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# ON THE INDEX THEOREM FOR SYMPLECTIC ORBIFOLDS

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## 1. Introduction.

This paper deals with deformation quantization on a symplectic orbifold. As is shown in [14], the method of [8] may be generalized almost without changes to the orbifold case. Our aim here is to construct traces on the algebra of observables and to introduce the corresponding indices. For compact orbifolds the latter are, by definition, the traces of the identity element in the algebra. In contrast to the smooth case, the trace on orbifolds is not unique. We give a particular construction and then obtain its different copies induced by the action of the so-called Picard group. This is the main result of the paper. It is not clear, however, whether this construction exhausts the set of traces.

We also planned to prove an index theorem for deformation quantization on symplectic orbifolds which gives an index formula in terms of characteristic classes and the symplectic form. Our hope was that combining the methods of Kawasaki [13] and Vergne [15] for the index theorem for elliptic operators on orbifolds on the one hand, with the methods of [9] for the index theorem for deformation quantization on smooth symplectic manifolds on the other hand, we would be able to obtain the desired formula. Unfortunately, we did not succeed so far. The matter is that Atiyah's

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theory of transversally elliptic operators [1] playing a key role in the proof of Kawasaki is missing for deformation quantization. Thus, we need either to develop such a theory from the very beginning or to invent some new tools. So, the index theorem still remains a conjecture which we hope to prove in future.

However, we decided to give such a formula as a conjecture in this paper, since it sheds some new light on the whole concept of deformation quantization. There are many facts which support it, for instance, a formula for contributions of fixed point manifolds to the $G$-index [12], or a direct calculation of the first three terms. Moreover, for a virtual bundle with compact support over an orbifold cotangent bundle our index formula coincides with Kawasaki's topological index. An example of the two-dimensional harmonic oscillator in a resonance case considered in the last section is also an argument in favor of the conjecture. It shows that the index theorem gives the right spectrum in this case, moreover, the example clarifies the role of the Picard group: it is responsible for different series of eigenvalues.

Thinking over the role of the index theorem one encounters a rather philosophical question: what is the place of the deformation quantization among other quantization theories. We mention here geometric quantization and a semiclassical approach to quantum problems. The latter, in particular, has a long history summarized in the book [3], recent developments are presented in the thesis [4]. In particular, in [3] multiplicities were expressed as integrals of characteristic classes, which is nothing but the index theorem for a discrete series of admissible Planck constants. Of course, the multidimensional harmonic oscillator is a touchstone for most of these theories, see for instance the recent paper [5]. As a rule, the results obtained by semiclassical methods are extremely strong and complete. They usually are based on a delicate analysis of Fourier integral operators and often require hard work.

On the contrary, deformation quantization is the most coarse version of quantization. It would be at least naive to expect here as strong results as in the semiclassical approach, for nobody will even try to compare the sportsmen of different weight categories. The fact itself that the spectral information has been obtained by means of deformation quantization only, that is what seems really surprising. Indeed, even a statement of an eigenvalue problem is meaningless in the deformation quantization framework since the observables are not operators. The example shows, however, that the spectral information is hidden somewhere even in purely deformation quantization approach, and the index theorem becomes a tool to give the eigenvalue problem a precise meaning and, simultaneously, to extract the hidden information. Let us explain how this tool works. First of all, the index in simplest cases is a polynomial in $1/h$, so, although $h$ is a formal parameter, we can calculate the index for positive values of $h$, obtaining a numerical value of it. Requiring the obtained value of $\operatorname{Tr} 1$ to be a positive integer, in analogy to the operator theory, we obtain a constraint for $h$ which may be considered as a characteristic equation for eigenvalues. Taking different traces and different non-trivial coefficient bundles, we obtain more constraints giving more and more precise information about the spectrum. Hence, the role played by the index theorem in deformation quantization is quite opposite to the multiplicity formula of [3]: the latter, knowing the spectrum beforehand, gives the values for multiplicities which turn out to be positive integers. The former gives conditions which ensure the integrality of multiplicities.

A natural question arises: If the semiclassical approach gives stronger results why do we need deformation quantization at all? The answer is that deformation quantization as a most coarse theory has a wider field of applications. Indeed, the observables in deformation quantization have a very simple nature: They are simply classical observables with successive quantum corrections. So, no prequantization conditions are required, many important constructions with classical observables such as symplectic reduction may easily be lifted to the quantum level, and so on. Of course, we have to pay for such a freedom: we lose essential information concerning the Schrödinger dynamics, eigenvalues, etc. But having an index theorem at our disposal, we may reconstruct, at least partly, the lost spectral information. There are examples coming from purely physical problems [7] where the coarse methods based on the index theorem still give a good description of complex molecular spectra, while semiclassical analysis gives no additional information because of the complexity of the system.

Passing to the content of the paper, let us first describe for the reader's convenience a strategy for most of the calculation schemes in deformation quantization. In particular, we apply this strategy for a trace construction. It consists of the following steps:

1° A global construction of the algebra of observables.

2° Localization to a Darboux coordinate chart.

3° Semiclassical representation of the localized algebra.

4° Calculation with operators leading to an asymptotic power series in $h$ for the quantity in question (if it exists).

5° Returning to deformation quantization setting.

The latter item means that we forget about the asymptotic character of the series treating it as a formal one. But we also need to prove that the result is well defined, that is it does not depend on the choices made in items 2° and 3°.

The local structure of orbifolds and orbifold vector bundles is investigated in Sections 2 and 3. We also give here a brief description of the Picard group. The first and the second items of our above-mentioned program are briefly discussed in Section 4. Section 5 deals with a local operator representation of the localized algebra of observables in the Fock space. Note that the global operator representation may not exist in general. It is our luck that the local representation which always exists is sufficient for the trace calculation. We prove its independence of the special choice of the orbifold chart using the homotopy considered in Section 2. In Section 6 we formulate our conjecture on the index formula. First we give it in a form which is a natural generalization of the index formula for smooth manifolds (more precisely, for a smooth component of a fixed point set) and then reduce the integrand to Kawasaki's form. The last Section 7 deals with an example of a two-dimensional harmonic oscillator. As was mentioned above, this is in favor of the conjecture. We also consider it as a hint that there should be a reasonable spectral theory for deformation quantization quite different from the standard operator spectral theory and based on index theorems and their modifications.

We would like to thank the referee for his critical remarks. Following them we have essentially shortened the exposition omitting excessive descriptions (the Picard group, symplectic reduction).

## 2. Symplectic orbifolds.

As a background on orbifolds we recommend the book [6].

A *symplectic orbifold* is a Hausdorff topological space $B$ which admits a locally finite covering $\{O_i\}$ with the following properties:

1° For each $O_i$ there exists a contractible domain $\widetilde{O}_i$ in the standard symplectic space $(\mathbb{R}^{2n}, \omega)$ and a finite group $G_i$ of symplectomorphisms of $\widetilde{O}_i$, such that $O_i$ is homeomorphic to the orbit space $\widetilde{O}_i/G_i$. We denote by $p_i$ the corresponding projection $p_i : \widetilde{O}_i \to \widetilde{O}_i/G_i = O_i$. The domains $\widetilde{O}_i$ with the given group action are called *orbifold charts*.

2° If for two orbifold charts $\{O_j, \widetilde{O}_j, G_j, p_j\}$ and $\{O_k, \widetilde{O}_k, G_k, p_k\}$ the intersection $O_j \cap O_k$ is not empty, then there exists an orbifold chart $\{O, \widetilde{O}, G, p\}$ such that $x \in O \subset O_j \cap O_k$ and $\widetilde{O}$ is subordinate to both $\widetilde{O}_j$ and $\widetilde{O}_k$ in the following sense: There are symplectomorphic open embeddings

$$\widetilde{O}_j \xleftarrow{\varphi_j} \widetilde{O} \xrightarrow{\varphi_k} \widetilde{O}_k \tag{2.1}$$

and group embeddings

$$G_j \xleftarrow{i_j} G \xrightarrow{i_k} G_k \tag{2.2}$$

such that the maps (2.1) are equivariant with respect to the homomorphisms (2.2).

For a point $x \in O_i$ take one of its preimages $\tilde{x} \in \widetilde{O}_i$ and consider a subgroup $G(\tilde{x}) \subset G_i$ leaving $\tilde{x}$ fixed. It is called a *stabilizer* or *isotropy subgroup* of $\tilde{x}$. Any other preimage has the form $\gamma\tilde{x}$ for some $\gamma \in G_i$, so that

$$G(\gamma\tilde{x}) = \gamma\, G(\tilde{x})\, \gamma^{-1}. \tag{2.3}$$

We see that $G(\gamma\tilde{x})$ may be obtained by conjugation from $G(\tilde{x})$. In particular, $G(\gamma\tilde{x})$ and $G(\tilde{x})$ are isomorphic, and we may introduce the stabilizer of $x \in O_i$ as a group isomorphic to any of $G(\tilde{x})$. Moreover, the conditions (2.1), (2.2) imply that $G(x)$ is independent of the chart up to an isomorphism, so the notion of the isotropy group $G(x)$ makes sense for a point $x \in B$.

A similar situation holds if we replace a point $\tilde{x}$ by a small orbifold chart $\widetilde{O}$ subordinate to $\widetilde{O}_j$. Then we have an equivariant embedding

$$\varphi : \widetilde{O} \to \widetilde{O}_j,$$
$$i : G \to G_j.$$

Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be representatives of the cosets $G_j/i\,(G)$. Then, if $\widetilde{O}$ is small enough, the domains $\gamma_k\varphi(\widetilde{O})$ do not intersect each other, and we have $m$ distinct equivariant embeddings

$$\begin{array}{lclcccc} \varphi_k & = & \gamma_k\varphi & : & \widetilde{O} & \to & \widetilde{O}_j, \\ i_k & = & \gamma_k\, i\, \gamma_k^{-1} & : & g & \mapsto & \gamma_k\, i(g)\, \gamma_k^{-1}, \end{array}$$

so that the projection $p_j : \widetilde{O}_j \to O_j$ over $O = p(\widetilde{O})$ is a covering map with $m$ leaves.

THEOREM 2.1. — *For any compact symplectic orbifold $B$ there exists a covering $\{O_j, \widetilde{O}_j, G_j, p_j\}$ by orbifold charts with the following properties:*

*1° In each $O_j$ there exists a point $x_j$ called the center, such that $G(x_j)$ coincides with $G_j$.*

*2° There is a $G_j$-invariant complex structure on $\widetilde{O}_j$, such that $\widetilde{O}_j$ is a neighborhood of the origin in $\mathbb{C}^n$ with the symplectic form*

$$\omega = \frac{1}{2\imath}(d\bar{z}_1 \wedge dz_1 + \cdots + d\bar{z}_n \wedge dz_n). \tag{2.4}$$

*3° $G_j$ acts on $\widetilde{O}_j$ by unitary transformation, that is $G_j$ is a finite subgroup of $\mathrm{U}(n)$.*

This is the so-called *linearization theorem.*

*Proof.* — Take a point $x_0 \in B$ and an orbifold chart $\{O, \widetilde{O}, G, p\}$ containing $x_0$. Pick one of the preimages $\tilde{x}_0 \in \widetilde{O}$ and a smaller neighborhood $\widetilde{O}_1 \subset \widetilde{O}$, such that $\gamma\widetilde{O}_1 \cap \widetilde{O}_1 \neq \emptyset$ implies $\gamma \in G(\tilde{x}_0)$. Then $\{p(\widetilde{O}_1) = O_1, \widetilde{O}_1, G(\tilde{x}_0), p\}$ is a smaller orbifold chart. Because of compactness we may choose a finite covering of such a form, proving 1°. Further we will consider only orbifold charts together with their centers.

Let $\{O, \widetilde{O}, G, p\}$ be an orbifold chart with the center $x_0$, so that $\tilde{x}_0$ coincides with $0 \in \mathbb{R}^{2n}$. Introduce a $G$-invariant Riemannian metric $\alpha$ on $\widetilde{O}$ (averaging over a finite group $G$ makes any metric $G$-invariant). On the tangent space $T_0\mathbb{R}^{2n} = \mathbb{R}^{2n}$ the group $G$ acts by orthogonal linear transformations. The exponential map defined by $\alpha$

$$\exp_0 : \; T_0\mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

intertwines this linear action with the original action on $\widetilde{O} \subset \mathbb{R}^{2n}$. Thus, in normal coordinates the inclusion $G \subset \mathrm{SO}(2n)$ is valid, in other words, we have linearized the group action. The determinant is equal to 1 because the orientation given by the form $\omega$ is preserved.

