prime factors, and $\mu(m) = 0$ if m is not squarefree). It amounts to the same to assume that \mathcal{D}_a , or \mathcal{D}_b , has at most a polynomial increase on vertical strips.

PROPOSITION 2.6. — *Assume that the sequence $a = (a_n)_{n \geq 1}$ has at most polynomial increase, and that the function \mathcal{D}_a has at most a polynomial increase on vertical strips. Let b be the sequence linked to a by (2.22) or, equivalently, by the equation $\mathcal{D}_b(s) = \frac{\mathcal{D}_a(s)}{\zeta(s)}$. Assume that the function \mathcal{D}_b extends as a meromorphic function in the plane, with no pole with a real part $\geq \frac{d}{2}$ except, possibly, the point $s = d$. Then the comb \mathfrak{S}_a introduced in (2.5) admits the weak decomposition in $\mathcal{S}'(\mathbb{R}^d)$:*

$$(2.23) \quad \mathfrak{S}_a = \int_{-\infty}^{\infty} \mathcal{D}_b\left(\frac{d}{2} - i\lambda\right) \mathfrak{E}_{i\lambda}^d d\lambda + 2\pi \text{Res}_{s=d} \left(\mathcal{D}_b(s) \mathfrak{E}_{\frac{d}{2}-s}^d \right).$$

The right-hand side of the equation that precedes still makes sense when tested against any function h lying in some space $\mathcal{C}_\varepsilon^{[\infty]}$ as introduced in Definition 2.4. By convention, we shall also set

$$(2.24) \quad 2\pi \sum_{m \in \mathbb{Z}^d \setminus \{0\}} a_{r(m)} h(m) := \int_{-\infty}^{\infty} \mathcal{D}_{\mathbf{b}}\left(\frac{d}{2} - i\lambda\right) \langle \mathfrak{E}_{i\lambda}^d, h \rangle d\lambda \\ + 2\pi \operatorname{Res}_{s=d} \left(\mathcal{D}_{\mathbf{b}}(s) \langle \mathfrak{E}_{\frac{d}{2}-s}^d, h \rangle \right)$$

whenever h satisfies the condition just indicated, defining in this way the unique extension of the measure $\mathfrak{S}_{\mathbf{a}}$ as a continuous linear form on the space $\mathcal{C}_\varepsilon^{[\infty]}$.

Proof. — The convergence of the integral comes from the fact that arbitrary powers of λ can be compensated since arbitrary powers of the Euler operator are applicable to the test function in $\mathcal{S}(\mathbb{R}^d)$; also, the Dirichlet series $\mathcal{D}_{\mathbf{b}}(s)$, has at most a polynomial increase on vertical lines.

If $h \in \mathcal{S}(\mathbb{R}^d)$, its decomposition into homogeneous components of degree $-\frac{d}{2} - i\lambda$ is given as

$$(2.25) \quad h = \int_{-\infty}^{\infty} h_\lambda d\lambda$$

with

$$(2.26) \quad h_\lambda(x) = \frac{1}{2\pi} \int_0^\infty t^{i\lambda + \frac{d}{2}} h(tx) \frac{dt}{t}, \quad x \neq 0.$$

One can also define, for $\mu \in \mathbb{C}$ with $\operatorname{Im} \mu > 0$, the function $h_{-\mu}$ by the same formula, leading for all $b > 0$ and $x \neq 0$ to

$$(2.27) \quad h(x) = \int_{ib-\infty}^{ib+\infty} h_{-\mu}(x) d\mu.$$

Now, it is immediate from (2.10) and (2.26) that, if $\operatorname{Im} \mu > \frac{d}{2}$, one has

$$(2.28) \quad \langle \mathfrak{D}, h_{-\mu} \rangle = \langle \mathfrak{E}_{i\mu}^d, h \rangle,$$

also, from the definitions (2.7) and (2.5), that

$$(2.29) \quad \langle \mathfrak{D}^{\text{prime}}, h_{-\mu} \rangle = \left(\zeta\left(\frac{d}{2} - i\mu\right) \right)^{-1} \langle \mathfrak{D}, h_{-\mu} \rangle$$

and

$$(2.30) \quad \langle \mathfrak{S}_{\mathbf{a}}, h_{-\mu} \rangle = \frac{\mathcal{D}_{\mathbf{a}}\left(\frac{d}{2} - i\mu\right)}{\zeta\left(\frac{d}{2} - i\mu\right)} \langle \mathfrak{D}, h_{-\mu} \rangle.$$

Then, starting from

$$(2.31) \quad \begin{aligned} \langle \mathfrak{S}_{\mathbf{a}}, h \rangle &= \int_{ib-\infty}^{ib+\infty} \frac{\mathcal{D}_{\mathbf{a}}(\frac{d}{2} - i\mu)}{\zeta(\frac{d}{2} - i\mu)} \langle \mathfrak{E}_{i\mu}^d, h \rangle d\mu \\ &= \int_{ib-\infty}^{ib+\infty} \mathcal{D}_{\mathbf{b}}\left(\frac{d}{2} - i\mu\right) \langle \mathfrak{E}_{i\mu}^d, h \rangle d\mu, \end{aligned}$$

we only have to apply the residue theorem, moving the line of integration $ib + \mathbb{R}$ to \mathbb{R} , and taking into account the residue at the pole $s = d$ which, whether it is a pole of $\mathcal{D}_{\mathbf{b}}$ or not, arises anyway as a pole of $\mathfrak{E}_{\frac{d}{2}-s}^d$.

Note that, if d is not a pole of $\mathcal{D}_{\mathbf{b}}$, the corresponding special term in the decomposition (2.23) reduces to $2\pi\mathcal{D}_{\mathbf{b}}(d)$.

That the right-hand side still makes sense when h , instead of lying in $\mathcal{S}(\mathbb{R}^d)$, only satisfies the less demanding conditions stated at the end of the theorem, is a consequence of Theorem 2.5. □

Aside from the special case of the distribution $\mathfrak{D} + 2\pi\delta$, combs are generally not transformed into combs by the Fourier transformation. However, this may happen if the Fourier transformation \mathcal{F} is replaced by some modification $\mathcal{F}_{d,w}$ just as nice in all aspects not involving the additive group structure of \mathbb{Z}^d , only the structure of $\mathbb{R}^d \setminus \{0\}$ as a homogeneous space of $SL(d, \mathbb{Z})$: here, d denotes the dimension as before, and the *weight* w , to be borrowed later from the consideration of some modular form, is assumed, for the time being, to be just some real number $\geq \frac{1}{2}$.

Set, in the spectral-theoretic sense,

$$(2.32) \quad \begin{aligned} \mathfrak{K}_{d,w} &= \pi^{4i\pi w\mathcal{E}} \frac{\Gamma(w(\frac{d}{2} - 2i\pi\mathcal{E}))}{\Gamma(w(\frac{d}{2} + 2i\pi\mathcal{E}))} \\ &= 2\pi \int_0^\infty t^{-4i\pi w\mathcal{E}} J_{wd-1}(2\pi t) dt, \end{aligned}$$

where the second equation is to be found in [9, p. 91]: in other words, for every $h \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d \setminus \{0\}$, one sets

$$(2.33) \quad (\mathfrak{K}_{d,w}h)(x) = 2\pi \int_0^\infty t^{-wd} h(t^{-2w}x) J_{wd-1}(2\pi t) dt,$$

which makes $\mathfrak{K}_{d,w}h$ a C^∞ function on $\mathbb{R}^d \setminus \{0\}$. The following is an immediate consequence of the spectral definition (2.32) of $\mathfrak{K}_{d,w}$ and of the equation $\mathcal{F}\mathcal{E} = -\mathcal{E}\mathcal{F}$.

PROPOSITION 2.7. — *Set $\mathcal{F}_{d,w} = \mathfrak{K}_{d,w}\mathcal{F}$. The operator $\mathcal{F}_{d,w}$ extends as a unitary operator on $L^2(\mathbb{R}^d)$, and one may also write $\mathcal{F}_{d,w} = \mathcal{F}\mathfrak{K}_{d,w}^{-1}$. Also, $\mathcal{F}_{d,w}^2$, just like \mathcal{F}^2 , is the symmetry operator $\mathfrak{S} \mapsto \check{\mathfrak{S}}$.*

The transformation $\mathcal{F}_{d,\frac{1}{2}}$ is especially easy to analyze on radial functions:

PROPOSITION 2.8. — *Let $\|\cdot\|$ and $\|\cdot\|$ be two Euclidean norms on \mathbb{R}^d defined by quadratic forms of discriminant 1, dual of each other with respect to the canonical pairing $\langle \cdot, \cdot \rangle$. If $(\mathcal{F}\Phi)(\xi) = \chi(\|\xi\|)$, one has $\mathcal{F}_{d,\frac{1}{2}}\Phi = \Psi$ with $(\mathcal{F}\Psi)(x) = \|x\|^{-d}\chi(\|x\|^{-1})$.*

Proof. — Rewriting (2.32) as

$$(2.34) \quad \mathfrak{R}_{d,\frac{1}{2}}^{-1} = 2\pi \int_0^\infty t^{2i\pi\varepsilon} J_{\frac{d-2}{2}}(2\pi t) dt,$$

we see that, if $\Phi(x) = \phi(\|x\|)$, one has $(\mathfrak{R}_{d,\frac{1}{2}}^{-1}\Phi)(x) = \psi(\|x\|)$ with

$$(2.35) \quad \psi(r) = 2\pi \int_0^\infty t^{\frac{d}{2}} \phi(tr) J_{\frac{d-2}{2}}(2\pi t) dt.$$

On the other hand, one has the well-known formula (radial Fourier transformation)

$$(2.36) \quad \chi(r) = 2\pi r^{-d} \int_0^\infty t^{\frac{d}{2}} \phi\left(\frac{t}{r}\right) J_{\frac{d-2}{2}}(2\pi t) dt.$$

Since $\mathcal{F}^{-1}\mathcal{F}_{d,\frac{1}{2}} = \mathfrak{R}_{d,\frac{1}{2}}^{-1}$, we are done. □

We need to study pairs of functions Φ, Ψ linked by the more general $\mathcal{F}_{d,w}$ -transform in some detail: this transformation does not preserve the space $\mathcal{S}(\mathbb{R}^d)$.

LEMMA 2.9. — *Let Φ and Ψ be two radial functions on \mathbb{R}^d : set $\Phi(x) = \phi(|x|)$, $\Psi(x) = \psi(|x|)$. Assume that Φ and Ψ lie in the space \mathcal{C}_ε introduced in Definition 2.4: recall that this means that Φ and Ψ are continuous on \mathbb{R}^d , are both $\mathcal{O}(|x|^{-d-\varepsilon})$ at infinity, and that the same holds for the Fourier transforms of Φ and Ψ . Assume that, for some real number $w \geq \frac{1}{2}$, Φ and Ψ are linked by the equation $\Psi = \mathcal{F}_{d,w}\Phi$. Define, for $0 < \text{Re } s < d + \varepsilon$,*

$$(2.37) \quad \begin{aligned} V_\Phi(s) &= \frac{1}{2\pi} \int_0^\infty t^{s-1} \phi(t) dt, \\ V_\Psi(s) &= \frac{1}{2\pi} \int_0^\infty t^{s-1} \psi(t) dt, \end{aligned}$$

a pair of convergent integrals. Then V_Φ and V_Ψ extend as meromorphic functions throughout the strip $-\varepsilon < \text{Re } s < d + \varepsilon$, without any poles

except possibly at $s = 0$: this pole can only be simple, with residues $\frac{\Phi(0)}{2\pi}$ and $\frac{\Psi(0)}{2\pi}$ respectively. On the other hand, V_Φ and V_Ψ vanish at $s = d$ (in other words, $\int_{\mathbb{R}^d} \Phi(x)dx = \int_{\mathbb{R}^d} \Psi(x)dx = 0$). Finally, these two functions are linked by the functional equation

$$(2.38) \quad \frac{\pi^{(w+\frac{1}{2})s}}{\Gamma(ws)\Gamma(\frac{s}{2})} V_\Phi(s) = \frac{\pi^{(w+\frac{1}{2})(d-s)}}{\Gamma(w(d-s))\Gamma(\frac{d-s}{2})} V_\Psi(d-s),$$

and one has

$$(2.39) \quad \left(\frac{d}{ds} V_\Phi(s)\right)(s=d) = -\frac{w}{4} \pi^{-wd-\frac{d}{2}-1} \Gamma(wd)\Gamma\left(\frac{d}{2}\right) \Psi(0).$$

If $\Phi \in \mathcal{S}(\mathbb{R}^d)$, if Φ is radial and satisfies $\int_{\mathbb{R}^d} \Phi(x)dx = 0$, the function Ψ defined as $\Psi = \mathcal{F}_{d,w}\Phi$ lies in $C_\varepsilon^{[\infty]}$ for every ε such that $0 < \varepsilon < w^{-1}$.

Proof. — The hypotheses are symmetric — except for the last sentence — with respect to the pair Φ, Ψ . Since the Fourier transforms of these functions are $O(|x|^{-d-\varepsilon})$ as $|x| \rightarrow \infty$, Φ and Ψ have some Hölder regularity: in particular there exists some $C > 0$ such that

$$(2.40) \quad \begin{aligned} |\Phi(x) - \Phi(0)| &\leq C|x|^\varepsilon, \\ |\Psi(x) - \Psi(0)| &\leq C|x|^\varepsilon \text{ as } |x| \rightarrow 0. \end{aligned}$$

Writing, when $0 < \text{Re } s < d + \varepsilon$,

$$(2.41) \quad V_\Phi(s) = \frac{1}{2\pi} \left[\int_0^1 t^{s-1}(\phi(t) - \phi(0)) dt + \frac{\phi(0)}{s} + \int_1^\infty t^{s-1}\phi(t) dt \right],$$

one finds the meromorphic continuation of V_Φ in the strip indicated, as well as its residue at 0. The decomposition into homogeneous terms

$$(2.42) \quad \Phi = \int_{-\infty}^\infty \Phi_\lambda d\lambda$$

of a radial function only involves functions of the type

$$(2.43) \quad \Phi_\lambda(x) = \text{spec}(\Phi; \lambda) |x|^{-\frac{d}{2}-i\lambda} :$$

the scalar function $\text{spec}(\Phi; \cdot)$ has been systematically computed, for certain special functions Φ , in Section 5. According to (2.26) (or (5.2)), one has

$$(2.44) \quad \text{spec}(\Phi; \lambda) = \frac{1}{2\pi} \int_0^\infty t^{\frac{d}{2}+i\lambda} \phi(t) \frac{dt}{t},$$

so that

$$(2.45) \quad V_\Phi(s) = \text{spec}\left(\Phi; i\left(\frac{d}{2} - s\right)\right)$$

when $\text{Re } s = \frac{d}{2}$. In Lemma 5.3, the following is established:

$$(2.46) \quad \text{spec}(\Phi; \lambda) = \pi^{-(2w+1)i\lambda} \frac{\Gamma(w(\frac{d}{2} + i\lambda)) \Gamma(\frac{d}{4} + \frac{i\lambda}{2})}{\Gamma(w(\frac{d}{2} - i\lambda)) \Gamma(\frac{d}{4} - \frac{i\lambda}{2})} \text{spec}(\Psi; -\lambda).$$

The first part of the present lemma follows, since the equation (2.39) can be derived from the functional equation (2.38), itself a consequence of the last equation.

