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Yves COLIN DE VERDIÈRE

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THE LEVEL CROSSING PROBLEM IN SEMI-CLASSICAL ANALYSIS, II: THE HERMITIAN CASE

by Yves COLIN DE VERDIÈRE

Introduction.

This paper is the second part of [3]. We want to study microlocally the solutions of a self-adjoint system of semi-classical pseudo-differential operators using normal forms. In our paper [3], we studied the case where the principal symbol (called the dispersion matrix) is a real symmetric matrix. We will consider here the case where the dispersion matrix H_{class} is complex Hermitian. There are several cases to consider depending on the rank of the restriction of the symplectic form to the codimension 4 singular manifold Σ:

1. The *symplectic* case.

2. The *elliptic* corank 2 case.

3. The *hyperbolic* corank 2 case.

4. The case of *one degree of freedom* with some parameters (*avoided crossings*).

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Our goal is to get local normal forms for these systems both for the principal symbol (classical normal form) and for the pseudo-differential system (semi-classical normal form). The classical normal form uses canonical transformations and gauge transforms while the semi-classical normal form uses quantized versions of the previous transformations (Fourier integral operators and pseudo-differential gauge transforms). The semi-classical normal form can be used in order to describe the solutions of the system near the singular manifold (Landau-Zener type formulae, propagation of localized states, semi-classical measures). The reader is supposed to have already a knowledge of [3]. Some arguments work the same way and are only sketched.

1. The geometric setting.

We will consider a $d \times d$ Hermitian system of pseudo-differential equations
$$\widehat{H}\vec{U} = 0$$
in $\mathbb{R}^n$ near some point $z_0 \in T^\star\mathbb{R}^n$ (with the symplectic form $\Omega = \sum_{j=1}^n d\xi_j \wedge dx_j$) where the kernel of the principal symbol $H_{\rm class}$ is of dimension 2. We will denote by $p = \det(H_{\rm class})$; the manifold $\{p = 0\}$ (more precisely the principal ideal $C^\infty.p$) is the *dispersion relation*. We can reduce the system near the point z_0 to a 2×2 system for which the principal symbol vanishes at the point z_0. We will assume that

($\star$) **the mapping $z \to H_{\rm class}(z)$ is transversal at z_0 to $W_2 = \{A \,|\, \dim\ker A = 2\} \subset \mathrm{Herm}(\mathbb{C}^d)$.**

The inverse image $H_{\rm class}^{-1}(W_2)$ is then a codimension 4 manifold Σ of the phase space $T^\star\mathbb{R}^n$, the *singular locus*; we have
$$\Sigma = \{z \in T^\star\mathbb{R}^n \mid \dim\ker H_{\rm class}(z) = 2\}.$$

Assuming always ($\star$), we will study 4 cases:

- The **symplectic case**, denoted by $(\mathcal{HE})$, where Σ is a symplectic submanifold of $T^\star\mathbb{R}^n$. It implies (see Lemma 1) that the linearization M of $\mathcal{X}_p$ at z_0 admits 2 pairs of non vanishing eigenvalues $\pm\lambda$, $\pm i\omega$ with $\lambda > 0$, $\omega > 0$.
- The **hyperbolic corank** 2 case, denoted by $(\mathcal{H})$, where $\Omega_{|\Sigma}$ is of CONSTANT corank 2 and M admits 1 pair of real nonzero eigenvalues $\pm\lambda$ with $\lambda > 0$.

- The **elliptic corank 2 case**, denoted by $(\mathcal{E})$, where $\Omega_{|\Sigma}$ is of CONSTANT corank 2 and M admits 1 pair of imaginary nonzero eigenvalues $\pm i\omega$ with $\omega > 0$.
- The case of **one degree of freedom with parameters**, denoted by $(\mathcal{P})$, which splits into an elliptic case $(\mathcal{P}_{\mathcal{E}})$ and an hyperbolic one $(\mathcal{P}_{\mathcal{H}})$: in that case, it is needed to have parameters in order to get the transversality assumption $(\star)$.

An important remark: *in the cases $(\mathcal{H})$ and $(\mathcal{E})$, the manifold Σ is not symplectic as should be (locally) a generic codimension 4 submanifold of a symplectic one. It implies that these cases are* **not structurally stable.**

Nevertheless it is usefull to study these cases because they occur in Born-Oppenheimer type systems: a system $\hat{H}u = 0$ is of Born-Openheimer type if

$$\hat{H} = \mathrm{Op}_{\mathrm{Weyl}}(P) \otimes \mathrm{Id} + Q(x)$$

where P is a scalar Hamiltonian. The set Σ is then defined by

$$\Sigma = \{\lambda_j(Q)(x) = \lambda_{j+1}(Q)(x) = -P(x,\xi)\},$$

where $\lambda_1(x) \leqslant \lambda_2(x) \leqslant \cdots$ are the eigenvalues of $Q(x)$. The condition $(\star)$ is then equivalent to a transversal crossing of λ_j and λ_{j+1} along a codimension 3 submanifold S of $\mathbb{R}^n$ and the non vanishing of the differential of P restricted to Σ. If $(\partial P/\partial\xi)\partial_x$ is transversal to S, we are in the $\mathcal{H}$ case (see Theorem 1).

