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Anne BOUTET DE MONVEL & Dmitry SHEPELSKY

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INITIAL BOUNDARY VALUE PROBLEM FOR THE mKdV EQUATION ON A FINITE INTERVAL

by **Anne BOUTET de MONVEL and Dmitry SHEPELSKY**

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1. Introduction.

The general method that was announced in [4] for analyzing initial-boundary value problems for two-dimensional linear and integrable nonlinear PDEs and was further developed in [5, 6, 7, 10] is based on the simultaneous spectral analysis of the two eigenvalue equations of the associated Lax pair. It expresses the solution in terms of the solution of a matrix Riemann-Hilbert (RH) problem formulated in the complex plane of the spectral parameter. The spectral functions determining the RH problem are expressed in terms of the initial and boundary values of the solution. The fact that these initial and boundary values are in general related can be expressed in a simple way in terms of an algebraic relation satisfied by the corresponding spectral functions. This relation can be used to characterize a part of boundary values which, on the one hand, are involved in the construction of the solution of the boundary value problem but, on the other hand, are not given as boundary data for a well-posed boundary value problem [2].

The rigorous implementation of the method to the modified Korteweg-de Vries (mKdV) equation

$$q_t - q_{xxx} + 6\lambda q^2 q_x = 0, \quad \lambda = \pm 1 \tag{1.1}$$

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on the half-line $0 < x < \infty$ is presented in [1]. In the present paper, this methodology is applied to the mKdV equation on a finite interval, $0 < x < L$. The main result was announced in [3]. The similar problem for the nonlinear Schrödinger equation is studied in [9].

The problem we are dealing with is the initial-boundary value problem for the mKdV equation in the domain $\{0 < x < L,\ 0 < t < T\}$ with $L < \infty$, $T \leq \infty$. We follow the scheme below.

Step 1. Assuming that the solution of the mKdV equation, $q(x,t)$, exists, carry out the direct spectral analysis. For this purpose:

• Define appropriate solutions of (2.5) (eigenfunctions) analytic and bounded (in k) in domains forming a partition of the Riemann sphere $\overline{C} = C \cup \{\infty\}$.

• Define spectral functions $s(k)$, $S(k)$, and $S_1(k)$ such that:

1. $s(k)$ is determined by the initial conditions $q(x,0) = q_0(x)$, $0 < x < L$.
2. $S(k)$ is determined by the boundary values $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$, $q_{xx}(0,t) = g_2(t)$, $0 < t < T$.
3. $S_1(k)$ is determined by the boundary values $q(L,t) = f_0(t)$, $q_x(L,t) = f_1(t)$, $q_{xx}(L,t) = f_2(t)$, $0 < t < T$.

• Show that the spectral functions satisfy an algebraic relation, called *global relation*, expressing the fact that $q_0(x)$, $g_0(t)$, $g_1(t)$, $g_2(t)$, $f_0(t)$, $f_1(t)$, and $f_2(t)$ being the initial and boundary values for the mKdV equation, cannot be chosen arbitrary.

Step 2. Given $s(k)$, $S(k)$, and $S_1(k)$, determine a regular Riemann-Hilbert problem, the solution of which gives a solution of the mKdV equation.

Step 3. Assuming that $\{g_j(t)\}_{j=0}^2$ and $\{f_j(t)\}_{j=0}^2$ are such that the associated $S(k)$ and $S_1(k)$ together with $s(k)$ satisfy the "global relation", prove that the function $q(x,t)$ obtained from the solution of the RH problem solves the initial boundary value problem for the mKdV equation with initial data $q(x,0) = q_0(x)$ and boundary data $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$, $q_{xx}(0,t) = g_2(t)$, $q(L,t) = f_0(t)$, $q_x(L,t) = f_1(t)$, $q_{xx}(L,t) = f_2(t)$.

In Section 2 we define appropriate eigenfunctions of the associated Lax pair and spectral functions and study their properties. In particular, we show that the spectral functions satisfy an algebraic relation, the "global relation". In Section 3, we express the solution of the initial boundary value problem for the mKdV equation in terms of a matrix-valued Riemann-Hilbert (RH) problem. We show that the solution of this RH problem gives the solution of the mKdV equation with prescribed initial and boundary values provided that the spectral functions satisfy the "global relation".

2. Eigenfunctions and spectral functions.

2.1. Lax pair

The modified Korteweg-de Vries equation

$$q_t - q_{xxx} + 6\lambda q^2 q_x = 0, \quad \lambda = \pm 1 \tag{1.1}$$

is the compatibility condition for the Lax pair

$$\psi_x - \mathrm{i}k\sigma_3\psi = Q(x,t)\psi, \tag{2.1a}$$

$$\psi_t + 4\mathrm{i}k^3\sigma_3\psi = \tilde{Q}(x,t,k)\psi \tag{2.1b}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \lambda q(x,t) & 0 \end{pmatrix}, \tag{2.2}$$

(2.3)
$$\tilde{Q}(x,t,k) = \begin{pmatrix} -2\mathrm{i}\lambda q^2 k & -4qk^2+2\mathrm{i}q_x k-2\lambda q^3+q_{xx} \\ \lambda\left(-4qk^2-2\mathrm{i}q_x k-2\lambda q^3+q_{xx}\right) & 2\mathrm{i}\lambda q^2 k \end{pmatrix}.$$

We denote $\hat{\sigma}_3$ the commutation operation $\operatorname{ad}\sigma_3$:

$$\hat{\sigma}_3 A := [\sigma_3, A] = \sigma_3 A - A\sigma_3$$

for any 2×2 matrix A. The Lax pair (2.1) can be rewritten as

$$\mu_x - \mathrm{i}k\hat{\sigma}_3\mu = Q(x,t)\mu, \tag{2.4a}$$

$$\mu_t + 4\mathrm{i}k^3\hat{\sigma}_3\mu = \tilde{Q}(x,t,k)\mu, \tag{2.4b}$$

where

$$\mu := \psi e^{\mathrm{i}(-kx+4k^3t)\sigma_3}.$$

In turn, (2.4) can be written as

$$\mathrm{d}\left(\mathrm{e}^{\mathrm{i}(-kx+4k^3t)\hat{\sigma}_3}\mu\right) = W, \tag{2.5}$$

where

$$\mathrm{e}^{\hat{\sigma}_3}A = \mathrm{e}^{\sigma_3}A\mathrm{e}^{-\sigma_3},$$

and W is the exact 1-form defined by

$$W(x,t,k) = \mathrm{e}^{\mathrm{i}(-kx+4k^3t)\hat{\sigma}_3}\big(Q\mu\mathrm{d}x + \tilde{Q}\mu\mathrm{d}t\big). \tag{2.6}$$