For the second part, that starts with a rather general function $\Phi \in \mathcal{S}(\mathbb{R}^d)$, we set $(\mathcal{F}\Phi)(x) = \tilde{\phi}(|x|)$, so that

$$(2.47) \quad \text{spec}(\mathcal{F}\Phi; \lambda) = \frac{1}{2\pi} \int_0^\infty t^{\frac{d}{2} + i\lambda} \tilde{\phi}(t) \frac{dt}{t}$$

extends as a holomorphic function to complex values of λ in the domain $\text{Re}(i\lambda) > -\frac{d}{2} - 1$, as it follows from the assumption that $(\mathcal{F}\Phi)(0) = 0$; also, when $\text{Im } \lambda$ is kept fixed within the half-line just introduced, this function is rapidly decreasing as a function of the real part of λ as $|\lambda| \rightarrow \infty$. Then, we write, using (2.46) and the last formula of Lemma 5.1,

$$(2.48) \quad \Psi(x) = \int_{-\infty}^\infty \pi^{-2wi\lambda} \frac{\Gamma(w(\frac{d}{2} + i\lambda))}{\Gamma(w(\frac{d}{2} - i\lambda))} \text{spec}(\mathcal{F}\Phi; \lambda) |x|^{-\frac{d}{2} - i\lambda} d\lambda$$

and

$$(2.49) \quad (\mathcal{F}\Psi)(x) = \int_{-\infty}^\infty \pi^{(2w-1)i\lambda} \frac{\Gamma(w(\frac{d}{2} - i\lambda)) \Gamma(\frac{d}{4} + \frac{i\lambda}{2})}{\Gamma(w(\frac{d}{2} + i\lambda)) \Gamma(\frac{d}{4} - \frac{i\lambda}{2})} \text{spec}(\mathcal{F}\Phi; -\lambda) |x|^{-\frac{d}{2} - i\lambda} d\lambda.$$

So far as the integral defining $\Psi(x)$ is concerned, we can use a deformation of contour, substituting $i\lambda + M$ for $i\lambda$ with $M > 0$ arbitrarily large, ending up with the fact that $\Psi(x)$ is rapidly decreasing as $|x| \rightarrow \infty$. Something similar works for $(\mathcal{F}\Psi)(x)$: only, we are constrained by the condition $M < \frac{d}{2} + w^{-1}$ in view of what has been said above regarding the complex continuation of the function $\text{spec}(\mathcal{F}\Phi; \lambda)$. □

An especially nice pair satisfying the assumptions of Lemma 2.9 is the pair, for which $\Phi = \Psi$, denoted as $\Phi_{d,w}$ and introduced in Proposition 5.4. It may be thought of as playing, with respect to the transformation $\mathcal{F}_{d,w}$, essentially the canonical role played by the Gaussian function $x \mapsto \exp(-\pi|x|^2)$ with respect to the usual Fourier transformation. When $w > \frac{1}{2}$, this function lies in the space $\mathcal{C}_w^{[\infty]}$. In the case when $w = 1$ (the only case in which we have completed the calculations), one can also use the pair $\Phi_{d,1}^j, \Psi_{d,1}^j$ provided by Lemma 5.6: the relationship of the sequence

$(\Phi_{d,1}^j)_{j \geq 0}$ to the function $\Phi_{d,1}$ can be compared to that of the sequence of Hermite (radial) functions to the Gaussian function.

3. Combs and holomorphic modular forms.

We now connect the question of the construction of $\mathcal{F}_{d,w}$ -invariant combs to holomorphic modular form theory, relying on the first result of Hecke's theory as explained in section 1 of [10]. The starting point of the construction is a modular form of weight wd , where w is an integer ≥ 1 .

THEOREM 3.1. — *Let f be a modular form of weight wd , with $w = 1, 2, \dots$, for the group $G(2)$ of fractional-linear transformations of the upper half-plane generated by the translation $z \mapsto z + 2$ and the inversion $z \mapsto -\frac{1}{z}$. More precisely, assume that f admits the Fourier expansion*

$$(3.1) \quad f(z) = f_0 + \sum_{n \geq 1} f_n e^{i\pi n z},$$

where the sequence (f_n) is bounded for large n by some power of n , and that f satisfies the equation

$$(3.2) \quad f(z) = \kappa \left(\frac{z}{i}\right)^{-wd} f\left(-\frac{1}{z}\right)$$

with $\kappa = \pm 1$. Let Φ and Ψ be two radial functions on \mathbb{R}^d related by the equation $\Psi = \mathcal{F}_{d,w} \Phi$, and lying in the space $C_\varepsilon^{[\infty]}$ for some $\varepsilon > 0$. For every $m \in \mathbb{Z}^d$, set

$$(3.3) \quad c_m = \sum_{\substack{n \geq 1 \\ n^w | (m_1, \dots, m_d)}} f_n \text{ if } m \neq 0$$

and $c_0 = -f_0$. Then, for every $t > 0$, one has

$$(3.4) \quad \sum_{m \in \mathbb{Z}^d} c_m \Phi(tm) = \kappa t^{-d} \sum_{m \in \mathbb{Z}^d} c_m \Psi\left(\frac{m}{t}\right),$$

where each of the two sides should be ascribed the meaning defined in (2.24).

Proof. — Recall that

$$(3.5) \quad \mathcal{D}_f(s) = \sum_{n \geq 1} \frac{f_n}{n^s}$$

when $\text{Re } s$ is large, and set

$$(3.6) \quad v(s) = \pi^{-s} \Gamma(s) \mathcal{D}_{\mathbf{f}}(s).$$

According to [10], the function

$$(3.7) \quad s \mapsto v(s) + \frac{f_0}{s} + \kappa \frac{f_0}{wd - s}$$

extends as an entire function, bounded in vertical strips, which implies first that $f_0 = -\mathcal{D}_{\mathbf{f}}(0)$, and satisfies the functional equation

$$(3.8) \quad v(s) = \kappa v(wd - s).$$

We then set, for $r = 1, 2, \dots$,

$$(3.9) \quad b_r = \begin{cases} f_n & \text{if } r = n^w \text{ for some } n \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and consider the function

$$(3.10) \quad \mathcal{D}_{\mathbf{b}}(s) = \sum_{n \geq 1} \frac{b_n}{n^s} = \mathcal{D}_{\mathbf{f}}(ws).$$

It follows from [10, I-5] that the function $\mathcal{D}_{\mathbf{b}}$ is meromorphic in the whole plane, with only one simple pole at d , with residue $\frac{\kappa f_0}{w} \frac{\pi^{wd}}{\Gamma(wd)}$, and satisfies the equation

$$(3.11) \quad \mathcal{D}_{\mathbf{b}}(s) \pi^{w(d-2s)} \frac{\Gamma(ws)}{\Gamma(w(d-s))} = \kappa \mathcal{D}_{\mathbf{b}}(d-s).$$

Next, recall from Theorem 2.5 that the distribution-valued function $s \mapsto \mathfrak{E}_{\frac{d}{2}-s}^d$ has a simple pole at $s = d$ with residue 1 (the Lebesgue measure of \mathbb{R}^d). From (2.15)–(2.17) we find the next term in the expansion of $\mathfrak{E}_{\frac{d}{2}-s}^d$ near $s = d$, namely

$$(3.12) \quad \lim_{s \rightarrow d} \left[\mathfrak{E}_{\frac{d}{2}-s}^d - \frac{1}{s-d} \right] = -d^{-1} \delta + \left(\mathfrak{E}_{-\frac{d}{2}}^d \right)_{\text{res}} + \mathcal{F} \left(\mathfrak{E}_{\frac{d}{2}}^d \right)_{\text{res}}.$$

We may, *a priori*, need a second term as explained since we must consider the residue, at $s = d$, of the product $\mathcal{D}_{\mathbf{b}}(s) \mathfrak{E}_{\frac{d}{2}-s}^d$, where the first factor also has a simple pole at d : that the measure on the right-hand side of the last equation extends as a continuous linear form on the space of functions C_ε is, however, sufficient for our purpose.

Setting $\mathbf{a} = (a_r)_{r \geq 1}$ with $a_r = \sum_{1 \leq n | r} b_n$, we shall consider the automorphic distribution

$$(3.13) \quad \mathfrak{I}_{\mathbf{a}} = \mathfrak{S}_{\mathbf{a}} - 2\pi \text{Res}_{s=d} \left(\mathcal{D}_{\mathbf{b}}(s) \mathfrak{E}_{\frac{d}{2}-s}^d \right)$$

and show that it transforms to $\kappa\mathfrak{T}_{\mathbf{a}}$ under the modified Fourier transformation $\mathcal{F}_{d,w}$.

This is a consequence of (2.23),

$$(3.14) \quad \mathfrak{T}_{\mathbf{a}} = \int_{-\infty}^{\infty} \mathcal{D}_{\mathbf{b}}\left(\frac{d}{2} - i\lambda\right) \mathfrak{E}_{i\lambda}^d d\lambda :$$

applying first the fact that $\mathfrak{E}_{i\lambda}^d$ transforms to $\mathfrak{E}_{-i\lambda}^d$ under the usual Fourier transformation (Theorem 2.5), next that $\mathcal{F}_{d,w} = \mathfrak{K}_{d,w}\mathcal{F}$ together with the spectral definition (2.32) of $\mathfrak{K}_{d,w}$, one finds

$$(3.15) \quad \mathcal{F}_{d,w}\mathfrak{T}_{\mathbf{a}} = \int_{-\infty}^{\infty} \mathcal{D}_{\mathbf{b}}\left(\frac{d}{2} - i\lambda\right) \pi^{2iw\lambda} \frac{\Gamma(w(\frac{d}{2} - i\lambda))}{\Gamma(w(\frac{d}{2} + i\lambda))} \mathfrak{E}_{-i\lambda}^d d\lambda,$$

which is the same as $\kappa\mathfrak{T}_{\mathbf{a}}$ in view of the functional equation (3.11).

We now test the distribution identity just established against the function Φ^t defined as

$$(3.16) \quad \Phi^t(x) = \Phi(tx).$$

The $\mathcal{F}_{d,w}$ -transform of this function is the function $x \mapsto t^{-d}\Psi(t^{-1}x)$: indeed, the Euler operator \mathcal{E} commutes with the operator $\mathfrak{K}_{d,w}$ introduced in (2.32), so that it anticommutes with $\mathcal{F}_{d,w}$ just as it does with the usual Fourier transformation. Thus

$$(3.17) \quad \begin{aligned} \langle \mathfrak{T}_{\mathbf{a}}, \Phi^t \rangle &= \langle \mathfrak{K}_{d,w}\mathcal{F}^{-1}\mathcal{F}_{d,w}\mathfrak{T}_{\mathbf{a}}, \Phi^t \rangle \\ &= \langle \mathcal{F}^{-1}\mathcal{F}_{d,w}\mathfrak{T}_{\mathbf{a}}, \mathfrak{K}_{d,w}^{-1}\Phi^t \rangle \\ &= \langle \mathcal{F}_{d,w}\mathfrak{T}_{\mathbf{a}}, \mathcal{F}_{d,w}\Phi^t \rangle \\ &= \kappa t^{-d} \langle \mathfrak{T}_{\mathbf{a}}, \Psi^{t^{-1}} \rangle. \end{aligned}$$

The computation of the second term from the decomposition of $\langle \mathfrak{T}_{\mathbf{a}}, \Phi^t \rangle$ that arises from (3.13) can be obtained as an application of (2.28), which gives if $-\varepsilon < \text{Re } s < d + \varepsilon$ the equation

$$(3.18) \quad \langle \mathfrak{E}_{\frac{d}{2}-s}^d, \Phi^t \rangle = \langle \mathfrak{D}, (\Phi^t)_{i(\frac{d}{2}-s)} \rangle.$$

Note that

$$(3.19) \quad \begin{aligned} (\Phi^t)_{i(\frac{d}{2}-s)}(x) &= t^{-s} \text{spec}\left(\Phi; i\left(\frac{d}{2} - s\right)\right) |x|^{-s} \\ &= t^{-s} V_{\Phi}(s) |x|^{-s} \end{aligned}$$

vanishes at $s = d$ according to Lemma 2.9. One has, for $\text{Re } s > d$,

$$(3.20) \quad \frac{1}{4\pi} \langle \mathfrak{D}, |x|^{-s} \rangle = Z^d(s)$$

as defined in (2.20), and it has been recalled in the remark that follows the proof of Theorem 2.5 that this function has a unique pole at $s = d$ and that this pole is simple, with the residue $\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$.

From (3.19) and (2.39) (which gives the derivative of $V_{\Phi}(s)$ at $s = d$ in terms of the value of Ψ at 0, whereas $V_{\Phi}(d) = 0$), one then finds that the function $\langle \mathfrak{E}_{\frac{d}{2}-s}^d, \Phi^t \rangle$ is regular at $s = d$ with the value

$$(3.21) \quad \left(\langle \mathfrak{E}_{\frac{d}{2}-s}^d, \Phi^t \rangle \right) (s = d) = -w\pi^{-wd}\Gamma(wd)\Psi(0)t^{-d},$$

from which it is immediate (using a result stated immediately before (3.11)) that

$$(3.22) \quad \text{Res}_{s=d} \left(\mathcal{D}_{\mathbf{b}}(s) \langle \mathfrak{E}_{\frac{d}{2}-s}^d, \Phi^t \rangle \right) = -\kappa f_0 \Psi(0) t^{-d}.$$

Thus

$$(3.23) \quad \langle \mathfrak{I}_{\mathbf{a}}, \Phi^t \rangle = \langle \mathfrak{S}_{\mathbf{a}}, \Phi^t \rangle + 2\pi\kappa f_0 \Psi(0) t^{-d},$$

so that (3.17) yields

$$(3.24) \quad 2\pi \sum_{m \in \mathbb{Z}^d \setminus \{0\}} a_{r(m)} [\Phi(tm) - \kappa t^{-d} \Psi(t^{-1}m)] = 2\pi f_0 [\Phi(0) - \kappa t^{-d} \Psi(0)].$$

This completes the proof of Theorem 3.1. □

Remarks.

(i) In some sense, Theorem 3.1 can also be understood as follows. Let $\mathfrak{R} = \mathfrak{S}_{\mathbf{c}} - 2\pi f_0 \delta$, an automorphic measure supported by \mathbb{Z}^d : the image of \mathfrak{R} under the operator $\pi^{-w(\frac{d}{2} - 2i\pi\mathcal{E})}\Gamma(w(\frac{d}{2} - 2i\pi\mathcal{E}))$ is multiplied by κ under the Fourier transformation. However, we found it preferable to let the Gamma-like function of the Euler operator act on the test function rather than the discrete measure for the following two reasons: appropriate spaces of test functions are easier to define in this way (*cf.* Lemma 2.9); also, the role of combs (which carry all the arithmetic of the situation, the extra operator under study being just the Archimedean factor which usually appears in functional equations) is more immediately apparent.

(ii) The assumption that Φ is radial is far from necessary, and should be considered as purely technical. Indeed, nothing would be changed if another Euclidean norm were substituted for the standard one: only, in Lemma 2.9, if $\Phi(x) = \phi(\|x\|)$, one should set $\Psi(x) = \psi(\| \|x\| \|)$, where the two Euclidean norms $\| \|$ and $\| \| \|$ are defined by quadratic forms of discriminant one and dual to each other in the sense of Proposition 2.8. Consequently, integral superpositions of functions Φ , radial with respect

to various Euclidean norms, should be equally possible. The sole difficulty stems from the fact that the regularity assumptions to be made are more difficult to state in general than in the radial case: in particular, if ellipsoids with large radii become very flattened, the approximation (by the volume) of the number of points with integral coordinates within fails to hold in a uniform way.