The following Lemma, used in the case $(\mathcal{HE})$, is easy to check using the normal forms of [1] (Appendix 6) based on results on [20] (see the 4 dimensional case at page 163); in fact one checks that in all other non degenerate cases the Morse index of Q is even.

LEMMA 1. — *Let Q be a quadratic form on $T^\star\mathbb{R}^2$ with signature $(+,-,-,-)$. The Hamiltonian linear vector field $\mathcal{X}_Q$ associated to Q admits $(\pm\lambda, \pm i\omega)$ with $\lambda > 0$, $\omega > 0$ as eigenvalues.*

If $d = 2$ and

$$H_{\mathrm{class}} = \begin{pmatrix} p_1 + p_2 & p_3 + ip_4 \\ p_3 - ip_4 & p_1 - p_2 \end{pmatrix},$$

we define:

$$\omega_{i,j} = dp_j(\mathcal{X}_i) = \{p_i, p_j\},$$

$$\Pi = \omega_{1,2}\omega_{3,4} - \omega_{1,3}\omega_{2,4} + \omega_{1,4}\omega_{2,3} \tag{1}$$

(Π is the Pfaffian of the antisymmetric matrix $(\omega_{i,j})$) and

$$\delta = \frac{1}{8}\mathrm{Tr}(M^2) = \omega_{1,2}^2 + \omega_{1,3}^2 + \omega_{1,4}^2 - \omega_{2,3}^2 - \omega_{2,4}^2 - \omega_{3,4}^2.$$

THEOREM 1. — *We get the following classification:*

1. *The symplectic case* $(\mathcal{HE})$ *corresponds to* $\Pi(z_0) \neq 0$. *The ratio*

$$K = \frac{\omega^2 - \lambda^2}{\lambda\omega},$$

which is a function of $z' \in \Sigma$, *called the* Ray Helicity *in [16], is given by*

$$K(z') := -\frac{\delta}{|\Pi|}(z').$$

2. *The hyperbolic corank* 2 *case* $(\mathcal{H})$ *corresponds to the vanishing of* Π *on some* neighbourhood of z_0 in Σ and $\delta(z_0) > 0$.

3. The elliptic corank 2 case $(\mathcal{E})$ corresponds to the vanishing of Π on some neighbourhood of z_0 in Σ and $\delta(z_0) < 0$.

Proof. — A basis of the image of M is the set of Hamiltonian vector fields $\mathcal{X}_{p_j}$, $j = 1, \cdots, 4$. The restriction of Ω to $\mathrm{Im}M$ admits the matrix $(\omega_{i,j})$ in this basis. We have $\det(\omega_{i,j}) = \Pi^2$. This gives the first condition. The map M is given by

$$M = 2\Big(dp_1 \otimes \mathcal{X}_{p_1} - \sum_{j=2}^{4} dp_j \otimes \mathcal{X}_{p_j}\Big)$$

so that the matrix of the restriction of M to $\mathrm{Im}M$ is $2(\varepsilon_{i,j}\omega_{i,j})$ with $\varepsilon_{1,j} = 1$ and $\varepsilon_{i,j} = -1$ if $i \geqslant 2$. We get $4|\Pi| = \lambda\omega$ and $4\delta = \lambda^2 - \omega^2$, hence K. The ratio $r := \omega/\lambda$ is given from K by $r^2 - Kr - 1 = 0,\ r > 0$.

In the corank 2 case, the square of any of the nonzero eigenvalues of M is $4\delta(z_0)$. □

Examples of these cases have been studied in various papers:

1. The symplectic case $(\mathcal{HE})$ in [9] and [11]: it is the case where $E.B \neq 0$ with the notations of these papers.
2. The hyperbolic corank 2 $(\mathcal{H})$ in Born-Oppenheimer approximation, [3], [12] and [10]. In [9], it is the case where $E.B = 0$ and $|E| > |B|$.
3. The case of one degree of freedom with parameters is studied in [5] (adiabatic limit, hyperbolic case $(\mathcal{P}_{\mathcal{H}})$) (see also [15] and [19]) and in [7] and [8] (elliptic case: band crossings, $(\mathcal{P}_{\mathcal{E}})$).

2. The general strategy.

We will proceed for each case along the same lines:

1. Reduction to a 2×2 system. This part is always the same and is recalled in Section 3.
2. Finding a normal form for the dispersion relation: this part works by
   - Finding a "Birkhoff normal form" along the singular manifold Σ.
   - Using Sternberg's Theorem in order to get a normal form for the hyperbolic part.
3. Using a general result stated in Section 4, we pass from a normal form for the dispersion relation to a normal form for the dispersion matrix.
4. In order to get the semi-classical normal form, we need to solve the following type of homological equation
$$\{S, H_0\} + C^{\star} H_0 + H_0 C = R$$
where H_0 is the classical normal form, R is given, S is an unknown real valued function and C an unknown matrix valued function. Fortunately, this equation is the linearization of the classical normal form, so that we can solve it for free!