2.2. Eigenfunctions

Assume that there exists a smooth real-valued function $q(x,t)$ satisfying (1.1) in $\{0 < x < L, 0 < t < T\}$ (if $T = \infty$ then it is also assumed a sufficient decay of $q(x,t)$ as $t \to \infty$). Define $\mu_n(x,t,k)$, $n = 1,2,3,4$ as 2×2-matrix-valued solutions of the integral equations

$$\mu_n(x,t,k) = I + \int_{(x_n,t_n)}^{(x,t)} \mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat{\sigma}_3}W(y,\tau,k), \tag{2.7}$$

where $(x_1,t_1) = (0,T)$, $(x_2,t_2) = (0,0)$, $(x_3,t_3) = (L,0)$, $(x_4,t_4) = (L,T)$, and the paths of integration are chosen to be parallel to the x and t axes:

$$\begin{aligned}\mu_1(x,t,k) = I &+ \int_0^x \mathrm{e}^{\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_1)(y,t,k)\mathrm{d}y \\ &- \mathrm{e}^{\mathrm{i}kx\hat{\sigma}_3}\int_t^T \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat{\sigma}_3}(\tilde{Q}\mu_1)(0,\tau,k)\mathrm{d}\tau,\end{aligned} \tag{2.8a}$$

$$\begin{aligned}\mu_2(x,t,k) = I &+ \int_0^x \mathrm{e}^{\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_2)(y,t,k)\mathrm{d}y \\ &+ \mathrm{e}^{\mathrm{i}kx\hat{\sigma}_3}\int_0^t \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat{\sigma}_3}(\tilde{Q}\mu_2)(0,\tau,k)\mathrm{d}\tau,\end{aligned} \tag{2.8b}$$

$$\begin{aligned}\mu_3(x,t,k) = I &- \int_x^L \mathrm{e}^{\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_3)(y,t,k)\mathrm{d}y \\ &+ \mathrm{e}^{\mathrm{i}k(x-L)\hat{\sigma}_3}\int_0^t \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat{\sigma}_3}(\tilde{Q}\mu_3)(L,\tau,k)\mathrm{d}\tau,\end{aligned} \tag{2.8c}$$

$$\begin{aligned}\mu_4(x,t,k) = I &- \int_x^L \mathrm{e}^{\mathrm{i}k(x-y)\hat{\sigma}_3}(Q\mu_4)(y,t,k)\mathrm{d}y \\ &- \mathrm{e}^{\mathrm{i}k(x-L)\hat{\sigma}_3}\int_t^T \mathrm{e}^{-4\mathrm{i}k^3(t-\tau)\hat{\sigma}_3}(\tilde{Q}\mu_4)(L,\tau,k)\mathrm{d}\tau.\end{aligned} \tag{2.8d}$$

The domains where the exponentials are bounded, are separated by the three lines L_0, L_1, L_2 where $\operatorname{Im} k^3$ vanishes: $L_0 \cup L_1 \cup L_2 = \{k \in C \mid \operatorname{Im} k^3 = 0\}$. The relevant domains in the k-plane are labelled $\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI}$, see Fig. 1.

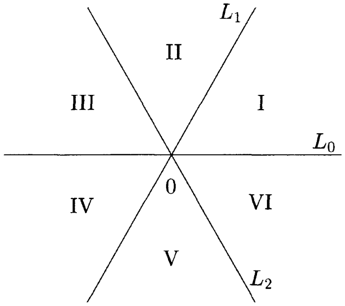

Figure 1: Domains of analyticity and boundedness of eigenfunctions

Let us denote $\mu^{(1)}$, $\mu^{(2)}$ the columns of a 2×2 matrix $\mu = (\,\mu^{(1)} \quad \mu^{(2)}\,)$. Then the columns of μ_n are analytic and bounded in the following domains in the complex k-plane (where the exponentials involved in the corresponding integral equations are bounded):

eigenfunctions	domains of analyticity and boundedness
$\mu_1^{(1)}$ and $\mu_3^{(2)}$	$\{k \mid \operatorname{Im} k \leq 0,\ \operatorname{Im} k^3 \leq 0\} = \mathrm{IV} \cup \mathrm{VI}$,
$\mu_1^{(2)}$ and $\mu_3^{(1)}$	$\{k \mid \operatorname{Im} k \geq 0,\ \operatorname{Im} k^3 \geq 0\} = \mathrm{I} \cup \mathrm{III}$,
$\mu_2^{(1)}$ and $\mu_4^{(2)}$	$\{k \mid \operatorname{Im} k \leq 0,\ \operatorname{Im} k^3 \geq 0\} = \mathrm{V}$,
$\mu_2^{(2)}$ and $\mu_4^{(1)}$	$\{k \mid \operatorname{Im} k \geq 0,\ \operatorname{Im} k^3 \leq 0\} = \mathrm{II}$.

Thus, in each domain $\mathrm{I}, \ldots, \mathrm{VI}$, one has an analytic and bounded 2×2 matrix-valued eigenfunction, consisting of the appropriate vectors $\mu_n^{(j)}$, $n = 1, 2, 3, 4$, $j = 1, 2$.

For particular values of x or t, the domains of boundedness of the eigenfunctions are larger than indicated above. In particular, for $t = 0$, the domains of boundedness of μ_2 and μ_3 are:

eigenfunctions	domains of boundedness
$\mu_2^{(1)}(x,0,k)$ and $\mu_3^{(2)}(x,0,k)$	$\{k \mid \operatorname{Im} k \leq 0\} = \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI}$,
$\mu_2^{(2)}(x,0,k)$ and $\mu_3^{(1)}(x,0,k)$	$\{k \mid \operatorname{Im} k \geq 0\} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$,

for $x = 0$, the domains of boundedness of μ_1 and μ_2 are:

eigenfunctions	domains of boundedness
$\mu_1^{(1)}(0,t,k)$ and $\mu_2^{(2)}(0,t,k)$	$\{k \mid \operatorname{Im} k^3 \leq 0\} = \mathrm{II} \cup \mathrm{IV} \cup \mathrm{VI}$,
$\mu_1^{(2)}(0,t,k)$ and $\mu_2^{(1)}(0,t,k)$	$\{k \mid \operatorname{Im} k^3 \geq 0\} = \mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$,

for $x = L$, the domains of boundedness of μ_3 and μ_4 are:

eigenfunctions	domains of boundedness
$\mu_3^{(2)}(L,t,k)$ and $\mu_4^{(1)}(L,t,k)$	$\{k \mid \operatorname{Im} k^3 \leq 0\} = \mathrm{II} \cup \mathrm{IV} \cup \mathrm{VI}$,
$\mu_3^{(1)}(L,t,k)$ and $\mu_4^{(2)}(L,t,k)$	$\{k \mid \operatorname{Im} k^3 \geq 0\} = \mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$.

