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On the exact WKB analysis of microdifferential operators of WKB type

ON THE EXACT WKB ANALYSIS OF
MICRODIFFERENTIAL OPERATORS OF WKB TYPE

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0. Introduction.

The purpose of this article is to introduce a new class of integral operators which we call microdifferential operators of WKB type and to show how our previous results for differential operators of WKB type (see [AKKT1], [AKKT2]) can be extended for such operators. Our introduction of such a new class of operators was motivated by the WKB analysis of plasma wave propagation in inhomogeneous media (see [BRS], [BB]). A typical example which we want to understand from the WKB-theoretic viewpoint is given at the end of this introduction (cf. (0.11) below). Besides an important property that its WKB solutions may have infinitely many phases, we observe that the operator contains "a differential operator of a negative order", i.e., a microdifferential operator. As the exact WKB analysis of genuine microdifferential (i.e., not differential) operators with a large parameter seems to have been rarely discussed in mathematical literature, we begin our discussion by a heuristic explanation of what we mean by the WKB analysis of a microdifferential operator. (See [Sj2], [Mar] and references cited there for the WKB analysis in C^∞-category

of microdifferential operators. See also [Sj1] which the reasoning of [Sj2] makes essential use of.) Here and in what follows, exact WKB analysis means WKB analysis based on the Borel resummation. (See [V], [S], [DDP], [KT] and references cited there.) The heuristic discussion given below will serve the reader as a guide to the mathematically rigorous discussions based on symbol calculus of microdifferential operators (see [A]), which are given in Sections 1 and 2.

Let us consider the situation where $\exp(\eta\phi(x))$ vanishes sufficiently rapidly as x tends to $-\infty$, where η is a large parameter and $\phi(x)$ is an analytic function of x. By ignoring the contribution from the endpoint $x = -\infty$ in the integral $\int_{-\infty}^{x} \exp(\eta\phi(x))\mathrm{d}x$, we find the following relation (0.1) by the repeated applications of the integration by parts:

$$(0.1) \quad \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \exp(\eta\phi(x))$$

$$= \int_{-\infty}^{x} \exp(\eta\phi(x))\,\mathrm{d}x$$

$$= \frac{1}{\eta\phi'(x)} \exp(\eta\phi(x)) + \int_{-\infty}^{x} \frac{\phi''(x)}{\eta\phi'(x)^2} \exp(\eta\phi(x))\,\mathrm{d}x$$

$$= \left(\frac{1}{\eta\phi'(x)} + \frac{\phi''(x)}{\eta^2\phi'(x)^3}\right) \exp(\eta\phi(x))$$

$$+ \int_{-\infty}^{x} \frac{3\phi''(x)^2 - \phi'(x)\phi'''(x)}{\eta^2\phi'(x)^4} \exp(\eta\phi(x))\,\mathrm{d}x$$

$$= \cdots.$$

Here $\phi'(x)$, etc. respectively denote $\mathrm{d}\phi/\mathrm{d}x$, etc. Otherwise stated, we may develop the WKB analysis of microdifferential equations by using the following relation (0.2) repeatedly:

$$(0.2) \quad \left(\eta^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \exp(\eta\phi(x)) = \Gamma(x,\eta) \exp(\eta\phi(x)),$$

where $\Gamma(x,\eta)$ is a formal power series of η^{-1} of the following form

$$(0.3) \quad \gamma_0(x) + \eta^{-1}\gamma_1(x) + \eta^{-2}\gamma_2(x) + \cdots$$

with

$$(0.4) \quad \gamma_0(x) = \phi'(x)^{-1}.$$

Note also that we can similarly determine the action of $(\eta^{-1}\mathrm{d}/\mathrm{d}x)^{-1}$ on a series

$$(0.5) \quad \Delta(x,\eta) \exp(\eta\phi(x))$$

where $\Delta(x, \eta)$ is a formal power series of η^{-1} of the form

$$(0.6) \qquad \delta_0(x) + \eta^{-1}\delta_1(x) + \eta^{-2}\delta_2(x) + \cdots.$$

This time we determine $\Gamma(x, \eta)$ that satisfies

$$(0.7) \qquad \left(\eta^{-1}\frac{d}{dx}\right)^{-1}\left(\Delta(x, \eta)\exp(\eta\phi(x))\right) = \Gamma(x, \eta)\exp\left(\eta\phi(x)\right)$$

through the differential equation

$$(0.8) \qquad \Delta(x, \eta) = \eta^{-1}\frac{\partial\Gamma}{\partial x} + \Gamma(x, \eta)\phi'(x).$$

Assuming that $\Gamma(x, \eta)$ is again of the form (0.3), we find

$$(0.9) \qquad \gamma_0 = \delta_0\phi'^{-1},$$

$$(0.10) \qquad \gamma'_{\ell-1} = -\phi'\gamma_\ell + \delta_\ell \quad (\ell \geq 1).$$

If we let $\mathcal{F}\exp(\eta\phi(x))$ denote the totality of formal power series of the form (0.3) multiplied by $\exp(\eta\phi(x))$, the above relations entail that $(\eta^{-1}d/dx)^{-1}$ determines a well-defined isomorphism from $\mathcal{F}\exp(\eta\phi(x))$ to $\mathcal{F}\exp(\eta\phi(x))$. It is now reasonable to imagine that some appropriate systematization of the above observations will give us the exact WKB analysis of microdifferential operators with a large parameter. As a matter of fact, the framework of differential equations of WKB type and the construction of their WKB solutions given in [AKKT1] provide us with a neat way for such systematization. Adopting the same approach as in [AKKT1], i.e., the approach to the exact WKB analysis through microlocal analysis, we first introduce the notion of microdifferential operators of WKB type using some analytic properties of the symbols of their Borel transforms as their characterizations, and then construct their WKB solutions with the help of symbol calculus of microdifferential operators. One can then readily find that the construction is a straightforward generalization of (0.2) (cf. Example 2.1 in Section 2). Since we can prove a Weierstrass-type division theorem for microdifferential operators of WKB type, we can develop the exact WKB analysis of microdifferential equations near their turning points (Section 3).

In ending this introduction we show the example that motivated our study; except for some simplification of numerical factors, etc. the operator given below is the same as that discussed by Berk and Book [BB], (18):

$$(0.11) \quad \left[-2\left(\eta^{-1}\frac{d}{dx}\right)^{-2} + 4\left(\eta^{-1}\frac{d}{dx}\right)^{-3}\exp\left(-\left(\eta^{-1}\frac{d}{dx}\right)^{-2}\right)\right.$$

$$\left. \times \int_0^{(\eta^{-1}d/dx)^{-1}} \exp t^2 dt\right] - \gamma^2\exp x^2,$$

where γ is a real parameter and η is a purely imaginary parameter

with $\eta/i \gg 1$. Note that the symbol of the first term of (0.11) is given by

$$(0.12) \qquad U = -2\zeta^{-2} + 4\zeta^{-3}\exp(-\zeta^{-2})\int_0^{\zeta^{-1}} \exp t^2 \mathrm{d}t$$

and that it is holomorphic for $\zeta \neq 0$. Note also that for real and positive ζ, U has the asymptotic expansion

$$(0.13) \qquad 1 + \frac{3}{2}\zeta^2 + \cdots + \frac{(2p-1)!!}{2^{p-1}}\zeta^{2(p-1)} + \cdots$$

as $\zeta \to 0$, but the coefficients of the series (0.13) grow so rapidly that the formal sum

$$(0.14) \qquad 1 + \frac{3}{2}\left(\eta^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \cdots + \frac{(2p-1)!!}{2^{p-1}}\left(\eta^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2(p-1)} + \cdots$$

cannot determine a differential operator of WKB type despite the fact that it is free from the negative powers of $\eta^{-1}(\mathrm{d}/\mathrm{d}x)$. Some detailed study of complex analytic properties of the function U is given in Appendix.

An announcement [AKKT3] of a part of the results in this paper was published in RIMS Kôkyûroku No. 1316 (2003).

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1. Microdifferential operators of WKB type.

In order to incorporate the equations like (0.11) into the framework of the exact WKB analysis we generalize the notion of differential operators of WKB type that was introduced in [AKKT1] so that ordinary differential operators of negative orders with a large parameter η may be equally treated. An important feature of such operators is that the balance between the multiplication by η and the differentiation with respect to x (or rather the integration) should be maintained. Thus in view of the results in [AKKT1] an intuitive idea of such an operator is given by the operator of the form

$$(1.1) \qquad P\left(x, \eta^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\right),$$

where $P(x, \zeta)$ is holomorphic on $U \times \{\zeta \in \mathbb{C}\,;\, \zeta \neq 0\}$ for an open set U in $\mathbb{C}_x$.