One of the simplest examples is obtained if we start from the modular form [10, I-42]

$$(3.25) \quad \left(\sum_{n \in \mathbb{Z}} e^{i\pi n^2 z} \right)^{2d} = 1 + \sum_{n \geq 1} \text{Sq}_{2d}(n) e^{i\pi n z},$$

in which $\text{Sq}_{2d}(n)$ is the number of ways the number n can be decomposed as the sum of squares of $2d$ integers (of any sign; the order is taken into account): the associated L -function is

$$(3.26) \quad \mathcal{D}_{\mathbf{b}}(s) = \sum_{n \geq 1} \text{Sq}_{2d}(n) n^{-s},$$

and the conditions of the preceding theorem relative to the modular form f are satisfied with $w = 1$ and $\kappa = 1$. On the other hand, we choose $\Phi = \Phi_{d,1}$, as defined in Proposition 5.5, ending up with the Poisson-like formula shown in the introduction (1.4), (1.6). One can do the same with $w = 2, 3, \dots$, substituting the coefficients $\text{Sq}_{2wd}(n)$ for the coefficients above, in which case the $\mathcal{F}_{d,w}$ -invariant function $\Phi_{d,w}$, computed in Proposition 5.4, is an explicit generalized hypergeometric series.

In the one-dimensional case, consider more generally the Dirichlet L -function $L(s, \chi)$ associated with an even, primitive character modulo N . Set [1, p. 4]

$$(3.27) \quad \tau(\chi) = \sum_{n \bmod N} \chi(n) e^{\frac{2i\pi n}{N}},$$

so that $\frac{|\tau(\chi)|}{\sqrt{N}} = 1$. If $a_r = \chi(r)$ for $r \geq 1$, the Dirichlet series $\mathcal{D}_{\mathbf{b}}(s) = \frac{\mathcal{D}_{\mathbf{a}}(s)}{\zeta(s)} = \frac{L(s, \chi)}{\zeta(s)}$ and the series $\mathcal{D}_{\bar{\mathbf{b}}}(s) = \frac{L(s, \bar{\chi})}{\zeta(s)}$ are linked by the functional equation

$$(3.28) \quad N^{\frac{s}{2}} \mathcal{D}_{\mathbf{b}}(s) = \frac{\tau(\chi)}{\sqrt{N}} N^{\frac{1-s}{2}} \mathcal{D}_{\bar{\mathbf{b}}}(1-s).$$

The argument that precedes then shows that, for every $\Phi \in \mathcal{S}(\mathbb{R})$, the equation

$$(3.29) \quad \sum_{n \neq 0} \chi(n) \Phi(n) = \frac{\tau(\chi)}{N} \sum_{n \neq 0} \bar{\chi}(n) (\mathcal{F}\Phi) \left(\frac{n}{N} \right)$$

holds: in *loc.cit.*, p. 8, this *twisted Poisson summation formula* is proved in an independent way, then applied towards a derivation of (3.28).

Theorem 3.1 can be used in particular in the one-dimensional case, and associates a summation formula of the Poisson style to each modular form of weight $w = 1, 2, \dots$ for the group $G(2)$. However, there is another possibility in this case. Using the already mentioned fact that, in dimension 1, $\mathfrak{E}_\nu^1(x) = \zeta(\frac{1}{2} - \nu)|x|^{-\frac{1}{2}-\nu}$ for $\nu \neq \pm\frac{1}{2}$, and the link between \mathcal{D}_a and \mathcal{D}_b recalled in Proposition 2.6, one can write

$$(3.30) \quad \mathcal{D}_b\left(\frac{1}{2} - i\lambda\right)\mathfrak{E}_{i\lambda}^1(x) = \mathcal{D}_a\left(\frac{1}{2} - i\lambda\right)|x|^{-\frac{1}{2}-i\lambda}.$$

With \mathfrak{T}_a still defined by (3.13), the equation (3.14) can now be written as

$$(3.31) \quad \mathfrak{T}_a(x) = \int_{-\infty}^{\infty} \mathcal{D}_a\left(\frac{1}{2} - i\lambda\right)|x|^{-\frac{1}{2}-i\lambda}d\lambda.$$

We may then assume that it is \mathcal{D}_a , not \mathcal{D}_b , that satisfies the functional equation (3.11), with $d = w = 1$. Granted that we have not strived for the weakest assumptions regarding the function Φ , we here essentially recover a well-known result of Voronoï [19] (see also [4]), who considered the case when Φ is the characteristic function of $[0, 1]$. Note that the existence of pairs (Φ, Ξ) of functions satisfying the hypotheses of the proposition that follows is easily ascertained: it suffices for instance to start from $\Phi_\alpha(x) = (1 - x^2)_+^{\alpha-1}$, with $\alpha > 2$, using the Lemmas 5.2 and 5.3 together with some contour deformation.

PROPOSITION 3.2. — *Let Φ and Ξ be two even functions on the real line, in the space $\mathcal{C}_\varepsilon^{[\infty]}$ for some $\varepsilon > 0$, linked by the equation $\Xi = \mathcal{G}_{1,1}\Phi$ where $\mathcal{G}_{1,1} := \frac{\mathfrak{K}_{1,1}}{\mathfrak{K}_{1,\frac{1}{2}}}\mathcal{F}$ with $\mathfrak{K}_{d,w}$ as defined in (2.32). Then, for every $t > 0$, one has*

$$(3.32) \quad \sum_{n \geq 1} \text{Sq}_2(n)\Phi(tn) + \Phi(0) = t^{-1} \left[\sum_{n \geq 1} \text{Sq}_2(n)\Xi\left(\frac{n}{t}\right) + \Xi(0) \right].$$

Proof. — Set $a_n = \text{Sq}_2(n)$ for every $n \geq 1$: this is the sequence of Fourier coefficients of the function $f(z) = 1 + \sum_{n \geq 1} \text{Sq}_2(n)e^{i\pi nz}$, the (unique, up to multiplication by a constant) modular form of weight 1 for the group $G(2)$. The associated Dirichlet series \mathcal{D}_a verifies the equation

$$(3.33) \quad \mathcal{D}_a(s)\pi^{1-2s} \frac{\Gamma(s)}{\Gamma(1-s)} = \mathcal{D}_a(1-s).$$

From (2.32), one has

$$(3.34) \quad \mathcal{G}_{1,1} = \pi^{2i\pi\mathcal{E}} \frac{\Gamma(\frac{1}{2} - 2i\pi\mathcal{E}) \Gamma(\frac{1}{4} + i\pi\mathcal{E})}{\Gamma(\frac{1}{2} + 2i\pi\mathcal{E}) \Gamma(\frac{1}{4} - i\pi\mathcal{E})} \mathcal{F}$$

so that, starting from (3.31) and using (5.5), one finds

$$(3.35) \quad (\mathcal{G}_{1,1}\mathfrak{T}_{\mathbf{a}})(x) = \frac{\mathfrak{K}_{1,1}}{\mathfrak{K}_{1,\frac{1}{2}}} \int_{-\infty}^{\infty} \mathcal{D}_{\mathbf{a}}\left(\frac{1}{2} - i\lambda\right) \pi^{i\lambda} \frac{\Gamma(\frac{1}{4} - \frac{i\lambda}{2})}{\Gamma(\frac{1}{4} + \frac{i\lambda}{2})} |x|^{-\frac{1}{2} + i\lambda} d\lambda$$

$$= \int_{-\infty}^{\infty} \mathcal{D}_{\mathbf{a}}\left(\frac{1}{2} - i\lambda\right) \pi^{2i\lambda} \frac{\Gamma(\frac{1}{2} - i\lambda)}{\Gamma(\frac{1}{2} + i\lambda)} |x|^{-\frac{1}{2} + i\lambda} d\lambda$$

or, using (3.33),

$$(3.36) \quad (\mathcal{G}_{1,1}\mathfrak{T}_{\mathbf{a}})(x) = \int_{-\infty}^{\infty} \mathcal{D}_{\mathbf{a}}\left(\frac{1}{2} + i\lambda\right) |x|^{-\frac{1}{2} + i\lambda} d\lambda :$$

thus $\mathcal{G}_{1,1}\mathfrak{T}_{\mathbf{a}} = \mathfrak{T}_{\mathbf{a}}$. With the same notation as in the proof of Theorem 3.1, we now have the functional equation

$$(3.37) \quad V_{\Phi}(s) \frac{\pi^s}{\Gamma(s)} = V_{\Xi}(1-s) \frac{\pi^{1-s}}{\Gamma(1-s)},$$

a consequence of (3.19) and (5.11). It follows, using also the equation (2.41) relative to V_{Ξ} , that $V_{\Phi}(1) = \frac{\Xi(0)}{2\pi^2}$. Now, as $s \rightarrow 1$, $\pi^{-s}\Gamma(s)\mathcal{D}_{\mathbf{a}}(s) \sim (s-1)^{-1}$ as recalled in (3.7) and $\zeta(s) \sim (s-1)^{-1}$ so that $\mathcal{D}_{\mathbf{b}}(1) = \pi$ and, using also (2.37),

$$(3.38) \quad \text{Res}_{s=1} \left(\mathcal{D}_{\mathbf{b}}(s) \langle \mathfrak{E}_{\frac{1}{2}-s}^1, \Phi^t \rangle \right) = \pi \int_{-\infty}^{\infty} \Phi^t(x) dx$$

$$= 4\pi^2 t^{-1} V_{\Phi}(1) = 2t^{-1} \Xi(0),$$

from which the proposition follows.

Starting from (3.34) and using [9, pp. 3,91], one may write

$$(3.39) \quad \mathcal{G}_{1,1} = \left(\frac{\pi}{4}\right)^{2i\pi\mathcal{E}} \frac{\Gamma(\frac{3}{4} - i\pi\mathcal{E})}{\Gamma(\frac{3}{4} + i\pi\mathcal{E})} \mathcal{F}$$

$$= \frac{\pi}{2} \int_0^{\infty} t^{-2i\pi\mathcal{E}} J_{\frac{1}{2}}\left(\frac{\pi t}{2}\right) dt \cdot \mathcal{F},$$

which can be put into a more classical form: for $x > 0$, one has

$$(3.40) \quad (\mathcal{G}_{1,1}\Phi)(x) = \frac{\pi x^{\frac{1}{2}}}{4} \int_{-\infty}^{\infty} |t|^{-\frac{3}{2}} J_{\frac{1}{2}}\left(\frac{\pi x}{2|t|}\right) (\mathcal{F}\Phi)(t) dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} |t|^{-1} \left(\sin \frac{\pi x}{2|t|}\right) (\mathcal{F}\Phi)(t) dt$$

$$= \frac{\pi}{2} \int_{-\infty}^{\infty} J_0(2\pi\sqrt{x|t|}) \Phi(t) dt,$$

where we have used [9, p. 405]; one thus recovers the expression given in [4, p. 449]. □

Our last example in this section may be introduced by the consideration, in the case when $d = 7, 11, \dots$, of the holomorphic Eisenstein series

$$(3.41) \quad G_{\frac{d+1}{4}}(z) = \sum_{|m|+|n| \neq 0} (mz + n)^{-\frac{d+1}{2}} :$$

one has the following expansion [13, p. 150]:

$$(3.42) \quad \frac{1}{2}(2i\pi)^{-\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right) G_{\frac{d+1}{4}}(z) = a_0 + \sum_{r \geq 1} a_r e^{2i\pi r z}$$

with

$$(3.43) \quad \begin{aligned} a_0 &= (2i\pi)^{-\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right) \zeta\left(\frac{d+1}{2}\right) \\ &= \frac{1}{2} \pi^{-\frac{d}{2}} \frac{\Gamma\left(\frac{1+d}{4}\right)}{\Gamma\left(\frac{1-d}{4}\right)} \zeta\left(\frac{d+1}{2}\right) = \frac{1}{2} \zeta\left(\frac{1-d}{2}\right) \end{aligned}$$

and, for $r \geq 1$,

$$(3.44) \quad a_r = \sigma_{\frac{d-1}{2}}(r) = \sum_{1 \leq n|r} n^{\frac{d-1}{2}}.$$

THEOREM 3.3. — Assume $d = 4q + 3$ with $q = 0, 1, \dots$ and set $2a_0 = \zeta\left(\frac{1-d}{2}\right)$. Let Φ be a radial function in the space $\mathcal{S}(\mathbb{R}^d)$, orthogonal in the space $L^2(\mathbb{R}^d)$ to the functions $|x|^{-\frac{d+1}{2}}, |x|^{-\frac{d+1}{2}-2}, \dots, |x|^{-\frac{d+1}{2}-2q} = |x|^{-d+1}$. Let Ψ be the function defined as

$$(3.45) \quad \Psi = \pi^{2i\pi\mathcal{E}} \frac{\Gamma\left(\frac{1}{4} - i\pi\mathcal{E}\right)}{\Gamma\left(\frac{1}{4} + i\pi\mathcal{E}\right)} \mathcal{F}\Phi.$$

For every $t > 0$, one has the relation

$$(3.46) \quad \begin{aligned} 2a_0\Phi(0) + \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \sigma_{\frac{d-1}{2}}(r(m))\Phi(tm) \\ = t^{-d} \left[2a_0\Psi(0) + \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \sigma_{\frac{d-1}{2}}(r(m))\Psi\left(\frac{m}{t}\right) \right]. \end{aligned}$$

Proof. — Starting just as in (2.31), we set [3, p. 250]

$$(3.47) \quad \mathcal{D}_{\mathbf{a}}(s) = \sum_{r \geq 1} \frac{a_r}{r^s} = \zeta(s)\zeta\left(s + \frac{1-d}{2}\right)$$

so that

$$(3.48) \quad \mathcal{D}_{\mathbf{b}}(s) = \frac{\mathcal{D}_{\mathbf{a}}(s)}{\zeta(s)} = \zeta\left(s + \frac{1-d}{2}\right).$$

In our present case, the second term on the right-hand side of (2.24) reduces since $d \geq 2$ to

$$(3.49) \quad 2\pi \operatorname{Res}_{s=d} \left(\zeta\left(s + \frac{1-d}{2}\right) \langle \mathfrak{E}_{\frac{d}{2}-s}^d, h \rangle \right) = 2\pi \zeta\left(\frac{d+1}{2}\right) \langle 1, h \rangle$$

so that

$$(3.50) \quad \langle \mathfrak{S}_{\mathbf{a}}, h \rangle - 2\pi \zeta\left(\frac{d+1}{2}\right) \langle 1, h \rangle = \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2} - i\lambda\right) \langle \mathfrak{E}_{i\lambda}^d, h \rangle d\lambda :$$

now, given Φ and Ψ , where Φ is a radial function in the space $\mathcal{S}(\mathbb{R}^d)$, linked by (3.45), there is the same link between the functions $x \mapsto \Phi^t(x) = \Phi(tx)$ and $x \mapsto \Psi_t(x) = t^{-d}\Psi\left(\frac{x}{t}\right)$. Applying the equation just obtained with h replaced by Φ^t or Ψ_t , we observe that the right-hand sides will coincide as a consequence of the functional equation of the zeta function

$$\zeta\left(\frac{1}{2} + i\lambda\right) = \pi^{i\lambda} \frac{\Gamma\left(\frac{1}{4} - \frac{i\lambda}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{i\lambda}{2}\right)} \zeta\left(\frac{1}{2} - i\lambda\right) :$$

the argument is just the same as the one used in the proof of Theorem 3.1. What remains to be done so as to complete the proof of Theorem 3.3 is analyzing the transform in (3.45), in particular getting a link between $\langle 1, \Phi \rangle$ and $\Psi(0)$ (or $\langle 1, \Psi \rangle$ and $\Phi(0)$) as a consequence: this is done in the next lemma. □

LEMMA 3.4. — *Under the assumptions of Theorem 3.3 relative to the dimension d and to the radial function $\Phi \in \mathcal{S}(\mathbb{R}^d)$, the function Ψ linked to Φ by (3.45) is continuous throughout \mathbb{R}^d , rapidly decreasing at infinity, and satisfies the equation*

$$(3.51) \quad \Psi(0) = \pi^{\frac{d}{2}} \frac{\Gamma\left(\frac{1-d}{4}\right)}{\Gamma\left(\frac{1+d}{4}\right)} \langle 1, \Phi \rangle$$

as well as the same equation in which the roles of Φ and Ψ are exchanged.