2.3. Spectral functions

Since the eigenfunctions μ_j are solutions of the system of differential equations (2.4), they are simply related (in the domains where they are defined):

$$\mu_3(x,t,k) = \mu_2(x,t,k)\mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat\sigma_3}\mu_3(0,0,k), \tag{2.9a}$$

$$\begin{aligned}\mu_1(x,t,k) &= \mu_2(x,t,k)\mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat\sigma_3}\mu_1(0,0,k)\\ &= \mu_2(x,t,k)\mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat\sigma_3}\left(\mathrm{e}^{4\mathrm{i}k^3T\hat\sigma_3}\mu_2(0,T,k)\right)^{-1},\end{aligned} \tag{2.9b}$$

$$\begin{aligned}\mu_4(x,t,k) &= \mu_3(x,t,k)\mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat\sigma_3}\mathrm{e}^{-\mathrm{i}kL\hat\sigma_3}\mu_4(L,0,k)\\ &= \mu_3(x,t,k)\mathrm{e}^{\mathrm{i}(kx-4k^3t)\hat\sigma_3}\mathrm{e}^{-\mathrm{i}kL\hat\sigma_3}\left(\mathrm{e}^{4\mathrm{i}k^3T\hat\sigma_3}\mu_3(L,T,k)\right)^{-1}.\end{aligned} \tag{2.9c}$$

Relations (2.9) suggest the introduction of the 2×2-matrix-valued spectral functions

$$s(k) := \mu_3(0,0,k), \tag{2.10a}$$

$$S(k) = S(k;T) := \mu_1(0,0,k) = \left(\mathrm{e}^{4\mathrm{i}k^3T\hat\sigma_3}\mu_2(0,T,k)\right)^{-1}, \tag{2.10b}$$

$$S_1(k) = S_1(k;T) := \mu_4(L,0,k) = \left(\mathrm{e}^{4\mathrm{i}k^3T\hat\sigma_3}\mu_3(L,T,k)\right)^{-1}. \tag{2.10c}$$

From (2.9) and (2.10) it follows that $s(k)$ is determined by the initial values of $q(x,t)$, whereas $S(k)$ and $S_1(k)$ are determined by the boundary values at $x = 0$ and $x = L$. Namely,

$$s(k) = I - \int_0^L \mathrm{e}^{-\mathrm{i}ky\hat\sigma_3}(Q\mu_3)(y,0,k)\mathrm{d}y, \tag{2.11}$$

where $\mu_3(x,0,k)$, $0<x<L$ is the solution of the integral equation

$$\mu_3(x,0,k) = I - \int_x^L e^{ik(x-y)\hat\sigma_3}(Q\mu_3)(y,0,k)dy. \tag{2.12}$$

Similarly,

$$S(k;T) = \left(I + \int_0^T e^{4ik^3\tau\hat\sigma_3}(\tilde Q\mu_2)(0,\tau,k)d\tau\right)^{-1}, \tag{2.13a}$$

$$S_1(k;T) = \left(I + \int_0^T e^{4ik^3\tau\hat\sigma_3}(\tilde Q\mu_3)(L,\tau,k)d\tau\right)^{-1}, \tag{2.13b}$$

where $\mu_2(0,t,k)$ and $\mu_3(L,t,k)$, $0<t<T$ are the solutions of the integral equations

$$\mu_2(0,t,k) = I + \int_0^t e^{-4ik^3(t-\tau)\hat\sigma_3}(\tilde Q\mu_2)(0,\tau,k)d\tau, \tag{2.14a}$$

$$\mu_3(L,t,k) = I + \int_0^t e^{-4ik^3(t-\tau)\hat\sigma_3}(\tilde Q\mu_3)(L,\tau,k)d\tau, \tag{2.14b}$$

respectively. Note that $Q(x,0)$ is determined by $q(x,0)$, whereas $\tilde Q(0,t,k)$ is determined by $q(0,t)$, $q_x(0,t)$, and $q_{xx}(0,t)$, and $\tilde Q(L,t,k)$ is determined by $q(L,t)$, $q_x(L,t)$, and $q_{xx}(L,t)$.

In what follows, $s(k)$, $S(k)$, and $S_1(k)$ will be used to construct a Riemann-Hilbert problem (more precisely, a family of Riemann-Hilbert problems parametrized by x and t), whose solution gives the eigenfunctions μ_n and hence $q(x,t)$, the solution of the mKdV problem.

2.4. Symmetry properties

Due to the particular symmetries of the equations of the Lax pair (2.1), the spectral matrices $s(k)$, $S(k)$, and $S_1(k)$ can be written as

$$s(k) = \begin{pmatrix} \overline{a(\bar k)} & b(k) \\ \lambda\overline{b(\bar k)} & a(k) \end{pmatrix}, \tag{2.15a}$$

$$S(k) = \begin{pmatrix} \overline{A(\bar k)} & B(k) \\ \lambda\overline{B(\bar k)} & A(k) \end{pmatrix}, \tag{2.15b}$$

$$S_1(k) = \begin{pmatrix} \overline{A_1(\bar k)} & B_1(k) \\ \lambda\overline{B_1(\bar k)} & A_1(k) \end{pmatrix}. \tag{2.15c}$$

2.5. Global relation

Evaluating eqs. (2.9c) and (2.9a) at $x = 0$, $t = T$, we find

$$\mu_4(0,T,k) = \mu_3(0,T,k)\mathrm{e}^{-\mathrm{i}kL\hat\sigma_3}\mathrm{e}^{-4\mathrm{i}k^3T\hat\sigma_3}\mu_4(L,0,k), \tag{2.16a}$$

$$\mu_3(0,T,k) = \mu_2(0,T,k)\mathrm{e}^{-4\mathrm{i}k^3T\hat\sigma_3}\mu_3(0,0,k). \tag{2.16b}$$

Writing $\mu_3(0,0,k)$, $\mu_2(0,T,k)$, and $\mu_4(L,0,k)$ in terms of $s(k)$, $S(k)$, and $S_1(k)$, respectively (see equations (2.10)), and using eq. (2.8d) to evaluate $\mu_4(0,T,k)$, eq. (2.16a) becomes

$$\begin{aligned} S^{-1}(k)s(k)\big[\mathrm{e}^{-\mathrm{i}kL\hat\sigma_3}S_1(k)\big] &= \mathrm{e}^{4\mathrm{i}k^3T\hat\sigma_3}\mu_4(0,T,k) \\ &= I - \mathrm{e}^{4\mathrm{i}k^3T\hat\sigma_3}\int_0^L \mathrm{e}^{-\mathrm{i}ky\hat\sigma_3}(Q\mu_4)(y,T,k)\mathrm{d}y. \end{aligned} \tag{2.17}$$

• For $T < \infty$, the $(1,2)$ coefficient of (2.17) is, for $k \in C$,

$$\begin{aligned} \mathrm{e}^{-2\mathrm{i}kL}\big(\overline{a(\bar k)}A(k) - \lambda\overline{b(\bar k)}B(k)\big)B_1(k) - \big(a(k)B(k) \\ -b(k)A(k)\big)A_1(k) = \mathrm{e}^{8\mathrm{i}k^3T}c(k), \end{aligned} \tag{2.18}$$

where

$$c(k) = c(k;T) = \int_0^L \mathrm{e}^{-2\mathrm{i}ky}(Q\mu_4)_{12}(y,T,k)\mathrm{d}y$$

is an entire function which is $\mathrm{O}\big((1+\mathrm{e}^{-2\mathrm{i}kL})/k\big)$ as $k\to\infty$.