As is usual in the exact WKB analysis, we consider its Borel transform $P_B(x, \partial_y^{-1}\partial_x)$, where ∂_x and ∂_y respectively denote $\partial/\partial x$ and $\partial/\partial y$, and

we give the precise definition of the required class of operators using the properties of Borel transformed operators. In what follows we often use the simplified notation $P(x, \partial_y^{-1}\partial_x)$, instead of $P_B(x, \partial_y^{-1}\partial_x)$. (Since the operators P we discuss in this article are independent of y, i.e.,

$$[P_B, \partial_y] \underset{\text{def}}{=} P_B\partial_y - \partial_y P_B = 0,$$

no confusions will be caused by this simplification. Note, however, that here and in what follows we use the normal ordering in the notations $P(x, \eta^{-1}d/dx)$ and $P_B(x, \partial_y^{-1}\partial_x)$, that is, all the multiplication operators by functions of x stand to the left of all the differential operators in x in these operators.) Although the intuitive idea (1.1) of the operator in question requires that the operator $P_B(x, \partial_y^{-1}\partial_x)$ is exactly of order 0 as a microdifferential operator on

$$\overset{\circ}{\Omega} \underset{\text{def}}{=} \{(x, y; \xi, \eta) \in T^*(U \times \mathbb{C}_y) ; \ \xi \neq 0, \eta \neq 0\},$$

we choose somewhat more general operators as our target, that is, we consider the totality of microdifferential operators on $\overset{\circ}{\Omega}$ that are independent of y and of order at most or equal to 0. Although restricting our consideration to the operators of order exactly 0 might not be too restrictive from the practical viewpoint, the theoretical completeness seems to be better attained by including operators of negative order into consideration. (See e.g., Theorem 3.1 below.)

DEFINITION 1.1. — Let U be an open set in $\mathbb{C}_x$ and let $\overset{\circ}{\Omega}$ denote the subset of $T^*(U \times \mathbb{C}_y)$ defined by $\{(x, y; \xi, \eta) \in T^*(U \times \mathbb{C}); \xi \neq 0, \eta \neq 0\}$. A microdifferential operator of WKB type on U is, by definition, the inverse Borel transform of a microdifferential operator defined on $\overset{\circ}{\Omega}$ that is free from y. The totality of microdifferential operators of WKB type on U is denoted by $\overset{\circ}{\mathcal{E}}_{\text{WKB}}(U)$. The composition of microdifferential operators of WKB type is defined through the composition of their Borel transforms.

Remark 1.1. — It follows from the above definition that the total symbol $\sigma(P_B)$ of the Borel transform P_B of a microdifferential operator P of WKB type is a formal series of the form

$$(1.2) \qquad \sigma(P_B) = \sigma(P_B)(x, \xi/\eta, \eta) = \sum_{j=0}^{\infty} \eta^{-j} P_j(x, \xi/\eta),$$

where

(1.3) $P_j(x, \xi/\eta)$ is a holomorphic function on $\overset{\circ}{\Omega}$, that is, homogeneous of degree 0 in (ξ, η),

(1.4) for each compact set K of $\overset{\circ}{\Omega}$ there exists a constant C_K for which the following holds:

$$\sup_K \left| P_j(x, \xi/\eta) \right| \leq j! C_K^{j+1} \qquad (j \geq 0).$$

We note that the pioneering work [BK] of Boutet de Monvel and Krée is one of the earliest works that make effective use of the estimation of the type (1.4) in developing the theory of pseudo-differential operators (see [BK]).

2. WKB solutions of microdifferential equations of WKB type.

To construct a WKB solution for a microdifferential operator P of WKB type we use the notion of a (Borel transformable) WKB symbol introduced in [AKKT1]. It is a formal series of the form

$$(2.1) \qquad f = \exp\big(\eta\phi(x)\big) \sum_{j=0}^{\infty} \eta^{-j-\alpha} f_j(x)$$

for some real number α, where $\phi(x)$ and $f_j(x)$ are holomorphic functions on an open set V in $\mathbb{C}$, and it is said to be *Borel transformable* if

(2.2) for each compact set K in V there exists a constant C_K for which the following holds:

$$\sup_{x \in K} \left| f_j(x) \right| \leq j! C_K^{j+1}.$$

See [AKKT1] for the detailed discussions of the estimation of the type (2.2) for the several series we construct below; the reasoning given in [AKKT1] can be readily found to apply to our case. Note also that, although the estimation is done only on the domain of analyticity of functions $f_j(x)$, we often consider a WKB symbol on a domain containing singular points of $f_j(x)$.

Since the principal symbol of the Borel transform of a microdifferential operator of WKB type has a singularity at $\zeta = 0$, the actual construction of WKB solutions for such an operator becomes somewhat more delicate than that for a differential operator of WKB type discussed in [AKKT1]. Hence we begin our reasoning by showing the following

PROPOSITION 2.1. — *Let P be a microdifferential operator of WKB type defined on an open subset U of $\mathbb{C}$ and let*

$$S(x,\eta) = \sum_{j \geq -1} S_j(x)\eta^{-j}$$

be a formal series in η^{-1} with $S_j(x)$ being holomorphic on an open subset V of U. Suppose that $S_j(x)$ ($j \geq 0$) satisfies the condition (2.2). Let δ be a sufficiently small positive number and denote by $R(x,z,\eta)$ the following integral

$$(2.3) \qquad \int_0^1 S(x + zt,\eta)\mathrm{d}t,$$

where x belongs to an open subset W of V for which $x + zt$ ($0 \leq t \leq 1$) belongs to V for any z in $\mathbb{C}$ with $|z| < \delta$, and let $T(x,\eta)$ denote

$$(2.4) \qquad \int_{x_0}^x S(x,\eta)\mathrm{d}x,$$

where x_0 is a generically fixed point in W. Then we find

$$(2.5) \quad \sigma\Big(\exp\big(-T(x,\partial_y)\big)P_B(x,\partial_x\partial_y^{-1},\partial_y) \exp\big(T(x,\partial_y)\big)\Big)$$
$$= \Big[\exp(\eta^{-1}\partial_\zeta\partial_z)\sigma(P_B)(x,\zeta + \eta^{-1}R(x,z,\eta),\eta)\Big]\Big|_{z=0}$$

on $\omega = \{(x,y;\xi,\eta) \in T^*(W \times \mathbb{C}); x \in W, \xi \neq 0, \xi/\eta \neq -S_{-1}(x), \eta \neq 0\}$. Here ζ stands for ξ/η and $\sigma(Q)$ denotes the symbol of a microdifferential operator Q.

Proof. — Using the idea of Malgrange [M] to neatly write down the composition of microdifferential operators in terms of their symbols, we find

$$(2.6) \quad \sigma\big(P_B(x,\partial_x\partial_y^{-1},\partial_y) \exp(T(x,\partial_y))\big)$$
$$= \big[\exp(\eta^{-1}\partial_\zeta\partial_z)\sigma(P_B)(x,\zeta,\eta) \exp(T(x+z,\eta))\big]\big|_{z=0}$$
$$= \exp\big(T(x,\eta)\big)\Big[\exp(\eta^{-1}\partial_\zeta\partial_z)\sigma(P_B)(x,\zeta,\eta)$$
$$\times \exp(z \int_0^1 S(x+tz,\eta)\mathrm{d}t)\Big]\Big|_{z=0}.$$

To relate this expression with the required form (2.5), we note the following:

$$(2.7) \quad \exp(\eta^{-1}\partial_\zeta\partial_z)\big(\exp(zR(x,z,\eta)) - \exp\big(R(x,z,\eta)\eta^{-1}\partial_\zeta\big)\big)$$
$$= \exp(\eta^{-1}\partial_\zeta\partial_z)(z - \eta^{-1}\partial_\zeta)R(x,z,\eta)F(zR, R\eta^{-1}\partial_\zeta)$$
$$= z\exp(\eta^{-1}\partial_\zeta\partial_z)R(x,z,\zeta)F(zR, R\eta^{-1}\partial_\zeta),$$

where $F = (\exp(zR) - \exp(R\eta^{-1}\partial_\zeta))/(zR - R\eta^{-1}\partial_\zeta)$. We then combine (2.6) and (2.7) to find

$$(2.8) \quad \exp\big(-T(x,\eta)\big)\sigma\big(P_B(x,\partial_x\partial_y^{-1},\partial_y)\exp\big(T(x,\partial_y)\big)\big)$$
$$= \big[\exp(\eta^{-1}\partial_\zeta\partial_z)\sigma(P_B)(x,\zeta + \eta^{-1}R(x,z,\eta),\eta)\big]\big|_{z=0}$$

on ω. Note that $R_{-1}(x,z)|_{z=0} = S_{-1}(x)$.   □

Remark 2.1. — The above reasoning is, essentially speaking, adopted from the proof of Sublemma of [A], p. 509, although the statement of the sublemma only refers to differential operators.

The above relation (2.5) describes the composition of the operators $P_B(x,\partial_x\partial_y^{-1},\partial_y)$ and $\exp(\partial_y T(x,\partial_y))$. However, what we really want to know is the resulting WKB symbol when $\exp(\partial_y T(x,\partial_y))$ regarded as a function of x is acted upon by the operator $P_B(x,\partial_x\partial_y^{-1},\partial_y)$. To find the required WKB symbol, we note that the right-hand side of (2.5) makes sense at $\xi = 0$ on the condition that $S_{-1}(x)$ is different from 0. Hence the microdifferential operator whose symbol is given by the right-hand side of (2.5) can be expanded in non-negative powers of $\eta^{-1}d/dx$ and η^{-1} if we arrange the multiplication operator by a function of x always stands left to the differential operators d^k/dx^k ($k \geq 1$) in the expansion. Then the part free from the differential operator d/dx is the required WKB symbol. Hence in order to find the required WKB symbol it suffices to evaluate the right-hand side of (2.5) at $\zeta = 0$.