The realization of this program is more difficult than in [3], especially in the corank 2 case which is not structurally stable.

3. Reduction to a 2×2 system.

It is well known (see [2] or [3]) that near a point $z_0 \in T^{\star}\mathbb{R}^n$ such that
$$\dim \ker H_{\text{class}}(z_0) = 2,$$
we can split microlocally the system into a direct sum of a $(d-2) \times (d-2)$ *elliptic block* and a 2×2 block whose principal symbol vanishes at z_0.

The dispersion relations of the initial system and the small one are the same. In what follows, we will always assume that this splitting has been done and therefore we have a 2×2 system to study.

For convenience, the transversality hypothesis $(\star)$ has been formulated in Section 1 for the big system.

4. A lemma about gauge transforms.

The following Lemma will be used several times:

LEMMA 2. — *Let $H : \mathbb{R}^4_X \times \mathbb{R}^N_\lambda \to \mathrm{Herm}(2)$ be a smooth map such that*

$$\det(H(X,\lambda)) = X_1X_2 - (X_3^2 + X_4^2).$$

There exist uniquely defined $\varepsilon = \pm 1$, $\alpha = \pm 1$ and a smooth germ of map $J : \mathbb{R}^4 \times \mathbb{R}^N \to GL(2,\mathbb{C})$ such that

$$J^\star H(X,\lambda)J = \begin{pmatrix} \alpha X_1 & X_3 + i\varepsilon X_4 \\ X_3 - i\varepsilon X_4 & \alpha X_2 \end{pmatrix}$$

The proof follows exactly the same lines as the proof of Lemma 5 in [3].

5. The symplectic case.

5.1. The normal form for the dispersion relation.

PROPOSITION 1. — *Assuming $(\star)$ and $(\mathcal{HE})$ (we are in the symplectic case), near the point z_0 of the singular set Σ, there exists a canonical transformation χ and two invertible positive (> 0) germs $e(z)$ and $b(\tau, z')$ so that:*

$$\det(H_{\text{class}}) \circ \chi = e(z)\left(x_1\xi_1 - b^2(x_2^2 + \xi_2^2, z')(x_2^2 + \xi_2^2)\right) + O_Y(\infty)$$

where $Y = \{x_2 = \xi_2 = 0\}$, $z = (x_1, \xi_1, x_2, \xi_2, z' = (x', \xi'))$ are canonical coordinates near $0 \in T^\star\mathbb{R}^n$. We have $b^2(0, z') = \frac{1}{2}r(z')$ (see Theorem 1), so that Ray Helicity defined in Theorem 1 is given by $K = b - \frac{1}{b}$ and hence is only (a function of) the first term of a complete Birkhoff series.

Proof. — We start using the same kind of arguments as in the proof of Theorem 2 in [3]. We get then a (formal) Birkhoff normal form along the singular set Σ of the form:

$$A(x_1\xi_1, x_2^2 + \xi_2^2, z') + O_\Sigma(\infty),$$

with A a smooth function which satisfies

$$A(\tau_1, \tau_2, z') = \lambda(z')\tau_1 - \frac{\omega(z')}{2}\tau_2 + O(\tau_1^2 + \tau_2^2).$$

There is a minus sign in front of the τ_2 term because it is the only way to get the appropriate signature $(+,-,-,-)$ for p'' along Σ. Using Taylor formula, we can rewrite A as follows

$$A(\tau_1,\tau_2,z') = F(\tau_1,\tau_2,z')(\tau_1 - \tau_2 b^2(\tau_2,z')).$$

Using Sternberg's linearization as in [3], we get the result. We use the following version of Sternberg's Theorem whose proof can be given using the same arguments as in Nelson's book [18]:

THEOREM (Sternberg). — *Let X be a smooth vector field on $T^\star\mathbb{R}^n$ and $\Sigma = \{x_1 = \xi_1 = x_2 = \xi_2 = 0\}$. Let us assume that $X = X_0 + X_1$ with X_1 is compactly supported and $X_1 = O_\Sigma(\infty)$. We assume*

$$X_0 = x_1\partial_{x_1} - \xi_1\partial_{\xi_1} + Y_0(x_2,\xi_2,z'),$$

with Y_0 tangent to all codimension 2 subspaces $x_1 = a,\ \xi_1 = b$. There exists a diffeomorphism χ which is tangent of order ∞ to the identity along Σ such that

$$\chi^\star(X_0 + X_1) = X_0 + O_Y(\infty).$$

Moreover, if X and X_0 are Hamiltonian vector fields, χ can be choosen to be symplectic.

□

Remark 1. — The normal form is convergent in the case of 2 degrees of freedom and analytic data. This is implied by a result of Moser [17].