Consider now the form $\omega$ in normal coordinates

$$\omega = \frac{1}{2}\,\omega_{ij}(x)\,dx^i \wedge dx^j.$$

We apply a $G$-invariant version of the Weinstein-Moser trick to reduce $\omega$ to the constant form

$$\omega_0 = \frac{1}{2}\,\omega_{ij}(0)\,dx^i \wedge dx^j.$$

Consider the family of forms $\omega(t) = \omega_0 + t(\omega - \omega_0)$. For each $t \in [0,1]$ they are $G$-invariant and

$$\omega - \omega_0 = d\lambda, \tag{2.5}$$

where $\lambda = \lambda_i(x)\,dx^i$ is a one-form. We may assume that $\lambda$ is $G$-invariant since the averaging defines a new form $\lambda$ also satisfying (2.5). Moreover, we may assume that $\lambda_i(0) = 0$, otherwise we replace it by $(\lambda_i(x) - \lambda_i(0))\,dx^i$, which also satisfies (2.5) and is $G$-invariant. It gives a $G$-invariant vector field $X_t$ vanishing at the origin and satisfying

$$i(X_t)\omega(t) = -\lambda. \tag{2.6}$$

The flow $f_t(x)$ of this vector field is defined on the whole interval $t \in [0,1]$ for sufficiently small $x$, because $X_t(0) = 0$. Moreover, the flow commutes with the linear $G$-action since $X_t$ is $G$-invariant. Then $f_1^*\omega = \omega_0$, because, by virtue of the Cartan homotopy formula,

$$\begin{aligned}
\frac{d}{dt}\,f_t^*\omega(t) &= f_t^*\Big(\frac{\partial}{\partial t}\,\omega(t) + \mathcal{L}_{X_t}\omega(t)\Big)\\
&= f_t^*\big(\omega - \omega_0 + di(X_t)\omega(t)\big)\\
&= f_t^*(d\lambda - d\lambda)\\
&= 0.
\end{aligned}$$

At this step we have a constant symplectic form $\omega_0$ preserved by the group $G \subset \mathrm{SO}(2n)$.

Taking any constant $G$-invariant metric $\alpha$ and applying a standard construction (see for instance [9, Ch. 2]), we obtain a $G$-invariant positive complex structure $J$ and a new metric $\alpha_0 = \omega_0(J\cdot,\cdot)$. The symplectic space $\mathbb{R}^{2n}$ with the complex structure $J$ becomes $\mathbb{C}^n$, the form $\alpha_0 + \imath\omega_0$ gives a Hermitian metric and the group $G$ which preserves this Hermitian metric is thus a subgroup of $\mathrm{U}(n)$. □

For paracompact orbifolds the existence of such a linearized covering will be assumed. In some cases (for example, for vector bundles over a compact base $B$) this assumption may be proved.

Next, we consider the whole set of symplectic linearizations of a fixed orbifold chart $\widetilde{O}$. We will show that this set is connected, provided that $\widetilde{O}$ is small enough. In other words, any two linearizations may be linked by a smooth one-parameter family. This fact will be crucial for the trace construction in Section 5.

Let $\widetilde{O}$ be an orbifold chart with the finite group $G$ of symplectomorphisms, $\widetilde{x}_0 = 0$ being its center. Suppose we have two Darboux coordinate systems in $\widetilde{O}$, namely $x = (x^1, \dots, x^{2n})$ and $y = (y^1, \dots, y^{2n})$, so that $G$ acts by linear symplectomorphisms in both coordinate systems

$$g : \begin{array}{l} x \mapsto gx, \\ y \mapsto \tilde{g}y, \end{array} \tag{2.7}$$

where $g, \tilde{g} \in \mathrm{Sp}(2n)$. To put it differently, $x$ and $y$ are related by a non-linear symplectomorphism

$$y = f(x) \tag{2.8}$$

with the property

$$f(gx) = \tilde{g} f(x), \quad g \in G. \tag{2.9}$$

THEOREM 2.2. — *There exists a smooth family $f_t(x)$, $t \in [0,1]$, of symplectomorphisms defined in a smaller neighborhood of the origin such that $f_0(x) = x$, $f_1(x) = f(x)$ and $f_t(x)$ satisfies the relation (2.9) for each $t \in [0,1]$.*

*Remark 2.3.* — Geometrically this theorem means that there exists an equivariant interpolation between two linearized orbifold charts. It is, however, not quite obvious. Our proof gives a way to linearize this procedure using generating functions.

*Proof.* — Consider first the case when the linear part of (2.8) is identity, that is

$$f(x) = x + \varphi(x) = x + O(|x|^2). \tag{2.10}$$

Comparing the linear parts on both sides of (2.10), we see that $g = \tilde{g}$. To construct the homotopy $f_t(x)$ we will use generating functions (see for instance [9, Section 2.4] or [10]).

Let the symplectomorphism (2.8) correspond to the Cayley generating function $S$, that is, the relation $y = f(x)$ is obtained by elimination of the auxiliary variable $z$ from the following two equations

$$\begin{aligned} x &= z - \nabla S(z) \\ y &= z + \nabla S(z). \end{aligned} \tag{2.11}$$

Here $\nabla S$ means the symplectic gradient, it is a vector satisfying the relation $i(\nabla S)\,\omega = -dS$, or in coordinates

$$\nabla^i S = \omega^{ij}\,\frac{\partial S}{\partial z^j}.$$

From (2.11) it follows that

$$\begin{aligned} z &= \frac{x+y}{2} = \frac{x+f(x)}{2}, \\ \nabla S &= \frac{x-y}{2}, \end{aligned} \tag{2.12}$$

and further, taking $S(0)=0$, we get

$$S(z) = \int_0^z i\Big(\frac{x-y}{2}\Big)\omega. \tag{2.13}$$

It is easy to verify that equations (2.11) actually define a symplectomorphism provided we can express $z$ as an implicit function of $x$ from the first equation (2.11). This is always the case in a neighborhood of the origin if $S(z)$ has a third order zero at $z=0$. Vice versa, if the linear part of $f(x)$ is identity then $x$ may be defined as an implicit function of $z$ and the integrand in (2.13) is a closed form.

LEMMA 2.4. — *A symplectomorphism $y = f(x)$ defined by a generating function $S(z)$ satisfies (2.9) if and only if the generating function is invariant, i.e.*

$$S(gz) = S(z), \quad g \in G. \tag{2.14}$$

*Proof.* — Differentiating (2.14), we get

$$\nabla^i S(z) = \omega^{ij} g_j^k\,\frac{\partial S}{\partial z^k}(gz),$$

and further, applying $g$ to both sides,

$$g_i^l\,\nabla^i S(z) = g_i^l\,\omega^{ij} g_j^k\,\frac{\partial S}{\partial z^k}(gz) = \omega^{lk}\,\frac{\partial S}{\partial z^k}(gz).$$

Here we have used the fact that $g$ is a symplectic matrix, thus $g_i^l\,\omega^{ij} g_j^k = \omega^{lk}$. In other words, the vector $\nabla S(z)$ satisfies the relation

$$g\,(\nabla S(z)) = \nabla S\,(gz).$$

Then (2.11) yields

$$\begin{aligned} gx &= gz - \nabla S(gz) &= \widetilde{z} - \nabla S(\widetilde{z}) \\ gy &= gz + \nabla S(gz) &= \widetilde{z} + \nabla S(\widetilde{z}), \end{aligned}$$

where we have denoted $gz$ by $\tilde{z}$. Eliminating $\tilde{z}$ from these equations, we see that $gx$ and $gy$ satisfy the same relation $gy = f(gx)$ as $x$ and $y$, whence

$$gf(x) = f(gx). \tag{2.15}$$

Vice versa, let (2.9) be fulfilled. Then the action $x \mapsto gx$ implies

$$\begin{array}{rcccc} y & = & f(x) & \mapsto & gy, \\ z & = & \dfrac{x+y}{2} & \mapsto & gz, \\ \nabla S & = & \dfrac{x-y}{2} & \mapsto & g\,(\nabla S). \end{array}$$

We thus get

$$\nabla S\,(gz) = g\,(\nabla S(z)),$$

and this, in turn, implies by virtue of (2.14) that $S(z)$ is invariant □

Now, to construct the homotopy $f_t(z)$, we take the generating function $S(z)$ of the original symplectomorphism $f(x)$, multiply it by $t \in [0,1]$ and then consider the symplectomorphisms defined by the generating functions $tS(z)$. Because of (2.15) $S(z)$ is invariant, hence so is $tS(z)$, implying that (2.15) is fulfilled for any $t$. This proves the theorem in the special case (2.10).

In the general case we rewrite the symplectomorphism $f(x)$ in the form

$$y = a\,(b + x + \varphi(x)) \tag{2.16}$$

where $ab + ax$ is the linear part and $\varphi(x) = O(|x|^2)$. Thus, $a = f'(0)$, $ab = f(0)$, and the property (2.9) gives us

$$\tilde{g}\,a\,(b + x + \varphi(x)) = a\,(b + gx + \varphi(gx)).$$

Hence it follows that $\tilde{g} = aga^{-1}$, $gb = b$ and $\varphi(gx) = g\varphi(x)$. The group $\mathrm{Sp}(2n)$ is connected, thus there exists a path $a_t \in \mathrm{Sp}(2n)$ linking $a$ with the identity matrix 1. After this homotopy the symplectomorphism (2.16) takes the form $y = b + x + \varphi(x)$ and moreover, $\tilde{g} = g$. The vector $b$ belongs to $\ker(g - 1)$ for any $g \in G$, and so does $tb$ for any $t \in [0,1]$. Thus, we can pull $b$ to zero. Our symplectomorphism then becomes

$$y = x + \varphi(x),$$

and this can be linked to the identity, as was already proved. □

*Remark 2.5.* — The homotopy $f_t(x)$ constructed in the theorem may be extended to a positive complex structure $J$. Indeed, let $J$ be a constant complex structure making the symplectic space $\mathbb{R}^{2n}$ into $\mathbb{C}^n$ and the group $G \subset \mathrm{Sp}(2n)$ into a subgroup of $\mathrm{U}(n)$. It defines a constant metric $\alpha = \omega(J\cdot,\cdot)$. Having a homotopy $y = f_t(x)$ of the constant symplectic structure and the corresponding group $G_t \subset \mathrm{Sp}(2n)$ we define a new metric $\alpha_t$ by averaging $\alpha$ with respect to $G_t$ and then construct a positive complex structure $J_t$ in a standard way starting with two bilinear forms $\omega$ and $\alpha_t$.

The group $G$ for an orbifold chart $\widetilde{O}$ is not uniquely defined. For example, the chart $O = \mathbb{C}/\mathbb{Z}_3$, where $\mathbb{Z}_3$ acts by multiplication by

$$\exp\left(2\pi\imath\frac{k}{3}\right), \quad k \in \mathbb{Z} \mod 3, \tag{2.17}$$

may be replaced by the chart $\mathbb{C}/\mathbb{Z}_6$ with the same action as in (2.17) but with $k \in \mathbb{Z} \mod 6$. In the second case the action is not effective: the subgroup with $k = 3l \mod 6$ acts as identity. In general, denoting by $G_0 \subset G$ a subgroup which acts on $\widetilde{O}$ as identity, we can pass on to the effective action by replacing $G$ by the quotient $G/G_0$ (clearly, $G_0$ is a normal subgroup). So, we will assume as a rule that the actions of $G_i$ on $\widetilde{O}_i$ are effective. In this case we have an open dense set $B_0 \subset B$, the so-called *principal stratum*, such that each point $x \in B_0$ has a trivial stabilizer. This is a smooth part of the orbifold $B$. The remaining points form singularities, which in general may be very complicated. In the above example (2.17) we have the only singular point $z = 0$ which is a conical point.