• For $T = \infty$, the $(1,2)$ coefficient of (2.17) becomes

$$\begin{aligned} \mathrm{e}^{-2\mathrm{i}kL}\big(\overline{a(\bar k)}A(k) - \lambda\overline{b(\bar k)}B(k)\big)B_1(k) \\ -\big(a(k)B(k) - b(k)A(k)\big)A_1(k) = 0, \end{aligned} \tag{2.19}$$

which is valid for $k \in \mathrm{I}\cup\mathrm{III}\cup\mathrm{V}$.

Equation (2.18) for $T < \infty$, resp. (2.19) for $T = \infty$, is an algebraic relation between the spectral functions. We call it “global relation”, because it express, in spectral terms, the relations between the initial and boundary values of a solution of the mKdV equation.

2.6. Direct and inverse spectral maps

• **The spectral maps $\mathcal{S}$, $\mathcal{Q}$.** The direct spectral map

$$\mathcal{S}\colon \{q_0(x)\} \longmapsto \{a(k), b(k)\} \tag{2.20}$$

is defined following (2.10a), (2.15a), by the second column of the solution $\mu_3(x,0,k)$ of equation (2.12) where $Q = Q(x,0)$ is given by (2.2) with $q_0(x)$ instead of $q(x,t)$:

$$Q = \begin{pmatrix} 0 & q_0(x) \\ \lambda q_0(x) & 0 \end{pmatrix}.$$

Thus, this direct spectral map is the same as in the inverse scattering theory on the whole line for the Dirac equation

$$\psi_x - \mathrm{i}k\sigma_3\psi = \begin{pmatrix} 0 & q_0(x) \\ \lambda q_0(x) & 0 \end{pmatrix}\psi$$

restricted to the potentials with support on the finite interval $[0, L]$. The analysis of the linear Volterra integral equation (2.12) gives the following properties of $a(k)$ and $b(k)$:

- $a(k)$ and $b(k)$ are entire functions, bounded for $\operatorname{Im} k \leq 0$;
- they have the asymptotic behavior, as $k \to \infty$:

$$(2.21) \qquad a(k) = 1 + \mathrm{O}\left(\frac{1 + \mathrm{e}^{-2\mathrm{i}kL}}{k}\right), \qquad b(k) = \mathrm{O}\left(\frac{1 + \mathrm{e}^{-2\mathrm{i}kL}}{k}\right).$$

The fact that Q has zero trace together with the behavior as $k \to \infty$ given above imply that $\det s(k) = 1$ which, in terms of $a(k)$ and $b(k)$, reads

- $a(k)\overline{a(\bar{k})} - \lambda b(k)\overline{b(\bar{k})} = 1$, $k \in C$;
- $a(-k) = \overline{a(\bar{k})}$, $\quad b(-k) = \overline{b(\bar{k})}$.

Define a sectionally holomorphic, matrix-valued function $M^{(x)}(x, k)$:

$$(2.22) \quad M^{(x)}(x,k) = \begin{cases} \begin{pmatrix} \mu_3^{(1)}(x,0,k) & \dfrac{\mu_2^{(2)}(x,0,k)}{\overline{a(\bar{k})}} \end{pmatrix} & |k| > R \ \operatorname{Im} k > 0, \\ \begin{pmatrix} \dfrac{\mu_2^{(1)}(x,0,k)}{a(k)} & \mu_3^{(2)}(x,0,k) \end{pmatrix} & |k| > R, \ \operatorname{Im} k < 0, \\ \mu_2(x,0,k) & |k| < R, \end{cases}$$

where R is chosen to be sufficiently large such that the disk $\{k \in C \mid |k| < R\}$ contains all the zeros of $a(k)$ with $\operatorname{Im} k \leq 0$.

Let $\Sigma^{(x)}$ be the oriented contour in the complex k-plane consisting of the real axis and the circle $\Gamma_R = \{k \mid |k| = R\}$ of radius R, and let $\Omega_\pm^{(x)}$ be the sectors bounded by $\Sigma^{(x)}$, see Fig. 2.

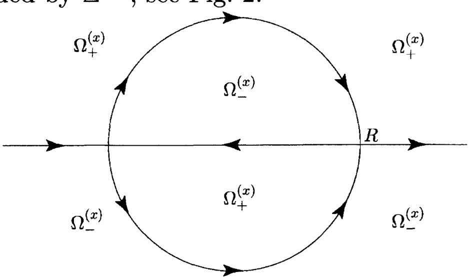

Figure 2: The contour $\Sigma^{(x)}$

Similarly to $\mu_3(x,0,k)$, $\mu_2(x,0,k)$ in (2.22) is determined by (2.8b) for $t=0$, where $q(x,0)$ in $Q(x,0)$ is replaced by $q_0(x)$. Then the limits $M_\pm^{(x)}(x,\zeta)$ of $M^{(x)}(x,k)$ as k approaches $\zeta\in\Sigma^{(x)}$ from $\Omega_\pm^{(x)}$ are related by the jump matrix $J^{(x)}(x,\zeta)$:

$$M_-^{(x)}(x,\zeta)=M_+^{(x)}(x,\zeta)J^{(x)}(x,\zeta),\quad \zeta\in\Sigma^{(x)}, \tag{2.23}$$

where

$$J^{(x)}(x,k)=\begin{cases}\begin{pmatrix}1 & -\dfrac{b(k)}{\bar a(k)}\,\mathrm{e}^{2\mathrm{i}kx}\\ \lambda\dfrac{\bar b(k)}{a(k)}\,\mathrm{e}^{-2\mathrm{i}kx} & 1-\lambda\left|\dfrac{b(k)}{\bar a(k)}\right|^2\end{pmatrix} & |k|>R,\ \operatorname{Im}k=0,\\ I & |k|<R,\ \operatorname{Im}k=0,\\ \begin{pmatrix}\overline{a(\bar k)} & 0\\ \lambda\overline{b(\bar k)}\,\mathrm{e}^{-2\mathrm{i}kx} & \dfrac{1}{\overline{a(\bar k)}}\end{pmatrix} & |k|=R,\ \operatorname{Im}k>0,\\ \begin{pmatrix}a(k) & -b(k)\,\mathrm{e}^{2\mathrm{i}kx}\\ 0 & \dfrac{1}{a(k)}\end{pmatrix} & |k|=R,\ \operatorname{Im}k<0.\end{cases} \tag{2.24}$$

The inverse spectral map

$$\mathcal{Q}\colon \{a(k),b(k)\}\longmapsto\{q_0(x)\}$$

is defined as

$$q_0(x)=-2\mathrm{i}\lim_{k\to\infty}k\big(M^{(x)}(x,k)\big)_{12}, \tag{2.25}$$

where $M^{(x)}(x,k)$ is the solution of the following RH problem:

- $M^{(x)}(x,k)$ is a sectionally holomorphic function relative to the contour $\Sigma^{(x)}$.