5.2. The gauge transform.

Using Lemma 2 with $X_1 = x_1,\ X_2 = \xi_1,\ X_3 = bx_2, X_4 = b\xi_2$, we get a gauge transform. The value of α can be changed to $+1$ using the canonical transformation $(x_1,\xi_1) \to (-x_1,-\xi_1)$.

Both signs of ε in the classical normal form give non equivalent Hamiltonians. Using the notations of Equation (1) in Section 1, we have:

$$\varepsilon = \mathrm{sign}(\Pi(z_0)).$$

So $\varepsilon = 1$ if the orientations of the normal bundle to Σ given by $dp_1\wedge\cdots\wedge dp_4$ and $\Omega\wedge\Omega$ are the same and $\varepsilon = -1$ if they are not the same. It is clear that ε is invariant by gauge transform, the group $GL(2,\mathbb{C})$ being connected.

Remark 2. — We can see that in a more topological way: let us denote by $\lambda_- \leqslant \lambda_+$ the eigenvalues close to 0 of the dispersion matrix. The open cones $C_\pm \subset p^{-1}(0)$ which correspond respectively to $\lambda_- = 0 < \lambda_+$ ($\lambda_- < \lambda_+ = 0$) are well defined near Σ: Morse indices differs by 1 on those cones. Moreover, both cones are oriented by $p > 0$. The spaces $\{z' = \textit{constant}\}$ are co-oriented by the z' symplectic structure, hence oriented. It follows that the basis of the cone $C_+ \cap \{z' = \textit{constant}\}$ (a 2-sphere) is a well defined homology class of the germ of C_+. Hence the polarization bundle have a well defined first Chern class on C_+ and both signs in the normal form gives both signs in the Chern class.

5.3. The classical normal form.

Using the previous results, we get:

THEOREM 2. — *We assume that H_{class} satisfies $(\star)$ and $(\mathcal{HE})$ (the symplectic case). Then there exists a canonical transformation χ and a gauge transform $J \in GL(2, \mathbb{C})$ such that:*

$$J^\star \left(H_{\text{class}} \circ \chi\right) J := H_{\text{symp}} + O_Y(\infty)$$

where

$$H_{\text{symp}} = \begin{pmatrix} \xi_1 & b(x_2^2 + \xi_2^2, z')(x_2 \pm i\xi_2) \\ b(x_2^2 + \xi_2^2, z')(x_2 \mp i\xi_2) & x_1 \end{pmatrix}$$

and $b = b(\tau, z') > 0$ is smooth.

5.4. The semi-classical normal form.

THEOREM 3. — *We assume that H_{class} satisfies $(\star)$ and 1. (the symplectic case). Using FIO and gauge transform, we get the following microlocal normal form:*

$$\widehat{H} = \begin{pmatrix} \widehat{\xi_1} & \widehat{B}a \\ a^\star \widehat{B}^\star & x_1 \end{pmatrix} + R$$

where

- *$\widehat{B}$ is an elliptic ΨDO whose total symbol is > 0 and depends only on $x_2^2 + \xi_2^2$ and z'.*

- $a = \widehat{x_2 \pm i\xi_2}$.
- *The full symbol of R is flat on Y.*

Proof. — Using the same method as in the proof of Theorem 3 in [3] (Lemma 4), we need to solve the following homological equation:

$$\{S, H_{\text{symp}}\} + C^\star H_{\text{symp}} + H_{\text{symp}} C = R$$

where R is given and S (real valued) and C (matrix valued) are unknown functions. This can be done directly by solving the normal form problem for $H_{\text{symp}} + \varepsilon R$ which satisfy our basic hypothesis (($\star$) and 2.) for all small ε (this case is *structurally stable*) and taking the first order term in ε. □

5.5. Microlocal solutions.

A study of the solutions of the normal form in terms of 2-scale semi-classical measures is given in [9] and [11].

For simplicity we will restrict our discussion to the + case (of the normal form) and 2 degrees of freedom ($n = 2$). The − case is similar. In this case, the singular manifold Σ is the point $z_0 = (0,0)$ in $T^\star\mathbb{R}^2$. The classical dynamics is integrable and easy to discuss. There is a stable (resp. unstable) manifold of dimension 1. The trajectories which are not included in these manifold are helices.

Let us denote by $B_h(x_2^2 + \xi_2^2)$ the symbol of $\hat{B}$. Following [9], we can solve the normal form as follows:

Proposition 2. — *Let us denote by φ_j the usual orthonormal basis of $L^2(\mathbb{R}, dx_2)$ such that $\varphi_j = c_j \alpha^j \varphi_0$ with $c_j > 0$, $\varphi_0 = c_0 \exp(-x_2^2/2h)$ and $\alpha = -h\partial_{x_2} + x_2$ is the* creation operator.