Example 2.1. — To exemplify the above procedure let us consider the following case:

$$(2.9) \qquad\qquad P_p = \Big(\eta^{-1}\frac{d}{dx}\Big)^{-p} \quad (p = 1,2,3,\ldots)$$

$$(2.10) \qquad\qquad S = \eta\frac{d\phi}{dx},$$

where $\phi(x)$ is a holomorphic function on a neighborhood of a point x_0. Let us suppose $\phi'(x)$ $(= \mathrm{d}\phi/\mathrm{d}x)$ does not vanish at x_0. In this case we find

$$(2.11) \qquad R(x, z, \eta) = \eta R_{-1}(x, z) = \eta \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} \frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}} \phi(x).$$

Hence we have

$$(2.12) \qquad z R_{-1}(x, z) = \phi(x + z) - \phi(x).$$

Let $\Gamma_p = \sum_{n \geq 0} \gamma_n^{(p)}(x) \eta^{-n}$ denote

$$(2.13) \qquad \left[\exp(\eta^{-1} \partial_\zeta \partial_z)\left(\zeta + R_{-1}(x, z) \right)^{-p} \right]\Big|_{z=\zeta=0}.$$

Here $\gamma_n^{(p)}$ does not mean the p-th derivative of $\gamma_n(x)$; the symbol (p) designates just an index. Then it follows from the definition of Γ_p and (2.12) that

$$(2.14) \qquad \gamma_\ell^{(p)} = (-1)^\ell \frac{(\ell + p - 1)!}{\ell!(p-1)!} \partial_z^\ell R_{-1}(x, z)^{-\ell - p}\Big|_{z=0}$$

$$= \frac{(-1)^\ell}{2\pi i} \frac{(\ell + p - 1)!}{(p-1)!} \oint_C \frac{\mathrm{d}z}{z^{\ell+1} R_{-1}^{\ell + p}}$$

$$= \frac{(-1)^\ell}{2\pi i} \frac{(\ell + p - 1)!}{(p-1)!} \oint_C \frac{z^{p-1} \mathrm{d}z}{(\phi(x + z) - \phi(x))^{\ell + p}}$$

where C is the boundary of a sufficiently small disk centered at the origin on which $\phi(x + z) - \phi(x)$ vanishes only at $z = 0$. Note that it follows from the assumption that $R_{-1}(x, 0) = \phi'(x)$ is different from 0 at the point in question. We then find

$$(2.15) \qquad \gamma_0^{(p)} = \phi'(x)^{-p} \quad (p = 1, 2, 3, \ldots)$$

and

$$(2.16) \quad \frac{\mathrm{d}}{\mathrm{d}x} \gamma_\ell^{(p)} = \frac{(-1)^{\ell+1}}{2\pi i} \frac{(\ell + p)!}{(p-1)!} \oint_C \frac{z^{p-1}(\phi'(x + z) - \phi'(x))\mathrm{d}z}{(\phi(x + z) - \phi(x))^{\ell + p + 1}}$$

$$= \frac{(-1)^{\ell+2}}{2\pi i} \frac{(\ell + p)!}{(p-1)!} \phi'(x) \oint_C \frac{z^{p-1} \mathrm{d}z}{(\phi(x + z) - \phi(x))^{\ell + p + 1}}$$

$$+ \frac{(-1)^{\ell+1}}{2\pi i} \frac{(\ell + p)!}{(p-1)!} \oint_C \frac{z^{p-1}\phi'(x + z)\mathrm{d}z}{(\phi(x + z) - \phi(x))^{\ell + p + 1}}$$

$$= -\phi'(x)\gamma_{\ell+1}^{(p)}$$

$$+ \frac{(-1)^{\ell+1}}{2\pi i} \frac{(\ell + p)!}{(p-1)!} \oint_C \frac{z^{p-1}\phi'(x + z)\mathrm{d}z}{(\phi(x + z) - \phi(x))^{\ell + p + 1}}.$$

On the other hand,

$$(2.17) \quad \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^{p-1}}{(\phi(x+z)-\phi(x))^{\ell+p}}\right)$$

$$= \frac{(p-1)z^{p-2}}{(\phi(x+z)-\phi(x))^{\ell+p}} - \frac{(\ell+p)z^{p-1}\phi'(x+z)}{(\phi(x+z)-\phi(x))^{\ell+p+1}}$$

holds and the contour integral along C of the left-hand side of (2.17) vanishes. Hence it follows from (2.14) that

$$(2.18) \quad \frac{(-1)^{\ell+1}}{2\pi i}\frac{(\ell+p)!}{(p-1)!}\oint_C \frac{z^{p-1}\phi'(x+z)\mathrm{d}z}{(\phi(x+z)-\phi(x))^{\ell+p+1}}$$

$$= \frac{(-1)^{\ell+1}}{2\pi i}\frac{(\ell+p-1)!}{(p-1)!}(p-1)\oint_C \frac{z^{p-2}\mathrm{d}z}{(\phi(x+z)-\phi(x))^{\ell+p}} = \gamma_{\ell+1}^{(p-1)}.$$

Thus we obtain

$$(2.19) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\gamma_\ell^{(p)} = -\phi'(x)\gamma_{\ell+1}^{(p)} + \gamma_{\ell+1}^{(p-1)}.$$

Comparing the relations (0.9) and (0.10) with (2.15) and (2.19), we immediately find $\Gamma_p\exp(\eta\phi(x))$ constructed here coincides with $(\eta^{-1}\mathrm{d}/\mathrm{d}x)^{-p}\exp(\eta\phi(x))$ in the Introduction.

Example 2.1 clearly shows that considering the restriction of the right-hand side of (2.5) to $\{\zeta = 0\}$ gives us the required systematization of the observations given in the Introduction. Thus we arrive at the following definition (2.1) of a WKB solution of microdifferential equations of WKB type; it naturally extends the definition of WKB solutions of differential equations of WKB type given in [AKKT1], Definition 3.1. Note that [AKKT1] first introduces the notion of WKB solutions in a somewhat more sophisticated manner and then uses the relation corresponding to (2.21) to write down the Riccati-type equation that the logarithmic derivative of a WKB solution should satisfy (cf. [AKKT1], Proposition 4.1).

DEFINITION 2.1. — For a microdifferential operator $P(x,\eta^{-1}\mathrm{d}/\mathrm{d}x,\eta)$ of WKB type and a WKB symbol $\widetilde{S} = \widetilde{S}_0(x) + \eta^{-1}\widetilde{S}_1(x) + \cdots$ that satisfies

$$(2.20) \qquad \widetilde{S}_0(x) \text{ is holomorphic and different from 0 at } x = x_0,$$

$\psi(x,\eta) = \exp(\eta\int_{x_0}^x \widetilde{S}(x,\eta)\mathrm{d}x)$ is said to be a WKB solution of the equation $P\psi = 0$ near x_0 if the following relation holds:

$$(2.21) \quad \exp(\eta^{-1}\partial_\zeta\partial_z)\sigma(P_B)\left(x,\zeta + \sum_{k=0}^\infty \frac{z^k}{(k+1)!}\frac{\partial^k}{\partial x^k}\widetilde{S}(x,\eta),\eta\right)\Big|_{z=\zeta=0} = 0.$$

Equation (2.21) is called a Riccati-type equation for a WKB solution ψ of the equation $P\psi = 0$.

Remark 2.2. — In order to conform to the traditional numbering in WKB analysis we should shift the index by 1; that is, defining

(2.22) $$S_j(x) = \widetilde{S}_{j+1}(x) \qquad (j = -1, 0, 1, \ldots)$$

and setting $S = \sum_{j \geq -1} \eta^{-j} S_j(x)$, we consider a WKB solution

$$\psi = \exp\left(\int_{x_0}^{x} S \, dx\right).$$

Since in the computation below our numbering is more convenient, we use this non-traditional numbering. To avoid the possible confusion we use the symbol $\widetilde{S}_j$ to emphasize the fact that a non-traditional numbering is used. We present the final result (Theorem 2.1) using the traditional numbering.