Singular points (the points with non-trivial stabilizer) admit further stratification but we will not touch this subject here.

Although $B$ may have rather complicated singularities, the notion of a smooth function still makes sense for orbifolds. Namely, $f \in C^\infty(B)$ if its lifting to any orbifold chart $\widetilde{O}$ is smooth and necessarily $G$-invariant, that is

$$f(\gamma\tilde{x}) = f(\tilde{x}). \tag{2.18}$$

Thus, a possibility appears to develop analysis and differential geometry on orbifolds.

## 3. Orbifold vector bundles.

The notion of orbifold vector bundle requires some precautions. A naive definition is that $E$ is a continuous vector bundle over a topological space $B$, which may be described by means of smooth functions. For example, any matrix-valued function

$$P(x) = \big(p_{ij}(x)\big) \tag{3.1}$$

whose values are projectors and whose entries $p_{ij}(x)$ are smooth functions defines an orbifold vector bundle. But this definition is too restrictive, even tangent and cotangent bundles do not fit into this scheme.

DEFINITION 3.1. — *An orbifold vector bundle $E$ is an object which in an orbifold chart $\{O, \widetilde{O}, G, p\}$ is given by a $G$-equivariant vector bundle $\widetilde{E}_G$. These local equivariant bundles should be compatible with respect to symplectomorphisms (2.1) and homomorphisms (2.2).*

Thus, the total space of $E$ is an orbifold itself, with the same local groups $G_i$ as for the base $B$. For the orbifold bundles we have a linearization theorem similar to Theorem 2.1.

THEOREM 3.2. — *Let $E$ be an orbifold vector bundle. Then for any linearized orbifold chart $\{O, \widetilde{O}, G, p\}$, $G \subset \mathrm{U}(n)$, there exists a frame of the bundle $\widetilde{E}_G$ and a complex representation $g \mapsto T(g) \in \mathrm{End}(E_0)$, such that the sections $s(z, \bar{z})$ are vector-valued functions with values in $E_0$, and the group $G$ acts on sections as follows*

$$s(gz, \overline{gz}) = T(g)\, s(z, \bar{z}). \tag{3.2}$$

*Proof.* — The point $\tilde{x} \in \widetilde{O}$ may be written as a pair $z$, $\bar{z}$ with $z \in \mathbb{C}^n$, and we will use both designations $\tilde{x}$ and $z$, $\bar{z}$. Choosing any frame of $\widetilde{E}_G$ over $\widetilde{O}$, we consider the sections $s(\tilde{x})$ as vector-valued functions. By the definition of the equivariant vector bundle we have the action of the group $G$ on sections by $s(g\tilde{x}) = T(g)(\tilde{x})\, s(\tilde{x})$, where the matrix-valued function $T(g)(\tilde{x})$ defines a linear map

$$T(g)(\tilde{x}) : \widetilde{E}_{\tilde{x}} \to \widetilde{E}_{g\tilde{x}}. \tag{3.3}$$

At the origin $0 \in \widetilde{O}$ which is a fixed point we have thus endomorphisms $T(g)(0) : \widetilde{E}_0 \to \widetilde{E}_0$, and this is the desired representation. Clearly, we can make it unitary by introducing a Hermitian metric and averaging it over the finite group.

In order to construct the needed frame we now choose a $G$-invariant Hermitian connection on $E$ (again using the averaging), take any unitary frame in $\widetilde{E}_0$ and spread it over the whole $\widetilde{O}$ by parallel transports along the rays $t\tilde{x}$, $t \in [0, 1]$. □

*Remark 3.3.* — For linearized coordinates $z$, $\bar{z}$ and frames (3.2), the simplest invariant connection is given by the de Rham differential $ds(z, \bar{z})$ of vector-valued functions.

Now, smooth vector fields, differential forms and other geometric objects are defined as sections of corresponding orbifold vector bundles $TB$, $T^*B$ and so on.

The integral of a differential form of top degree over the orbifold $B$ is defined in an obvious way. We take a smooth partition of unity $\rho_i(x)$ on $B$. When lifted to an orbifold chart $\widetilde{O}_i$, the function $\rho_i(\tilde{x})$ is smooth and $G_i$-invariant, and we define

$$\int_B \alpha = \sum_i \frac{1}{|G_i|} \int_{\widetilde{O}_i} \rho_i(\tilde{x})\, \alpha_i$$

where $\alpha_i$ is a local expression of the form $\alpha$. Now, since $\gamma^*(\rho_i \alpha_i) = \chi_i(\gamma)\, \rho_i \alpha_i$ where $\chi_i(\gamma)$ is a one-dimensional character, the integrals are equal to zero unless $\chi_i(\gamma) \equiv 1$. In the latter case we may consider $\rho_i(\tilde{x})\alpha_i$ as the form coming from $B$, and the integral will be simply equal to

$$\int_{B_0} \alpha,$$

that is the integral over the smooth part $B_0$ of $B$.

We have tacitly assumed that the group $G_i$ acts effectively. In general, when the action is not effective, we pass to the quotient $\widetilde{G}_i = G_i/G_0$ where $G_0$ is a normal subgroup of $G_i$ acting as identity. By compatibility conditions (2.1) and (2.2), the number $m(B) = |G_0|$ does not depend on the chart (for connected orbifolds), it is called the *multiplicity* of the orbifold. Then one defines

$$\begin{aligned} \int_B \alpha &= \sum_i \frac{1}{|\widetilde{G}_i|} \int_{\widetilde{O}_i} \rho_i(\tilde{x})\, \alpha_i \\ &= m(B) \sum_i \frac{1}{|G_i|} \int_{\widetilde{O}_i} \rho_i(\tilde{x})\, \alpha_i. \end{aligned} \tag{3.4}$$

We finish this section by introducing an important notion of the *Picard group.* To this end, consider the set $\mathcal{E}^1$ of flat one-dimensional orbifold vector bundles. Flatness means that any bundle $E^1 \in \mathcal{E}^1$ is equipped with a connection $\partial$ whose curvature vanishes. Such bundles form an Abelian group with respect to tensor product. In linearized charts $\widetilde{O}_i$ the action (3.2) defines one-dimensional representations of local groups $G_i$, that is characters of $G_i$. Thus, we have a homomorphism

$$\chi : E^1 \to \prod_i \chi(G_i),$$

where $\chi(G_i)$ means the group of characters of $G_i$. We define the Picard group by

$$\operatorname{Pic} = \mathcal{E}^1 / \ker \chi \cong \operatorname{im} \chi. \tag{3.5}$$

A subgroup $\mathcal{E}^1_B = \ker\chi$ consists of those one-dimensional flat bundles $E^1$, for which the actions of all local groups are trivial. It means that they come from the base, that is have a description (3.1).

The group $\mathcal{E}^1$ acts by tensoring on any orbifold vector bundle $E$. We would like to mention two obvious properties of this action, they will be of great importance in the sequel.

1° If $E$ is a bundle with a connection $\partial$ then $E \otimes E^1$ has a connection with the same curvature for any $E^1 \in \mathcal{E}^1$.

2° In linearized orbifold charts this action consists in the replacement

$$T_i(g) \mapsto \chi_i(g)\, T_i(g) \tag{3.6}$$

of local representations $T_i$ by $\chi_i T_i$. In particular, it depends only on the class of $E^1$ in the Picard group.

## 4. Deformation quantization.

We use the standard scheme of deformation quantization described in [8]. For symplectic orbifolds it was generalized in [14]. The scheme was discussed many times in the literature, so we need not repeat it here in detail. Yet, to fix notation, we give a brief survey.

The starting data are:

1° a symplectic orbifold $B$ with a symplectic connection $\partial_B$;

2° an orbifold vector bundle $E$ with a connection $\partial_E$.

We emphasize that the connections $\partial_B$ and $\partial_E$ are global objects. They give rise to a global procedure of deformation quantization, although the construction is given in terms of local charts of Theorems 2.1 and 3.2.

We will consider quantization with a non-trivial coefficient bundle $K$, namely $K = \operatorname{End} E$.

The algebra $A^0$ of classical observables with coefficients in $K$ consists, by definition, of all sections of the bundle $K$, that is $A^0 = C^\infty(B, K)$. In an orbifold chart $\widetilde{O}_i$ a section $a \in A^0$ is given by a matrix-valued function $a(\tilde{x})$ satisfying equivariance relations similar to (3.2),

$$a(g\tilde{x}) = T_i(g)\, a(\tilde{x})\, T_i^{-1}(g) \tag{4.1}$$

for $g \in G_i$. Note that $A^0$ is an algebra (non-commutative in general) with respect to the pointwise matrix product. The connection $\partial_E$ on $E$ defines an associated connection on $K$ which we will denote by the same symbol $\partial_E$.

To construct an algebra $A^h$ of quantum observables and a quantization map

$$Q : A^0 \to A^h,$$

consider the so-called Weyl bundle with coefficients in $K$. Its sections over linearized orbifold charts $\{O_i, \widetilde{O}_i, G_i, p_i\}$ are "functions"

$$\begin{aligned} a &= a(\tilde{x}, y, h) \\ &= \sum_{k,|\alpha|=0}^{\infty} h^k \, a_{k,\alpha}(\tilde{x}) \, y^\alpha. \end{aligned} \tag{4.2}$$

Here $\tilde{x} \in \widetilde{O}_i$, $y \in T_{\tilde{x}}\widetilde{O}_i$, and $h$ is a formal parameter. The coefficients $a_{k,\alpha}$ are matrix-valued functions, and the series (4.2) is understood as a formal one whose terms are ordered by total degrees $2k + |\alpha|$. These "functions" should possess the invariance property

$$\begin{aligned} g^* a &:= T_i(g) \, a(g^{-1}\tilde{x}, g^{-1}y, h) \, T_i^{-1}(g) \\ &= a(\tilde{x}, y, h) \end{aligned}$$

for all $g \in G_i$, similar to (4.1). The space of sections $C^\infty(B, W \otimes K)$ is an algebra with respect to the pointwise Weyl product

$$a \circ b = \exp\Big(-\frac{\imath h}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\Big) a(\tilde{x}, y, h) b(\tilde{x}, z, h)\Big|_{z=y}. \tag{4.3}$$

We will also need an extension $W^+ \otimes K$ of the bundle $W \otimes K$ obtained by dropping the requirement $k \geqslant 0$ in (4.2). We assume instead that $2k + |\alpha| \geqslant 0$ while $k$ may be negative. A typical example of a section of $W^+$ not belonging to $W$ is

$$\exp\Big(\frac{1}{h}\, a_{ijk} y^i y^j y^k\Big).$$

Finally, the connections $\partial_B$ and $\partial_E$ give rise to a connection on the bundle $W \otimes K$, that is a derivation of the algebra of sections with a local expression

$$\partial a = d_{\tilde{x}} a + \Big[\frac{\imath}{2h}\, \Gamma_{ijk} y^i y^j d\tilde{x}^k + \Gamma_k d\tilde{x}^k,\, a\Big].$$

Our aim is to construct a connection

$$Da = -\delta a + \partial a + [r, a] \tag{4.4}$$

with the property $D^2 a = 0$ for any section $a$ (the so-called Abelian property). Here

$$\begin{aligned} \delta a &:= d\tilde{x}^i \, \frac{\partial a}{\partial y^i}(\tilde{x}, y, h) \\ &= -\Big[\frac{\imath}{h}\, \omega_{ij} y^i d\tilde{x}^j,\, a\Big] \end{aligned}$$

and

$$r = \frac{i}{h}\, r_i(\tilde{x}, y, h)\, d\tilde{x}^i$$

with $\deg r_i \geqslant 3$.

The simplest equation for $r$ to ensure the Abelian property $D^2 = 0$ reads

$$\delta r = R + \partial r + r \circ r \tag{4.5}$$

where

$$R = \frac{i}{4h} R_{ijkl} y^i y^j d\tilde{x}^k \wedge d\tilde{x}^l + \frac{1}{2} R_{ij} d\tilde{x}^k \wedge d\tilde{x}^l$$

is the curvature of $\partial$.

The following theorem describes the global construction of deformation quantization (the first step in the program mentioned in the introduction).

THEOREM 4.1.