- The limits $M_\pm^{(x)}(x,\zeta)$ of $M^{(x)}(x,k)$ as k approaches $\zeta\in\Sigma^{(x)}$ from $\Omega_\pm$ are related by (2.23), where the jump matrix $J^{(x)}(x,\zeta)$ is constructed from $a(k)$ and $b(k)$ following (2.24).

- $M^{(x)}(x,k)=I+\mathrm{O}\left(\frac{1}{k}\right),\quad k\to\infty.$

Remark 1. — In [1] the corresponding inverse spectral map was defined via the solution of a *singular* RH problem, where $M^{(x)}(x,k)$ can have poles (at possible zeros of a and $\bar a$). In that case, residue relations were also added to the formulation of the RH problem. Here we give a formulation of the inverse map which is based on the solution of a *regular* RH problem relative to a contour containing the circle Γ_R as additional part (see [13]). In such a formulation, we do not need any assumption on the zeros of $a(k)$. Notice that the cyclic conditions for the jump matrices at $k = R$ and $k = -R$, the intersection points of $\Sigma^{(x)}$, which are necessary for the solvability of the Riemann-Hilbert problem, are automatically satisfied by the construction of $J^{(x)}$.

• **The spectral maps $\tilde{\mathcal{S}}$, $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{S}}_1$, $\tilde{\mathcal{Q}}_1$.** The direct spectral map

$$\tilde{\mathcal{S}}\colon \{g_0(t), g_1(t), g_2(t)\} \longmapsto \{A(k), B(k)\} \tag{2.26}$$

is defined following (2.10b), (2.15b), by the second column of the solution $\mu_2(0,t,k)$ of equation (2.14a) where $\tilde{Q} = \tilde{Q}(0,t,k)$ is determined by (2.3) with q, q_x, and q_{xx} replaced by $g_0(t)$, $g_1(t)$, and $g_2(t)$, respectively:

$$\begin{pmatrix} -\mathrm{e}^{-8ik^3T}B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \mu_2^{(2)}(0,T,k). \tag{2.27}$$

Similarly, the direct spectral map

$$\tilde{\mathcal{S}}_1\colon \{f_0(t), f_1(t), f_2(t)\} \longmapsto \{A_1(k), B_1(k)\} \tag{2.28}$$

is defined following (2.10c), (2.15c), by the second column of the solution $\mu_3(L,t,k)$ of equation (2.14b) where $\tilde{Q} = \tilde{Q}(L,t,k)$ is determined by (2.3) with q, q_x, and q_{xx} replaced by $f_0(t)$, $f_1(t)$, and $f_2(t)$, respectively:

$$\begin{pmatrix} -\mathrm{e}^{-8ik^3T}B_1(k) \\ \overline{A_1(\bar{k})} \end{pmatrix} = \mu_3^{(2)}(L,T,k). \tag{2.29}$$

The properties of A and B are discussed in [1], and A_1 and B_1 have the same properties. In particular:

• $A(-k) = \overline{A(\bar{k})}$, $\quad B(-k) = \overline{B(\bar{k})}$.

• If $T < \infty$, $A(k)$ and $B(k)$ are entire functions, bounded in $\mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$. If $T = \infty$, $A(k)$ and $B(k)$ are only defined in $\mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$, being analytic and bounded there.

• $A(k)$ and $B(k)$ have the asymptotic behavior, as $k \to \infty$:

$$A(k) = 1 + \mathrm{O}\Big(\frac{1+\mathrm{e}^{8ik^3T}}{k}\Big), \quad B(k) = \mathrm{O}\Big(\frac{1+\mathrm{e}^{8ik^3T}}{k}\Big). \tag{2.30}$$

• $A(k)$ and $B(k)$ are related by the relation

$$A(k)\overline{A(\bar{k})} - \lambda B(k)\overline{B(\bar{k})} = 1, \; k \in C \text{ (with } \operatorname{Im} k^3{=}0 \text{ if } T{=}\infty). \tag{2.31}$$

Define a sectionally holomorphic, matrix-valued function

$$(2.32)\quad M^{(t)}(t,k)=\begin{cases}\begin{pmatrix}\mu_1^{(1)}(0,t,k) & \dfrac{\mu_2^{(2)}(0,t,k)}{\overline{A(\bar{k})}}\end{pmatrix} & |k|>R,\ \operatorname{Im}k^3<0,\\ \begin{pmatrix}\dfrac{\mu_2^{(1)}(0,t,k)}{A(k)} & \mu_1^{(2)}(0,t,k)\end{pmatrix} & |k|>R,\ \operatorname{Im}k^3>0,\\ \mu_2(0,t,k) & |k|<R,\end{cases}$$

where R is again chosen to be sufficiently large such that the disk $\{k\in C \mid |k|<R\}$ contains all the zeros of $A(k)$ with $\operatorname{Im}k^3\geq 0$.

Let Σ be the oriented contour in the complex k-plane consisting of the lines L_0, L_1, L_2 and the circle $\Gamma_R=\{k\mid |k|=R\}$ of radius R, and let $\Omega_\pm$ be the sectors bounded by Σ, see Fig. 3.

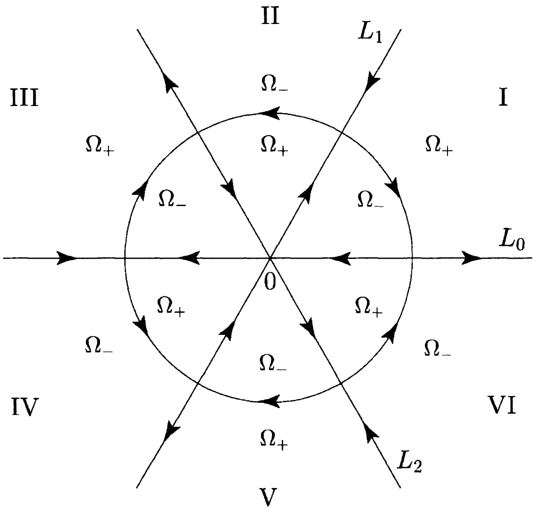

Figure 3: The contour Σ

Similarly to $\mu_2(0,t,k)$, $\mu_1(0,t,k)$ in (2.32) is determined by (2.8a) for $x=0$, where $q(0,t)$, $q_x(0,t)$, and $q_{xx}(0,t)$ in $\tilde{Q}(0,t,k)$ are replaced by $g_0(t)$, $g_1(t)$, and $g_2(t)$, respectively. Then the limits $M_\pm^{(t)}(t,\zeta)$ of $M^{(t)}(t,k)$ as k approaches $\zeta\in\Sigma$ from the domains $\Omega_\pm$ are related by the jump matrix $J^{(t)}(t,\zeta)$:

$$(2.33)\qquad M_-^{(t)}(t,\zeta)=M_+^{(t)}(t,\zeta)J^{(t)}(t,\zeta),\quad \zeta\in\Sigma,$$

where

(2.34)

$$J^{(t)}(t,k)=\begin{cases}\begin{pmatrix}1 & -\dfrac{B(k)}{\overline{A(\bar k)}}\,\mathrm{e}^{-8\mathrm{i}k^3t}\\ \dfrac{\lambda\overline{B(\bar k)}}{A(k)}\,\mathrm{e}^{8\mathrm{i}k^3t} & \dfrac{1}{A(k)\overline{A(\bar k)}}\end{pmatrix} & \operatorname{Im}k^3=0,\ |k|>R\\ I & \operatorname{Im}k^3=0,\ |k|<R\\ \begin{pmatrix}A(k) & -B(k)\,\mathrm{e}^{-8\mathrm{i}k^3t}\\ 0 & \dfrac{1}{A(k)}\end{pmatrix} & \operatorname{Im}k^3>0,\ |k|=R\\ \begin{pmatrix}\overline{A(\bar k)} & 0\\ \lambda\overline{B(\bar k)}\,\mathrm{e}^{8\mathrm{i}k^3t} & \dfrac{1}{\overline{A(\bar k)}}\end{pmatrix} & \operatorname{Im}k^3<0,\ |k|=R.\end{cases}$$

The map

$$\tilde{\mathcal{Q}}\colon\{A(k),B(k)\}\longmapsto\{g_0(t),g_1(t),g_2(t)\}$$

inverse to $\tilde{\mathcal{S}}$ is defined as follows:

(2.35a) $g_0(t)=-2\mathrm{i}(M_1^{(t)})_{12}(t),$

(2.35b) $g_1(t)=4(M_2^{(t)})_{12}(t)-2\mathrm{i}g_0(t)(M_1^{(t)})_{22}(t),$

(2.35c) $g_2(t)=\lambda g_0^3(t)+8\mathrm{i}(M_3^{(t)})_{12}(t)+4g_0(t)(M_2^{(t)})_{22}(t)$
$\qquad\qquad -2\mathrm{i}g_1(t)(M_1^{(t)})_{22}(t),$

where $M_j^{(t)}(t,k)$, $j=1,2,3$ are given by the large k-asymptotic expansion

$$M^{(t)}(t,k)=I+\frac{M_1^{(t)}(t)}{k}+\frac{M_2^{(t)}(t)}{k^2}+\frac{M_3^{(t)}(t)}{k^3}+\mathrm{O}\Big(\frac{1}{k^4}\Big),\quad k\to\infty,$$

of the solution $M^{(t)}(t,k)$ of the following RH problem:

- $M^{(t)}(t,k)$ is a sectionally holomorphic function relative to the contour Σ.

- The limits $M_\pm^{(t)}(t,\zeta)$ of $M^{(t)}(t,k)$ as k approaches $\zeta\in\Sigma$ from $\Omega_\pm$ are related by (2.33), where the jump matrix $J^{(t)}(t,\zeta)$ is constructed from $A(k)$ and $B(k)$ following (2.34).

- $M^{(t)}(t,k)=I+\mathrm{O}\left(\frac{1}{k}\right),\quad k\to\infty.$

The spectral map

$$\tilde{\mathcal{Q}}_1\colon\{A_1(k),B_1(k)\}\longmapsto\{f_0(t),f_1(t),f_2(t)\},$$

inverse to $\tilde{\mathcal{S}}_1$ is defined via the solution of a similar RH problem, with g_k, A, B replaced by f_k, A_1, B_1.

Remark 2. — As in the case of the x-problem, the two inverse spectral t-problems described above are based on *regular* RH problems: the residue conditions at zeros of A in the *singular* RH problem [1] are replaced by additional jump conditions at the auxiliary part Γ_R of the contour. Here again, the cyclic conditions for the jump matrices at the points of self-intersection of Σ are satisfied by the construction of $J^{(t)}$.

3. The Riemann-Hilbert problem.

Relating the vector solutions of (2.4a)–(2.4b) (analytic in domains I, ..., VI) by using (2.9) and the definitions of the spectral functions (2.11)–(2.13), we find

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \Sigma, \tag{3.1}$$

where $M_\pm(x,t,k)$ are the limit values (as k approaches Σ from $\Omega_\pm$) of a sectionally holomorphic function $M(x,t,k)$ defined as follows:

$$M = \begin{cases} \left(\mu_3^{(1)} \quad \dfrac{\mu_1^{(2)}}{\overline{d(\bar{k})}}\right), & \text{for } |k|>R, \quad k \in \mathrm{I} \cup \mathrm{III}, \\ \left(\dfrac{\mu_4^{(1)}\overline{a(\bar{k})}}{\overline{d_1(\bar{k})}} \quad \dfrac{\mu_2^{(2)}}{\overline{a(\bar{k})}}\right), & \text{for } |k|>R, \quad k \in \mathrm{II}, \\ \left(\dfrac{\mu_1^{(1)}}{d(k)} \quad \mu_3^{(2)}\right), & \text{for } |k|>R, \quad k \in \mathrm{IV} \cup \mathrm{VI}, \\ \left(\dfrac{\mu_2^{(1)}}{a(k)} \quad \dfrac{\mu_4^{(2)}a(k)}{d_1(k)}\right), & \text{for } |k|>R, \quad k \in \mathrm{V}, \\ \mu_2, & \text{for } |k|<R, \end{cases} \tag{3.2}$$

where

$$d(k) = a(k)\overline{A(\bar{k})} - \lambda b(k)\overline{B(\bar{k})}, \qquad k \in \mathrm{II} \cup \mathrm{IV} \cup \mathrm{VI}, \tag{3.3a}$$

$$d_1(k) = a(k)A_1(k) + \lambda\, \mathrm{e}^{-2\mathrm{i}kL}\overline{b(\bar{k})}B_1(k), \quad k \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{V}. \tag{3.3b}$$

The jump matrix $J(x,t,k)$ has an explicit (x,t)-dependence:

$$J(x,t,k) = \mathrm{e}^{(\mathrm{i}kx - 4\mathrm{i}k^3t)\hat{\sigma}_3} J_0(k), \tag{3.4}$$

where $J_0(k)$ is constructed from the elements of the spectral functions:

(3.5)