Proof. — From Lemma 5.1 and (3.45), we get

$$(3.52) \quad \operatorname{spec}(\Psi; \lambda) = \pi^{-2i\lambda} \frac{\Gamma\left(\frac{1}{4} + \frac{i\lambda}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{i\lambda}{2}\right)} \frac{\Gamma\left(\frac{d}{4} + \frac{i\lambda}{2}\right)}{\Gamma\left(\frac{d}{4} - \frac{i\lambda}{2}\right)} \operatorname{spec}(\Phi; -\lambda)$$

and, from (5.1),

$$(3.53) \quad \Psi(x) = \frac{1}{i} \int_{\operatorname{Re} z=0} \pi^{-2z} \frac{\Gamma\left(\frac{1}{4} + \frac{z}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{z}{2}\right)} \frac{\Gamma\left(\frac{d}{4} + \frac{z}{2}\right)}{\Gamma\left(\frac{d}{4} - \frac{z}{2}\right)} \operatorname{spec}(\Phi; iz) |x|^{-\frac{d}{2}-z} dz.$$

As already noted in (2.44), one has

$$(3.54) \quad \text{spec}(\Phi; iz) = \frac{1}{2\pi} \int_0^\infty t^{\frac{d}{2}-z} \phi(t) \frac{dt}{t}$$

if $\Phi(x) = \phi(|x|)$: now the Taylor expansion of $\phi(t)$ near $t = 0$ only involves even-order terms and makes it possible to show that $\text{spec}(\Phi; iz)$ extends as a meromorphic function in the complex plane, with only simple poles at $z = \frac{d}{2} + 2j$, $j = 0, 1, \dots$. These poles correspond to poles of $\Gamma(\frac{d}{4} - \frac{z}{2})$ so that, using also the fact that $\text{spec}(\Phi; iz)$ is rapidly decreasing as a function of $\text{Im } z$, one sees by a contour deformation $z \mapsto z + \frac{d}{2} + 2k + 1$ that $\Psi(x)$ is rapidly decreasing as a function of x for large $|x|$. For $|x| < 1$, we must on the contrary perform the change of contour $z \mapsto z - \frac{d}{2} - 2k - \frac{1}{2}$, thus coming across the simple poles $z = -\frac{d}{2} - 2j$ ($j = 0, 1, \dots$) or $z = -\frac{1}{2} - 2\ell$ ($\ell = 0, 1, \dots$) of the integrand of (3.53). Each pole of the first type will then contribute to $\Psi(x)$ a term in $|x|^{2j}$, but a pole of the second type would contribute a term in $|x|^{-\frac{d}{2} + \frac{1}{2} + 2\ell}$ discontinuous at the origin for $\ell = 0, \dots, q$ unless the conditions $\text{spec}(\Phi; i(\frac{1}{2} + 2\ell)) = 0$ are satisfied in this range of values of ℓ : now, from (3.54), such a set of conditions can be expressed as

$$(3.55) \quad \int_0^\infty t^{\frac{d-1}{2}-2\ell} \phi(t) \frac{dt}{t} = 0, \quad \ell = 0, \dots, q,$$

which is just the assumption made in the statement of Theorem 3.3.

Computing a residue at $z = -\frac{d}{2}$, we easily get

$$(3.56) \quad \begin{aligned} \Psi(0) &= 4\pi^{d+1} \frac{\Gamma(\frac{1-d}{4})}{\Gamma(\frac{1+d}{4})\Gamma(\frac{d}{2})} \text{spec}\left(\Phi; -\frac{id}{2}\right) \\ &= \pi^{\frac{d}{2}} \frac{\Gamma(\frac{1-d}{4})}{\Gamma(\frac{1+d}{4})} \langle 1, \Phi \rangle. \end{aligned}$$

□

4. Combs and non-holomorphic modular forms.

Non-holomorphic modular forms f also give rise to a generalization of the Poisson summation formula, analogous to that developed in the preceding section. There are, however, two major differences, which we shall briefly report to start with. Again, everything will be centered on the functional equation satisfied by the associated L -function. In the non-holomorphic case, the Archimedean factor making such an equation valid

involves the product of two Gamma factors. As a consequence, the modified Fourier transformation $\Phi \mapsto \Psi$ taking the place of the one which appeared in Lemma 2.9 or in (3.45) will have twice as many Gamma factors, as in (4.31) or (4.58). Another difference is more fundamental: if one goes back to (2.23), one will remember that it was essential, so as to obtain Theorem 3.1, to construct a Dirichlet series \mathcal{D}_b with a functional equation (3.11) concerned with the symmetry $s \mapsto d - s$: the way we were able, in the holomorphic case, to manage this was to appeal to a modular weight of a high level (a multiple of d). No such trick is available any more in the non-holomorphic case: instead, we shall make use of a certain expression (4.28) of the tensor product $L(s_1, f)L(s_2, f)$. This will force us to understand in a deeper way, in the proof of Proposition 4.2, the action on combs of Dirichlet series in the Euler operator. Then, the coefficients, in Theorem 4.7, of the generalized Poisson formula built in this way will no longer be read directly from the Fourier coefficients of the non-holomorphic modular form f , but will involve the value, at the symmetry point $s = \frac{1}{2}$, of a slightly generalized version of the L -function $L(s, f)$. In another version (Proposition 4.8), more closely related to the method used in the proof of Theorem 3.2, the coefficients of the Poisson formula will be quite simple, but this version is only available in dimension 1.

For simplicity, we shall consider only non-holomorphic (Maass) modular forms for the group $\Gamma = SL(2, \mathbb{Z})$: these are functions f on the upper half-plane Π , Γ -invariant for the usual fractional-linear action of Γ on Π , at the same time generalized eigenfunctions of the hyperbolic Laplacian

$$(4.1) \quad \Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

on Π . A Maass cusp-form is a non-holomorphic modular form which lies in the Hilbert space $L^2(\Gamma \backslash \Pi)$: this is the space of Γ -invariant functions on Π , the restriction of which to any fundamental domain for the action of Γ on Π is square-integrable. The theory of non-holomorphic modular forms, another name for which is the spectral theory of the modular Laplacian, is developed in many places, including [7, 15, 5, 1].

Even though explicit examples of Maass cusp-forms are known only for some (congruence) subgroups of Γ distinct from Γ , it is very well known that the discrete spectrum of Δ in $L^2(\Gamma \backslash \Pi)$ (i.e. the set of genuine eigenvalues of Δ) constitutes a sequence $(\frac{1+\lambda_k^2}{4})$ with $\lambda_k \rightarrow \infty$: we are only interested in even cusp-forms, i.e. cusp-forms invariant under the symmetry $z \mapsto -\bar{z}$. Recall [15, p. 208] that, as shown by a simple separation of

variables, any cusp-form f admits a Fourier expansion (with respect to x) of the kind

$$(4.2) \quad f(z) = y^{\frac{1}{2}} \sum_{r \neq 0} f_r K_{\frac{i\lambda_k}{2}}(2\pi|r|y) e^{2i\pi r x} :$$

if f is even, which we assume from now on, one has $f_{-r} = f_r$ for all r .

A fundamental concept, in relation with cusp forms, is that of L -function: the L -function associated with f is the Dirichlet series

$$(4.3) \quad L(s, f) = \sum_{r \geq 1} f_r r^{-s} :$$

it is convergent when $\operatorname{Re} s$ is large and admits a continuation [1, p. 106] as an entire function satisfying some functional equation of the usual type: setting

$$(4.4) \quad L^*(s, f) = \pi^{-s} \Gamma\left(\frac{s}{2} + \frac{i\lambda_k}{4}\right) \Gamma\left(\frac{s}{2} - \frac{i\lambda_k}{4}\right) L(s, f),$$

this is again an entire function and one has (if f is an even cusp-form)

$$(4.5) \quad L^*(s, f) = L^*(1 - s, f).$$

We need a slight extension of this notion:

LEMMA 4.1. — Given an even cusp-form f with the Fourier expansion (4.2), and an integer $n \geq 1$, the function

$$(4.6) \quad L^*(s; n; f) = \pi^{-s} \Gamma\left(\frac{s}{2} + \frac{i\lambda_k}{4}\right) \Gamma\left(\frac{s}{2} - \frac{i\lambda_k}{4}\right) L(s; n; f),$$

with

$$(4.7) \quad L(s; n; f) = \sum_{\substack{r \geq 1 \\ r \text{ divisible by } n}} f_r r^{-s},$$

extends as an entire function of s .

Proof. — We show that

$$(4.8) \quad L^*(s; n; f) = \frac{2}{n} \sum_{q=0}^{n-1} \int_0^\infty f\left(\frac{q}{n} + iy\right) y^{s-\frac{1}{2}} \frac{dy}{y}.$$

First, each integral on the right-hand side converges for all s : indeed, for large y , $f(x + iy)$ is rapidly decreasing as shown by the expansion (4.2); next, in the case when $q = 0$, the automorphy condition $f(iy) = f(iy^{-1})$ takes care of the integrand near $y = 0$. Finally, to estimate $f\left(\frac{q}{n} + iy\right)$ when

$q \neq 0$ and y is small, we set $q = a(q, n)$, $n = c(q, n)$, complete the pair a, c to a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, so that $an - cq = 0$, $bn - dq \neq 0$ and

$$(4.9) \quad \frac{d\left(\frac{q}{n} + iy\right) - b}{-c\left(\frac{q}{n} + iy\right) + a} = \frac{-dny + (dq - bn)i}{cny},$$

a point in the upper half-plane congruent to $\frac{q}{n} + iy$, with an imaginary part that goes to ∞ as $y \rightarrow 0$ just like y^{-1} .

Next, we compute the right-hand side of (4.8) under the assumption that $\text{Re } s$ is large. Starting from (4.2), we find

$$(4.10) \quad \frac{2}{n} \sum_{q=0}^{n-1} f\left(\frac{q}{n} + iy\right) = 2y^{\frac{1}{2}} \sum_{r \neq 0} f_r K_{\frac{i\lambda_k}{2}}(2\pi|r|y) S_n(r)$$

with

$$(4.11) \quad S_n(r) = n^{-1} \sum_{q=0}^{n-1} e^{2i\pi r \frac{q}{n}},$$

a number which is 1 if $n|r$, 0 otherwise. The proof is concluded, just as for the usual L -function, by the use of the equation [9, p. 91]

$$(4.12) \quad \int_0^\infty y^{s-1} K_{\frac{i\lambda_k}{2}}(2\pi|r|y) dy = 2^{s-2} (2\pi|r|)^{-s} \Gamma\left(\frac{s}{2} + \frac{i\lambda_k}{4}\right) \Gamma\left(\frac{s}{2} - \frac{i\lambda_k}{4}\right).$$

□

The construction that follows associates a comb with a Poisson-style equations to any non-holomorphic cusp-form f under the condition that $L\left(\frac{1}{2}, f\right) \neq 0$. One may note that the same non-vanishing condition occurs in [17, theorem 15.2], so that the values $\frac{1-i\lambda_k}{2}$ of interest are exactly the poles with real value $\frac{1}{2}$ of the continuation to the half-plane $\text{Re } s > 0$ of the Dirichlet series

$$(4.13) \quad \sum_{m \geq 1} \left(\sum_{\substack{r \bmod m \\ r(1-r) \equiv 0}} e^{2i\pi \frac{r}{m}} \right) m^{-s}$$

(the other poles of this function in the given half-plane are just the numbers $\frac{\rho}{2}$ with ρ a non-trivial zero of the zeta function). Let us remind the reader that nothing is known, at present, about the arithmetic nature of the sequence (λ_k) .

Maass even cusp-forms corresponding to some given eigenvalue $\frac{1+\lambda_k^2}{4}$ constitute a finite-dimensional vector space with a basis consisting of *Maass-Hecke forms*, also called Maass eigenforms, with especially nice

properties ([1, p. 119] or [15, p. 241]): they are not only eigenfunctions of Δ , but also joint eigenfunctions of the sequence $(T_N)_{N \geq 1}$ of Hecke operators (*loc.cit.*). It is useful (and customary) to normalize Maass-Hecke forms by the requirement that the first coefficient f_1 in their Fourier expansion (4.2) should be $f_1 = 1$: in this way, the L -function of f has the Eulerian product

$$(4.14) \quad L(s, f) = \prod_{p \text{ prime}} (1 - f_p p^{-s} + p^{-2s})^{-1}.$$

When dealing with Maass-Hecke forms, we shall always assume that the normalization above has been chosen.

We start with the following analogue of Proposition 2.6.

PROPOSITION 4.2. — *Let $d = 1, 2, \dots$ and let f be a Maass-Hecke cusp-form for the group Γ . Set, for $r \geq 1$,*

$$(4.15) \quad a_r = \sum_{1 \leq n | r} n^{\frac{d}{2}} \sigma_{\frac{d-2}{2}}\left(\frac{r}{n}\right) L\left(\frac{1}{2}; n; f\right),$$

where $\sigma_\mu(n)$ always denotes the sum of all positive divisors of n taken to the power μ , and consider the comb \mathfrak{S}_a as defined in (2.5). It is a tempered distribution, and admits the following weak decomposition in $\mathcal{S}'(\mathbb{R}^d)$:

$$(4.16) \quad \mathfrak{S}_a = L\left(\frac{1}{2}, f\right) \left[\int_{-\infty}^{\infty} L\left(\frac{1}{2} - i\lambda, f\right) \mathfrak{E}_{i\lambda}^d d\lambda + 2\pi L\left(\frac{d+1}{2}, f\right) \right].$$

Proof. — Dirichlet series in $2i\pi\mathcal{E}$ the coefficients of which constitute a slowly increasing sequence transform (tempered) combs into (tempered) combs, for

$$(4.17) \quad \left(\sum_{n \geq 1} \gamma(n) \cdot n^{-\frac{d}{2} - 2i\pi\mathcal{E}} \right) \mathfrak{S}_a = \mathfrak{S}_b$$

with

$$(4.18) \quad b(r) = \sum_{1 \leq n | r} \gamma(n) a\left(\frac{r}{n}\right).$$

Indeed, given $h \in \mathcal{S}(\mathbb{R}^d)$, one has

$$(4.19) \quad \begin{aligned} & \left\langle \sum_{n \geq 1} \gamma(n) \cdot n^{-\frac{d}{2} - 2i\pi\mathcal{E}} \mathfrak{S}_a, h \right\rangle \\ &= \sum_{n \geq 1} \gamma(n) \langle \mathfrak{S}_a, x \mapsto h(nx) \rangle \\ &= 2\pi \sum_{n \geq 1} \gamma(n) \sum_{\substack{m \in \mathbb{Z}^d \\ m \neq 0}} a(r(m)) h(nm) \\ &= 2\pi \sum_{n \geq 1} \gamma(n) \sum_{\substack{m' \in \mathbb{Z}^d \setminus \{0\} \\ n | r(m')}} a(n^{-1}r(m')) h(m'). \end{aligned}$$

For instance, every tempered comb $\mathfrak{S}_{\mathbf{a}}$ is the image of $\mathfrak{D}^{\text{prime}}$ (cf. (2.7)) under the Dirichlet series $\sum_{n \geq 1} a_n n^{-\frac{d}{2}-2i\pi\mathcal{E}}$: it is also the image under some appropriate Dirichlet series in $2i\pi\mathcal{E}$ of the Dirac comb itself, as it follows from the Möbius formula (2.22). Note (applying (2.10)) the relation between tempered distributions, valid, for a given sequence $(\gamma(n))$, if $-\text{Re } \nu$ is large enough,

$$(4.20) \quad \left(\sum_{n \geq 1} \gamma(n) \cdot n^{-\frac{d}{2}-2i\pi\mathcal{E}} \right) \mathfrak{E}_{\nu}^d = \left(\sum_{n \geq 1} \gamma(n) n^{-\frac{d}{2}+\nu} \right) \mathfrak{E}_{\nu}^d :$$

this is more easily remembered as the rule of thumb that the operator $2i\pi\mathcal{E}$ acts on \mathfrak{E}_{ν}^d like the multiplication by $-\nu$, a reminder of the fact that the distribution \mathfrak{E}_{ν}^d is homogeneous of degree $-\frac{d}{2} - \nu$.