Any solution $\vec{U}$ of the normal form is of the form:

$$\vec{U} = \sum_{j=-1}^{\infty} \vec{U}_j \tag{2}$$

where

$$\vec{U}_{-1} = \begin{pmatrix} 0 \\ b_0\varphi_0 \end{pmatrix}$$

and, for $j \geqslant 0$,

$$\vec{U}_j = \begin{pmatrix} a_j\varphi_j \\ b_{j+1}\varphi_{j+1} \end{pmatrix},$$

with $x_1 b_0 = 0$ *and, for* $j \geqslant 0$*:*
(3)
$$\begin{pmatrix} \frac{h}{i}\partial_{x_1} & \sqrt{2(j+1)h}B_h\left((2j+1)h\right) \\ \sqrt{2(j+1)h}B_h\left((2j+1)h\right) & x_1 \end{pmatrix} \begin{pmatrix} a_j \\ b_{j+1} \end{pmatrix} = 0.$$

The solutions of Equation (3) obeys a 1 dimensional normal form and then are well described by a Landau-Zener type formula (see [15] and [5]) with transmission coefficient

$$t_j = e^{-2\pi(j+1)B_h((2j+1)h)}.$$

Let $J(h)$ be given so that

$$1 << J(h) << \frac{1}{h}.$$

We can decompose

$$\vec{U} = \vec{U}_{LZ} + \vec{U}_r$$

by cutting the sum in Equation (2) at the index $J(h)$.

The solution $\vec{U}_{LZ}$ is microlocalized on the union of the stable and unstable manifold according to a Landau-Zener type rule, while the remainder $\vec{U}_r$ obeys usual propagation along nonsingular trajectories.

6. The corank 2 case.

6.1. Singular perturbations and homological equations.

6.1.1. Introduction.

The corank 2 case is more difficult, because it is not *structurally stable*: a generic perturbation of the dispersion matrix will be in the symplectic case. In the subsection 6.1.3, we will find the space of infinitesimal deformations of the corank 2 case. In the subsection 6.1.4, we will look at a Birkhoff normal form for the Taylor expansion along Σ of the dispersion relation. In the subsection 6.1.5, we will look at the homological equation needed for finding the semi-classical normal form.

6.1.2. Singular deformations of Σ.

DEFINITION 1. — *Let $\Sigma = \{x_1 = \xi_1 = x_2 = x_3 = 0\} \subset (T^\star\mathbb{R}^n, \Omega)$. A smooth deformation Σ_ε of Σ is called* singular *if the corank of $\Omega_{|\Sigma_\varepsilon}$ is constant ($\equiv 2$).*

A deformation (F, G, A, B) given by

$$\Sigma_\varepsilon = \{x_1 = \varepsilon F(\sigma),\ \xi_1 = \varepsilon G(\sigma),\ x_2 = \varepsilon A(\sigma),\ x_3 = \varepsilon B(\sigma) \mid \sigma \in \Sigma\}$$

is called infinitesimally singular *if it can be modified by $O(\varepsilon^2)$ terms so that the new deformation is singular.*

LEMMA 3. — *The space of infinitesimally singular deformations is the space*

$$\left(F, G, \frac{\partial T}{\partial \xi_2}, \frac{\partial T}{\partial \xi_3}\right)$$

where F, G, T are arbitrary functions on Σ.

Proof. — Let us start with a singular deformation whose infinitesimal deformation is given by (F, G, A, B). We see that the pull-back of $\Omega_{|\Sigma_\varepsilon}$ on Σ is given by:

$$\Omega_\varepsilon = d\xi' \wedge dx' - \varepsilon(dA \wedge d\xi_2 + dB \wedge d\xi_3) + O(\varepsilon^2).$$

We have

$$\Omega_\varepsilon^{n-1} = -(n-1)\varepsilon(d\xi' \wedge dx')^{n-2} \wedge (dA \wedge d\xi_2 + dB \wedge d\xi_3) + O(\varepsilon^2)$$

whose vanishing implies there exists T such that

$$A = \frac{\partial T}{\partial \xi_2},\ B = \frac{\partial T}{\partial \xi_3}.$$

Conversely, let us start with the infinitesimal deformation given by (F, G, T). Let $S = T + \xi_1 F - x_1 G$. Let χ_ε the flow at time ε of the Hamiltonian vector field $\mathcal{X}_S$ generated by S. The deformation $\Sigma_\varepsilon = \chi_\varepsilon(\Sigma)$ is singular. It is easy to check that the infinitesimal deformation associated to Σ_ε is $(F, G, \frac{\partial T}{\partial \xi_2}, \frac{\partial T}{\partial \xi_3})$. □

6.1.3. Singular perturbations of the dispersion matrix.

Let us denote by H_{hyp} (resp. H_{ell}) some dispersion matrices given by

$$H_{\text{hyp}} = \begin{pmatrix} \xi_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_1 \end{pmatrix} + O_\Sigma(2)$$

$$\left(\text{resp. } H_{\text{ell}} = \begin{pmatrix} x_2 & x_1 + i\xi_1 \\ x_1 - i\xi_1 & x_3 \end{pmatrix} + O_\Sigma(2)\right).$$

DEFINITION 2.