• for $k\in\Sigma$, $\operatorname{Im}k\geq 0$, $|k|>R$,

$$J_0(k)=\begin{cases}\begin{pmatrix}1 & -\lambda\overline{\Gamma(\bar k)}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \lambda\overline{\Gamma_1(\bar k)} & 1\end{pmatrix}, & \arg k=\frac{\pi}{3},\ \frac{2\pi}{3},\\ \begin{pmatrix}1 & -\lambda\overline{\Gamma(\bar k)}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1-\lambda|\gamma(k)|^2 & \gamma(k)\\ -\lambda\bar\gamma(k) & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \Gamma(k) & 1\end{pmatrix}, & \arg k=0,\ \pi,\end{cases}$$

• for $k\in\Sigma$, $\operatorname{Im}k\geq 0$, $|k|=R$,

$$J_0(k)=\begin{cases}\begin{pmatrix}\frac{A(k)}{\overline{d(\bar k)}} & -\frac{B(k)}{\overline{d(\bar k)}}\\ -\lambda\overline{b(\bar k)} & \overline{a(\bar k)}\end{pmatrix}, & \arg k\in\left(0,\frac{\pi}{3}\right)\cup\left(\frac{2\pi}{3},\pi\right),\\ \begin{pmatrix}\overline{a(\bar k)} & 0\\ \lambda\overline{\Gamma_2(\bar k)} & \frac{1}{\overline{a(\bar k)}}\end{pmatrix}, & \arg k\in\left(\frac{\pi}{3},\frac{2\pi}{3}\right),\end{cases}$$

• for $k\in\Sigma$, $\operatorname{Im}k<0$,

$$J_0(k)=\operatorname{diag}\{-1,\lambda\}J_0^*(\bar k)\operatorname{diag}\{-1,\lambda\},$$

• for $k\in\Sigma$, $|k|<R$.

$$J_0(k)=I.$$

Here

$$\gamma(k)=\frac{b(k)}{\bar a(k)},\tag{3.6a}$$

$$\Gamma(k)=\lambda\frac{\lambda\overline{B(\bar k)}}{a(k)d(k)},\tag{3.6b}$$

$$\Gamma_1(k)=\frac{\mathrm{e}^{-2ikL}a(k)B_1(k)}{d_1(k)},\tag{3.6c}$$

$$\Gamma_2(k)=\frac{a(k)}{d_1(k)}\left(\mathrm{e}^{-2ikL}\overline{a(\bar k)}B_1(k)+b(k)A_1(k)\right),\tag{3.6d}$$

and R is such that all zeros of $a(k)$, $d(k)$, and $d_1(k)$ with $\operatorname{Im}k\leq 0$ are in the disk $|k|<R$.

Remark 3. — Since

$$\Gamma(k) = \lambda \frac{\overline{B(\bar{k})}/\overline{A(\bar{k})}}{a(k)\left(a(k) - \lambda b(k)(\overline{B(\bar{k})}/\overline{A(\bar{k})})\right)}, \tag{3.7a}$$

$$\Gamma_1(k) = \frac{\mathrm{e}^{-2\mathrm{i}kL} a(k)(B_1(k)/A_1(k))}{a(k) + \lambda\, \mathrm{e}^{-2\mathrm{i}kL}\overline{b(\bar{k})}(B_1(k)/A_1(k))}, \tag{3.7b}$$

$$\Gamma_2(k) = a(k) \frac{\mathrm{e}^{-2\mathrm{i}kL}\overline{a(\bar{k})}(B_1(k)/A_1(k)) + b(k)}{a(k) + \lambda\, \mathrm{e}^{-2\mathrm{i}kL}\overline{b(\bar{k})}(B_1(k)/A_1(k))}, \tag{3.7c}$$

the jump data for the RH problem (3.4) are determined by:

- $a(k)$ and $b(k)$ for $k \in C$, $|k| \geq R$,
- $\dfrac{B(k)}{A(k)}$ and $\dfrac{B_1(k)}{A_1(k)}$ for $k \in \mathrm{I} \cup \mathrm{III} \cup \mathrm{V}$, $|k| \geq R$.

The main result on the inverse spectral problem is the following:

THEOREM. — *Let $q_0(x) \in C^\infty([0,L])$. Let $\{g_j(t)\}_{j=0}^2$ and $\{f_j(t)\}_{j=0}^2$ be smooth functions such that:*

- $(\partial_x)^j q_0(0) = g_j(0), \quad (\partial_x)^j q_0(L) = f_j(0), \quad j = 0,1,2,$
- *the associated spectral functions $s(k)$, $S(k)$, and $S_1(k)$ satisfy the global relation (2.18) for $T < \infty$, or (2.19) for $T = \infty$, where $c(k)$ is an entire function such that $c(k) = \mathrm{O}\left((1+\mathrm{e}^{-2\mathrm{i}kL})/k\right)$ as $|k| \to \infty$.*

Let $M(x,t,k)$ be a solution of the following 2×2 matrix RH problem:

- *M is sectionally holomorphic in $k \in C \setminus \Sigma$.*
- *At $k \in \Sigma$, M satisfies the jump conditions (3.1), where the jump matrix J is defined in terms of the spectral functions a, b, A, B, A_1 and B_1 by equations (3.4)–(3.6).*
- *As $k \to \infty$,*

$$M(x,t,k) = I + \mathrm{O}\Big(\frac{1}{k}\Big). \tag{3.8}$$

Then:

1. *$M(x,t,k)$ exists and is unique;*
2. *$q(x,t)$ defined in terms of $M(x,t,k)$ by*

$$q(x,t) = -2\mathrm{i} \lim_{k\to\infty} (kM(x,t,k))_{12} \tag{3.9}$$

satisfies the mKdV equation (1.1);

3. *$q(x,t)$ satisfies the initial and boundary conditions*

$$q(x,0)=q_0(x), \tag{3.10a}$$

$$q(0,t)=g_0(t),\quad q_x(0,t)=g_1(t),\quad q_{xx}(0,t)=g_2(t), \tag{3.10b}$$

$$q(L,t)=f_0(t),\quad q_x(L,t)=f_1(t),\quad q_{xx}(L,t)=f_2(t). \tag{3.10c}$$

Proof. — The proof follows the same lines as in the case of nonlinear integrable equations on the half-line. For the mKdV equation on the half-line, see [1]; for a more detailed presentation, see [10] in the case of the NLS equation. The main steps of the proof include the following.

Step 1. The RH problem in question is regular; its unique solvability is a consequence of a "vanishing lemma" for the associated RH problem with the vanishing condition at infinity $M=\mathrm{O}(1/k)$, $k\to\infty$ (see [12]).

Step 2. The proof that the constructed $q(x,t)$ solves the mKdV equation is straightforward and follows the proof in the case of the whole line problem; see [11].