The above equation (2.21) can be solved in a recursive manner once the top order term $\widetilde{S}_0(x)$ is given; the top order term $\widetilde{S}_0(x)$ is a characteristic root of the equation $P\psi = 0$, namely,

(2.23) $$P_0\big(x, \widetilde{S}_0(x)\big) = 0.$$

Let us suppose $\partial_\zeta P_0(x, \widetilde{S}_0(x)) \neq 0$ holds if $P_0(x, \widetilde{S}_0(x)) = 0$. Suppose further

(2.24) $$\widetilde{S}_0(x_0) \neq 0.$$

To find $\widetilde{S}_m(x)(m \geq 1)$ let us calculate the coefficient of η^{-p} $(p \geq 1)$ in (2.21). Recalling that P has the form $\sum_{n \geq 0} \eta^{-n} P_n(x, \zeta)$, we first consider each contribution from $\eta^{-n} P_n$ in (2.21) separately:

(2.25) $$\exp(\eta^{-1}\partial_\zeta\partial_z)\sigma(P_{n,B})\left(x, \zeta + \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} \frac{\partial^k}{\partial x^k} \widetilde{S}(x, \eta)\right)$$
$$= \sum_{\ell \geq 0} \eta^{-\ell} \frac{1}{\ell!} \partial_\zeta^\ell \partial_z^\ell \sigma(P_{n,B})\left(x, \zeta + \widetilde{S}_0 + \sum_{\substack{k+m>0 \\ k \geq 0, m \geq 0}} \frac{z^k}{(k+1)!} \frac{\partial^k \widetilde{S}_m}{\partial x^k} \eta^{-m}\right).$$

We then use the assumption (2.24) to consider the Taylor expansion of $\sigma(P_{n,B})$ at $(x, \zeta + \widetilde{S}_0)$ with $|\zeta| < |\widetilde{S}_0(x)|$; we then find

(2.26) $$\sum_{\ell \geq 0} \eta^{-\ell} \frac{1}{\ell!} \partial_\zeta^\ell \partial_z^\ell \left(\sum_{j \geq 0} \frac{1}{j!} \frac{\partial^j \sigma(P_{n,B})}{\partial \zeta^j}(x, \zeta + \widetilde{S}_0)\right.$$
$$\left.\left(\sum_{\substack{k+m>0 \\ k, m \geq 0}} \frac{z^k}{(k+1)!} \frac{\partial^k \widetilde{S}_m}{\partial x^k} \eta^{-m}\right)^j\right)$$

$$= \sum_{\ell \geq 0, j \geq 0} \eta^{-\ell} \frac{1}{\ell!} \frac{1}{j!} \frac{\partial^{j+\ell} \sigma(P_{n,B})}{\partial \zeta^{j+\ell}} (x, \zeta + \widetilde{S}_0)$$

$$\times \partial_z^\ell \Big(\sum_{I(j)} \frac{z^{k_1}}{(k_1+1)!} \cdots \frac{z^{k_j}}{(k_j+1)!} \frac{\partial^{k_1} \widetilde{S}_{m_1}}{\partial x^{k_1}} \cdots \frac{\partial^{k_j} \widetilde{S}_{m_j}}{\partial x^{k_j}} \eta^{-m_1 - \cdots - m_j} \Big),$$

where $I(j)$ $(j \geq 1)$ denotes the set of $2j$-tuple indices given by

(2.27) $I(j) = \big\{ (k_1, \ldots, k_j; m_1, \ldots, m_j) \, ; \, k_i, m_i \geq 0,$

$$k_i + m_i > 0 \ (i = 1, \ldots, j) \big\}.$$

For $j = 0$, $I(j)$ is by definition the void set, and the summation over $I(0)$ is conventionally defined to be 1.

The surviving terms in

(2.28) $\partial_z^\ell \Big(\sum_{I(j)} \frac{z^{k_1}}{(k_1+1)!} \cdots \frac{z^{k_j}}{(k_j+1)!} \frac{\partial^{k_1} \widetilde{S}_{m_1}}{\partial x^{k_1}} \cdots \frac{\partial^{k_j} \widetilde{S}_{m_j}}{\partial x^{k_j}} \Big) \Big|_{z=0}$

are only those satisfying

(2.29) $$k_1 + \cdots + k_j = \ell,$$

and the outcome is

(2.30) $\displaystyle \ell! \sum_{\substack{I(j) \\ k_1 + \cdots + k_j = \ell}} \frac{1}{(k_1+1)!} \cdots \frac{1}{(k_j+1)!} \frac{\partial^{k_1} \widetilde{S}_{m_1}}{\partial x^{k_1}} \cdots \frac{\partial^{k_j} \widetilde{S}_{m_j}}{\partial x^{k_j}}.$

Hence (2.25) evaluated at $\{z = \zeta = 0\}$ is

(2.31) $\displaystyle \sum_{\substack{\ell \geq 0 \\ j \geq 0}} \eta^{-\ell} \frac{1}{j!} \frac{\partial^{j+\ell} \sigma(P_{n,B})}{\partial \zeta^{j+\ell}} (x, \widetilde{S}_0)$

$$\times \Big(\sum_{\substack{I(j) \\ \Sigma k_i = \ell}} \frac{1}{(k_1+1)!} \cdots \frac{1}{(k_j+1)!} \frac{\partial^{k_1} \widetilde{S}_{m_1}}{\partial x^{k_1}} \cdots \frac{\partial^{k_j} \widetilde{S}_{m_j}}{\partial x^{k_j}} \eta^{-\Sigma_{i=1}^j m_i} \Big).$$

Thus, taking into account the extra-factor η^{-n} coupled with P_n, we find the coefficient of η^{-p} in (2.22) is given by

(2.32) $\displaystyle \sum_{j,\ell,n} \frac{1}{j!} \frac{\partial^{j+\ell} \sigma(P_{n,B})}{\partial \zeta^{j+\ell}} (x, \widetilde{S}_0)$

$$\times \Big(\sum_{\substack{I(j) \\ \Sigma k_i = \ell}} \frac{1}{(k_1+1)!} \cdots \frac{1}{(k_j+1)!} \frac{\partial^{k_1} \widetilde{S}_{m_1}}{\partial x^{k_1}} \cdots \frac{\partial^{k_j} \widetilde{S}_{m_j}}{\partial x^{k_j}} \Big),$$

where either

$$(2.33) \qquad\qquad p = \ell + m + n$$

with

$$(2.34) \quad \ell = \sum_{i=1}^{j} k_i, \ m = \sum_{i=1}^{j} m_i, \ k_i, m_i \geq 0, \ k_i + m_i > 0 \quad (i = 1, \ldots, j)$$

for some $j \geq 1$, or

$$(2.35) \qquad\qquad p = n$$

with $j = 0$ (and hence $\ell = m = 0$).

Because of the non-negativity of k_i, m_i and n, (2.32) consists of finitely many terms. Furthermore the term containing $\widetilde{S}_p$ in (2.32) is only the term with $j = 1, k_1 = 0, m_1 = p$ and $n = 0$, i.e.,

$$(2.36) \qquad\qquad \partial_\zeta \sigma(P_{0,B})(x, \widetilde{S}_0) \widetilde{S}_p.$$

Other terms in (2.32) depends only on $\widetilde{S}_m$ or its derivatives with $m < p$. To illustrate them let us write down $\widetilde{S}_1$ and $\widetilde{S}_2$: here P_j stands for $\sigma(P_{j,B})$ and $\widetilde{S}_0'$, etc. mean $d\widetilde{S}_0/dx$, etc.

$$(2.37) \quad \widetilde{S}_1 = -\big(\partial_\zeta P_0(x, \widetilde{S}_0)\big)^{-1}\big(\partial_\zeta^2 P_0(x, \widetilde{S}_0)\frac{1}{2!}\widetilde{S}_0' + P_1(x, \widetilde{S}_0)\big),$$

$$(2.38) \quad \widetilde{S}_2 = -\big(\partial_\zeta P_0(x, \widetilde{S}_0)\big)^{-1}\big(s_{0,0} + s_{0,1} + \big(s_{0,2} - \partial_\zeta P_0(x, \widetilde{S}_0)\widetilde{S}_2\big)$$
$$+ s_{1,0} + s_{1,1} + s_{2,0}\big),$$

where

$$(2.39) \ s_{0,0} = \partial_\zeta^3 P_0(x, \widetilde{S}_0)\frac{1}{3!}\widetilde{S}_0'' + \frac{1}{2!}\partial_\zeta^4 P_0(x, \widetilde{S}_0)\Big(\frac{1}{2!}\Big)^2 \widetilde{S}_0'^{\,2},$$

$$(2.40) \ s_{0,1} = \partial_\zeta^2 P_0(x, \widetilde{S}_0)\frac{1}{2!}\widetilde{S}_1' + 2\Big(\frac{1}{2!}\partial_\zeta^3 P_0(x, \widetilde{S}_0)\frac{1}{2!}\widetilde{S}_0'\widetilde{S}_1\Big),$$

$$(2.41) \ s_{0,2} = \partial_\zeta P_0(x, \widetilde{S}_0)\widetilde{S}_2 + \frac{1}{2!}\partial_\zeta^2 P_0(x, \widetilde{S}_0)\widetilde{S}_1^2,$$

$$(2.42) \ s_{1,0} = \frac{1}{2!}\partial_\zeta^2 P_1(x, \widetilde{S}_0)\widetilde{S}_0',$$