- *We say that a smooth deformation*

$$H_\varepsilon = H_{\text{hyp}} \ (\text{resp. } H_{\text{ell}}) + \varepsilon K_0 + O(\varepsilon^2) \tag{4}$$

is singular *if it satisfies the hypothesis* $(\mathcal{H})$ *(resp.* $(\mathcal{E})$*) of Section 1 for* ε *small enough.*

- *An* infinitesimal deformation K_0 *is* singular *if it can be embedded into a smooth singular deformation.*

LEMMA 4.

- *In the case* $(\mathcal{H})$*, an* infinitesimal deformation K_0 *is* singular *if and only if there exists* $T : \Sigma \to \mathbb{R}$ *so that*

$$((K_0)_{1,2})_{|\Sigma} = \frac{\partial T}{\partial \xi_2} + i \frac{\partial T}{\partial \xi_3}.$$

- *The same result holds in the* elliptic $(\mathcal{E})$ *case by replacing the previous condition by:*

$$\begin{cases} ((K_0)_{1,1})_{|\Sigma} = \dfrac{\partial T}{\partial \xi_2} \\ ((K_0)_{2,2})_{|\Sigma} = \dfrac{\partial T}{\partial \xi_3} \end{cases}$$

Proof. — Lemma 4 is an easy consequence of Lemma 3. □

6.1.4. Homological equation to high order.

We will give a Lemma in the hyperbolic case, the elliptic case works similarly:

LEMMA 5. — *Let* H_N *be the space of function homogeneous of degree* N *w.r. to* (x_1, ξ_1, x_2, x_3) *and* $p_0 = x_1\xi_1 - (x_2^2 + x_3^2)$*. We can solve the following equation:*

$$\{U + W, p_0\} + V p_0 + x_3^N \tau(\sigma) = \rho + O_\Sigma(N+1) \tag{5}$$

where $\rho \in H_N$ *is given. The unknowns functions are:*

- $U \in H_N$.

- *$W \in H_{N-1}$ an homogeneous polynomial of degree $N-1$ w.r. to the variables (x_2, x_3) with coefficients in $C^\infty(\Sigma)$.*
- *$V \in H_{N-2}$.*
- *$\tau \in C^\infty(\Sigma)$.*

Proof. — The proof is very close to the proof of Lemma 2 in [3]. We decompose everything into sums of monomial terms in (x_2, x_3). At the last step, we fail to be able to solve unless we add a term $x_3^N \tau(\sigma)$ to ρ. A bit more specifically, we decompose every function F into monomial w.r. to (x_2, x_3):

$$F(x_1, \xi_1, x_2, \xi_2, x_3, \xi_3, z') = \sum F_{i,j}(x_1, \xi_1, \xi_2, \xi_3, z') x_2^i x_3^j.$$

We then decompose equation (5) according to the powers of $x_2^i x_3^j$ into a system of equations $(E_{i,j})$, $i+j \leqslant N$. We first solve equations $(E_{i,j})$, $i+j \leqslant N-1$ recursively by increasing the values of $i+j$:

$$(E_{i,j})\ \{U_{i,j}, x_1\xi_1\} = -V_{i,j} x_1 \xi_1 + \rho_{i,j} + V_{i-2,j} + V_{i,j-2}$$

by choosing $V_{i,j}$ so that there is no *resonant term* (powers of $x_1\xi_1$) in the right-hand side.

Then we are left with the following system:

$$\begin{cases} (E_{0,N}) & -2\frac{\partial W_{0,N-1}}{\partial \xi_3} + \tau & = \rho_{0,N} + V_{0,N-2} \\ (E_{1,N-1}) & -2\frac{\partial W_{0,N-1}}{\partial \xi_2} - 2\frac{\partial W_{1,N-2}}{\partial \xi_3} & = \rho_{1,N-1} + V_{1,N-3} \\ \cdots & \cdots & = \cdots \\ (E_{N,0}) & -2\frac{\partial W_{N-1,0}}{\partial \xi_2} & = \rho_{N,0} + V_{N-2,0} \end{cases}$$

All equations involve only functions on Σ. We solve them recursively from the last. The first one defines τ. □

6.1.5. Matrix homological equation.

LEMMA 6. — *Let us consider the homological equation*

$$\{S, H_0\} + C^\star H_0 + H_0 C = R + T \tag{6}$$

where R (self-adjoint) is given and S (real-valued), B, T are the unknowns.

- *In the hyperbolic case $H_0 = H_{\text{hyp}}$, equation (6) can be solved with*

$$T = i\begin{pmatrix} 0 & t(x_3, \xi_2, \xi_3, z') \\ -t(x_3, \xi_2, \xi_3, z') & 0 \end{pmatrix}$$

with t real valued.