1° *There exists a unique solution of equation (4.5), such that* $r_i y^i = 0$.

2° *For any classical observable* $a(\tilde{x}) \in A^0$ *there exists a unique flat section* $a(\tilde{x}, y, h)$ *of the bundle* $W \otimes K$, *such that* $a(\tilde{x}, 0, h) \equiv a(\tilde{x})$.

Thus, our global algebra of quantum observables is

$$A^h = \{a \in C^\infty(B, W \otimes K) : Da = 0\}$$

and the quantization map $Q$ is given by the second item of the theorem. Note that $Q$ may be extended by linearity to formal power series in $h$

$$Q : C^\infty(B, K)[[h]] \to A^h.$$

Clearly, this extension is a bijection, hence $A^h$ may be also thought of as the space $C^\infty(B, K)[[h]]$ with a so-called star-product $*$, which is obtained from the pointwise Weyl product $\circ$ by the bijection $Q$.

*Remark 4.2.* — The section $r$ in (4.4) depends only on curvatures of the connections $\partial_B$ and $\partial_E$. If these curvatures vanish then $r = 0$. Moreover, for any one-dimensional bundle $E^1 \in \mathcal{E}^1$ the replacement of $E$ by $E \otimes E^1$ leads to the same (up to an isomorphism) algebra of quantum observables.

Now, let us pass to the second step of the program, namely, localization to a linearized chart. In fixed Darboux coordinates and in a fixed frame of $E$ the simplest choice of connections $\partial_B$ and $\partial_E$ is the de Rham differential $d$ with respect to $\tilde{x}$. Since $d$ is a flat connection we obtain

$D_0 = -\delta + d_{\tilde{x}}$ for the Abelian connection (4.4). The quantization map $Q$ has a simple explicit form

$$Q : a(\tilde{x}, h) \mapsto a(\tilde{x} + y, h), \tag{4.6}$$

where by the right-hand side is meant a formal Taylor series. Thus, we have two algebras of observables, namely the global algebra $A^h$ restricted to $\widetilde{O}$, and the algebra $A_0^h$ of flat sections (4.6) with respect to $D_0$. The following lemma gives an isomorphism of these algebras.

LEMMA 4.3. — *There exists an invertible section*

$$\begin{aligned} U(\tilde{x}, y, h) &= 1 - \frac{\imath}{6h}\Gamma^B_{ijk} y^i y^j y^k - \Gamma^E_k y^k + \cdots \\ &\in C^\infty(\widetilde{O}, W^+ \otimes K), \end{aligned}$$

*such that for any flat section $a \in A^h \mid_{\widetilde{O}}$ its image under the conjugation*

$$a_0 = Ia = U \circ a \circ U^{-1} \tag{4.7}$$

*belongs to $A_0^h$.*

Next, consider a homotopy $f_t(\tilde{x})$ of the linearized orbifold chart $\widetilde{O}$ as in Theorem 2.2. The change of variables and the conjugation automorphism (4.7) applied to a flat section $a_0 = a_0(\tilde{x} + y, h) \in A_0^h$ define an automorphism of the algebra $A_0^h$, namely

$$\begin{aligned} a_0(\tilde{x} + y, h, t) &= I_t\, a_0(\tilde{x}, y, h) \\ &= U_t \circ a\left(f_t(\tilde{x}) + f_t'(\tilde{x}) y, h\right) \circ U_t^{-1} \\ &\in A_0^h. \end{aligned} \tag{4.8}$$

Differentiating in $t$, we arrive at the following Heisenberg equation (for more details see [9, Section 5.4])

$$\dot{a}_0 = [H, a_0], \tag{4.9}$$

where $H$ is a section of $W^+ \otimes K$ depending on $t$.

LEMMA 4.4. — *There exists a flat section $H_0 \in A_0^h$, such that (4.9) may be rewritten in the form*

$$\dot{a}_0 = \left[\frac{\imath}{h} H_0,\, a_0\right].$$

*Proof.* — Applying $D_0$ to both sides of (4.9) and using the fact that $a_0$, $\dot{a}_0$ are flat with respect to $D_0$, we get

$$[D_0 H, a_0] = 0.$$

Being valid for any flat section $a_0$, this equality implies that $D_0 H$ is a central one-form, that is independent of $y$. This yields

$$D_0 H = \varphi = \varphi_i(\tilde{x}, h, t)\, d\tilde{x}^i.$$

Applying $D_0$ to both sides, we get 0 on the left since $D_0(D_0 H) \equiv 0$. On the other hand, $D_0 \varphi = d\varphi$ since $\varphi$ is a central form. Hence, $\varphi = d\psi$ in the local chart. It means that the section $\tilde{H} = H - \psi$ belongs to $A_0^h$. Clearly, the section $H$ in (4.9) may be replaced by $\tilde{H}$, for they differ by a central function $\psi(\tilde{x}, h, t)$.

Comparing degrees on both sides of (4.9), one can see that $\tilde{H}$ must have degree $-2$, thus it can be rewritten in the form $(\imath/h)\, H_0$ with $H_0 \in A_0^h$. □

Finally, we can simplify the algebra $A_0^h$ putting $y = 0$ everywhere and introducing a $*$-product in its simplest form

$$(4.10)\quad a(\tilde{x}, h) * b(\tilde{x}, h) = \exp\Big( -\frac{\imath h}{2} \omega^{ij} \frac{\partial}{\partial s^i} \frac{\partial}{\partial t^j} \Big) a(\tilde{x} + s, h) b(\tilde{x} + t, h) \Big|_{s=t=0}$$

instead of the Weyl product $\circ$, *cf.* (4.3).

## 5. Orbifold trace.

To define the trace we have to consider operator representations of the quantum algebra $A_0^h$ and start with the so-called Weyl correspondence between symbols and pseudodifferential operators. Classical observables with support in a linearized orbifold chart $\widetilde{O}$ will be called here Weyl symbols and denoted by $W = W(\widetilde{O})$. In particular, we assume that the Weyl symbols satisfy equivariance relation (4.1). Choose a splitting of the symplectic space $\mathbb{R}^{2n}$ into a direct sum $\mathbb{R}^n_q \oplus \mathbb{R}^n_p$ of two Lagrangian subspaces, so that the symplectic form becomes $\omega = dp \wedge dq$. The Weyl pseudodifferential operator corresponding to a symbol $a(\tilde{x}) = a(q, p) \in W$ acts in $L^2(\mathbb{R}^n)$ as

$$(5.1)\qquad \mathrm{Op}(a(\tilde{x}))u\,(q) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} \exp\Big(\frac{\imath}{h} \langle q - q', p\rangle\Big) \cdot a\Big(\frac{q + q'}{2}, p\Big) u(q')\, dq' dp.$$

Here $h$ is understood as a numerical parameter in the interval $(0, 1]$. It is well known that the product of two such operators $\mathrm{Op}(a(\tilde{x}))$ and $\mathrm{Op}(b(\tilde{x}))$ is again an operator of the form $\mathrm{Op}(c(\tilde{x}, h))$ with

$$(5.2)\qquad c(\tilde{x}, h) = \frac{1}{(\pi h)^{2n}} \int_{\mathbb{R}^{4n}} \exp\Big(\frac{2\imath}{h}\, \omega(s, t)\Big) a(\tilde{x} + s)\, b(\tilde{x} + t)\, ds dt.$$

The integral is understood as an oscillatory one. Its asymptotic expansion at $h \to 0$ is known to coincide with the formal series (4.10) for the star-product. Thus, we obtain an operator interpretation of the star-product.

More pedantically, the star-product $a * b$ of two formal series $a(\tilde{x}, h)$ and $b(\tilde{x}, h)$ from the algebra $A_0^h$ with supports in a chart $\widetilde{O} \subset \mathbb{R}^{2n}$ has the following operator description. To find a finite segment $a * b\,|_N$ of the formal series $a * b$ up to $h^N$, consider the truncated series

$$a\,|_N = \sum_{k=0}^{N} h^k\, a_k(\tilde{x})$$

and $b\,|_N$. In these finite sums $h$ may be regarded as a numerical parameter, so that the operators $\mathrm{Op}(a|_N)$ and $\mathrm{Op}(b|_N)$ are meaningful. Their product provides a symbol $c$ which has an asymptotic expansion in powers of $h$ as $h \to 0$. The $N$-th segment of this asymptotic series is precisely the $N$-th segment of $a * b$. This kind of reasoning is often used in deformation quantization. For instance, the associativity of the star-product (4.10) immediately follows from the associativity of the operator product.

We will need also the expression for the star-product in complex coordinates,

$$\begin{aligned} a(z,\bar{z},h) * b(z,\bar{z},h) &= \frac{1}{(\pi h)^{2n}} \int\limits_{\mathbb{C}^{2n}} \exp\Big(\frac{1}{h}(u^* v - v^* u)\Big) \\ &\quad \cdot a(z+u, \bar{z}+\bar{u}, h) b(z+v, \bar{z}+\bar{v}, h)\, du d\bar{u} dv d\bar{v} \\ &= \sum_{\alpha,\beta} (-1)^{|\beta|} \frac{h^{|\alpha|+|\beta|}}{\alpha!\beta!} \frac{\partial^{\alpha+\beta} a}{\partial z^\alpha \partial \bar{z}^\beta} \frac{\partial^{\alpha+\beta} b}{\partial \bar{z}^\alpha \partial z^\beta}. \end{aligned} \tag{5.3}$$

Here, using a complex structure, we write $\tilde{x} = (z, \bar{z})$ with $z \in \mathbb{C}^n$, the notation $du d\bar{u}$ means the Lebesgue measure in $\mathbb{C}^n$. We often treat $z$, $u$, $v$ as column vectors, then $z^*$, $u^*$, $v^*$ mean the rows of complex conjugate elements, so that $u^* v = \bar{u}_1 v_1 + \cdots + \bar{u}_n v_n$ is a Hermitian scalar product. Again we have two expressions for the star-product, namely the integral one in which by $h$ is meant a numerical parameter, and the formal one which can be extended to formal power series in $h$.

We will need another correspondence between symbols and operators, the so-called representation in the Fock space. It is defined for a fixed numerical value of $h \in (0,1]$ and for a larger symbol class than $W$. Let $S$ denote the space of matrix-valued functions $a(\tilde{x})$ defined on all of $\mathbb{R}^{2n} = \mathbb{C}^n$ and satisfying the following estimates: There exists $N$ depending on $a$, such that

$$|\partial^\alpha a(\tilde{x})| \leqslant C_\alpha \langle \tilde{x} \rangle^{N-|\alpha|}$$

holds for any multi-index $\alpha$. Equivariance relations (4.1) are not assumed in general for symbols from $S$.

For such symbols the Weyl correspondence $\operatorname{Op}(a)$ is still defined with the product $*$ given by the integral form of (5.3) or (5.2). Note, however, that if one of the symbols $a$ or $b$ is a polynomial then the integral in (5.3) coincides with the series which terminates in this case. Clearly, we have $W \subset S$.

Consider an operator on $L^2(\mathbb{R}^n)$ with the Weyl symbol

$$p(z, \bar{z}) = 2^n \exp\left(-\frac{1}{h}|z|^2\right) \tag{5.4}$$

which belongs to $S$ for a fixed positive $h$. The integral form of the $*$-product shows that $p * p = p$. It means that the operator $P = \operatorname{Op}(p)$ is a projector in $L^2(\mathbb{R}^n)$. Furthermore, we have

$$\operatorname{Tr} P = \frac{1}{(2\pi h)^n} \int_{\mathbb{C}^n} p(z, \bar{z})\, dz d\bar{z} = 1,$$

which means that $P$ is a one-dimensional projector. Now, it follows from (5.3) that

$$z^i * p = 0,$$
$$\bar{z}^i * p \neq 0.$$

For this reason $z^i$ are called annihilation operators while $\bar{z}^i$ creation operators. As a linear space the *Fock space* is generated by the symbols

$$u = a(z, \bar{z}) * p(z, \bar{z}) \tag{5.5}$$

where $a(z, \bar{z})$ are polynomials in $z$, $\bar{z}$.