Introduce, for $\text{Re } s$ large, the following operator:

$$(4.21) \quad \mathcal{A}_s = \sum_{N \geq 1} f_N N^{-s} \sigma_{-2i\pi\mathcal{E}}(N).$$

This becomes a Dirichlet series in the argument $2i\pi\mathcal{E}$ after resummation:

$$(4.22) \quad \mathcal{A}_s = \sum_{n \geq 1} \gamma(n) \cdot n^{-\frac{d}{2}-2i\pi\mathcal{E}}$$

with

$$(4.23) \quad \gamma(n) = n^{\frac{d}{2}} \sum_{\substack{N \text{ divisible by } n}} f_N N^{-s} = n^{\frac{d}{2}} L(s; n; f),$$

as a consequence of which it operates on combs: if $\mathbf{c} = (c(1), c(2), \dots)$, one has, according to (4.19),

$$(4.24) \quad \mathcal{A}_s \mathfrak{S}_{\mathbf{c}} = \mathfrak{S}_{\mathbf{c}'_s}$$

with

$$(4.25) \quad c'_s(r) = \sum_{1 \leq n | r} n^{\frac{d}{2}} c\left(\frac{r}{n}\right) L(s; n; f).$$

By Lemma 4.1, \mathcal{A}_s , viewed for each s as an operator on temperate combs, extends as an entire function of s : we are especially interested in the value $s = \frac{1}{2}$. We shall apply what precedes with $c(r) = \sigma_{\frac{d-2}{2}}(r)$, in which case $c'_{\frac{1}{2}}(r) = a_r$ as defined in (4.15), and [3, p. 250]

$$(4.26) \quad \mathcal{D}_{\mathbf{c}}(s) = \sum_{r \geq 1} c(r) r^{-s} = \zeta(s) \zeta\left(s - \frac{d-2}{2}\right)$$

for $\text{Re } s > \max(1, \frac{d}{2})$. Then, from Proposition 2.6, next Theorem 2.5,

$$(4.27) \quad \begin{aligned} \mathfrak{S}_{\mathbf{c}} &= \int_{-\infty}^{\infty} \zeta(1-i\lambda) \mathfrak{E}_{i\lambda}^d d\lambda + 2\pi \text{Res}_{s=d} \left(\zeta\left(s - \frac{d-2}{2}\right) \mathfrak{E}_{\frac{d}{2}-s}^d \right) \\ &= \int_{-\infty}^{\infty} \zeta(1-i\lambda) \mathfrak{E}_{i\lambda}^d d\lambda + 2\pi \zeta\left(\frac{d+2}{2}\right). \end{aligned}$$

Now, if ν_1 and $\nu_2 \in \mathbb{C}$ are such that $-\nu_1 \pm \nu_2$ has a large real part, one has the equation [17, p. 140]

$$(4.28) \quad \sum_{N \geq 1} N^{-\frac{1+\nu_1-\nu_2}{2}} \sigma_{\nu_2}(N) f_N = \frac{L(\frac{1-\nu_1+\nu_2}{2}, f)L(\frac{1-\nu_1-\nu_2}{2}, f)}{\zeta(1-\nu_1)} :$$

this equation, easily proved by an application of the Eulerian product (4.14), really belongs to the theory of so-called convolution L -functions ([1, p. 73] or [6, p. 231], though in these two places it is, rather, the case of a convolution L -function associated with a pair of holomorphic modular forms that is considered). From this, together with (4.20) and the fact that the operator $2i\pi\mathcal{E}$ acts on constants like the multiplication by $\frac{d}{2}$, it follows that, when $\text{Re } s$ is large,

$$(4.29) \quad \begin{aligned} \mathcal{A}_s \mathfrak{E}_{i\lambda}^d &= \frac{L(s, f)L(s-i\lambda, f)}{\zeta(2s-i\lambda)} \mathfrak{E}_{i\lambda}^d, \\ \mathcal{A}_s .1 &= \frac{L(s, f)L(s+\frac{d}{2}, f)}{\zeta(2s+\frac{d}{2})}, \end{aligned}$$

which yields the spectral decomposition

$$(4.30) \quad \mathfrak{S}_{\mathfrak{c}_s} = L(s, f) \left[\int_{-\infty}^{\infty} \frac{\zeta(1-i\lambda)}{\zeta(2s-i\lambda)} L(s-i\lambda, f) \mathfrak{E}_{i\lambda}^d d\lambda + 2\pi \frac{\zeta(\frac{d+2}{2})}{\zeta(2s+\frac{d}{2})} L(s+\frac{d}{2}, f) \right],$$

a detailed proof of which would follow the lines of that of Proposition 2.6.

Taking the complex continuation with respect to s , we obtain Proposition 4.2 as the case $s = \frac{1}{2}$ of the last equation. Concerning the convergence of the integral with respect to λ involved, what was said at the beginning of the proof of Proposition 2.6 still applies. \square

So that the integral term from the spectral decomposition (4.16) should remain invariant under the operator $\mathcal{H}_{d, \lambda_k}$ to be introduced now, we rely again on the equation $\mathcal{F} \mathfrak{E}_{i\lambda}^d = \mathfrak{E}_{-i\lambda}^d$ together with (4.5), expressed as the invariance under $\lambda \mapsto -\lambda$ of the function

$$\pi^{\frac{1}{2}+i\lambda} \Gamma\left(\frac{1+i\lambda_k}{4} - \frac{i\lambda}{2}\right) \Gamma\left(\frac{1-i\lambda_k}{4} - \frac{i\lambda}{2}\right) L\left(\frac{1}{2} - i\lambda, f\right),$$

and set

$$(4.31) \quad \mathcal{H}_{d, \lambda_k} = \pi^{4i\pi\mathcal{E}} \frac{\Gamma(\frac{1}{4} + \frac{i\lambda_k}{4} - i\pi\mathcal{E})\Gamma(\frac{1}{4} - \frac{i\lambda_k}{4} - i\pi\mathcal{E})}{\Gamma(\frac{1}{4} + \frac{i\lambda_k}{4} + i\pi\mathcal{E})\Gamma(\frac{1}{4} - \frac{i\lambda_k}{4} + i\pi\mathcal{E})} \mathcal{F},$$

a definition to be compared to the definition of $\mathcal{F}_{d,w} = \mathfrak{K}_{d,w} \mathcal{F}$ with $\mathfrak{K}_{d,w}$ as introduced in (2.32): recall from the introduction to the present section,

however, the reason for the larger number of Gamma factors appearing now.

The space $\mathcal{S}(\mathbb{R}^d)$ or its subspace consisting of radial functions, is not invariant under the operator $\mathcal{H}_{d,\lambda_k}$. There is a larger space that is, but of necessity it has to depend on λ_k , which may give a somewhat technical appearance to the definition that follows. Until we come to Theorem 4.7, λ_k could be any non-zero real number.

DEFINITION 4.3. — *The linear space E_{λ_k} consists of all radial functions Φ on \mathbb{R}^d with the following list of properties. The functions Φ and $\mathcal{F}\Phi$ are locally summable on \mathbb{R}^d and C^∞ outside 0. There exist constants α_j, β_j ($j = 0, 1, \dots$) such that, for every $\ell = 0, 1, \dots$, and every $M = 0, 1, \dots$, the function*

$$(4.32) \quad \mathcal{E}^\ell \left[(\mathcal{F}\Phi)(x) - \sum_{j=0}^{M-1} \alpha_j |x|^{-\frac{d+1}{2} - \frac{i\lambda_k}{2} - 2j} - \sum_{j=0}^{M-1} \beta_j |x|^{-\frac{d+1}{2} + \frac{i\lambda_k}{2} - 2j} \right]$$

is a $O(|x|^{-\frac{d+1}{2} - 2M})$ as $|x| \rightarrow \infty$. On the other hand, the function $\Phi(x)$ is rapidly decreasing as $|x| \rightarrow \infty$.

The space E_{λ_k} has a natural Fréchet topology.

LEMMA 4.4. — *The tempered distribution \mathfrak{E}_ν^d extends for $\nu \neq \pm \frac{d}{2}$, $\nu \neq \frac{1 \pm i\lambda_k}{2} + 2j$ ($0 \leq j < \infty$), as a continuous linear functional on E_{λ_k} . As a function of ν , regarded as valued in the dual of E_{λ_k} , \mathfrak{E}_ν^d is a meromorphic function in the entire complex plane, with only simple poles at the points mentioned.*

Proof. — First, the behaviour of Φ and $\mathcal{F}\Phi$ near 0 is clear: $\mathcal{F}\Phi$ is C^∞ throughout \mathbb{R}^d since Φ is the sum of a function in $\mathcal{S}(\mathbb{R}^d)$ and of a summable function with compact support; next, $\Phi(x)$ is a linear combination of functions $|x|^{-\frac{d-1}{2} \pm \frac{i\lambda_k}{2} + 2j}$, and of a function which is as regular as one may wish at $0 \in \mathbb{R}^d$. Set

$$(4.33) \quad \Xi = \left(\frac{1}{4} - i\pi\mathcal{E} + \frac{i\lambda_k}{4} \right)_M \left(\frac{1}{4} - i\pi\mathcal{E} - \frac{i\lambda_k}{4} \right)_M \Phi,$$

using in an operator-theoretic context Pochhammer’s notation

$$(4.34) \quad \left(\frac{1}{4} - i\pi\mathcal{E} + \frac{i\lambda_k}{4} \right)_M = \prod_{j=0}^{M-1} \left(\frac{1}{4} - i\pi\mathcal{E} + \frac{i\lambda_k}{4} + j \right).$$

Since one of the factors of the differential operator $(\frac{1}{4} - i\pi\mathcal{E} + \frac{i\lambda_k}{4})_M$, to wit $\frac{1}{4} - i\pi\mathcal{E} + \frac{i\lambda_k}{4} + j$, annihilates the function $|x|^{\frac{1-d}{2} + \frac{i\lambda_k}{2} + 2j}$, and since the operator

$$(4.35) \quad \mathcal{F}\left(\frac{1}{4} - i\pi\mathcal{E} - \frac{i\lambda_k}{4} + j\right)\mathcal{F}^{-1} = \frac{1}{4} + i\pi\mathcal{E} - \frac{i\lambda_k}{4} + j$$

annihilates the term $|x|^{-\frac{d+1}{2} + \frac{i\lambda_k}{2} - 2j}$ from $(\mathcal{F}\Phi)(x)$, it is easy to verify that Ξ belongs to the space $C_\varepsilon^{[\infty]}$ introduced in Definition 2.4, with $\varepsilon = 2M + \frac{1-d}{2}$, as soon as $2M > \frac{d-1}{2}$: note that ε is as large as one pleases. One may then define, provided that $|\operatorname{Re} \nu| < \frac{d}{2} + \varepsilon$, $\nu \neq \pm \frac{d}{2}$, $\nu \neq \frac{1 \pm i\lambda_k}{2} + 2j$ for $j = 0, 1, \dots$,

$$(4.36) \quad \langle \mathfrak{E}_\nu^d, \Phi \rangle := \left[\left(\frac{1}{4} - \frac{\nu}{2} + \frac{i\lambda_k}{4} \right)_M \left(\frac{1}{4} - \frac{\nu}{2} - \frac{i\lambda_k}{4} \right)_M \right]^{-1} \langle \mathfrak{E}_\nu^d, \Xi \rangle.$$

What remains to be seen is that this definition extends the one already known in the case when $\Phi \in \mathcal{S}(\mathbb{R}^d)$: now, this follows from the equation

$$(4.37) \quad \langle i\pi\mathcal{E}\mathfrak{E}_\nu^d, \Phi \rangle = -\langle \mathfrak{E}_\nu^d, i\pi\mathcal{E}\Phi \rangle,$$

together with the fact that the distribution \mathfrak{E}_ν^d is homogeneous of degree $-\frac{d}{2} - \nu$. □

LEMMA 4.5. — *The operator $\mathcal{H}_{d,\lambda_k}$ preserves the space E_{λ_k} .*

Proof. — Let $\Phi \in E_{\lambda_k}$, $\Phi(x) = \phi(|x|)$. From the beginning of the proof of the preceding lemma, it is square-integrable, so that $\Psi = \mathcal{H}_{d,\lambda_k}\Phi$ is well-defined in the spectral-theoretic sense. Using (2.42) and (2.43), we write

$$(4.38) \quad \Phi(x) = \int_{-\infty}^{\infty} \operatorname{spec}(\Phi; \lambda) |x|^{-\frac{d}{2} - i\lambda} d\lambda,$$

with

$$(4.39) \quad \operatorname{spec}(\Phi; \lambda) = \frac{1}{2\pi} \int_0^{\infty} t^{\frac{d}{2} + i\lambda} \phi(t) \frac{dt}{t}.$$

The integral from 1 to ∞ extends as an entire function of $i\lambda$. Let M be an integer ≥ 1 . From the proof of Lemma 4.4, one sees that, applying the operator $(\frac{1}{4} - i\pi\mathcal{E} + \frac{i\lambda_k}{4})_M$ to Φ , one gets a function which is continuous at 0 as soon as $M > \frac{d-1}{4}$, which makes it possible, since M can be taken arbitrarily large, to extend the function $\operatorname{spec}(\Phi; \lambda)$ as a meromorphic function of $i\lambda$ in the half-plane $\operatorname{Re}(i\lambda) > -\frac{d}{2}$, setting

$$(4.40) \quad \operatorname{spec}(\Phi; \lambda) = \left(\frac{1}{4} + \frac{i\lambda}{2} + \frac{i\lambda_k}{4} \right)_M^{-1} \operatorname{spec} \left(\left(\frac{1}{4} - i\pi\mathcal{E} + \frac{i\lambda_k}{4} \right)_M \Phi; \lambda \right).$$