- *In the elliptic case* $H_0 = H_{\mathrm{ell}}$, *equation (6) can be solved with*

$$T = \begin{pmatrix} 0 & 0 \\ 0 & t(x_3, \xi_2, \xi_3, z') \end{pmatrix}$$

with t *real valued.*

Proof. — It is enough to choose T so that $R+T$ is an infinitesimal singular deformation and to take the term in ε^1 in the classical normal form result for a singular deformation $H_{\mathrm{hyp}} + \varepsilon(R+T) + O(\varepsilon^2)$. □

6.2. The normal form for the dispersion relation.

PROPOSITION 3. — *Assuming* $(\star)$ *and* $(\mathcal{H})$ *or* $(\mathcal{E})$ *(we are in the case where one pair of eigenvalues does not vanish), near any point* z_0 *of the singular set* Σ, *there exists a canonical transformation* χ, *a smooth function* $a(x_3, \sigma)$ *and an invertible positive germ* e *so that:*

- *In the hyperbolic case* $(\mathcal{H})$:

$$\det(H_{\mathrm{class}}) \circ \chi = e(z)\left(x_1\xi_1 - \left(x_2^2 + x_3^2(1 + x_3 a(x_3,\sigma))^2\right)\right),$$

where $z = (x_1, \xi_1, x_2, \xi_2, x_3, \xi_3, z' = (x', \xi'))$ *are canonical coordinates near* $0 \in T^\star\mathbb{R}^n$ *and* $\sigma \in \Sigma$.

- *In the elliptic case* $(\mathcal{E})$:

$$\det(H_{\mathrm{class}}) \circ \chi = e(z)\left(x_2 x_3\left(1 + x_3 a(x_3,\sigma)\right) - (x_1^2 + \xi_1^2)\right) + O_\Sigma(\infty).$$

The proof follows exactly the same lines as in [3].

6.3. The classical normal form.

Using the same tools as before and [3], we get

THEOREM 4. — *Assuming* $(\star)$ *and* $(\mathcal{H})$ *or* $(\mathcal{E})$ *(we are in the case where one pair of eigenvalues does not vanish), near any point* z_0 *of the singular set* Σ, *there exists a canonical transformation* χ, *a* $GL(2,\mathbb{C})$ *valued gauge transform* $J(z)$ *and a smooth real valued function* $a(x_3, \sigma)$ *so that:*

• *In the hyperbolic case* ($\mathcal{H}$):

$$J^\star(H_{\text{class}} \circ \chi)\, J = \begin{pmatrix} \xi_1 & x_2 + ix_3(1 + x_3 a(x_3,\sigma)) \\ x_2 - ix_3(1+x_3 a(x_3,\sigma)) & x_1 \end{pmatrix} \quad (= H_{\text{hyp}})$$

where $z = (x_1, \xi_1, x_2, \xi_2, x_3, \xi_3, z' = (x', \xi'))$ *are canonical coordinates near* $0 \in T^\star \mathbb{R}^n$.

• *In the elliptic case* ($\mathcal{E}$):

$$J^\star\,(H_{\text{class}} \circ \chi)\, J = \begin{pmatrix} x_2 & x_1 + i\xi_1 \\ x_1 - i\xi_1 & x_3(1 + x_3 a(x_3,\sigma)) \end{pmatrix} + O_\Sigma(\infty) \ (= H_{\text{ell}}).$$

6.4. The semi-classical normal form.

From the previous subsections, we deduce the following semi-classical normal forms

THEOREM 5.

- *In the hyperbolic case* ($\mathcal{H}$):

$$\begin{pmatrix} \frac{h}{i}\partial_{x_1} & x_2 + ix_3(1+x_3 a(x_3,\sigma)) + ih\gamma \\ x_2 - ix_3(1 + x_3 a(x_3,\sigma)) - ih\gamma & x_1 \end{pmatrix}$$

  where γ *is a self-adjoint pseudo-differential operator of order* 0 *whose Weyl-symbol is independent of* (x_1, ξ_1, x_2).

- *In the elliptic case* ($\mathcal{E}$):

$$\begin{pmatrix} x_2 & x_1 + h\partial_{x_1} \\ x_1 - h\partial_{x_1} & x_3(1 + x_3 a(x_3,\sigma)) + h\gamma \end{pmatrix} + O_\Sigma(\infty)$$

where γ *is a self-adjoint pseudo-differential operator of order* 0 *whose Weyl-symbol is independent of* (x_1, ξ_1, x_2).

The microlocal solutions of the previous models can be studied following the same lines as in [3]. The main property is that they look like:

- $$\begin{pmatrix} \frac{h}{i}\frac{\partial}{\partial x_1} & Q \\ Q^\star & x_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

  where Q commutes with x_1 and $\frac{\partial}{\partial x_1}$ in the hyperbolic case

- $$\begin{pmatrix} Q & x_1 + h\frac{\partial}{\partial x_1} \\ x_1 - h\frac{\partial}{\partial x_1} & R \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$
  where Q and R commute with x_1 and $\frac{\partial}{\partial x_1}$ in the elliptic case.

7. One dimensional systems with parameters.

7.1. Normal forms.

In this section we will consider the case of a system

$$\widehat{H(\lambda)}\vec{U} = 0$$

where $\widehat{H(\lambda)}$ is a $d \times d$ self-adjoint system in one variable x_1 and depending smoothly of an external parameter $\lambda \in \mathbb{R}^N$, $N \geqslant 2$. Usually λ contains some spectral parameter.