For two vectors $u$, $v$ of the form (5.5) define the scalar product

$$\begin{aligned}(u, v) &= \operatorname{Tr} \operatorname{Op}(v^*)\ \operatorname{Op}(u) \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{C}^n} v^*(z, \bar{z}) u(z, \bar{z})\, dz d\bar{z}.\end{aligned} \tag{5.6}$$

Writing $a(z, \bar{z})$ in (5.5) in the *normal form* $a_{\alpha,\beta} \bar{z}^\alpha * z^\beta$ (first creation, then annihilation operators) we see that non-zero vectors (5.5) are linear combinations of

$$e_\alpha = \frac{\bar{z}^\alpha * p}{\sqrt{\alpha!\, (2h)^{|\alpha|}}}. \tag{5.7}$$

These define an orthonormal basis with respect to the scalar product (5.6). The vector $e_0 = p$ is called the *vacuum*. Thus, the Fock space $\mathbb{F}$ is the left ideal of the symbol algebra $S$ generated by the vacuum vector. To obtain a Hilbert space, we need a completion with respect to the scalar product.

We next define an action of the symbols from $S$ and the group $\mathrm{U}(n)$ on the space $\mathbb{F}\otimes E$. For a symbol $b(z,\bar z)$ the corresponding operator $\hat b$ is given by the star-product in integral form

$$\hat b u = b(z,\bar z) * u(z,\bar z) = b(z,\bar z) * a(z,\bar z) * p(z,\bar z). \tag{5.8}$$

The group $\mathrm{U}(n)$ acts by pulling back the symbols (5.5) which represent vectors in $\mathbb{F}$ that is

$$\hat g u := T(g)\, u(g^{-1}z, \overline{g^{-1}z}). \tag{5.9}$$

LEMMA 5.1. — *The correspondence*

$$S \ni a \mapsto \hat a,$$
$$\mathrm{U}(n) \ni g \mapsto \hat g$$

*possesses the following important properties:*

1° *The homomorphism property*

$$\hat a\hat b = \widehat{a\circ b},$$
$$\hat g_1\hat g_2 = \widehat{g_1 g_2}; \tag{5.10}$$

2° *The conjugation property*

$$\widehat{g^{-1}}\hat a\hat g = \widehat{g^* a}, \tag{5.11}$$

*where the pull-back $g^*a$ is given by $g^*a(z,\bar z) := T(g^{-1})a(gz,\overline{gz})T(g)$.*

*Proof.* — Direct check. □

Since $W\subset S$ the operators $\hat a$ are defined for $a\in W$. Moreover, the conjugation (5.11) is trivial for $a\in W$ and $g\in G\subset \mathrm{U}(n)$ because of the equivariance relations. Note that the Weyl correspondence does not allow a single-valued representation of the group $\mathrm{U}(n)$, that is why we need a representation in the Fock space.

We are now in a position to define an orbifold trace on symbols $a\in W$. Namely, set

$$\mathrm{Tr}_{\mathrm{orb}}\, a = \mathrm{Tr}\,\hat a\,|_{\mathrm{inv}}, \tag{5.12}$$

$\hat a|_{\mathrm{inv}}$ on the right-hand side meaning an operator restricted to the subspace of $G$-invariant elements of $\mathbb{F}\otimes E$. For a finite group $G$, the orthogonal projector to this subspace is given by the averaging

$$P_{\mathrm{inv}} u = \frac{1}{|G|}\sum_{g\in G}\hat g u,$$

and (5.12) thus becomes

$$\mathrm{Tr}_{\mathrm{orb}}\, a = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}\, \hat{g}\hat{a}. \tag{5.13}$$

The last expression makes sense for any rapidly decreasing symbol from $S$, not necessarily invariant with respect to $G$-action (5.9). We will use the notation $\mathrm{Tr}_g\, a = \mathrm{Tr}\, \hat{g}\hat{a}$ for any rapidly decreasing symbol $a \in S$.

LEMMA 5.2.

1° *For any $a \in W$ and any $b \in S$,*

$$\mathrm{Tr}_g\, a * b = \mathrm{Tr}_g\, b * a. \tag{5.14}$$

2° *For any $a \in W$ (not necessarily $G$-invariant)*

$$\mathrm{Tr}_g\, \gamma^* a = \mathrm{Tr}_{\gamma g \gamma^{-1}}\, a. \tag{5.15}$$

*In particular, for a $G$-invariant $a \in W$ it just amounts to* $\mathrm{Tr}_g\, a$.

*Proof.* — The first item follows from the evident chain of equalities

$$\mathrm{Tr}\, \hat{g}\, \hat{a}\, \hat{b} = \mathrm{Tr}\, \hat{g}\, \hat{a}\, \widehat{g^{-1}}\, \hat{g}\, \hat{b} = \mathrm{Tr}\, \hat{a}\, \hat{g}\, \hat{b} = \mathrm{Tr}\, \hat{g}\, \hat{b}\, \hat{a}.$$

We have used that $\hat{g}\, \hat{a}\, \widehat{g^{-1}} = \hat{a}$ since $a \in W$ satisfies the equivariance relations, and changed cyclically the order of factors under the trace sign.

For the second item write

$$\mathrm{Tr}\, \hat{\gamma}\, \hat{g}\, \widehat{\gamma^{-1}}\hat{a} = \mathrm{Tr}\, \hat{g}\, \widehat{\gamma^{-1}}\hat{a}\, \hat{\gamma} = \mathrm{Tr}\, \hat{g}\, \widehat{\gamma^* a}. \qquad \square$$

Next, we want to express the orbifold trace directly in terms of the symbol and, moreover, to extend it to formal symbols. Using the orthonormal basis (5.7) we obtain

$$\mathrm{Tr}_g\, a \;=\; \mathrm{Tr}\, \hat{g}\hat{a} \;=\; \sum_\alpha (\hat{g}\hat{a}e_\alpha, e_\alpha).$$

The sum may be calculated explicitly (see [12])

$$\mathrm{Tr}\, \hat{g}\hat{a} = \frac{1}{(\pi h)^n} \frac{1}{\det(1+g)} \int_{\mathbb{C}^n} \exp\Big( -\frac{1}{h} z^* \frac{1-g}{1+g} z \Big) \cdot \mathrm{tr}\, T(g) a(z,\bar{z})\, dz d\bar{z} \tag{5.16}$$

provided $\det(1+g) \neq 0$. Here $T(g)$ means the action of $G$ on the bundle $E$, tr means the coefficient trace. This is an integral form of the trace formula. To extend it to formal symbols, we first let $h$ vary on $(0, 1]$ and calculate the stationary phase expansion of the integral (5.16). To this end, decompose $\mathbb{C}^n$ into the direct sum of the fixed point subspace

$$\widetilde{F}(g) = \ker(1-g)$$

and its orthogonal complement $N(g)$. Writing $z = (z_1, z_2)$, $n_1 = \dim \widetilde{F}(g)$, $n_2 = \dim N(g)$, and $g = 1$ on $\widetilde{F}(g)$ and $g = g_2$ on $N(g)$, we get

$$\begin{aligned}\operatorname{Tr} \hat{g}\hat{a} = \frac{1}{(2\pi h)^{n_1}} \int_{\widetilde{F}(g)} dz_1 d\bar{z}_1 \frac{1}{(\pi h)^{n_2}} \frac{1}{\det(1+g_2)} \\ \cdot \int_{N(g)} \exp\Big(-\frac{1}{h} z_2^* \frac{1-g_2}{1+g_2} z_2\Big) \operatorname{tr} T(g) a(z_1, \bar{z}_1, z_2, \bar{z}_2) dz_2 d\bar{z}_2.\end{aligned}$$

To calculate the inner integral, we expand $a$ in a formal Taylor series at $z_2 = 0$, namely

$$a(z_1, \bar{z}_1, z_2, \bar{z}_2) = \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!} \frac{\partial^{\alpha+\beta}}{\partial z_2^\alpha \partial \bar{z}_2^\beta} a(z_1, \bar{z}_1, 0, 0)\, z_2^\alpha \bar{z}_2^\beta,$$

and integrate this expansion termwise, thus obtaining

$$\frac{1}{\det(1-g_2)} \exp\Big(h \frac{\partial}{\partial z_2} \frac{1+g_2}{1-g_2} \frac{\partial}{\partial z_2^*}\Big) \operatorname{tr} T(g)\, a(z_1, \bar{z}_1, z_2, \bar{z}_2)\Big|_{z_2=\bar{z}_2=0}.$$

Here $\partial/\partial z_2$ means the row $(\partial/\partial z_2^1, \ldots, \partial/\partial z_2^{n_2})$ and $\partial/\partial z_2^*$ the column of complex conjugate elements. Observe also that on $N(g)$ the matrix $1 - g_2$ is non-degenerate.

Consider further the symplectic form

$$\omega_{\widetilde{F}(g)} = \frac{1}{2\imath} (d\bar{z}_1^1 \wedge dz_1^1 + \cdots + d\bar{z}_1^{n_1} \wedge dz_1^{n_1}).$$

The measure $dz_1 d\bar{z}_1$ including the factor $(2\pi h)^{n_1}$ in the outer integral is defined by the top degree of the non-homogeneous form

$$\exp\Big(\frac{\omega_{\widetilde{F}(g)}}{2\pi h}\Big).$$

In the final expression we denote $(z_1, \bar{z}_1)$ by $\tilde{x}$, $(z_2, \bar{z}_2)$ by $(z, \bar{z})$ and $g_2$ by $g_N$. The trace formula reduces then to the so-called Weyl form

$$\begin{aligned}(5.17) \qquad \operatorname{Tr}_g a = \int_{\widetilde{F}(g)} \exp\Big(\frac{\omega_{\widetilde{F}(g)}}{2\pi h}\Big) \frac{1}{\det(1-g_N)} \\ \cdot \exp\Big(h \frac{\partial}{\partial z} \frac{1+g_N}{1-g_N} \frac{\partial}{\partial z^*}\Big) \operatorname{tr} T(g) a(\tilde{x}, z, \bar{z})\Big|_{z=\bar{z}=0}.\end{aligned}$$

We now return to deformation quantization setting. By formal symbols we mean here formal power series in $h$

$$a(z, \bar{z}, h) = \sum_{k=0}^{\infty} h^k a_k(z, \bar{z})$$

with coefficients $a_k$ belonging to $W$ or to $S$. We will use notation $W^h$ or $S^h$, respectively. We define $\mathrm{Tr}_g\ a$ by linearity treating the series (5.17) for $a_k$ as a formal power series in $h$. Our previous construction via Fock space representation serves only to prove the following lemma which is an analog of Lemma 5.2 in the deformation context.

LEMMA 5.3. — *The properties (5.14) and (5.15) hold for any formal symbols $a \in W^h$ and $b \in S^h$.*

*Proof.* — Since $\mathrm{Tr}_g$ is defined on formal symbols by linearity, it is sufficient to prove the lemma for $a$ and $b$ consisting of a single term only. Hence we may assume that $a$ is a symbol from $W$ and $b \in S$. By Lemma 5.2 the equalities (5.14) and (5.15) hold for any fixed $h \in (0, 1]$. Thus, the asymptotic series for both sides must coincide, hence the corresponding formal series in both sides coincide as well. □

In the rest of this section the representation in the Fock space is not needed any more. We will use only the definition of the $\mathrm{Tr}_g$ on the algebra $W^h$ given by (5.17) and its properties (5.14) and (5.15). The definition (5.13) for a local orbifold trace written in the form

$$\mathrm{Tr}_{\mathrm{orb}}\ a = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}_g\ a$$

makes sense now for any formal symbol $a \in W^h$.

Our aim is to globalize this definition. First, we show its independence of linearization $f_t$ and its quantum lifting $I_t$ given by (4.8).