At $-\frac{d}{2}$ or below, one also has to take into account the singularities in $\text{spec}(\Phi; \lambda)$ which arise from the regular (even) terms in the expansion of $\phi(t)$ near 0. Finally, the function under consideration extends as a holomorphic function in the entire plane, save at simple poles given as $i\lambda = -\frac{1}{2} - \frac{i\lambda_k}{2} - 2j$ and $i\lambda = -\frac{d}{2} - 2j$, $j = 0, 1, \dots$. Also, when the imaginary part of λ is kept fixed, the function under study is rapidly decreasing as a function of $\text{Re } \lambda$ as $|\text{Re } \lambda| \rightarrow \infty$, because an integration by parts permits to benefit from the fact that arbitrary powers of $t \frac{d}{dt}$ are applicable to the function $\phi(t)$ on the right-hand side of (4.39). Then, we write, using (4.31) and (5.4),

$$(4.41) \quad \Psi(x) = \int_{-\infty}^{\infty} \pi^{-3i\lambda} \frac{\Gamma(\frac{1}{4} + \frac{i\lambda_k}{4} + \frac{i\lambda}{2})\Gamma(\frac{1}{4} - \frac{i\lambda_k}{4} + \frac{i\lambda}{2})}{\Gamma(\frac{1}{4} + \frac{i\lambda_k}{4} - \frac{i\lambda}{2})\Gamma(\frac{1}{4} - \frac{i\lambda_k}{4} - \frac{i\lambda}{2})} \cdot \frac{\Gamma(\frac{d}{4} + \frac{i\lambda}{2})}{\Gamma(\frac{d}{4} - \frac{i\lambda}{2})} \text{spec}(\Phi; -\lambda)|x|^{-\frac{d}{2}-i\lambda} d\lambda$$

and

$$(4.42) \quad (\mathcal{F}\Psi)(x) = \int_{-\infty}^{\infty} \pi^{2i\lambda} \frac{\Gamma(\frac{1}{4} + \frac{i\lambda_k}{4} - \frac{i\lambda}{2})\Gamma(\frac{1}{4} - \frac{i\lambda_k}{4} - \frac{i\lambda}{2})}{\Gamma(\frac{1}{4} + \frac{i\lambda_k}{4} + \frac{i\lambda}{2})\Gamma(\frac{1}{4} - \frac{i\lambda_k}{4} + \frac{i\lambda}{2})} \cdot \text{spec}(\Phi; \lambda)|x|^{-\frac{d}{2}-i\lambda} d\lambda.$$

Next, we perform the change of contour that corresponds to $i\lambda \mapsto i\lambda + M$, simply paying attention to the poles that may appear in the study of the second of these two integrals: observe that the function

$$\lambda \mapsto \left(\Gamma\left(\frac{1}{4} + \frac{i\lambda_k}{4} - \frac{i\lambda}{2}\right) \Gamma\left(\frac{d}{4} - \frac{i\lambda}{2}\right) \right)^{-1} \text{spec}(\Phi; -\lambda)$$

is entire. □

Functions Φ in the space E_{λ_k} are rapidly decreasing, even though they usually lack regularity at 0 (in a very controlled way): thus, one may define $\langle \mathfrak{S}_{\mathbf{a}}, \Phi \rangle$ for any given slowly increasing sequence $\mathbf{a} = (a_1, a_2, \dots)$.

LEMMA 4.6. — *Under the assumptions of Proposition 3.2, the decomposition (4.16) is still valid when tested against any function Φ in the space E_{λ_k} .*

Proof. — The proof of Proposition 4.2 was based on that of Proposition 2.6; the latter one, starting from (2.31), was based on a deformation of contour. This applies equally well in our case: the whole matter reduces to a question of pole-chasing. Looking at the right-hand side of (4.16), one notices that the new poles that appear come from the behaviour of $\mathfrak{E}_{i\lambda}^d$ when

regarded as a continuous linear functional on the space E_{λ_k} : from Lemma 4.4, these new poles can be only the points $i\lambda = \frac{1 \pm i\lambda_k}{2} + 2j$, $j = 0, 1, \dots$. They do not belong to the half-plane $\text{Re}(i\lambda) \leq 0$ in which, following the proof of Proposition 2.6, the contour deformation takes place. Anyway, from (4.4) together with the fact that $L^*(s, f)$ is an entire function, the function $L(\frac{1}{2} - i\lambda, f)$ which appears in the integrand on the right-hand side of (4.16) vanishes at these points, which gives another reason why there are no new poles to worry about. \square

Before stating the main theorem of this section, it is convenient to define the *true value* at 0 of a function $\Phi \in E_{\lambda_k}$, denoted as $\Phi[[0]]$, which coincides with $\Phi(0)$ in the case when Φ is continuous at 0, and which is defined in general as the coefficient γ_0 from the asymptotic expansion (a consequence of (4.31))

$$(4.43) \quad \Phi(x) \sim \sum_{j \geq 0} \alpha'_j |x|^{\frac{1-d}{2} + \frac{i\lambda_k}{2} + 2j} + \sum_{j \geq 0} \beta'_j |x|^{\frac{1-d}{2} - \frac{i\lambda_k}{2} + 2j} + \sum_{j \geq 0} \gamma_j |x|^{2j}$$

of Φ near 0. One then has (still assuming that $\Psi = \mathcal{H}_{d, \lambda_k} \Phi$) the equation

$$(4.44) \quad \int_{\mathbb{R}^d} \Phi(x) dx = \pi^{-d} \frac{\Gamma(\frac{1+d+i\lambda_k}{4})\Gamma(\frac{1+d-i\lambda_k}{4})}{\Gamma(\frac{1-d+i\lambda_k}{4})\Gamma(\frac{1-d-i\lambda_k}{4})} \Psi[[0]].$$

To show this, note that, as a consequence of (4.39) with Ψ substituted for Φ ,

$$(4.45) \quad \frac{\text{spec}(\Psi; -\lambda)}{\Gamma(\frac{d}{4} - \frac{i\lambda}{2})} \rightarrow \frac{\Psi[[0]]}{4\pi} \text{ as } \lambda \rightarrow -\frac{id}{2}.$$

Next, from the same equation,

$$(4.46) \quad \int_{\mathbb{R}^d} \Phi(x) dx = \frac{4\pi^{\frac{d}{2}+1}}{\Gamma(\frac{d}{2})} \text{spec}\left(\Phi; -\frac{id}{2}\right) :$$

combining the last two equations with the link between $\text{spec}(\Phi; \lambda)$ and $\text{spec}(\Psi; -\lambda)$ provided by (4.41), we are done. The equation (4.44) plays in the present context the same role as the one played by the equation (2.39) when dealing with holomorphic modular forms.

THEOREM 4.7. — *Let f be an even Maass-Hecke cusp-form for the group $SL(2, \mathbb{Z})$, corresponding to the eigenvalue $\frac{1+\lambda_k^2}{4}$ of the Laplacian, and define the sequence $(a_r)_{r \geq 1}$ by the equation (4.15) involving the (slightly generalized) L -function of f . Assume that $L(\frac{1}{2}, f) \neq 0$, so that the measure \mathfrak{S}_a does not reduce to zero. Set*

$$(4.47) \quad a_0 = \pi^{-d} \frac{\Gamma(\frac{1+d+i\lambda_k}{4})\Gamma(\frac{1+d-i\lambda_k}{4})}{\Gamma(\frac{1-d+i\lambda_k}{4})\Gamma(\frac{1-d-i\lambda_k}{4})} L\left(\frac{1}{2}, f\right) L\left(\frac{d+1}{2}, f\right).$$

Let $\Phi \in E_{\lambda_k}$ (in particular, any radial function $\Phi \in \mathcal{S}(\mathbb{R}^d)$ would do), and set $\Psi = \mathcal{H}_{d,\lambda_k} \Phi$. For every $t > 0$, one has the equation

$$(4.48) \quad \sum_{m \in \mathbb{Z}^d \setminus \{0\}} a_{r(m)} \Phi(tm) + a_0 \Phi[[0]] = t^{-d} \left[\sum_{m \in \mathbb{Z}^d \setminus \{0\}} a_{r(m)} \Psi\left(\frac{m}{t}\right) + a_0 \Psi[[0]] \right].$$

Proof. — One starts from (4.16), noting that the measure

$$(4.49) \quad \mathfrak{I}_{\mathbf{a}} := \mathfrak{S}_{\mathbf{a}} - 2\pi L\left(\frac{1}{2}, f\right) L\left(\frac{d+1}{2}, f\right)$$

is invariant under the transform $\mathcal{H}_{d,\lambda_k}$ introduced in (4.31) precisely to that effect: then, the proof ends just like that of Theorem 2.10, except for the fact that we have no residue to compute. \square

The remark (ii) that follows the proof of Theorem 2.10, to the effect that a pair ($\|\cdot\|, \|\cdot\|$) of dual Euclidean norms can be used in place of the canonical norm $|\cdot|$, still applies.

In the one-dimensional case, there is a simpler way to associate a formula of the Poisson type to any given non-holomorphic cusp-form: it does not reduce to the case $d = 1$ of the one that precedes, though the Archimedean symmetry operator $\mathcal{H}_{1,\lambda_k}$ is the same.

PROPOSITION 4.8. — *Let f be the Maass cusp-form for the group $SL(2, \mathbb{Z})$ given by the Fourier expansion (4.2). Set*

$$(4.50) \quad a_r = \sum_{1 \leq n | r} f_n \text{ if } r \geq 1, \quad a_0 = \frac{1}{\pi} \frac{\Gamma(\frac{1}{2} + \frac{i\lambda_k}{4}) \Gamma(\frac{1}{2} - \frac{i\lambda_k}{4})}{\Gamma(\frac{i\lambda_k}{4}) \Gamma(-\frac{i\lambda_k}{4})} L(1, f).$$

Assume that $d = 1$, define the space E_{λ_k} accordingly, and let $\mathcal{H}_{1,\lambda_k}$ be still defined by (4.31). If $\Phi \in E_{\lambda_k}$ and $\Psi = \mathcal{H}_{1,\lambda_k} \Phi$, one has for every $t > 0$ the equation

$$(4.51) \quad \sum_{m \in \mathbb{Z}^\times} a_{|m|} \Phi(tm) + a_0 \Phi[[0]] = t^{-1} \left[\sum_{m \in \mathbb{Z}^\times} a_{|m|} \Psi\left(\frac{m}{t}\right) + a_0 \Psi[[0]] \right].$$

Proof. — In this case, $\mathcal{D}_{\mathbf{a}}(s) = \zeta(s)L(s, f)$ and $\mathcal{D}_{\mathbf{b}}(s) = L(s, f)$ so that, according to Proposition 2.6, $\mathfrak{I}_{\mathbf{a}} := \mathfrak{S}_{\mathbf{a}} - 2\pi L(1, f)$ is invariant under $\mathcal{H}_{1,\lambda_k}$. \square

There is no need to take for f a (Maass) cusp-form: one can generalize, say, Prop. 4.8, with the help of the (halved) Eisenstein series [15, p. 209]

$$\begin{aligned}
 (4.52) \quad f(z) &= \frac{1}{2} E_\rho^*(z) \\
 &= \frac{1}{2} (y^\rho \pi^{-\rho} \Gamma(\rho) \zeta(2\rho) + y^{1-\rho} \pi^{\rho-1} \Gamma(1-\rho) \zeta(2-2\rho)) \\
 &\quad + \sum_{n \neq 0} |n|^{\rho-\frac{1}{2}} \sigma_{1-2\rho}(|n|) y^{\frac{1}{2}} K_{\rho-\frac{1}{2}}(2\pi|n|y) e^{2i\pi n z}
 \end{aligned}$$

provided that some extra terms are granted admission to the summation formula. To start with, when $\text{Re } s$ is large,

$$(4.53) \quad L(s, f) = \sum_{r \geq 1} r^{\rho-\frac{1}{2}-s} \sigma_{1-2\rho}(r) = \zeta\left(s + \rho - \frac{1}{2}\right) \zeta\left(s - \rho + \frac{1}{2}\right)$$

so that

$$(4.54) \quad L(1, f) = \zeta\left(\frac{1}{2} + \rho\right) \zeta\left(\frac{3}{2} - \rho\right).$$

Set, for $r \geq 1$,

$$(4.55) \quad a_r = \sum_{1 \leq n|r} f_n = \sum_{1 \leq n|r} n^{\rho-\frac{1}{2}} \sigma_{1-2\rho}(n).$$

We need to generalize the definition 3.3 of the space E_{λ_k} , getting a new space \tilde{E}_ρ as a result: under the assumption that $\rho \neq \frac{1}{2}, 0 < \text{Re } \rho < 1$, the definition is entirely similar, only replacing $\frac{1+i\lambda_k}{2}$ by ρ (or, equivalently, $i\lambda_k$ by $2\rho - 1$).

DEFINITION 4.9. — Let $d = 1, \rho \in \mathbb{C}, \rho \neq \frac{1}{2}, 0 < \text{Re } \rho < 1$. Let $\Phi \in \tilde{E}_\rho$, and let

$$(4.56) \quad \Phi(x) \sim \sum_{j \geq 0} \alpha'_j |x|^{-\frac{1}{2} + \rho + 2j} + \sum_{j \geq 0} \beta'_j |x|^{\frac{1}{2} - \rho + 2j} + \sum_{j \geq 0} \gamma_j x^{2j}$$

be its asymptotic expansion near 0 (cf. (4.43)). One then sets

$$(4.57) \quad \Phi[[0]] = \gamma_0, \quad \text{res}_{-\frac{1}{2} + \rho}(\Phi) = \alpha'_0, \quad \text{res}_{\frac{1}{2} - \rho}(\Phi) = \beta'_0.$$

We also set, as a generalization of (4.31),

$$(4.58) \quad \tilde{\mathcal{H}}_{1,\rho} = \pi^{4i\pi\mathcal{E}} \frac{\Gamma(\frac{\rho}{2} - i\pi\mathcal{E}) \Gamma(\frac{1-\rho}{2} - i\pi\mathcal{E})}{\Gamma(\frac{\rho}{2} + i\pi\mathcal{E}) \Gamma(\frac{1-\rho}{2} + i\pi\mathcal{E})} \mathcal{F}$$

(an involutive transformation here considered only on even functions), and consider relations between the spectral densities of two even functions Φ and Ψ in the space \tilde{E}_ρ linked by the relation $\Psi = \tilde{\mathcal{H}}_{1,\rho} \Phi$.