Step 3. The proof that q satisfies the initial condition $q(x,0)=q_0(x)$ follows from the fact that it is possible to map the RH problem for $M(x,0,k)$ to that for $M^{(x)}(x,k)$ such that

$$M^{(x)}(x,k)=M(x,0,k)P^{(x)}(x,k) \tag{3.11}$$

where $P^{(x)}(x,k)$ is piecewise holomorphic relative to Σ and $P^{(x)}(x,k)=I+\mathrm{O}\left(\frac{1}{k}\right)$ as $k\to\infty$. For instance, for $\operatorname{Im}k>0$, one has

$$P^{(x)}(x,k)=\begin{cases}\begin{pmatrix}1 & -\lambda\overline{\Gamma(\bar{k})}\,\mathrm{e}^{2ikx}\\ 0 & 1\end{pmatrix}, & k\in\mathrm{I}\cup\mathrm{III},\ |k|>R,\\[2ex] \begin{pmatrix}1 & 0\\ -\lambda\overline{\Gamma_1(\bar{k})}\,\mathrm{e}^{-2ikx} & 1\end{pmatrix}, & k\in\mathrm{II},\ |k|>R.\end{cases} \tag{3.12}$$

Now (2.21), (2.30), and (3.7) imply that $\bar{\Gamma}$ is $\mathrm{O}\left(\frac{1}{k}\right)$ in I∪III and $\bar{\Gamma}_1\mathrm{e}^{-2ikx}=\mathrm{O}\left(\frac{1}{k}\right)\mathrm{e}^{2ik(L-x)}$ in II, respectively. Therefore, $P^{(x)}(x,k)=I+P^{(x)}_{\text{off}}(x,k)$, where $P^{(x)}_{\text{off}}(x,k)$ is off-diagonal and exponentially decaying as $k\to\infty$ for $\operatorname{Im}k\neq0$, so that the asymptotics (2.25) and (3.9) for $t=0$ yield $q_0(x)=q(x,0)$.

Step 4. The proof that q satisfies the boundary conditions $q(0,t)=g_0(t)$, $q_x(0,t)=g_1(t)$, $q_{xx}(0,t)=g_2(t)$ and $q(L,t)=f_0(t)$, $q_x(L,t)=f_1(t)$,

$q_{xx}(L,t) = f_2(t)$ is, in turn, based on the maps $M(0,t,k) \mapsto M^{(t)}(t,k)$ and $M(L,t,k) \mapsto M_1^{(t)}(t,k)$:

$$M^{(t)}(t,k) = M(0,t,k)P^{(t)}(t,k), \quad M_1^{(t)}(t,k) = M(L,t,k)P_1^{(t)}(t,k). \tag{3.13}$$

In this case, the fact that $P^{(t)}$ and $P_1^{(t)}$ in the different sectors I, ..., VI are $I + \mathrm{O}\left(\frac{1}{k}\right)$, is a consequence of the global relation. For instance, for $\operatorname{Im} k > 0$, one has

$$P^{(t)}(t,k) = \begin{cases} \begin{pmatrix} \dfrac{1}{\overline{d(\bar{k})}} & 0 \\ -\lambda \dfrac{\overline{b(\bar{k})}}{A(k)} \mathrm{e}^{8\mathrm{i}k^3 t} & \overline{d(\bar{k})} \end{pmatrix}, & |k| > R, \ k \in \mathrm{I} \cup \mathrm{III}, \\ \begin{pmatrix} \dfrac{\overline{A(\bar{k})}}{\overline{a(\bar{k})}} & 0 \\ -\dfrac{\overline{\mathrm{GR}(\bar{k}) a(\bar{k})}}{\overline{d_1(\bar{k})}} \mathrm{e}^{8\mathrm{i}k^3 t} & \dfrac{\overline{a(\bar{k})}}{\overline{A(\bar{k})}} \end{pmatrix}, & |k| > R, \ k \in \mathrm{II}, \end{cases} \tag{3.14}$$

where $\mathrm{GR}(k)$ is the l.h.s. of the global relation (2.18). Therefore, it is the global relation (2.18) for $k \in \mathrm{V}$ that gives $\overline{\mathrm{GR}(\bar{k})}\, \mathrm{e}^{8\mathrm{i}k^3 T} = \mathrm{O}\left(\frac{1}{k}\right)$ for $k \in \mathrm{II}$ and, consequently, $P_{21}^{(t)}(t,k) = \mathrm{O}\left(\frac{1}{k}\right)$ for $k \in \mathrm{II}$ and $t < T$. Similarly,

$$P_1^{(t)}(t,k) = \begin{cases} \begin{pmatrix} \dfrac{1}{A_1(k)} & \dfrac{\mathrm{GR}(k)}{\overline{d(\bar{k})}} \mathrm{e}^{2\mathrm{i}kL - 8\mathrm{i}k^3 t} \\ 0 & A_1(k) \end{pmatrix}, & |k| > R, \ k \in \mathrm{I} \cup \mathrm{III}, \\ \begin{pmatrix} \dfrac{\overline{d_1(\bar{k})}}{\overline{a(\bar{k})}} & \dfrac{b(k)}{\overline{a(\bar{k}) A_1(\bar{k})}} \mathrm{e}^{2\mathrm{i}kL - 8\mathrm{i}k^3 t} \\ 0 & \dfrac{\overline{a(\bar{k})}}{\overline{d_1(\bar{k})}} \end{pmatrix}, & |k| > R, \ k \in \mathrm{II}. \end{cases} \tag{3.15}$$

Now (2.18) in $\mathrm{I} \cup \mathrm{III}$ gives

$$\mathrm{GR}(k)\, \mathrm{e}^{2\mathrm{i}kL - 8\mathrm{i}k^3 T} = \mathrm{O}\left(\frac{1}{k}\right),$$

hence, $(P_1^{(t)})_{12}(t,k) = \mathrm{O}\left(\frac{1}{k}\right)$ for $t < T$.

Finally, one obtains $P^{(t)}(t,k) = P_{\mathrm{diag}}^{(t)}(t,k) + P_{\mathrm{off}}^{(t)}(t,k)$, where $P_{\mathrm{diag}}^{(t)}(t,k)$ is a diagonal matrix and $P_{\mathrm{off}}^{(t)}(t,k)$ is off-diagonal and exponentially decaying as $k \to \infty$, and $P_{\mathrm{diag}}^{(t)} = I + \mathrm{O}\left(\frac{1}{k}\right)$. Similarly, $P_1^{(t)}(t,k) = (P_1^{(t)})_{\mathrm{diag}}(t,k) + (P_1^{(t)})_{\mathrm{off}}(t,k)$ with the same properties. Now the fact that the multiplication by a diagonal matrix does not affect the r.h.s. of (2.35) provides the boundary conditions at $x = 0$ and $x = L$. □

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Anne BOUTET de MONVEL,
Université Paris 7
Institut de Mathématiques de Jussieu
case 7012
2 place Jussieu
75251 Paris (France)

Dmitry SHEPELSKY,
Mathematical Division
Institute for Low Temperature Physics
47 Lenin Avenue
61103 Kharkiv (Ukraine).

shepelsky@yahoo.com