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ON HOLOMORPHIC MAPS INTO COMPACT NON-KÄHLER MANIFOLDS

by Masahide KATO & Noboru OKADA

Introduction.

In this paper, we shall consider the extension problem of holomorphic maps of a Hartogs domain into compact complex manifolds. Fix an integer $n \geqslant 2$. By a *Hartogs domain*, we shall mean a domain $H^n_{r\rho}$ in complex n-dimensional Euclidean space $\mathbb{C}^n$ defined by

$$H^n_{r\rho} = G_1 \cup G_2,$$

where $0 < \rho < 1$, $0 < r < 1$, and

$$G_1 = \{z = (z_1, \dots, z_{n-1}, z_n) \in \mathbb{C}^n : |z_j| < \rho, 1 \leqslant j \leqslant n-1, |z_n| < 1\},$$
$$G_2 = \{z = (z_1, \dots, z_{n-1}, z_n) \in \mathbb{C}^n : |z_j| < 1, 1 \leqslant j \leqslant n-1, 1-r < |z_n| < 1\}.$$

Put

$$\Delta^k(s) = \{(z_1, \dots, z_k) \in \mathbb{C}^k : |z_j| < s, \quad 1 \leqslant j \leqslant k\}, \quad \Delta^k = \Delta^k(1).$$

Let X be a compact complex manifold and $\sigma : H^n_{r\rho} \to X$ a holomorphic map. We consider the holomorphic extendability of σ. Suppose

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that σ extends holomorphically to $\hat{\sigma} \ : \ \Delta^n \setminus A \to X$ outside a closed subset A of Δ^n. If X is a projective algebraic manifold, then it is well-known that for any σ, A can be chosen to be an analytic subset of complex codimension 2 in Δ^n. Ivashkovich [I1] proved the same fact for compact Kähler manifolds. If X is compact non-Kähler, then there are many examples where A are necessarily non-empty subsets with complicated structures. There are examples of compact complex manifolds for which σ cannot be extended across closed sets with interior points([Kt2]), or with various fractal dimensions([O]).

Motivated by Krachni [Kr], we think we may be able to obtain some informations of the complex structure of a given compact complex manifold, by considering holomorphic maps of Hartogs domains into the given manifold. This paper is our first attempt to probing manifolds by holomorphic maps. We shall give the definition of an index which measures the extendability of holomorphic maps and study its properties.

This paper is organized as follows. In section 1, we shall recall some facts on analytic continuations of holomorphic maps and give basic definitions. In section 2, we shall define *the holomorphic extension index.* In section 3, we shall give a kind of characterization of the index (Theorem 2). In section 4, we shall prove some properties of the indices (Theorems 4, 3, 4). In section 5, we shall give examples. Lastly in an appendix, we shall give proofs of some properties on Hausdorff dimensions in relation to holomorphic maps. Some of them are used in earlier sections. Applications of our results will be published elsewhere. Obviously, it is unavoidable to consider also meromorphic extendability, which we shall consider in the forthcoming papers.

1. The maximal extension of a holomorphic map.

Let Ω and M be complex manifolds and $\pi : \Omega \to M$ a holomorphic map. We say that (Ω, π) is an *étale domain* over M, if Ω is connected and π is locally biholomorphic. Fix a complex manifold X and a Stein manifold M. For an étale domain (Ω, π) over M and a holomorphic map $f : \Omega \to X$, we call the triple $(\Omega, \pi; f)$ a *map element* to X over M.

Definition 1 (Extension). — *Consider map elements $(\Omega_\alpha, \pi_\alpha; f_\alpha)$ and $(\Omega_\beta, \pi_\beta; f_\beta)$ over a Stein manifold M. We say that $(\Omega_\beta, \pi_\beta; f_\beta)$ is an extension of $(\Omega_\alpha, \pi_\alpha; f_\alpha)$ over M, if there exist an étale map $\lambda_{\beta\alpha}$:*

$\Omega_\alpha \to \Omega_\beta$ such that $f_\alpha = f_\beta \circ \lambda_{\beta\alpha}$, $\pi_\alpha = \pi_\beta \circ \lambda_{\beta\alpha}$. We denote this extension $(\Omega_\beta, \pi_\beta; f_\beta)$ by $(\Omega_\beta, \pi_\beta, \lambda_{\beta\alpha}; f_\beta)$. Further, we say that an extension $(\Omega, \pi, \lambda_\alpha; f)$ of $(\Omega_\alpha, \pi_\alpha; f_\alpha)$ is said to be maximal over M, if, for any extension $(\Omega_\beta, \pi_\beta, \lambda_{\beta\alpha}; f_\beta)$ of $(\Omega_\alpha, \pi_\alpha; f_\alpha)$ over M, there exists an étale map $\lambda_\beta : \Omega_\beta \to \Omega$ such that $\lambda_\alpha = \lambda_\beta \circ \lambda_{\beta\alpha}$, $\pi_\beta = \pi \circ \lambda_\beta$ and $f_\beta = f \circ \lambda_\beta$.