One has (cf. (4.39)) the relation

$$(4.59) \quad \text{spec}(\Psi; -\lambda) = \frac{1}{2\pi} \int_0^\infty t^{\frac{1}{2}-i\lambda} \Psi(t) \frac{dt}{t}$$

provided that $\text{Re}(i\lambda) < \text{Re } \rho$ and $\text{Re}(i\lambda) < 1 - \text{Re } \rho$. Applying Definition 4.9 to Ψ , one sees that, when $i\lambda \rightarrow 1 - \rho$, one has

$$(4.60) \quad \text{spec}(\Psi; -\lambda) \sim \frac{1}{2\pi} \text{res}_{\frac{1}{2}-\rho}(\Psi)(1 - \rho - i\lambda)^{-1}.$$

Using (5.4) and (4.58), or (4.41),

$$(4.61) \quad \text{spec}(\Phi; \lambda) = \pi^{-3i\lambda} \frac{\Gamma(\frac{\rho+i\lambda}{2})\Gamma(\frac{1-\rho+i\lambda}{2})}{\Gamma(\frac{\rho-i\lambda}{2})\Gamma(\frac{1-\rho-i\lambda}{2})} \frac{\Gamma(\frac{1}{4} + \frac{i\lambda}{2})}{\Gamma(\frac{1}{4} - \frac{i\lambda}{2})} \text{spec}(\Psi; -\lambda),$$

and the equation

$$(4.62) \quad \langle |x|^{\frac{1}{2}-\rho}, \Phi \rangle = 4\pi \text{spec}(\Phi; -i(1 - \rho)),$$

one gets the equation

$$(4.63) \quad \langle |x|^{\frac{1}{2}-\rho}, \Phi \rangle = \pi^{-\frac{5}{2}+3\rho} \frac{\Gamma(1 - \rho)\Gamma(\frac{3}{4} - \frac{\rho}{2})}{\Gamma(-\frac{1}{2} + \rho)\Gamma(-\frac{1}{4} + \frac{\rho}{2})} \text{res}_{\frac{1}{2}-\rho}(\Psi);$$

similarly,

$$(4.64) \quad \langle |x|^{\rho-\frac{1}{2}}, \Phi \rangle = \pi^{\frac{1}{2}-3\rho} \frac{\Gamma(\rho)\Gamma(\frac{1}{4} + \frac{\rho}{2})}{\Gamma(\frac{1}{2} - \rho)\Gamma(\frac{1}{4} - \frac{\rho}{2})} \text{res}_{\rho-\frac{1}{2}}(\Psi).$$

Also, generalizing (4.44),

$$(4.65) \quad \langle 1, \Phi \rangle = \frac{1}{\pi} \frac{\Gamma(\frac{1}{4} + \frac{\rho}{2})\Gamma(\frac{3}{4} - \frac{\rho}{2})}{\Gamma(-\frac{1}{4} + \frac{\rho}{2})\Gamma(\frac{1}{4} - \frac{\rho}{2})} \Psi[[0]].$$

THEOREM 4.10. — *Let $\rho \in \mathbb{C}, \rho \neq \frac{1}{2}, 0 < \text{Re } \rho < 1$. Let a_r be defined as in (4.55) if $r \geq 1$, and let*

$$(4.66) \quad a_0 = \frac{1}{\pi} \frac{\Gamma(\frac{1}{4} + \frac{\rho}{2})\Gamma(\frac{3}{4} - \frac{\rho}{2})}{\Gamma(-\frac{1}{4} + \frac{\rho}{2})\Gamma(\frac{1}{4} - \frac{\rho}{2})} \zeta\left(\frac{1}{2} + \rho\right) \zeta\left(\frac{3}{2} - \rho\right).$$

Also, set

$$(4.67) \quad a_+ = \pi^{-\frac{5}{2}+3\rho} \frac{\Gamma(1 - \rho)\Gamma(\frac{3}{4} - \frac{\rho}{2})}{\Gamma(-\frac{1}{2} + \rho)\Gamma(-\frac{1}{4} + \frac{\rho}{2})} \zeta(2 - 2\rho) \zeta\left(\frac{3}{2} - \rho\right),$$

$$a_- = \pi^{\frac{1}{2}-3\rho} \frac{\Gamma(\rho)\Gamma(\frac{1}{4} + \frac{\rho}{2})}{\Gamma(\frac{1}{2} - \rho)\Gamma(\frac{1}{4} - \frac{\rho}{2})} \zeta(2\rho) \zeta\left(\frac{1}{2} + \rho\right).$$

If $\Phi \in \tilde{E}_\rho$ and $\Psi = \tilde{\mathcal{H}}_{1,\rho}\Phi$, one has for every $t > 0$ the equation

$$(4.68) \quad \sum_{m \in \mathbb{Z}^\times} a_{|m|} \Phi(tm) + a_0 \Phi[[0]] + a_+ t^{\frac{1}{2}-\rho} \text{res}_{\frac{1}{2}-\rho}(\Phi) + a_- t^{\rho-\frac{1}{2}} \text{res}_{\rho-\frac{1}{2}}(\Phi)$$

$$= t^{-1} \left[\sum_{m \in \mathbb{Z}^\times} a_{|m|} \Psi\left(\frac{m}{t}\right) + a_0 \Psi[[0]] + a_+ t^{\rho-\frac{1}{2}} \text{res}_{\frac{1}{2}-\rho}(\Psi) + a_- t^{\frac{1}{2}-\rho} \text{res}_{\rho-\frac{1}{2}}(\Psi) \right].$$

Proof. — Proposition 2.6, together with its proof, extends as the equation

$$(4.69) \quad \mathfrak{S}_{\mathbf{a}} = \int_{-\infty}^{\infty} L\left(\frac{1}{2} - i\lambda, f\right) \mathfrak{E}_{i\lambda}^1 d\lambda + 2\pi \operatorname{Res}_{s=1} \left(L(s, f) \mathfrak{E}_{\frac{1}{2}-s}^1 \right) \\ + 2\pi \operatorname{Res}_{s=\frac{3}{2}-\rho} \left(L(s, f) \mathfrak{E}_{\frac{1}{2}-s}^1 \right) + 2\pi \operatorname{Res}_{s=\frac{1}{2}+\rho} \left(L(s, f) \mathfrak{E}_{\frac{1}{2}-s}^1 \right),$$

where we have been careful to take also the poles $\frac{3}{2} - \rho$ and $\frac{1}{2} + \rho$ of the function $s \mapsto L(s, f) = \zeta(s + \rho - \frac{1}{2})\zeta(s - \rho + \frac{1}{2})$, as made explicit in (4.53), into consideration. The other poles $s = \frac{1}{2} - \rho - 2j$ or $-\frac{1}{2} + \rho - 2j$ of $\mathfrak{E}_{\frac{1}{2}-s}^1$ when this distribution is regarded as a linear form on E_{ρ} (cf. Lemma 4.4), or the pole $s = 0$, do not enter the picture since, in the proof of Proposition 2.6, the contour deformation takes place entirely within the half-plane $\operatorname{Re} s \geq \frac{1}{2}$: anyway, just as in the proof of Lemma 4.6, these poles correspond to zeros of $L(s, f)$.

Now, from the equation just before (2.19), one has

$$(4.70) \quad \mathfrak{E}_{\rho-1}^1(x) = \zeta\left(\frac{3}{2} - \rho\right) |x|^{\frac{1}{2}-\rho}, \quad \mathfrak{E}_{-\rho}^1(x) = \zeta\left(\frac{1}{2} + \rho\right) |x|^{-\frac{1}{2}+\rho}.$$

The distribution

$$(2\pi)^{-1} \mathfrak{T}_{\mathbf{a}} = (2\pi)^{-1} \mathfrak{S}_{\mathbf{a}} - \zeta\left(\frac{1}{2} + \rho\right) \zeta\left(\frac{3}{2} - \rho\right) \\ - \zeta(2 - 2\rho) \zeta\left(\frac{3}{2} - \rho\right) |x|^{\frac{1}{2}-\rho} - \zeta(2\rho) \zeta\left(\frac{1}{2} + \rho\right) |x|^{-\frac{1}{2}+\rho}$$

is invariant under the operator $\tilde{\mathcal{H}}_{1,\rho}$: it thus only remains to apply the equations (4.63) to (4.65). □

5. The Archimedean factor.

Note. — The results of this section are to be used in the remainder of the paper, and their proof is independent of all that precedes, except for the definition of the operators $\mathcal{F}_{d,w}$ and $\mathcal{G}_{1,1}$.

We here compute the decomposition into their homogeneous components of a number of functions Φ, Ψ, \dots on \mathbb{R}^d . Also, given a transformation T of the type $\mathcal{F}_{d,w}$ with $w > \frac{1}{2}$ or the transformation $\mathcal{G}_{1,1}$, we construct explicit pairs of functions (Φ, Ψ) in the space $C_{\varepsilon}^{[\infty]}$ for some $\varepsilon > 0$ linked by the equation $\Psi = T\Phi$. One has in general, if $\Phi \in L^2(\mathbb{R}^d)$,

$$(5.1) \quad \Phi = \int_{-\infty}^{\infty} \Phi_{\lambda} d\lambda$$

with

$$(5.2) \quad \Phi_\lambda(x) = \frac{1}{2\pi} \int_0^\infty t^{i\lambda + \frac{d}{2}} \Phi(tx) \frac{dt}{t}.$$

The function Φ_λ is the spectral density of Φ at λ . Actually, we shall deal exclusively with $O(d)$ -invariant functions Φ : $\Phi(x) = \phi(|x|)$, and we set $\Phi(x) = \phi(|x|)$, $\Psi(x) = \psi(|x|)$. In this case the spectral density at λ is of necessity proportional to the function $x \mapsto |x|^{-\frac{d}{2} - i\lambda}$, and we set

$$(5.3) \quad \Phi_\lambda(x) = \text{spec}(\Phi; \lambda) |x|^{-\frac{d}{2} - i\lambda}.$$

In the present section, instead of the canonical norm $\| \cdot \|$ on \mathbb{R}^d , one can use throughout a pair $(\| \cdot \|, ||| \cdot |||)$ of Euclidean norms associated to quadratic forms of discriminant one, dual to each other with respect to the canonical pairing of $\mathbb{R}^d \times \mathbb{R}^d$. Of course, the coefficient $\text{spec}(\Phi; \lambda)$ in (5.3) should accompany the factor $\|x\|^{-\frac{d}{2} - i\lambda}$ in general and, at various places (e.g. the last of the five equations (5.4)), the definition of $\text{spec}(\Psi; \lambda)$ should be the one in relation to the dual norm $||| \cdot |||$. In Proposition 5.4, in the statement that $\Phi_{d,w}$ is invariant under $\mathcal{F}_{d,w}$, the norm that enters the definition (5.12) of $\Phi_{d,w}$ must be taken as self-dual; the same goes in Propositions 5.6 and 5.7.

First, some general transformation rules:

LEMMA 5.1.

$$(5.4) \quad \begin{aligned} \text{spec}(\Psi; \lambda) &= |\alpha|^{-\frac{d}{2} - i\lambda} \text{spec}(\Phi; \lambda) && \text{if } \psi(|x|) = \phi(|\alpha x|) \\ \text{spec}(\Psi; \lambda) &= \text{spec}(\Phi; -\lambda) && \text{if } \psi(|x|) = |x|^{-d} \phi(|x|^{-1}) \\ \text{spec}(\Psi; \lambda) &= \text{spec}(\Phi; \lambda - i\beta) && \text{if } \psi(|x|) = |x|^\beta \phi(|x|) \\ \text{spec}(\Psi; \lambda) &= |\alpha|^{-1} \text{spec}(\Phi; \frac{\lambda}{\alpha}) && \text{if } \psi(|x|) = |x|^{\frac{d(\alpha-1)}{2}} \phi(|x|^\alpha) \\ \text{spec}(\Psi; \lambda) &= \pi^{-i\lambda} \frac{\Gamma(\frac{d}{4} + \frac{i\lambda}{2})}{\Gamma(\frac{d}{4} - \frac{i\lambda}{2})} \text{spec}(\Phi; -\lambda) && \text{if } \Psi = \mathcal{F}\Phi. \end{aligned}$$

Proof. — The first four equations (in which it is of course assumed that $\alpha \in \mathbb{R}^\times$) are a trivial consequence of (5.1) or (5.2); in the third equation, it is assumed that the function $\text{spec}(\Phi; \cdot)$ extends as a holomorphic function to some appropriate strip around the real axis. The last one is a rephrasing of the classical equation

$$(5.5) \quad \mathcal{F}(\|x\|^{-\frac{d}{2} - i\lambda}) = \pi^{i\lambda} \frac{\Gamma(\frac{d}{4} - \frac{i\lambda}{2})}{\Gamma(\frac{d}{4} + \frac{i\lambda}{2})} |||x|||^{-\frac{d}{2} + i\lambda},$$

where the dual Euclidean norms $\| \cdot \|$ and $||| \cdot |||$ are assumed to be associated to quadratic forms of discriminant one. □

LEMMA 5.2. — *This is a short list of spectral densities at λ (in other words, a table of Mellin transforms)*

$$\begin{aligned}
 (5.6) \quad e^{-\pi|x|^2} & \qquad \frac{1}{4}\pi^{-\frac{d}{4}-1-\frac{i\lambda}{2}}\Gamma\left(\frac{d}{4} + \frac{i\lambda}{2}\right) \\
 |x|^{-d} \exp\left(-\frac{\pi}{|x|^2}\right) & \qquad \frac{1}{4}\pi^{-\frac{d}{4}-1+\frac{i\lambda}{2}}\Gamma\left(\frac{d}{4} - \frac{i\lambda}{2}\right) \\
 (1 - |x|^2)_+^{\alpha-1} & \qquad \frac{1}{4\pi} \frac{\Gamma\left(\frac{d}{4} + \frac{i\lambda}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{d}{4} + \alpha + \frac{i\lambda}{2}\right)} \\
 |x|^{-d}(1 - |x|^{-2})_+^{\alpha-1} & \qquad \frac{1}{4\pi} \frac{\Gamma\left(\frac{d}{4} - \frac{i\lambda}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{d}{4} + \alpha - \frac{i\lambda}{2}\right)} \\
 |x|^{-\nu} J_\nu(2\pi|x|) & \qquad \frac{1}{4}\pi^{-1-\frac{d}{2}+\nu-i\lambda} \frac{\Gamma\left(\frac{d}{4} + \frac{i\lambda}{2}\right)}{\Gamma\left(1 - \frac{d}{4} + \nu - \frac{i\lambda}{2}\right)}.
 \end{aligned}$$

It is assumed that $\text{Re } \alpha > 0$ in the third and fourth lines, that $\text{Re } \nu > \frac{d-1}{2}$ in the last one.

Proof. — The third of these identities, for instance, is obtained if one starts from

$$(5.7) \quad \text{spec}((1 - |x|^2)_+^{\alpha-1}; \lambda) = \frac{1}{2\pi} \int_0^1 t^{\frac{d}{2}+i\lambda}(1 - t^2)^{\alpha-1} \frac{dt}{t}.$$

One obtains the last one, starting from

$$(5.8) \quad \text{spec}(|x|^{-\nu} J_\nu(2\pi|x|); \lambda) = \frac{1}{2\pi} \int_0^\infty t^{\frac{d}{2}+i\lambda-\nu} J_\nu(2\pi t) \frac{dt}{t},$$

if one applies [9, p. 91]. The second and fourth equations are consequences of the first and third equations, together with Lemma 5.1. □

Recall that

$$(5.9) \quad \mathfrak{K}_{d,w} = \pi^{4i\pi w\mathcal{E}} \frac{\Gamma\left(w\left(\frac{d}{2} - 2i\pi\mathcal{E}\right)\right)}{\Gamma\left(w\left(\frac{d}{2} + 2i\pi\mathcal{E}\right)\right)},$$

that $\mathcal{F}_{d,w} = \mathfrak{K}_{d,w}\mathcal{F}$, and that we are also interested in the operator (in dimension 1) $\mathcal{G}_{1,1} = \frac{\mathfrak{K}_{1,1}}{\mathfrak{K}_{1,\frac{1}{2}}}\mathcal{F}$. Thus, in the radial case,

LEMMA 5.3.

$$(5.10) \quad \text{spec}(\mathcal{F}_{d,w}\Phi; \lambda) = \pi^{-(2w+1)i\lambda} \frac{\Gamma\left(w\left(\frac{d}{2} + i\lambda\right)\right)}{\Gamma\left(w\left(\frac{d}{2} - i\lambda\right)\right)} \frac{\Gamma\left(\frac{d}{4} + \frac{i\lambda}{2}\right)}{\Gamma\left(\frac{d}{4} - \frac{i\lambda}{2}\right)} \text{spec}(\Phi; -\lambda),$$

and, in the one-dimensional case,

$$(5.11) \quad \text{spec}(\mathcal{G}_{1,1}\Phi; \lambda) = \pi^{-2i\lambda} \frac{\Gamma\left(\frac{1}{2} + i\lambda\right)}{\Gamma\left(\frac{1}{2} - i\lambda\right)} \text{spec}(\Phi; -\lambda).$$

The following lemma defines the analogue of the Gaussian function associated with the transformation $\mathcal{F}_{d,w}$. One should note that $\text{spec}(\Phi_{d,w}; \lambda)$ vanishes at $\lambda = \frac{-id}{2}$, an important fact also explained by Lemma 2.9.