LEMMA 5.4. — *For any fixed $a \in W^h$ and $g \in G$, the functional $\mathrm{Tr}_g\ I_t a$ is independent of $t$.*

*Proof.* — The family $I_t$ of automorphisms is defined by the Heisenberg equation (Lemma 4.4). The quantum Hamiltonian $H$ is a formal symbol from $S^h$. Applying Lemma 5.3, we get

$$\mathrm{Tr}_g\ \dot{a} \;=\; -\frac{\imath}{h}\ \mathrm{Tr}_g\,[H, a] \;=\; 0,$$

proving the lemma. □

Consider the case of two subordinate orbifold charts

$$\begin{array}{rcccl} \varphi & : & \widetilde{O}_0 & \to & \widetilde{O}_1, \\ i & : & G_0 & \to & G_1. \end{array}$$

For $a \in W^h(\widetilde{O}_0)$ and $g \in G_0$, the functional $\mathrm{Tr}_g\, a$ is independent of the choice of the linearization, which is due to Lemma 5.4. Hence we may assume that $\widetilde{O}_0 \subset \widetilde{O}_1$, so that $\varphi$ is inclusion, and the group $G_0$ is a subgroup of $G_1$, mapping $\widetilde{O}_0$ to itself. Taking cosets $\gamma_k G_0$, for $k = 1, \dots, m$, we obtain non-intersecting subsets $\gamma_k \widetilde{O}_0 \subset \widetilde{O}_1$, their union

$$U = \bigcup_{k=1}^{m} \gamma_k \widetilde{O}_0$$

is a $G_1$-invariant subset of $\widetilde{O}_1$. By $a_0 \in W^h(\widetilde{O}_0)$ we denote a $G_0$-invariant Weyl symbol with support in $\widetilde{O}_0$, a similar notation $a_1 \in W^h(U)$ is used for a $G_1$-invariant Weyl symbol with support in $U$. There is a bijection between these symbols: for any $a_1 \in W^h(U)$ its restriction to $\widetilde{O}_0$ gives a symbol $a_0 \in W^h(\widetilde{O}_0)$. The inverse map is as follows: We first extend $a_0$ by 0 to the whole $\widetilde{O}_0$, thus obtaining a formal symbol in $\widetilde{O}_0$ which is only $G_0$-invariant but not $G_1$-invariant. Keeping for this symbol the same notation $a_0$, define

$$a_1 = \sum_{k=1}^{m} \gamma_k^* a_0 = \frac{1}{|G_0|} \sum_{g \in G_1} g^* a_0.$$

LEMMA 5.5. — *The local trace* $\mathrm{Tr}_{\mathrm{orb}}$ *is independent of orbifold charts.*

*Proof.* — For the subordinate charts considered above we have

$$\begin{aligned}
\mathrm{Tr}_{\mathrm{orb}}\, a_1 &= \frac{1}{|G_1|} \sum_{g \in G_1} \mathrm{Tr}_g\, a_1 \\
&= \frac{1}{m\,|G_0|} \sum_{g \in G_1} \sum_{k=1}^{m} \mathrm{Tr}_g\, \gamma_k^* a_0.
\end{aligned}$$

By Lemma 5.3,

$$\sum_{g \in G_1} \mathrm{Tr}_g\, \gamma_k^* a_0 = \sum_{g \in G_1} \mathrm{Tr}_{\gamma_k g \gamma_k^{-1}}\, a_0 = \sum_{g \in G_1} \mathrm{Tr}_g\, a_0$$

for each $k = 1, \dots, m$, so that the previous expression is equal to

$$\frac{1}{|G_0|} \sum_{g \in G_1} \mathrm{Tr}_g\, a_0.$$

But by virtue of the trace formula (5.17) non-zero summands may occur only for $g \in G_0$ since otherwise there are no fixed points of $g$ on the support of $a_0$.

A general case of two charts $\widetilde{O}_j$ and $\widetilde{O}_k$ with non-empty intersections $O_j \cap O_k$ may be reduced to the case of subordinate charts since the intersection may be covered by a finite number of orbifold charts subordinate to both $\widetilde{O}_j$ and $\widetilde{O}_k$. □

Now, to define the orbifold trace for a global quantum observable $a \in A^h$ with compact support we use a partition of unity $\{\rho_i(x)\}$ to decompose $a$ in a sum

$$a = \sum_i (Q\rho_i) \circ a$$

and then take traces of each summand in a linearized orbifold chart containing the support of $\rho_i$, no matter which if there are many. More precisely, let $I_i : A^h \to A_0^h(\widetilde{O}_i)$ be the isomorphisms of Lemma 4.3, and let

$$\begin{aligned} a_i &= I_i a \\ &\in A_0^h(\widetilde{O}_i). \end{aligned}$$

Then we set by definition

$$\begin{aligned} \operatorname{Tr}_{\mathrm{orb}} a &= \sum_i \operatorname{Tr}_{\mathrm{orb}}\ a_i \\ &= \sum_i \frac{1}{|G_i|} \sum_{g \in G_i} \int_{\widetilde{F}(g)} \exp\left(\frac{\omega_{\widetilde{F}(g)}}{2\pi h}\right) \frac{1}{\det(1-g_N)} \\ &\qquad \exp\left(h \frac{\partial}{\partial z} \frac{1+g_N}{1-g_N} \frac{\partial}{\partial z^*}\right) \operatorname{tr} T_i(g) a_i(\tilde{x}, z, \bar{z})\Big|_{z=\bar{z}=0}. \end{aligned} \tag{5.18}$$

This is actually the desired global trace functional since it vanishes on commutators.

LEMMA 5.6. — *The functional*

$$\operatorname{Tr}_{\mathrm{orb}} :\ A_c^h \to \mathbb{C}[h^{-1}, h]] \tag{5.19}$$

*possesses the trace property*

$$\operatorname{Tr}_{\mathrm{orb}}\ a \circ b = \operatorname{Tr}_{\mathrm{orb}}\ b \circ a, \tag{5.20}$$

*where $A_c^h \subset A^h$ means the ideal of compactly supported observables.*

*Proof.* — Using a partition of unity, we have

$$\begin{aligned} \operatorname{Tr}_{\mathrm{orb}}\ a \circ b &= \sum_{i,j} \operatorname{Tr}_{\mathrm{orb}}\ a_i * b_j \\ &= \sum_{i,j} \sum_{g \in G_i} \frac{1}{|G_i|} \operatorname{Tr}_g\ a_i * b_j \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} \sum_{g \in G_i} \frac{1}{|G_i|} \operatorname{Tr}_g b_j * a_i \\
&= \sum_{i,j} \sum_{g \in G_j} \frac{1}{|G_j|} \operatorname{Tr}_g b_j * a_i \\
&= \operatorname{Tr}_{\mathrm{orb}} b \circ a.
\end{aligned}
$$

The summation, of course, runs over those pairs $i$, $j$ for which the intersection $O_i \cap O_j$ is non-empty. In the second and third lines the local traces $\operatorname{Tr}_g$ are taken in the charts $\widetilde{O}_i$ while in the fourth line in $\widetilde{O}_j$. $\square$

We are now going to transform the formula (5.18) to a more invariant form. Note that the inner sum (over $g \in G_i$) contains many equal summands. Indeed, the integrals over $\widetilde{F}(g)$ and $\widetilde{F}(\gamma g \gamma^{-1}) = \gamma \widetilde{F}(g)$ are the same because of $G_i$-invariance of the integrand. Thus, decomposing $G_i$ into a union of conjugacy classes and choosing a representative $g_k$ in each conjugacy class $(g_k)$, we can rewrite the inner sum as

$$
\sum_{(g_k) \in (G_i)} \frac{|(g_k)|}{|G_i|} \int_{\widetilde{F}(g_k)} \cdots .
$$

Here $(G_i)$ denotes the set of conjugacy classes of $G_i$, $g_k$ runs over the whole set of representatives, and $|(g_k)|$ denotes the number of elements in the conjugacy class. We have dropped the integrand, it is the same as in (5.18) with $g$ replaced by $g_k$. To calculate the number $|(g_k)|$ observe that the elements $\gamma_1 g_k \gamma_1^{-1}$ and $\gamma_2 g_k \gamma_2^{-1}$ coincide if and only if $\gamma_1 \gamma_2^{-1}$ commutes with $g_k$, that is belongs to the *centralizer* of $g_k$ in $G_i$. (Recall that the centralizer $Z_{G_i}(g_k)$ is a subgroup in $G_i$ consisting of elements commuting with $g_k$). To obtain all distinct elements $\gamma g_k \gamma^{-1}$ the element $\gamma$ should run over all representatives of left cosets $G_i / Z_{G_i}(g_k)$. Thus,

$$
|(g_k)| = \frac{|G_i|}{|Z_{G_i}(g_k)|}
$$

and the sum takes the form

$$
\sum_{(g_k) \in (G_i)} \frac{1}{|Z_{G_i}(g_k)|} \int_{\widetilde{F}(g_k)} \cdots
$$

The summands here may be interpreted as integrals over linearized charts of some symplectic orbifolds. Indeed, $\widetilde{F}(g_k) = \ker(1 - (g_k)_N)$ is a complex space where the group $Z_{G_i}(g_k)$ acts by linear unitary transformations since

$$
\gamma \widetilde{F}(g_k) = \widetilde{F}(\gamma g_k \gamma^{-1}) = \widetilde{F}(g_k).
$$

Thus, introducing the notation $F(g_k)$ for the orbit space $\widetilde{F}(g_k)/Z_{G_\imath}(g_k)$ and $p(g_k)$ for the corresponding projection, we come to a quadruple

$$\{F(g_k), \widetilde{F}(g_k), Z_{G_\imath}(g_k), p(g_k)\} \tag{5.21}$$

which resembles an orbifold chart for some symplectic orbifold. In fact there are many connected symplectic orbifolds of different dimensions (and even for a given dimension there are many connected components) whose orbifold charts form the set (5.21) with all possible $k$ and $i$. We will denote these connected components by $F_1, F_2, \ldots, F_m$ and call them *fixed point orbifolds*. To define these orbifolds completely, we need to indicate what pairs from the set (5.21) should be glued. Each chart (5.21) is contained in a chart $\widetilde{O}_i$ of the original orbifold $B$. If $O_i \cap O_j = \emptyset$ then, clearly, there is no gluing conditions for the charts $\widetilde{F}(g_k) \in \widetilde{O}_i$ and $\widetilde{F}(g_l) \in \widetilde{O}_j$. Otherwise, if $O = O_i \cap O_j \neq \emptyset$, we glue $F(g_k) \in O_i$ and $F(g_l) \in O_j$ if and only if $F(g_k) \cap O = F(g_l) \cap O$. A connected component of the fixed point orbifold consists of those charts from the set (5.21) which can be connected with each other by a chain of pairwise glued charts (5.21).

The action of the centralizer $Z_{G_\imath}(g_k)$ on $\widetilde{F}(g_k)$ may be not effective, that is the components $F_k$ may have a multiplicity $m(F_k) > 1$, even if the multiplicity of the original orbifold $B$ was equal to 1. Then by the integration formula (3.4)

$$\operatorname{Tr}_{\text{orb}}\, a = \sum_k \frac{1}{m(F_k)} \int_{F_k} \alpha_k \tag{5.22}$$

where the integrand $\alpha_k$ is a differential form on $F_k$ which in an orbifold chart $\widetilde{F}(g)$ is defined by

$$\alpha = \exp\left(\frac{\omega_{\widetilde{F}(g)}}{2\pi h}\right) \frac{1}{\det(1-g_N)} \exp\left(h \frac{\partial}{\partial z} \frac{1+g_N}{1-g_N} \frac{\partial}{\partial z^*}\right) \cdot \operatorname{tr} T_i(g) a_i(\tilde{x}, z, \bar{z})\Big|_{z=\bar{z}=0}. \tag{5.23}$$

This finishes the construction of the orbifold trace.

The following simple observation shows that the trace is not unique in contrast to the smooth case. For any flat one-dimensional bundle $E^1 \in \mathcal{E}^1$ the replacement of $E$ by $E \otimes E^1$ does not change the algebra of quantum observables (see Remark 4.2). On the other hand, local representations $T_i(g)$ will be changed according to (3.6). So, in the final formula (5.23) the representation $T_i(g)$ should be replaced by $T_i(g)\,\chi_i(g)$. In other words, we have an action of the Picard group on the set of traces.

Summarizing, we arrive at the following theorem.