$$(2.43) \ s_{1,1} = \partial_\zeta P_1(x, \widetilde{S}_0)\widetilde{S}_1,$$

$$(2.44) \ s_{2,0} = P_2(x, \widetilde{S}_0);$$

here $s_{\alpha,\beta}$ consists of, by definition, terms corresponding to $n = \alpha$ and $m = \beta$ (and hence $\ell = 2 - \alpha - \beta$) in (2.32). Note that $\ell + m \geq j$ follows

from (2.34). Hence in our case, i.e., for $p = 2$, the situation with $j = 2$ is observed only when $n = 0$. Summing up (2.39)–(2.44) we find

$$(2.45) \quad \widetilde{S}_2 = -\left(\partial_\zeta P_0(x, \widetilde{S}_0)\right)^{-1} \Big\{ P_2(x, \widetilde{S}_0) + \partial_\zeta P_1(x, \widetilde{S}_0)\widetilde{S}_1$$

$$+ \frac{1}{2!}\partial_\zeta^2 P_1(x, \widetilde{S}_0)\widetilde{S}_0' + \frac{1}{2}\partial_\zeta^2 P_0(x, \widetilde{S}_0)(\widetilde{S}_1' + \widetilde{S}_1^2)$$

$$+ \frac{1}{2}\partial_\zeta^3 P_0(x, \widetilde{S}_0)\Big(\widetilde{S}_0'\widetilde{S}_1 + \frac{1}{3}\widetilde{S}_0''\Big) + \frac{1}{8}\partial_\zeta^4 P_0(x, \widetilde{S}_0)\widetilde{S}_0'^2 \Big\}.$$

Thus we can recursively determine $\widetilde{S}_m$ by (2.21) in spite of its formidable appearance. The Boutet de Monvel and Krée type estimation of S_m can be done in exactly the same way as in [AKKT1], proof of Theorem 4.1, and we finally obtain the following

THEOREM 2.1. — Let $P(x, \eta^{-1}\mathrm{d}/\mathrm{d}x, \eta) = \sum_{n\geq 0}\eta^{-n}P_n(x, \eta^{-1}\mathrm{d}/\mathrm{d}x)$ be a micro-differential operator of WKB type defined near $x = x_0$ and let $S_{-1}(x)$ be a holomorphic function that satisfies $\sigma(P_{0,B})(x, S_{-1}(x)) = 0$ near x_0. Suppose

$$(2.46) \qquad\qquad S_{-1}(x_0) \neq 0$$

and

$$(2.47) \qquad\qquad \partial_\zeta\sigma(P_{0,B})\big(x_0, S_{-1}(x_0)\big) \neq 0.$$

Then the WKB solution ψ of the equation $P\psi = 0$ can be constructed near $x = x_0$ and it is Borel transformable.

3. The local structure of a microdifferential equation of WKB type near its turning points.

Although a microdifferential operator of WKB type is singular at $\zeta = 0$, we can develop WKB analysis of such an operator near its turning point with a characteristic value different from 0; the argument can be done completely in parallel with the case of differential operators of WKB type discussed in [AKKT1]. To fix our notations let us first give the definition of a turning point of a microdifferential operator P of WKB type defined on an open subset U of $\mathbb{C}$, i.e.,

$$(3.1) \qquad\qquad P = \sum_{j\geq 0}\eta^{-j}P_j(x, \partial_x/\eta),$$

where $P_j(x, \zeta)$ is holomorphic on $U \times \mathbb{C}\backslash\{0\}$.

DEFINITION 3.1. — (i) Let $(x,\zeta) = (x_*,\zeta_*)$ be a point in $U \times \mathbb{C}\backslash\{0\}$ that satisfies

$$(3.2) \qquad P_0(x_*,\zeta_*) = \partial_\zeta P_0(x_*,\zeta_*) = 0.$$

Suppose that $P_0(x_*,\zeta)$ does not vanish identically as a function of ζ. Then we say that x_* is a turning point of the operator P (with a characteristic value ζ_*).

(ii) For a turning point x_* of the operator P with a characteristic value ζ_*, its rank is the smallest positive integer m such that $\partial_\zeta^m P_0(x_*,\zeta_*)$ does not vanish.

It follows from the Weierstrass preparation theorem in analytic function theory that $P_0(x,\zeta)$ can be uniquely decomposed near a turning point x_* into the following form

$$(3.3) \qquad P_0(x,\zeta) = q(x,\zeta)r(x,\zeta),$$

where $q(x,\zeta)$ is a holomorphic function that does not vanish at (x_*,ζ_*) and $r(x,\zeta)$ a Weierstrass polynomial of degree m in ζ centered at (x_*,ζ_*), i.e.,

$$(3.4) \qquad r(x,\zeta) = (\zeta - \zeta_*)^m + f_1(x)(\zeta - \zeta_*)^{m-1} + \cdots + f_m(x),$$

where $f_j(x)$ is holomorphic near x_* and vanishes at x_* for $j = 1,\ldots,m$.

DEFINITION 3.2. — For a turning point x_* of the operator P with a characteristic value ζ_* that has rank m, the Weierstrass polynomial $r(x,\zeta)$ in (3.3) is called the vanishing factor of P.

Let us consider the case where the rank of a turning point x_* with a characteristic value ζ_* is 2. Then we find two analytic functions $\zeta_\pm(x)$ that satisfy the following:

$$(3.5) \qquad P_0\big(x, \zeta_\pm(x)\big) = 0,$$

$$(3.6) \qquad \zeta_\pm(x_*) = \zeta_*.$$

Then a local Stokes curve emanating from x_* is, by definition, the following curve considered near x_*:

$$(3.7) \qquad \operatorname{Im} \int_{x_*}^x \big(\zeta_+(x) - \zeta_-(x)\big)\,\mathrm{d}x = 0.$$

Using the notion of a vanishing factor we can prove the following decomposition theorem. As the proof is exactly the same as that of Theorem 5.1 of [AKKT1], we omit it here.

THEOREM 3.1. — *Let P be a microdifferential operator of WKB type defined on U. Let x_* be a turning point of rank m with a characteristic value ζ_*. Let $r(x,\zeta)$ be the vanishing factor of P at (x_*,ζ_*). Then on a sufficiently small neighborhood U_0 of x_*, we find microdifferential operators Q and R defined on U_0 which satisfy the following:*

(3.8) $P = QR,$

(3.9)   *the principal symbol $R_0(x,\zeta)$ of R is $r(x,\zeta)$,*

(3.10)   *for each $j > 0$, the coefficient $R_j(x,\zeta)$ of η^{-j} of the operator R is of degree at most $m - 1$ in ζ,*

(3.11)   *the principal symbol $Q_0(x,\zeta)$ of Q does not vanish at (x_*,ζ_*).*

To show the utility of Theorem 3.1 let us consider the case where x_* is a simple turning point of rank 2, that is, the case where

(3.12) $\partial_x P_0(x_*, \zeta_*) \neq 0 \quad \text{and} \quad \partial_\zeta^2 P_0(x_*, \zeta_*) \neq 0.$

Then the operator R constructed in Theorem 3.1 has the following form (3.13) near x_*:

(3.13) $R = \eta^{-2}\partial_x^2 + A(x,\eta)\eta^{-1}\partial_x + B(x,\eta),$

where A and B are formal series of non-negative powers of η^{-1} with holomorphic coefficients defined on a neighborhood of x_* and the leading terms of A and B are $-(\zeta_+(x) + \zeta_-(x))$ and $\zeta_+(x)\zeta_-(x)$ respectively for analytic functions $\zeta_\pm(x)$ satisfying (3.5) and (3.6). Now let us consider the following two pairs of WKB solutions:

$\psi_\pm = \exp(\int^x S_\pm(x,\eta)\mathrm{d}x)$ of the equation $P\psi = 0$, where the leading term $S_{\pm,-1}(x)$ of $S_\pm(x,\eta)$ is respectively given by $\zeta_\pm(x)$;

$\varphi_\pm = \exp(\int^x T_\pm(x,\eta)\mathrm{d}x)$ of the equation $R\varphi = 0$ with $T_{\pm,-1}(x)$ being respectively given by $\zeta_\pm(x)$.

It then follows from (3.8) that $P\varphi_\pm = 0$. Since the logarithmic derivative of a WKB solution is uniquely determined by its leading term, we find that S_+ (resp., S_-) coincides with T_+ (resp., T_-). Although the concrete form of the operator R may be complicated, the WKB-theoretic structure of a differential operator of the second order has been completely analyzed near its simple turning points; it can be reduced to the Airy equation through some appropriate transformation (see [AY] for example). Thus we obtain the connection formula for WKB solutions $\psi_\pm$ across a

local Stokes curve emanating from x_* in exactly the same manner as in Theorem 5.3 of [AKKT1], despite the fact that the operator P may be very complicated (like (0.11)) and defined only outside $\{\zeta = 0\}$. Although the analysis of multiple turning points (except for the case of double turning points) has not yet been completed, the results of Pham [P] should be substantially useful to analyze the structure of the operator P near its turning points of rank 2.

Appendix. On the accumulation of simple turning points in the Berk-Book equation.