We will assume that

- $(x_1, \xi_1, \lambda) \to H_{\text{class}}(x_1, \xi_1, \lambda)$ satisfies the transversality hypothesis $(\star)$ of section 1 at $(0, 0, \lambda_0)$.
- $(x_1, \xi_1) \to \det(H_{\text{class}}(x_1, \xi_1, \lambda_0))$ admits at the origine a non degenerate critical point. We have two cases the *elliptic* one $(\mathcal{P}_{\mathcal{E}})$ and the *hyperbolic* one $(\mathcal{P}_{\mathcal{H}})$.

The hyperbolic case is strongly related to [5] (see also [19]) while the elliptic normal form has been introduced as a *model* in [7] and [8]. Using the previous methods, one can show the following:

THEOREM 6. — • *Elliptic case: near* $(0, 0, \lambda_0)$, *one can reduce the system using a* λ*-dependent gauge transform and FIO's to*

$$\begin{pmatrix} a_h(\lambda) & x_1 + i\widehat{\xi_1} \\ x_1 - i\widehat{\xi_1} & b_h(\lambda) \end{pmatrix} \vec{U} = 0.$$

- *Hyperbolic case: near* $(0, 0, \lambda_0)$, *one can reduce the system using* λ*-dependent gauge transform and FIO's to*

$$\begin{pmatrix} \widehat{\xi_1} & a_h(\lambda) \\ \bar{a}_h(\lambda) & x_1 \end{pmatrix} \vec{U} = 0.$$

The proof is as follows: first apply the isochoric Morse lemma [6] to the dispersion relation. The gauge transform is obtained from Lemma 2. We can then solve the homological equation by linearization of the classical normal from.

7.2. Solutions of the elliptic normal form.

For completness, we reproduce here the solution of the normal form in the elliptic case which is studied in [7] and [8].

We want to solve near $(0,0) \in T^\star\mathbb{R}$ the folllowing system:

$$\begin{cases} a_h u + (x_1 + h\partial_{x_1})v = 0 \\ (x_1 - h\partial_{x_1})u + b_h v = 0 \end{cases}$$

We get, using the notations of subsection 5.5:

- If $a_h = 0,\ b_h \neq 0$, no admissible solution
- If $b_h = 0,\ u = 0,\ v = c\varphi_0$
- If $a_h b_h \neq 0$,
  - If $a_h b_h \neq 2(n+1)h,\ n \geqslant 0,\ n \in \mathbb{N}$, no admissible solution
  - If $a_h b_h = 2(n+1)h,\ n \geqslant 0,\ n \in \mathbb{N}$,

$$u = c\varphi_n,\ v = -\frac{c\sqrt{2(n+1)h}}{b_h}\varphi_{n+1}$$

Let us assume that $N = 2$ and $\lambda = (E,t)$ where E is a spectral parameter and $\lambda_0 = (E_0, t_0)$. We have $a_h = f_h(E,t),\ b_h = g_h(E,t)$ where $(E,t) \to (a_h, b_h)$ is a diffeomorphism. We assume $\frac{\partial f_h}{\partial E}\frac{\partial g_h}{\partial E} > 0$. Then we have a *macroscopic* (h-independant) gap in the spectrum for $t < t_0$ as well as for $t > t_0$, but we get that one eigenvalue is moving from one band to the next one as t passes through t_0 (see Figure 1).

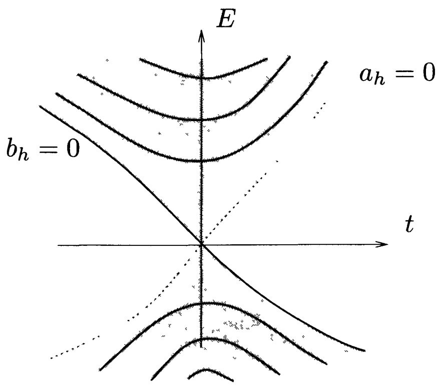

Figure 1: one eigenvalue is moving from the upper band to the lower one

7.3. Hyperbolic normal form and avoided crossings.

The hyperbolic case allows to recover the results of [5] (see also [15] and [19]) on the adiabatic limit. We consider a system:

$$\frac{h}{i}\frac{dX}{dt} = A(\lambda, t)X$$

where $A(\lambda, t)$ is an Hermitian matrix and $A(\lambda_0, t_0)$ admits an eigenvalue λ_0 of multiplicity 2. The previous results apply near the point (t_0, λ_0) of the phase space. We can recover that way a Landau-Zener formula.

Global computations including several crossings are presented in [4].

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Yves COLIN de VERDIÈRE,
Institut Fourier
Unité mixte de recherche CNRS-UJF 5582
BP 74
38402 Saint Martin d'Hères Cedex (France).
yves.colin-de-verdiere@ujf-grenoble.fr