THEOREM 5.7. — *There exist trace functionals (5.19) on the algebra $A^h$ satisfying the trace property (5.20). The Picard group acts on the set of traces by tensoring.*

One of the components in (5.22) coincides with the original orbifold $B$ whose multiplicity is equal to 1 according to the assumption in Section 2. This case corresponds to the conjugacy class (1) of the identity element in any local group $G_i$, the centralizer is then the whole group $G_i$. For $g = 1$ the integrand (5.23) takes a more simple form

$$\alpha = \exp\left(\frac{\omega}{2\pi h}\right) \operatorname{tr} a_i(\tilde{x}, h).$$

This gives us an integral over the principal stratum having the same form as in the case of smooth manifold. The other components are even-dimensional, and their dimension is at least by 2 less than the dimension of $B$.

## 6. An index formula.

In this section we propose a conjecture for the index formula prompted by the Kawasaki index theorem [13], the index theorem for deformation quantization [9] and the $G$-index formula [12]. For the time being we have a proof only in very particular cases, *cf.* [11]. We hope however to find a complete proof.

Restricting ourselves to the simplest case of a compact orbifold $B$, we define the *index* of the algebra of quantum observables $A^h$ as $\operatorname{Tr}_{\mathrm{orb}} 1$.

Let us look at the integral over one of the fixed point orbifolds $F_m$ in (5.22), (5.23). Assuming that the original coefficient bundle was $K = \operatorname{End}(E)$ over $B$, one can recognize in this integral an expression for the orbifold trace for a deformation quantization on $F_m$ with a coefficient bundle $K \otimes W(N)$ where $W(N)$ is the Weyl algebra in fibers of the normal bundle $N$ of $F_m$ with the coefficient trace on $W(N)$ equal to $\operatorname{Tr}_g$. Treating $W(N)$ as the isomorphism bundle of the Fock bundle $\mathbb{F}(N)$ and proceeding by analogy with the index theorem for deformation quantization, *cf.* [9], we come to the following conjecture

$$\operatorname{Tr}_{\mathrm{orb}} 1 = \sum_k \frac{1}{m(F_k)} \int_{F_k} \exp\left(\frac{\omega_{F_k}}{2\pi h}\right) \operatorname{ch}_g\left(E \otimes \mathbb{F}(N)\right) \widehat{A}(F_k).$$

Here $\mathrm{ch}_g$ means the character of the bundle $E \otimes \mathbb{F}(N)$ with respect to $\mathrm{Tr}_g$. In more detail,

$$\begin{aligned}\mathrm{ch}_g\,(E \otimes \mathbb{F}(N)) &= \mathrm{ch}_g\ E\ \mathrm{ch}_g\ \mathbb{F}(N)\\ &= \mathrm{tr}\, T(g) \exp\left(\frac{R^E}{2\pi\imath}\right) \mathrm{Tr}_g\ \exp\left(\frac{R^{\mathbb{F}}}{2\pi\imath}\right).\end{aligned}$$

To define the character of the Fock bundle, we introduce an Hermitian connection $\partial^N$ on the normal bundle $N$ and an associated connection on $\mathbb{F}(N)$. If

$$\Gamma^N = \Gamma^N_{\alpha\beta k} d\tilde{x}^k$$

is the connection form on $N$ then we take the form

$$\Gamma^{\mathbb{F}} = \frac{1}{2h}\, z^* \Gamma^N * z = \frac{1}{2h}\, \Gamma^N_{\alpha\beta k} \bar{z}^\alpha * z^\beta d\tilde{x}^k$$

as the connection form on $\mathbb{F}(N)$. The normal ordering in the last expression is very important, it implies that the vacuum is covariantly constant: $\partial^N e_0 = 0$. The curvature of this connection is

$$R^{\mathbb{F}} = \frac{1}{2h}\, z^* R^N * z = \frac{1}{4h}\, R^N_{\alpha\beta kl} \bar{z}^\alpha * z^\beta d\tilde{x}^k \wedge d\tilde{x}^l \tag{6.1}$$

where $R^N$ means the curvature of the normal bundle $N$.

LEMMA 6.1. — *The following formula holds*

$$\mathrm{Tr}_g\ \exp\left(\frac{R^{\mathbb{F}}}{2\pi\imath}\right) = 1/\det\left(1 - g^{-1} \exp\left(\frac{R^N}{2\pi\imath}\right)\right). \tag{6.2}$$

*Proof.* — The differential form

$$S = \exp\left(\frac{R^{\mathbb{F}}}{2\pi\imath}\right)$$

is meaningful as an operator in the Fock space where $h \in (0,1]$ is a number. Indeed, $S$ is a polynomial in $z$, $\bar{z}$, so the action (5.8) is well defined. Because of the normal ordering this action is trivial on the vacuum vector, *i.e.* $S * p = p$. Thus, for a vector $u = a(z,\bar{z}) * p \in \mathbb{F}$ we have

$$\begin{aligned}Su &= S * a * p\\ &= S * a * S^{-1} * p.\end{aligned}$$

Thus, we need to know the adjoint action of $S$ on $W$, and because it is an automorphism of $W$ it is sufficient to know the action on generators $z^i$, $\bar{z}^i$. It may be calculated using the well-known formula

$$\exp(A)\, B\, \exp(-A) = \exp([A,\cdot])\, B.$$

Since $R^{\mathbb{F}}$ is quadratic in generators we have

$$\Big[\frac{R^{\mathbb{F}}}{2\pi\imath}, z^i\Big] = \Big[\frac{z^* R^N * z}{2h\, 2\pi\imath}, z^i\Big] = -\Big(\frac{R^N z}{2\pi\imath}\Big)^i,$$

the superscript $i$ indicating the $i$-th coordinate, and similarly

$$\Big[\frac{R^{\mathbb{F}}}{2\pi\imath}, \bar{z}^i\Big] = \Big[\frac{z^* R^N * z}{2h\, 2\pi\imath}, \bar{z}^i\Big] = \Big(\frac{z^* R^N}{2\pi\imath}\Big)^i.$$

If $z$ denotes a column and $z^*$ a row, then it results in

$$\begin{aligned} S * z * S^{-1} &= \exp\Big(-\frac{R^N}{2\pi\imath}\Big) z &= s^{-1} z, \\ S * z^* * S^{-1} &= z^* \exp\Big(\frac{R^N}{2\pi\imath}\Big) &= z^* s \end{aligned}$$

where $s$ denotes the matrix $\exp(R^N/2\pi\imath)$. Now, by the definition of the trace $\mathrm{Tr}_g$, we get

$$\begin{aligned} \mathrm{Tr}_g\, S &= \mathrm{Tr}\, \hat{g} S \\ &= \sum_\alpha (\hat{g} S e_\alpha, e_\alpha) \\ &= \int_{\mathbb{C}^n} \sum_\alpha \frac{p * z^\alpha * (z^* g^{-1} s)^\alpha * p}{\alpha!\, (2h)^{|\alpha|}} \exp\Big(\frac{\omega}{2\pi h}\Big). \end{aligned}$$

The series (which in fact is a finite sum) under the integral sign may be simplified as follows. First, the last factor $p$ may be written in the first place, because the integral is a trace for the Weyl algebra on $\mathbb{C}^n$, and then may be omitted since $p$ is a projector with respect to the $*$-product on $W$. Further, for a polynomial $f(z)$ the following formula is true

$$p * f = \sum_\alpha \frac{(-h)^{|\alpha|}}{\alpha!} \frac{\partial^\alpha p}{\partial \bar{z}^\alpha} f^{(\alpha)}(z) = p \sum_\alpha \frac{z^\alpha}{\alpha!} f^{(\alpha)}(z) = p\, f(2z).$$

Finally, using the trace property of the integral, we may omit the last remaining $*$-product, obtaining

$$\int_{\mathbb{C}^n} a * b\, dz d\bar{z} = \int_{\mathbb{C}^n} ab\, dz d\bar{z},$$

so that the trace formula reduces to

$$\begin{aligned} \mathrm{Tr}_g\, S &= \int_{\mathbb{C}^n} p \sum_{k=0}^\infty \frac{(z^* g^{-1} s z)^k}{k!\, h^k} \exp\Big(\frac{\omega}{2\pi h}\Big) \\ &= \int_{\mathbb{C}^n} 2^n \exp\Big(-\frac{1}{h} z^* (1 - g^{-1} s) z\Big) \exp\Big(\frac{\omega}{2\pi h}\Big). \end{aligned}$$

Calculating this Gaussian integral, we come to (6.2). □

Thus, our conjecture takes the form

$$\mathrm{Tr}_{\mathrm{orb}}\, 1 = \sum_k \frac{1}{m(F_k)} \int_{F_k} \exp\left(\frac{\omega_{F_k}}{2\pi h}\right) \cdot \frac{\mathrm{ch}_g\, E}{\det(1 - g^{-1}\exp(R^N/2\pi\imath))} \widehat{A}(F_k). \tag{6.3}$$

## 7. Examples.

The purpose of this section is to interpret an index theorem for deformation quantization as an eigenvalue problem. In a particular case we come to our index theorem for symplectic orbifolds.

Consider the symplectic space $M = \mathbb{C}^2$ with the standard symplectic form

$$\omega = \frac{1}{2\imath}\,(d\bar{z}_1 \wedge dz_1 + d\bar{z}_2 \wedge dz_2), \tag{7.1}$$

and the Hamiltonian action of the group $G = \mathbb{R}$ with the Hamiltonian

$$H = \frac{1}{2}\,(|z_1|^2 + c\,|z_2|^2) \tag{7.2}$$

where $c > 0$ is a fixed number. The orbits are as follows

$$\begin{array}{ccc} z_1 & \mapsto & e^{-\imath t} z_1, \\ z_2 & \mapsto & e^{-\imath ct} z_2, \end{array} \tag{7.3}$$

for $t \in \mathbb{R}$, and we can distinguish two different cases:

1) $c = 1$;

2) $c$ is a rational number, $c \neq 1$ (we take further $c = 3$).

Take a number $\lambda$ which is a non-critical value of the Hamiltonian, that is $\lambda > 0$, and consider symplectic reduction at the level $\lambda$ in each of the two cases.

For $c = 1$ the level set $H = \lambda$ is a sphere $M_0 = S^3$ in $\mathbb{C}^2$ and all the orbits are periodic, so, taking $t \in \mathbb{R} \mod 2\pi$, we have a free action (7.3) of the group $G = \mathrm{U}(1) = \mathbb{S}^1$. The orbits are big circles, and we thus obtain the Hopf fibration

$$B = M_0/G = S^3/\mathbb{S}^1 = \mathbb{CP}^1 \tag{7.4}$$

whose base is a smooth manifold.

If $c = 3$ (this is our main example), the level set $M_0$ is an ellipsoid

$$|z_1|^2 + 3|z_2|^2 = \sqrt{2\lambda}. \tag{7.5}$$

Taking $t \in \mathbb{R} \mod 2\pi$ as in the previous case, we again obtain an action (7.3) of the group $G = \mathrm{U}(1)$ on the level manifold. This time, however, the action is only locally free. Indeed, each point of the type $(0, z_2)$ with $|z_2| = \sqrt{2\lambda}/3$ is a fixed point of the action (7.3) with $t = 0, \pm 2\pi/3$. The orbit space $B = M_0/G$ is an orbifold, as we shall see soon.

Let us try now to give a quantum interpretation to the reduction procedure. To this end, suppose first that our quantum observables are operators depending on a positive parameter $h$ and acting in some Hilbert space $E$. Algebraically, the reduction procedure goes in two steps. First we restrict ourselves to the subalgebra of invariant observables (let us denote temporarily this subalgebra by $A$). In quantum case the invariance means that $a$ commutes with $H - \lambda$

$$[H - \lambda, a] = 0. \tag{7.6}$$

This implies that each eigenspace of $H - \lambda$ is invariant with respect to any operator $a \in A$. At the second step we consider the restriction $a_0$ of the operator $a \in A$ to the zero eigenspace $E_0$ of $H - \lambda$. If the eigenspace is non-trivial the restriction gives an operator $a_0$ in $E_0$. We obtain thus an algebra $A_0 = A/(H - \lambda)$ of operators in the eigenspace $E_0$ which may be viewed as the reduced algebra of quantum observables. The multiplicity of the zero eigenvalue (which is a positive integer number) is given by the trace of the identity operator $1 \in A_0$

$$\dim E_0 = \operatorname{Tr} 1. \tag{7.7}$$

So, the eigenvalue problem may be reformulated in an equivalent way: for a fixed $\lambda$ find admissible values of $h \in \Lambda \subset (0, 1]$ for which the multiplicity $\operatorname{Tr} 1$ is non-zero and thus belongs to $\mathbb{N}$.