Berk and Book discuss the WKB analysis of an integral equation (0.11) in their pioneering work [BB]; they concentrate their attention to two "turning points" x_A and x_B, which are respectively defined by the following relations:

(A.1) $$\gamma^2 \exp x_A^2 = \max_{z \in \mathbb{R}_+} U(z) \quad (\underset{\text{def}}{=} U_A),$$

where γ is a positive constant and $U(z)$ is given by the following:

(A.2) $$U(z) = -2z^2 \left(1 - 2z \exp(-z^2) \int_0^z \exp t^2 \, dt\right),$$

and

(A.3) $$\gamma^2 \exp x_B^2 = 1.$$

(We choose $\gamma = 1/\sqrt{e} = 0.60653\ldots$ in concrete numerical computations below, e.g., in writing Figures A.2–A.5, so that x_B may be equal to 1.)
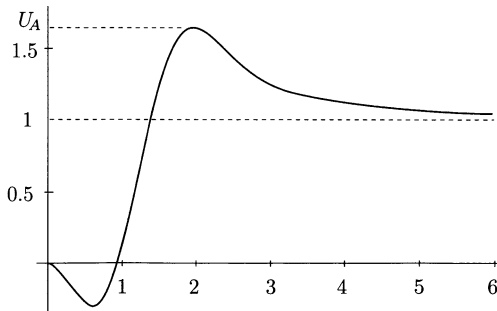

Figure A.1. Graph of $U(z)$ on the positive real axis. (*The same figure as* [BB], *Figure* 3, *p.* 656.)

We note that the function $\gamma^2 \exp x^2$ can be replaced by a more general function $\gamma^2 \exp \varphi(x)$ for some static potential $\varphi(x)$; WKB analysis of the equation (0.11) with this generalization should be an important subject. We wish to come back to the study of this generalized equation in some future, but in this paper we confine our consideration to the case where $\varphi(x) = x^2$.

Concerning the turning point x_A, all the assertions that [BB] makes are legitimate; it is a simple turning point in the sense of (3.12) and the study of the integral equation in question can be reduced to "the standard WKB turning point problem, which leads to the well-known connection formulas", as Berk and Book claim (cf. [BB], the paragraph following (9) in p. 653, and p. 656). For the convenience of the reader we present the local Stokes curves emanating from x_A in Figure A.2. We note that in Figure A.2 two Stokes curves sit on the same curve reflecting the fact that two turning points with different characteristic values sit on the same point x_A.
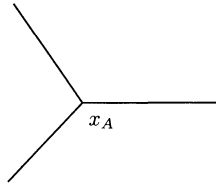

Figure A.2. Stokes curves near x_A (i.e., in the region $\{x; |\mathrm{Re}(x - x_A)| < 0.2, \ |\mathrm{Im}(x - x_A)| < 0.2\}$).

Unfortunately concerning the point x_B Berk and Book were too optimistic in apparently imagining that one can analyze the problem by solely making use of the real locus of the characteristic variety. As our analysis below shows, the analytic structure of the integral equation (0.11) near x_B can never be a standard WKB turning point problem, like that of the Airy equation or that of the Weber equation. To visualize the situation let us show Figure A.3 which illustrates the locus of $z(x) = \zeta(x)^{-1}$, i.e., the inverse of a characteristic root $\zeta(x)$ of the characteristic equation of (0.11), that is, $\gamma^2 \exp x^2 - U(\zeta(x)^{-1}) = 0$, supposing that $x = x_B + r \exp(i\theta)$ $(r = 0.1, -10\pi < \theta < 10\pi)$.

An important point to be observed in Figure A.3 is that, while $z(x)$ behaves approximately like a constant multiple of $(x - x_B)^{-1/2}$ for $|\theta|$ small (to be more precise, for $|\theta| < 0.45\pi < \frac{1}{2}\pi$, as our explicit computer-assisted computation indicates), the behavior of $z(x)$ suddenly changes as $|\theta|$ approaches the value $\frac{1}{2}\pi$. In particular, the value of $z(x)$
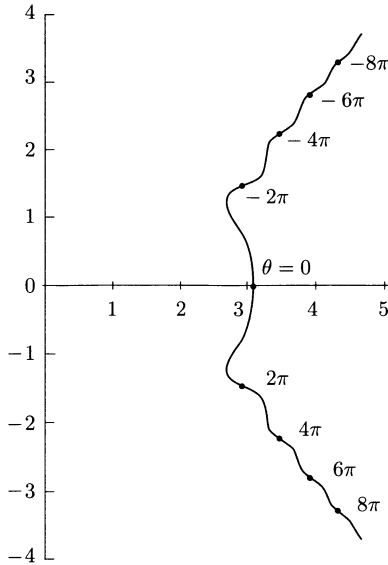
Figure A.3. Locus of $z(x)$ for $x = x_B + re^{i\theta}$ with $r = 0.1$.

for $\theta = \pm 2\pi, \pm 4\pi, \ldots$ differs very much from $z(x_B)$ (cf. [AKKT3]). Thus we clearly see that the structure of the Berk-Book equation (0.11) near x_B should be completely different from that of the Airy equation, although at first sight (i.e., if we study the equation only for $|\theta| \ll \frac{1}{2}\pi$) it might appear to be approximately the same as that of the Airy equation. Furthermore a more careful and detailed study of the locus of $z(x)$ with different r's gives us Figure A.4. See [KoT] for the more detailed study of several aspects of the behavior of $z(x)$ and some other related functions which Landau [L] studied in analyzing the penetration of an external electric field into the plasma.

The comparison of the two figures "left" and "middle" of Figure A.4 indicates that there exist turning points of the equation (0.11) in the region $\{x\,;\ 0.064 < |x - x_B| < 0.066\}$. We note that, if $x = x_*$ is a turning point of (0.11), so is its complex conjugate $x = \bar{x}_*$ since $U(z)$ is real-valued on the real axis. Thus it is expected that there are two turning points x_1 and $\bar{x}_1$ in $\{x\,;\ 0.064 < |x - x_B| < 0.066\}$. In a similar manner we can find another pair of turning points x_2 and $\bar{x}_2$ in $\{x\,;\ 0.040 < |x - x_B| < 0.042\}$, although the figure for $r = 0.042$ is omitted in Figure A.4 to save the space. (More careful numerical study shows that x_1 (resp., x_2) is approximately $1.04863 + 0.0425103i$ (resp., $1.02429 + 0.0339953i$).) This procedure of
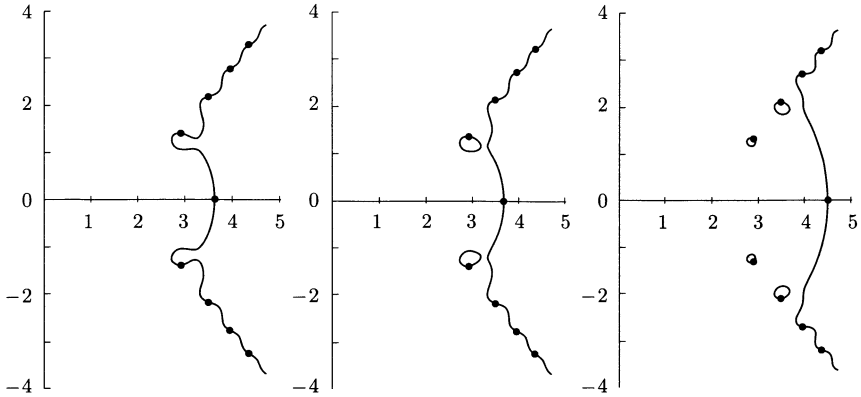
Figure A.4. Locus of $z(x)$ with different r's (left: $r = 0.066$, middle: $r = 0.064$, right: $r = 0.040$).

finding pairs of turning points can be continued further and such a computer-assisted study of the function $z(x)$ strongly suggests that the point x_B is an accumulation point of turning points of the equation (0.11). This observation was the starting point of this Appendix, and we can really validate this observation by the following

PROPOSITION A.1. — Let $P(X,z)$ denote

(A.4) $U(z) - (1 + 2X),$

where $U(z)$ is the entire function given by (A.2). Then there exists a sequence of points (X_n, z_n) $(n \geq 1)$ that satisfies the following:

(A.5) $P(X_n, z_n) = \dfrac{\partial P}{\partial z}(X_n, z_n) = 0,$

(A.6) $\dfrac{\partial P}{\partial X}(X_n, z_n) \neq 0,$

(A.7) $\dfrac{\partial^2 P}{\partial z^2}(X_n, z_n) \neq 0,$

(A.8) $X_n \longrightarrow 0.$

In fact, if we set

(A.9) $X = \dfrac{1}{2}\gamma^2(\exp x^2 - \exp x_B^2),$

the point x_n corresponding to X_n clearly converges to x_B with an appropriate choice of its sign. Hence Proposition A.1 entails the following

THEOREM A.1. — Let $D(x, \eta(\mathrm{d}/\mathrm{d}x)^{-1})$ denote the Berk-Book operator, that is,

(A.10) $\quad U\Big(\eta\Big(\dfrac{\mathrm{d}}{\mathrm{d}x}\Big)^{-1}\Big) - \gamma^2 \exp x^2$

$$\Big(= U\Big(\eta\Big(\dfrac{\mathrm{d}}{\mathrm{d}x}\Big)^{-1}\Big) - 1 - \gamma^2(\exp x^2 - \exp x_B^2)\Big).$$

Then there exists a sequence $\{x_n\}_{n \geq 1}$ of simple turning points of the Berk-Book operator which converges to x_B.