PROPOSITION 5.4. — Assuming that $w > \frac{1}{2}$, set $q = 2w$ and

$$(5.12) \quad \Phi_{d,w}(x) = \pi^{-1-\frac{dw}{2}+\frac{d}{4}} \int_{-\infty}^{\infty} \pi^{-(w+\frac{1}{2})i\lambda} \frac{\Gamma(w(\frac{d}{2} + i\lambda))}{\Gamma(\frac{d}{4} - \frac{i\lambda}{2})} |x|^{-\frac{d}{2}-i\lambda} d\lambda.$$

The function $\Phi_{d,w}$ is continuous on \mathbb{R}^d and rapidly decreasing at infinity. It can be made explicit as

$$(5.13) \quad \Phi_{d,w}(x) = 2\pi^{\frac{d}{2}} w^{-1} F(\pi^{\frac{q+1}{2}} |x|)$$

with

$$(5.14) \quad F(t) = \sum_{k \geq 0} \frac{(-1)^k t^{\frac{2k}{q}}}{k! \Gamma(\frac{d}{2} + \frac{k}{q})}.$$

It is invariant under the transformation $\mathcal{F}_{d,w}$.

In the case when $q = 2, 3, \dots$, F is a linear combination of generalized hypergeometric functions (cf. [9, p. 62] for the definition of the generalized hypergeometric series if needed):

$$(5.15) \quad F(t) = \sum_{p=0}^{q-1} \frac{(-1)^p t^{\frac{2p}{q}}}{p! \Gamma(\frac{d}{2} + \frac{p}{q})} {}_1F_{q+1}\left(1; \frac{d}{2} + \frac{p}{q}, \frac{p+1}{q}, \frac{p+2}{q}, \dots, \frac{p+q}{q}; -\frac{t^2}{2}\right).$$

Proof. — The integral that defines $\Phi_{d,w}(x)$ converges, in view of the asymptotics [9, p. 11]

$$(5.16) \quad |\Gamma(\sigma + it)| \sim (2\pi)^{\frac{1}{2}} |t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi}{2}|t|}, \quad |t| \rightarrow \infty$$

of the Gamma function on vertical lines, thanks to the condition $w > \frac{1}{2}$: it is even possible to use a contour deformation $\lambda \mapsto \lambda - iM$ with M arbitrarily large, without destroying the convergence, which shows that $\Phi_{d,w}(x)$ is rapidly decreasing as $|x| \rightarrow \infty$. Applying Cauchy's theorem, one transforms the integral into the power series in the variable $|x|^{\frac{2}{q}}$ indicated. The invariance of $\Phi_{d,w}$ under the transformation $\mathcal{F}_{d,w}$ is a consequence of Lemma 5.3 together with the decomposition (5.12).

In order to transform the series (5.14) into the sum of generalized hypergeometric functions (5.15), it suffices to set $k = p + qj, 0 \leq p < q, 0 \leq j < \infty$, and to use the multiplication formula [9, p. 3]

$$(5.17) \quad \Gamma(p + 1 + qj) = 2^{p+1-\frac{q}{2}+qj} \pi^{\frac{1}{2}-\frac{q}{2}} \prod_{\ell=0}^{q-1} \Gamma\left(\frac{p+1+\ell}{q} + j\right).$$

□

The Fourier transform of the function $\Phi_{d,1}$ is especially simple: at the same time, we show that, when $w > \frac{1}{2}$, the function $\Phi_{d,w}$ lies in the space \mathcal{C}_{w-1} : actually, it is immediate that it even lies in the space $\mathcal{C}_{w-1}^{[\infty]}$.

PROPOSITION 5.5. — *For every $w > \frac{1}{2}$, the function $(\mathcal{F}\Phi_{d,w})(x)$ is a $O(|x|^{-d-\frac{1}{w}})$ as $|x| \rightarrow \infty$. One has*

$$(5.18) \quad (\mathcal{F}\Phi_{d,1})(x) = |x|^{-d-1} \exp\left(-\frac{\pi}{4|x|^2}\right).$$

Proof. — Using (5.12) and Lemma 5.1, one gets

$$(5.19) \quad (\mathcal{F}\Phi_{d,w})(x) = \pi^{-1-\frac{dw}{2}+\frac{d}{4}} \int_{-\infty}^{\infty} \pi^{(w-\frac{1}{2})i\lambda} \frac{\Gamma(w(\frac{d}{2}-i\lambda))}{\Gamma(\frac{d}{4}-\frac{i\lambda}{2})} |x|^{-\frac{d}{2}-i\lambda} d\lambda.$$

So as to improve the estimate as $|x| \rightarrow \infty$, one uses the same contour deformation $z = i\lambda \mapsto z + M$ as in the proof of Proposition 5.4: the first pole one comes across is $z = \frac{d}{2} + \frac{1}{w}$, which corresponds to a term in $|x|^{-d-\frac{1}{w}}$: one can then choose M between $\frac{d}{2} + \frac{1}{w}$ and $\frac{d}{2} + \frac{2}{w}$.

In the case when $w = 1$, one can rewrite (5.19) as

$$(5.20) \quad \text{spec}(\mathcal{F}\Phi_{d,1}; \lambda) = \pi^{-1-\frac{d}{4}+\frac{i\lambda}{2}} \frac{\Gamma(\frac{d}{2}-i\lambda)}{\Gamma(\frac{d}{4}-\frac{i\lambda}{2})} \\ = 2^{\frac{d}{2}-1-i\lambda} \pi^{-\frac{d}{4}-\frac{3}{2}+\frac{i\lambda}{2}} \Gamma\left(\frac{d}{4} + \frac{1}{2} - \frac{i\lambda}{2}\right),$$

at which point one can use Cauchy's theorem: alternatively, one can write

$$(5.21) \quad \text{spec}(|x|^{-d} \exp\left(-\frac{\pi}{4|x|^2}\right); \lambda) = 2^{\frac{d}{2}-2-i\lambda} \pi^{-\frac{d}{4}-1+\frac{i\lambda}{2}} \Gamma\left(\frac{d}{4} - \frac{i\lambda}{2}\right)$$

and use Lemma 5.2 and Lemma 5.1 (twice). □

A more general class of test functions $\Phi_{d,1}^j$ can be obtained from $\Phi_{d,1}$ in a way which has some similarity to the way Hermite functions are built from the Gaussian function: note that, when $j = 0$, both $\Phi_{d,1}^j$ and $\Psi_{d,1}^j$, as defined in what follows, reduce to $\Phi_{d,1}$.

PROPOSITION 5.6. — *For every $j = 0, 1, \dots$, define the functions $\Phi_{d,1}^j$ and $\Psi_{d,1}^j$ on \mathbb{R}^d through their Fourier transforms:*

$$(\mathcal{F}\Phi_{d,1}^j)(x) = |x|^{-d-1} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+1}{2}-j)} {}_1F_1\left(\frac{d+1}{2}; \frac{d+1}{2}-j; -\frac{\pi}{4|x|^2}\right)$$

and

$$(5.22) \quad (\mathcal{F}\Psi_{d,1}^j)(x) = (-1)^j |x|^{-d-1} \frac{\Gamma(\frac{3}{2}+j)}{\Gamma(\frac{3}{2})} {}_1F_1\left(\frac{3}{2}+j; \frac{3}{2}; -\frac{\pi}{4|x|^2}\right)$$

(this is the elementary case [9, p. 264] of the confluent hypergeometric function ${}_1F_1(a + j; a; \cdot)$ since j is an integer). Each of the two functions $\mathcal{F}\Phi_{d,1}^j$ and $\mathcal{F}\Psi_{d,1}^j$ is the image of $\mathcal{F}\Phi_{d,1}$ under a polynomial of degree j in the Euler operator. With $\mathcal{F}_{d,1}$ as defined in Proposition 2.7, one has $\Psi_{d,1}^j = \mathcal{F}_{d,1}\Phi_{d,1}^j$. The functions $\Phi_{d,1}^j$ and $\Psi_{d,1}^j$ lie in the space \mathcal{C}_1 .

Proof. — In order to define $\Phi_{d,1}^j$, insert under the integral sign on the right-hand side of the definition (5.12) of $\Phi_{d,1}$ the extra factor $(\frac{d}{4} - \frac{i\lambda}{2} - j)_j$, as expressed as a Pochhammer symbol $(a)_j = a(a + 1) \cdots (a + j - 1)$. The rest is straightforward calculation: (5.19) immediately generalizes to a computation of $(\mathcal{F}\Phi_{d,1}^j)(x)$. On the other hand, defining $\Psi_{d,1}^j$ as $\mathcal{F}_{d,1}\Phi_{d,1}^j$, one has $\mathcal{F}\Psi_{d,1}^j = \mathfrak{K}^{-1}\Phi_{d,1}^j$ so that

$$(5.23) \quad (\mathcal{F}\Psi_{d,1}^j)(x) = \pi^{-\frac{d}{4}-1} \int_{-\infty}^{\infty} |x|^{-\frac{d}{2}-i\lambda} \pi^{\frac{3\lambda}{2}} \frac{\Gamma(\frac{d}{2} - i\lambda)}{\Gamma(\frac{d}{2} - i\lambda - j)} d\lambda,$$

from which (5.22) easily follows. □

PROPOSITION 5.7. — Let $\rho \in \mathbb{C}$, $\rho \notin 2\mathbb{Z}$, and let $d = 1$. The function

$$(5.24) \quad \begin{aligned} \Phi(x) &= |x|^{\rho-\frac{d}{2}} \sum_{j \geq 0} \frac{(\pi^{\frac{3}{2}}|x|)^{2j}}{j! \Gamma(\frac{d}{4} + \frac{\rho}{2} + j) \Gamma(\frac{1}{2} + \rho + j)} \\ &\quad - |x|^{1-\rho-\frac{d}{2}} \sum_{j \geq 0} \frac{(\pi^{\frac{3}{2}}|x|)^{2j}}{j! \Gamma(\frac{d}{4} + \frac{1-\rho}{2} + j) \Gamma(\frac{3}{2} - \rho + j)} \end{aligned}$$

is invariant under the transformation $\tilde{\mathcal{H}}_{1,\rho}$ in (4.58).

Proof. — Set

$$(5.25) \quad \Phi(x) = C \int_{-\infty}^{\infty} \pi^{-\frac{3\lambda}{2}} \frac{\Gamma(\frac{\rho}{2} + \frac{i\lambda}{2}) \Gamma(\frac{1-\rho}{2} + \frac{i\lambda}{2})}{\Gamma(\frac{d}{4} - \frac{i\lambda}{2})} |x|^{-\frac{d}{2}-i\lambda} d\lambda$$

with

$$(5.26) \quad C = \frac{\pi^{-1-\frac{3\rho}{2}}}{4\Gamma(\frac{1}{2} + \rho)\Gamma(\frac{1}{2} - \rho)}$$

and apply Cauchy's theorem. □

6. Conclusion: Automorphic distributions.

We have seen that a variety of summation formulas of the Poisson style can be associated to Dirichlet series with functional equations of the type which occurs in modular form theory. If such a series $\mathcal{D}(s)$ is invariant, up to some Archimedean (non arithmetic) factor, under the symmetry $s \mapsto k - s$, the method consists in constructing a comb with a decomposition into homogeneous parts given as an integral of Eisenstein distributions with respect to the density $\mathcal{D}(\frac{k}{2} - i\lambda)$. Though combs are precisely what is needed here, it is important to realize that combs do not exhaust the set of automorphic distributions: these are distributions on \mathbb{R}^d invariant under the linear action of the group $SL(d, \mathbb{Z})$.

In the two-dimensional case, they are exactly an unusual realization of *pairs of* automorphic functions on the upper half-plane, a function being automorphic if it is invariant under the action, by fractional-linear transformations, of some arithmetic subgroup of $SL(2, \mathbb{R})$, say $SL(2, \mathbb{Z})$. The map from automorphic distributions to such pairs is best constructed with the help of the (Weyl) pseudodifferential analysis [18]: an alternative construction [17, section 18] also realizes automorphic distributions as Cauchy data for the Lax-Phillips scattering theory for the automorphic wave equation [8]. Recalling that non-holomorphic modular forms are automorphic functions on the upper half-plane, at the same time (generalized) eigenfunctions of the Laplace-Beltrami operator Δ , let us define modular distributions as *homogeneous* automorphic distributions: that the two concepts are essentially equivalent is due to the fact that, under this map, the operator $\Delta - \frac{1}{4}$ on the half-plane corresponds to the square of the Euler operator on \mathbb{R}^2 .

Since the Euler operator commutes with linear automorphisms of \mathbb{R}^2 , the decomposition into homogeneous terms of any automorphic distribution will involve only modular distributions. Combs of the type considered in the present paper decompose as integral superpositions of Eisenstein distributions: needless to say, the terminology comes from the fact that, under the correspondence under discussion, Eisenstein distributions are related to (non-holomorphic) Eisenstein series. Discrete sums of special Eisenstein distributions may also arise in the decomposition of more general combs: cf. [18, Prop. 4.5] for an example in which all the Eisenstein distributions associated to parameters which are non-trivial zeros of the zeta function do appear. But Eisenstein series are only a part (that corresponding to the continuous spectrum of Δ) of the family of non-holomorphic modular forms: absent are the rather poorly understood

(Maass) cusp-forms. However, there is a canonical generating object for *all* modular distributions, including the cusp-distributions, to wit the Bezout automorphic distribution, introduced in (*loc.cit.*, section 3): it is the sum of a countable set of measures supported on straight lines, not points; again, there is a related concept (the non-holomorphic Poincaré series), due to Selberg [11], in the realm of automorphic functions.

The present generalization of Poisson's formula bears no relation to the non-Euclidean Poisson formulas [16, p. 246], originating with Selberg, which are concerned also with $GL(d, \mathbb{Z})$. For, on one hand, both sides of the formula discussed here are sums over \mathbb{Z}^d , whereas the two sides look quite different — and different from each other — in the formulas just alluded to. On the other hand, by concentrating on a specific homogeneous space of $GL(d, \mathbb{R})$, to wit $\mathbb{R}^d \setminus \{0\}$, we have of course avoided most of the difficulties of a harmonic analytic nature which arise from the fact that the symmetric space associated with $SL(d, \mathbb{R})$ has rank $d - 1$: this single fact explains why the general theory of Eisenstein series is already much more complicated when $d > 2$ (*loc.cit.*, p. 114 and 184). Again, in the case when $d = 2$, nothing is lost by the move from the half-plane to the plane: the whole theory of non-holomorphic modular forms can be described in terms of automorphic distributions on the plane. This is not the case when $d > 2$.

The concept of automorphic distribution made the development of automorphic pseudodifferential analysis possible, in the case of the two-dimensional phase space. It also led to a good understanding of bilinear operations (those corresponding to the composition of operators coupled with spectral decomposition) on some classes of non-holomorphic modular forms: one application of such [17, section 15] was to the construction of rather simple Dirichlet series, such as (4.13) (some better-known series are based on the use of so-called Kloosterman sums), with poles related to the discrete spectrum of Δ , or that of series of functions on the half-plane resembling Eisenstein series, at the same time meromorphic functions of some parameter, with Maass forms as residues. It would certainly be an interesting, if quite difficult, job to generalize part of all this to the higher-dimensional case.

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