In the deformation quantization framework we cannot pose an eigenvalue problem literally, since our algebra of quantum observables $\mathcal{A}^h$ on $M$ is not an operator algebra. Nevertheless we have good substitutes for the corresponding notions allowing us to reformulate the eigenvalue problem, so that it makes sense for deformation quantization. Namely, we replace the algebra $A_0$ by the algebra of flat sections $\mathcal{A}^h$ on the base manifold (or orbifold) $B$ and the multiplicity $\operatorname{Tr} 1$ by the index of $\mathcal{A}^h$. It is essential that the index is a polynomial $P(1/h)$ in $1/h$, thus numerical values of $h$ may be substituted. We now propose the following version of the eigenvalue problem in deformation quantization terms:

*For a given $\lambda \in \mathbb{R}$ find $h \in (0, 1]$ for which the index $P(1/h)$ takes positive integer values.*

Let us start with the simplest case $c = 1$. We compare two versions of the eigenvalue problem, namely the traditional eigenvalue problem for the quantum harmonic oscillator with the Weyl symbol

$$
\begin{aligned}
H &= \frac{1}{2}\left(\bar{z}_1 z_1 + \bar{z}_2 z_2\right) - \lambda \\
&= \frac{1}{2}\left(\bar{z}_1 * z_1 + \bar{z}_2 * z_2\right) + h - \lambda
\end{aligned} \tag{7.8}
$$

and the above deformation quantization version. In the sequel they will be referred to as the traditional and deformation versions. Note that in the former we consider $h$ as a number while in the latter $h$ is a formal parameter.

The traditional spectrum may be found explicitly using the Fock space representation. In fact, the complete set of eigenfunctions are obtained by acting on the vacuum by creation operators

$$u_{n_1,n_2} = \bar{z}_1^{n_1} \bar{z}_2^{n_2} * p, \tag{7.9}$$

with the corresponding eigenvalues equal to

$$h\left(n_1 + n_2 + 1\right) - \lambda,$$

where $n_1 \geqslant 0$, $n_2 \geqslant 0$ are integer numbers, *cf.* Fig. 1.

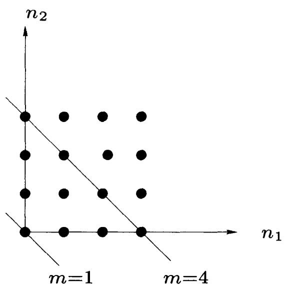


Fig. 1: $c = 1$

It follows that the Hamiltonian (7.8) has zero eigenvalue if and only if the ratio $m = \lambda/h$ is a positive integer number, this number gives us precisely the multiplicity of the eigenvalue.

The deformation version of the eigenvalue problem has the following form. First calculate the index of the reduced algebra $A^h$ on $\mathbb{CP}^1$ which gives us an a priori multiplicity of the zero eigenvalue of the Hamiltonian (7.8). This begins with the calculation of the reduced symplectic form $\omega_B$. Recall that the latter is uniquely defined from the equality

$$i^* \omega_M = p^* \omega_B$$

on the level manifold $M_0$, where $\imath$ and $p$ are inclusion and projection

$$M \xleftarrow{\imath} M_0 \xrightarrow{p} B.$$

Here $M_0$ is the sphere $|z_1|^2 + |z_2|^2 = 2\lambda$, and $B$ is the orbit space of the group action

$$\begin{aligned} z_1 &\mapsto e^{-\imath t} z_1, \\ z_2 &\mapsto e^{-\imath t} z_2, \end{aligned}$$

that is the projective space $\mathbb{CP}^1$. The orbits may be parametrized by the ratio $\zeta = z_1/z_2$ if $z_2 \neq 0$, and by the inverse ratio if $z_1 \neq 0$. Thus, the symplectic form $\omega_B$ should be $\alpha\, d\bar{\zeta} \wedge d\zeta/2\imath$. In polar coordinates

$$\zeta = \frac{r_1}{r_2}\, e^{\imath(\varphi_1 - \varphi_2)},$$

so that

$$\omega_B = \frac{\alpha}{2}\, d\Big(\frac{r_1^2}{r_2^2}\Big) \wedge d(\varphi_1 - \varphi_2).$$

Replacing $r_2^2 = 2\lambda - r_1^2$ on the sphere $M_0$, we find

$$\omega_B = \frac{\alpha\lambda}{(2\lambda - r_1^2)^2}\, dr_1^2 \wedge d(\varphi_1 - \varphi_2).$$

On the other hand, using polar coordinates, we get

$$\omega_M = \frac{1}{2}\,(dr_1^2 \wedge d\varphi_1 + dr_2^2 \wedge d\varphi_2)$$

or, eliminating $r_2^2$,

$$\frac{1}{2}\, dr_1^2 \wedge d(\varphi_1 - \varphi_2).$$

Equating these two expressions gives us

$$\begin{aligned} \alpha &= \frac{(2\lambda - r_1^2)^2}{2\lambda} \\ &= \frac{2\lambda}{(1 + |\zeta|^2)^2}. \end{aligned}$$

Thus, the reduced form is

$$\omega_B = \frac{\lambda}{\imath}\, \frac{d\bar{\zeta} \wedge d\zeta}{(1 + |\zeta|^2)^2}.$$

Integrating this form over $B = \mathbb{CP}^1$, we obtain

$$\operatorname{Tr} 1 = \int_B \frac{\omega_B}{2\pi h} = \frac{\lambda}{h}.$$

According to our deformation version of the eigenvalue problem, the spectrum is obtained by equating this ratio to positive integer numbers, the ratio itself being a multiplicity. We see that in this case both spectra, traditional and deformation, coincide.

Consider now the second case $c = 3$. The Hamiltonian in this case is

$$H = \frac{1}{2}\left(\bar{z}_1 * z_1 + 3\bar{z}_2 * z_2 + 4h\right) - \lambda. \tag{7.10}$$

Similarly to the case 1) the traditional spectrum may be calculated explicitly. The eigenfunctions are the same, namely

$$u_{n_1,n_2} = \bar{z}_1^{n_1}\bar{z}_2^{n_2} * p$$

with the eigenvalues

$$h\left(n_1 + 3n_2 + 2\right) - \lambda,$$

where $n_1 \geqslant 0$ and $n_2 \geqslant 0$ are integer numbers. Thus, the Hamiltonian (7.10) has zero eigenvalue if the ratio $\lambda/h$ is equal to $N + 2$, for $N = 0, 1, 2, \ldots$, and the multiplicity of this eigenvalue is equal to the number of lattice points on the line $n_1 + 3n_2 = N$, *cf.* Fig. 2.

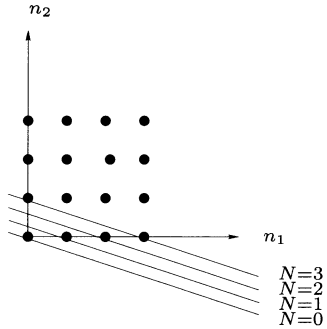


Fig. 2: $c = 3$

It is convenient to consider three series: $N = 3k$, $N = 3k + 1$ and $N = 3k + 2$ where $k = 0, 1, \ldots$. The multiplicity of the zero eigenvalue is equal to $k + 1$ for each of these series. Let us express the multiplicity as the function of the ratio $\lambda/h$ for each series. If $N = 3k$ then $\lambda/h - 2 = 3k$, and for the multiplicity $m = k + 1$ we obtain

$$m = \frac{\lambda}{3h} + \frac{1}{3}. \tag{7.11}$$

Similarly for the series $N = 3k + 1$ and $N = 3k + 2$ we find

$$m = \frac{\lambda}{3h} \tag{7.12}$$

and

$$m = \frac{\lambda}{3h} - \frac{1}{3}. \tag{7.13}$$

Thus, we have the following description of the traditional spectrum: $h$ belongs to the spectrum if one of the expressions $\lambda/3h - 1/3$, $\lambda/3h$, $\lambda/3h + 1/3$ takes a positive integer value.

Now, let us calculate the spectrum in deformation version. The zero level set $M_0$ is an ellipsoid

$$|z_1|^2 + 3\,|z_2|^2 = 2\lambda,$$

and the action of the group $\mathrm{U}(1)$ is

$$\begin{aligned} z_1 &\mapsto e^{-\imath t} z_1, \\ z_2 &\mapsto e^{-3\imath t} z_2. \end{aligned} \tag{7.14}$$

The orbifold charts for the orbit space are defined by two slices

$$S_1 = \left(z_1, \sqrt{\frac{2\lambda - |z_1|^2}{3}}\right), \quad |z_1| < \sqrt{2\lambda},$$

and

$$S_2 = \left(\sqrt{2\lambda - 3|z_2|^2}, z_2\right), \quad |z_2| < \sqrt{\frac{2\lambda}{3}}.$$

The group (7.14) maps $S_1$ into itself for $t = 0$, $\pm 2\pi/3$, and the only element which maps $S_2$ into itself is identity. Thus, for the orbifold charts one can take two discs

$$\widetilde{O}_1 = (|z_1| < \sqrt{2\lambda}), \quad G_1 = \mathbb{Z}_3,$$

with the standard action of $\mathbb{Z}_3$ by multiplication, and

$$\widetilde{O}_2 = (|z_2| < \sqrt{2\lambda/3}, \quad G_2 = 1.$$

The Picard group corresponds to the three different characters of the group $G_1$, namely,

$$\chi_0(g) = 1, \quad \chi_{\pm 1}(g) = g^{\pm 1}.$$

Indeed, the kernel of $\chi$ in (3.5) is trivial, so the Picard group coincides with the group of characters of $G_1$. It may be easily seen because our "teardrop" orbifold is topologically a 2-sphere which is smooth except one conical point. If monodromies of a flat bundle around this conical point are trivial then the bundle is trivial. Indeed, it has a non-vanishing flat section, defined on the smooth part and having a limit at the conical point, since there are no non-trivial monodromies.

The fixed point orbifolds $F_k$ consist of the orbifold $B = M_0/\,\mathrm{U}(1)$ itself corresponding to $g = 1$ in each chart, and a zero-dimensional component $F_0$ corresponding to non-trivial elements of the group $G_1 = \mathbb{Z}_3$ in the orbifold chart $\widetilde{O}_1$, that is $z_1 = 0$.

For a trivial bundle $E$ the term ch $E$ disappears, the term $\widehat{A}(F)$ also disappears since $\dim F \leqslant 2$. Thus for each of the three characters our index formula takes the form

$$\operatorname{ind}_\chi = \int_B \frac{\omega_B}{2\pi h} + \frac{1}{3} \sum_{g \neq 1} \frac{\chi(g)}{1-g}.$$

The integral term in the orbifold chart $\widetilde{O}_1$ reduces to

$$\frac{1}{3} \int_{|z_1| < \sqrt{2\lambda}} \frac{\omega_B}{2\pi h \, 2\imath} = \frac{\lambda}{3h}.$$

Now, for $\chi_0 = 1$ the additional term in the index formula is equal to

$$\frac{1}{3} \Big( \frac{1}{1 - e^{2\pi\imath/3}} + \frac{1}{1 - e^{-2\pi\imath/3}} \Big) = \frac{1}{3}.$$

Similarly, for $\chi_{-1} = g^{-1}$

$$\frac{1}{3} \Big( \frac{e^{-2\pi\imath/3}}{1 - e^{2\pi\imath/3}} + \frac{e^{2\pi\imath/3}}{1 - e^{-2\pi\imath/3}} \Big) = 0,$$

and for $\chi_1 = g$

$$\frac{1}{3} \left( \frac{e^{2\pi\imath/3}}{1 - e^{2\pi\imath/3}} + \frac{e^{-2\pi\imath/3}}{1 - e^{-2\pi\imath/3}} \right) = -\frac{1}{3}.$$

Thus, we see again that the deformation spectrum coincides with the traditional one.

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