Remark A.1. — As is shown in Lemma A.3 below (cf. (A.52)), the characteristic value $\zeta_n = 1/z_n$ of the turning point x_n tends to 0 as x_n tends to x_B. This fact clearly explains why the analysis of the Berk-Book operator is so difficult at x_B. (Recall that a point (x, ζ) with $\zeta = 0$ is outside the domain of definition of a microdifferential operator in general.)

For the reference of the reader we present Figure A.5 which describes the Stokes curves for the Berk-Book operator that emanate from x_1, $\bar{x}_1$, x_2 and $\bar{x}_2$. (As in Figure A.2, two Stokes curves sit on the same curve in Figure A.5 also.) We refer the reader to [KoT] for a more complete study of the Stokes geometry of the Berk-Book equation near x_B, which includes the added new Stokes curves. (See [AKKT2] for the notion of a new Stokes curve.)
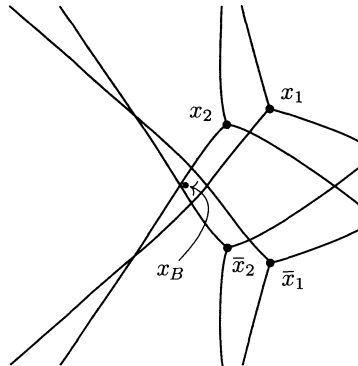

Figure A.5. Stokes curves emanating from x_1, $\bar{x}_1$, x_2 and $\bar{x}_2$ (in the region $\{x \, ; \, |\mathrm{Re}(x - x_B)| < 0.1, |\mathrm{Im}(x - x_B)| < 0.1\}$).

Let us now prove Proposition A.1. Since the argument is rather entangled, we divide it into several steps. We note that a crucially important

point in our reasoning is Lemma A.2 below; there exist infinitely many points z_n with $|\arg z_n| < \frac{1}{4}\pi$ such that $\exp(-z_n^2)$ is fairly large, i.e., of the magnitude z_n^{-k}.

First we note the following Lemma A.1.

LEMMA A.1. — *Let Ω be a convex and open subset of $\mathbb{R}^2_{(x,y)}$, and let $F_0 = {}^t(f_0, g_0)$ and $\widetilde{F} = {}^t(\tilde{f}, \tilde{g})$ denote C^2-mappings from Ω to $\mathbb{R}^2$. Set $F = F_0 + \widetilde{F}$ and suppose that $F_0, \widetilde{F}$ and F satisfy the following conditions:*

(A.11)   *There exists a positive constant α for which $|\widetilde{F}(x,y)| \leq \alpha|F_0(x,y)|$ holds for every (x,y) in Ω; here $|F_0|$ (resp., $|\widetilde{F}|$) denotes $|f_0| + |g_0|$ (resp., $|\tilde{f}| + |\tilde{g}|$).*

(A.12)   *For any ${}^t(a,b)$ in $\mathbb{R}^2\backslash\{0\}$, there exists (x_0, y_0) in Ω for which $F_0(x_0, y_0) = \begin{pmatrix} a \\ b \end{pmatrix}$ holds.*

(A.13)   *The Jacobian matrix $DF = \partial(f,g)/\partial(x,y)$ is invertible on Ω, where f (resp., g) denotes $f_0 + \tilde{f}$ (resp., $g_0 + \tilde{g}$).*

(A.14)   *There exists a constant M_1 for which $\|(DF)^{-1}\| \leq M_1$ holds on Ω. Here $\|(m_{ij})_{i,j=1,2}\|$ denotes $\max_{i,j} \sup_{\Omega} |m_{ij}(x,y)|$.*

(A.15)   *There exists a constant M_2 which dominates each second derivative of f or g on Ω; i.e.,*

$$\sup_{\Omega} \left(\left|\frac{\partial^2 f}{\partial x^2}\right|, \left|\frac{\partial^2 f}{\partial x \partial y}\right|, \left|\frac{\partial^2 f}{\partial y^2}\right|, \left|\frac{\partial^2 g}{\partial x^2}\right|, \left|\frac{\partial^2 g}{\partial x \partial y}\right|, \left|\frac{\partial^2 g}{\partial y2}\right| \right) \leq M_2.$$

Then, if

(A.16) $$8\alpha M_1^2 M_2 \left| \begin{pmatrix} a \\ b \end{pmatrix} \right| \leq \frac{1}{2}$$

holds, we can find $(\hat{x}, \hat{y})$ in Ω which satisfies $F(\hat{x}, \hat{y}) = {}^t(a,b)$.

The proof of this lemma is given by defining a sequence of points $z_n = (x_n, y_n)$ by the following relations:

(A.17) $z_0 = (x_0, y_0)$,

(A.18) $z_{n+1} = z_n + {}^t\left(DF(z_0)^{-1}\left({}^t(a,b) - F(z_n)\right)\right)$ $(n \geq 0)$.

Then by the induction on n we can confirm

(A.19) $$|z_n - z_0| \leq 2\alpha M_1 \left|{}^t(a,b)\right|$$

and

(A.20) $\qquad |z_{n+1} - z_n| \leq (8\alpha M_1^2 M_2|^t(a,b)|)|z_n - z_{n-1}|.$

We note that the convexity assumption on Ω is used to do the computations of the following type:

$$(A.21) \quad F(z_{n-1}) - F(z_n) = \int_0^1 \frac{d}{dt} F(tz_{n-1} + (1-t)z_n) dt$$

$$= \left(\int_0^1 DF(tz_{n-1} + (1-t)z_n) dt \right)(z_{n-1} - z_n).$$

The details of the computations are left to the reader.

Using Lemma A.1 we can show the following

LEMMA A.2. — *For any positive integer* k *and non-zero complex number* c, *we can find infinitely many solutions* $\{z_n\}$ *of the following equation*

$$(A.22) \qquad z^k e^{-z^2} = c$$

so that they satisfy for $n \gg 1$

$$(A.23) \qquad |z_n| = (2\pi n + \theta_0)^{1/2} + O(n^{-3/2}(\log n)^2),$$

$$(A.24) \qquad \arg z_n = -\frac{\pi}{4} + \frac{k\log(2\pi n + \theta_0) - 2\log r_0}{4(2\pi n + \theta_0)}$$
$$+ O(n^{-2}(\log n)^2).$$

Here r_0 *and* θ_0 *are positive constants that satisfy*

$$(A.25) \qquad r_0 e^{i\theta_0} = e^{\pi i k/4} c.$$

Proof. — Let (r, θ) be the polar coordinate of iz^2, i.e.,

$$(A.26) \qquad iz^2 = re^{i\theta} \quad (r \geq 0, \ 0 \leq \theta < 2\pi).$$

Then we can rewrite (A.22) in the following manner:

$$(A.27) \qquad \begin{cases} r\sin\theta - \frac{1}{2}k\log r = -\log r_0, \\ r\cos\theta + \frac{1}{2}k\theta = 2\pi n + \theta_0 \quad (n \in \mathbb{Z}). \end{cases}$$

For each positive integer n, we introduce the following notations so that we may employ Lemma A.1:

(A.28) $x = r, \; y = r\theta,$

(A.29) $\Omega = \{(x, y) \in \mathbb{R}^2 \; ; \; x > 2\pi n + \theta_0 - 1,$
$$0 < y < 2\log(2\pi n + \theta_0 - 1)\},$$

(A.30) $F_0 = \begin{pmatrix} x \\ y - \frac{1}{2}k\log x \end{pmatrix},$

(A.31) $\widetilde{F} = \begin{pmatrix} x(\cos(y/x) - 1) + \frac{1}{2}ky/x \\ x(\sin(y/x) - y/x) \end{pmatrix},$

(A.32) $(x_0, y_0) = (2\pi n + \theta_0, \frac{1}{2}k\log(2\pi n + \theta_0) - \log r_0),$

(A.33) $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2\pi n + \theta_0 \\ -\log r_0 \end{pmatrix}.$

It then follows from these definitions that

(A.34) $$F_0(x_0, y_0) = \begin{pmatrix} a \\ b \end{pmatrix}.$$

It is also immediate to see that, for $n \gg 1$,

(A.35) $$|\widetilde{F}| \le C_1 \left(\frac{\log n}{n} \right)^2 |F_0|$$

holds with a constant C_1 independent of n. Setting $F = F_0 + \widetilde{F} = {}^t(f, g)$, we find

(A.36) $\dfrac{\partial f}{\partial x} = \cos\dfrac{y}{x} + \dfrac{y}{x}\sin\dfrac{y}{x} - \dfrac{k}{2}\dfrac{y}{x^2},$

(A.37) $\dfrac{\partial f}{\partial y} = -\sin\dfrac{y}{x} + \dfrac{k}{2x},$

(A.38) $\dfrac{\partial g}{\partial x} = \sin\dfrac{y}{x} - \dfrac{y}{x}\cos\dfrac{y}{x} - \dfrac{k}{2x},$

(A.39) $\dfrac{\partial g}{\partial y} = \cos\dfrac{y}{x}.$

Thus, for $n \gg 1$, the Jacobian matrix DF is invertible on Ω and $\|DF^{-1}\|$ is bounded by a constant M_1 independent of n.

Similarly we compute the second derivatives of f and g explicitly so that we may find their bounds:

(A.40)
$$\frac{\partial^2 f}{\partial x^2} = -\frac{y^2}{x^3} \cos \frac{y}{x} + k \frac{y}{x^3},$$

(A.41)
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{y}{x^2} \cos \frac{y}{x} - \frac{1}{2} k \frac{1}{x^2},$$

(A.42)
$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{x} \cos \frac{y}{x},$$

(A.43)
$$\frac{\partial^2 g}{\partial x^2} = -\frac{y^2}{x^3} \sin \frac{y}{x} + \frac{1}{2} k \frac{1}{x^2},$$

(A.44)
$$\frac{\partial^2 g}{\partial x \partial y} = \frac{y}{x^2} \sin \frac{y}{x},$$

(A.45)
$$\frac{\partial^2 g}{\partial y^2} = -\frac{1}{x} \sin \frac{y}{x}.$$

Then it is obvious that, for $n \gg 1$, each of them is bounded by a constant M_2 of the form C_2/n with another constant C_2 independent of n. Combining all these estimations, we find

(A.46)
$$8\alpha M_1^2 M_2 |(a,b)| \le C \left(\frac{\log n}{n} \right)^2 \frac{1}{n} n$$

holds for $n \gg 1$, where C is a constant independent of n. Thus Lemma A.1 is applicable to our situation, and hence there exists a constant n_0 such that for $n \ge n_0$ the equation (A.22) has a family of solutions $\{z_n\}$ whose behavior is given as follows by (A.19):

(A.47)
$$|z_n| = (2\pi n + \theta_0)^{1/2} + O\big(n^{-3/2}(\log n)^2\big),$$

(A.48)
$$\arg z_n = -\frac{\pi}{4} + \frac{k \log(2\pi n + \theta_0) - 2 \log r_0}{4(2\pi n + \theta_0)}$$
$$+ O\big(n^{-2}(\log n)^2\big).$$

This completes the proof of Lemma A.2. $\square$

Now, in order to employ Lemma A.2 to prove Proposition A.1, we rewrite $dU(z)/dz$ in a form suited for our argument. Since $\exp t^2$ is bounded for $t = \exp(-\frac{1}{4}\pi i)s$ $(s > 0)$, we find the following asymptotic expansion that holds for z with $|\arg z| \le \frac{1}{4}\pi$:

(A.49)
$$\int_0^z e^{t^2} \, dt = \int_0^{\exp(-\frac{1}{4}\pi i)\infty} e^{t^2} \, dt + \int_{\exp(-\frac{1}{4}\pi i)\infty}^z e^{t^2} \, dt$$
$$= C_0 + e^{z^2} \left(\frac{1}{2z} + \frac{1}{4z^3} + \frac{3}{8z^5} + \frac{15}{16z^7} + \cdots \right),$$

where $C_0 = -\frac{1}{2}i\sqrt{\pi}$. Substituting this asymptotic expansion into dU/dz, we find

(A.50)
$$\frac{dU}{dz} = (12z^2 - 8z^4)C_0 e^{-z^2} - g(z),$$

where $g(z)$ is a holomorphic function on $\{|\arg z| < \frac{1}{4}\pi\}$ and it can be asymptotically expanded there in the form

(A.51)
$$-3z^{-3} + O(z^{-5}).$$

We then use Lemma A.2 to verify the following

LEMMA A.3. — *The equation $dU(z)/dz = 0$ has a family of solutions $\{z_n\}$ whose asymptotic behavior for $n \gg 1$ is given by*

(A.52) $|z_n| = (2\pi n + \theta_0)^{1/2} + O(n^{-1}),$

(A.53) $\arg z_n = -\dfrac{\pi}{4} + \dfrac{7\log(2\pi n + \theta_0) - 2\log r_0}{4(2\pi n + \theta_0)} + O(n^{-3/2})$

for some positive constants θ_0 and r_0.

Proof. — It follows from (A.50) and (A.51) that $dU(z)/dz = 0$ is equivalent to

(A.54) $\left(z^7 - \dfrac{3}{2}z^5\right)e^{-z^2} = \dfrac{3}{8C_0} + h(z)$ with $|h(z)| = O(z^{-2}),$

that is,

(A.55) $z^{-7}e^{z^2} - \dfrac{8}{3}C_0 + 4C_0 z^{-2} + \dfrac{8}{3}C_0 z^{-7}e^{z^2} h = 0.$

Let us now use Lemma A.2 with $k = 7$ and $c = 3/(8C_0)$ to find a family of solutions $\{z_n^0\}$ of (A.22) that satisfy (A.23) and (A.24). Let f and r respectively denote $z^{-7}e^{z^2} - \frac{8}{3}C_0$ and $4C_0 z^{-2} + \frac{8}{3}C_0 z^{-7}e^{z^2} h$. We then want to claim

(A.56)
$$|f(z)| > |r(z)|$$

holds on a circle centered at z_n^0 with the radius $1/n$. If (A.56) is confirmed, then the classical Rouché theorem finishes the proof of Lemma A.3. Since $f(z_n^0) = 0$ by the definition of z_n^0, we find

(A.57) $f(z) = f'(z_n^0)(z - z_n^0) + \displaystyle\int_0^1 (1-t)f''\big(z_n^0 + t(z - z_n^0)\big)\,dt(z - z_n^0)^2.$

On the other hand, we find

(A.58) $$f'(z) = (-7z^{-8} + 2z^{-6})e^{z^2},$$

(A.59) $$f''(z) = (56z^{-9} - 26z^{-7} + 4z^{-5})e^{z^2}.$$

Hence on the circle $C_n = \{z \, ; \, |z - z_n^0| = n^{-1}\}$ (A.22) and (A.23) entail that

(A.60) $$\left|f'(z_n^0)\right| = \left|2z_n^0 - \frac{7}{z_n^0}\right| \cdot \left|\frac{8}{3}C_0\right| \geq A_1\sqrt{n}$$

holds for some strictly positive constant A_1. Similarly we find that

(A.61) $$\left|f''(z)\right| \leq A_2 n$$

holds inside the circle C_n for some constant A_2. Thus (A.57), (A.60) and (A.61) imply the existence of a strictly positive constant A_0 for which

(A.62) $$|f(z)| \geq A_0 n^{-1/2}$$

holds on C_n for sufficiently large n. The asymptotic behavior of $|z_n^0|$ (cf. (A.23)) also guarantees that $|r(z)| = O(n^{-1})$ holds on C_n. Therefore we find that (A.56) holds on C_n. Thus the Rouché theorem finishes the proof of Lemma A.3.   □

Using the solution z_n of $dU(z)/dz = 0$ whose existence is guaranteed by the above lemma, we define

(A.63) $$X_n = \frac{1}{2}\big(U(z_n) - 1\big).$$

Then (X_n, z_n) clearly satisfies (A.5). We also find that (A.6) immediately follows from the definition of $P(X, z)$. We next show that (A.52) implies (A.8). Using (A.49) again, we first note

(A.64) $$X_n = 2C_0 z_n^3 e^{-z_n^2} + \frac{3}{4}z_n^{-2} + O(z_n^{-4}).$$

Furthermore (A.54) entails

(A.65) $$C_0 z_n^3 e^{-z_n^2} = \frac{3}{8z_n^4 - 12z_n^2} + O(z_n^{-6}).$$

Therefore (A.52) together with (A.64) and (A.65) guarantees that $\lim X_n = 0$, showing (A.8).

Finally we prove (A.7). It follows from the relation $\partial P(X_n, z_n)/\partial z = \mathrm{d}U(z_n)/\mathrm{d}z = 0$ that

$$(A.66) \qquad \mathrm{e}^{-z_n^2} \int_0^{z_n} \mathrm{e}^{t^2}\,\mathrm{d}t = \frac{4z_n^3 - 4z_n}{8z_n^4 - 12z_n^2}\,.$$

If $\partial^2 P/\partial z^2$ vanished at z_n, a simple computation shows that

$$(A.67) \qquad \mathrm{e}^{-z_n^2} \int_0^{z_n} \mathrm{e}^{t^2}\,\mathrm{d}t = \frac{8z_n^4 - 24z_n^2 + 4}{16z_n^5 - 56z_n^3 + 24z_n}$$

should hold. However, one can readily verify that the right-hand side of (A.66) and that of (A.67) cannot be equal. This is contradiction, showing that (A.7) holds for our choice of (X_n, z_n).

At long last, this completes the proof of Proposition A.1 (with an appropriate shift of the index n).   □

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