THEOREM 1 (Malgrange). — *Let X and M be complex manifolds. Then, for any map element $(\Omega_\alpha, \pi_\alpha; f_\alpha)$ to X over M, there exits a maximal extension of f_α over M, which is unique up to isomorphism.*

Proof. — Our proof is a copy of the argument of [M] pp.28-32. We define a sheaf of germs of holomorphic maps of M into X. Let $z \in M$. Consider the set of holomorphic maps into X of open neighborhoods of z. We introduce an equivalence relation in this set of maps by identifying two maps φ and ψ, if $\varphi = \psi$ on a neighborhood of z. The equivalence classes are called *germs of holomorphic maps* into X at z. The germ defined by φ at z is denoted by φ_z. The stalk of germs of holomorphic maps into X at z is denoted by $\mathcal{O}_z^X$. The sheaf of germs of holomorphic maps into X on M is defined by

$$\mathcal{O}_M^X := \bigcup_{z \in M} \mathcal{O}_z^X.$$

A complex analytic structure can be put on $\mathcal{O}_M^X$ in the following way. Let $z \in M$ and let $\varphi_z \in \mathcal{O}_z^X$. φ_z is represented by a holomorphic map φ defined on a neighborhood U of z. For every $w \in U$, φ defines a germ φ_w at w. We define

$$U_z := \bigcup_{w \in U} \varphi_w$$

to be a neighborhood of φ_z. It is easy to verify that this defines a topology on $\mathcal{O}_M^X$ and $\mathcal{O}_M^X$ is a Hausdorff space.

For the map element $(\Omega_\alpha, \pi_\alpha; f_\alpha)$ over M, take a point $\tilde{z} \in \Omega_\alpha$ and put $z = \pi_\alpha(\tilde{z})$. Then there is a neighborhood $\tilde{U}_{\tilde{z}}$ of $\tilde{z}$ in Ω_α such that $\pi_\alpha | \tilde{U}_{\tilde{z}}$ is biholomorphic. Put $U = \pi_\alpha(\tilde{U}_{\tilde{z}})$. Then the holomorphic map $\varphi = f_\alpha \circ \pi_\alpha^{-1}$ of U defines a germ in $\mathcal{O}_z^X \subset \mathcal{O}_M^X$. We define a map $\lambda_\alpha : \Omega_\alpha \to \mathcal{O}_M^X$ by setting

$$\lambda_\alpha(\tilde{z}) = (f_\alpha \circ \pi_\alpha^{-1})_z \in \mathcal{O}_M^X,$$

where $z = \pi_\alpha(\tilde{z})$. Let Ω be the connected component of $\lambda_\alpha(\Omega_\alpha)$ in $\mathcal{O}_M^X$. Let $\varpi : \mathcal{O}_M^X \to M$ be the natural projection and put $\pi = \varpi | \Omega$. Let $f(g_{\tilde{w}}) = g_{\tilde{w}}(\tilde{w}) \in X$ for $g_{\tilde{w}} \in \Omega \subset \mathcal{O}_M^X$. Then $(\Omega, \pi, \lambda_\alpha; f)$ is a maximal extension of $(\Omega_\alpha, \pi_\alpha; f_\alpha)$.

To verify this, take any extension $(\Omega_\beta, \pi_\beta, \lambda_{\beta\alpha}; f_\beta)$ of $(\Omega_\alpha, \pi_\alpha; f_\alpha)$. Then by the same argument as above, we have a map $\lambda_\beta : \Omega_\beta \to \mathcal{O}_M^X$ by setting

$$\lambda_\beta(\tilde{z}) = (f \circ \pi_\beta^{-1})_z \in \mathcal{O}_M^X,$$

where $z = \pi_\beta(\tilde{z})$. Since $\pi_\alpha = \pi_\beta \circ \lambda_{\beta\alpha}$ and $f_\alpha = f_\beta \circ \lambda_{\beta\alpha}$, regarding π_β^{-1} as a map of a neighborhood of $\pi_\alpha(\tilde{z})$ to a neighborhood of $\lambda_{\beta\alpha}(\tilde{z}) \in \Omega_\beta$, we have

$$\lambda_\alpha(\tilde{z}) = (f_\alpha \circ \pi_\alpha^{-1})_z = (f_\beta \circ \lambda_{\beta\alpha} \circ \pi_\alpha^{-1})_z = (f_\beta \circ \pi_\beta^{-1})_z = \lambda_\beta(\lambda_{\beta\alpha}(\tilde{z}))$$

for $\tilde{z} \in \Omega_\alpha$. Therefore $\lambda_\alpha(\Omega_\alpha) \subset \lambda_\beta(\Omega_\beta)$. Since Ω_β is connected by definition, we see that $\lambda_\beta(\Omega_\beta)$ is contained in Ω. Thus $(\Omega, \pi, \lambda_\alpha; f)$ is maximal.

Let $(\Omega_1, \pi_1, \lambda_1; f_1)$ and $(\Omega_2, \pi_2, \lambda_2; , f_2)$ be two maximal extensions of $(\Omega_\alpha, \pi_\alpha, f_\alpha)$. By the maximal property, there exist two étale maps $\lambda_{12} : \Omega_2 \to \Omega_1$, $\lambda_{21} : \Omega_1 \to \Omega_2$ such that $\lambda_2 = \lambda_{21} \circ \lambda_1$ and $\lambda_1 = \lambda_{12} \circ \lambda_2$. For $\tilde{z} \in \Omega_\alpha$, we have

$$(\lambda_{12} \circ \lambda_{21})(\lambda_1(\tilde{z})) = \lambda_{12}(\lambda_2(\tilde{z})) = \lambda_1(\tilde{z}),$$
$$(\lambda_{21} \circ \lambda_{12})(\lambda_2(\tilde{z})) = \lambda_{21}(\lambda_1(\tilde{z})) = \lambda_2(\tilde{z}).$$

Hence $\lambda_{12} \circ \lambda_{21} = \mathrm{id}$ on $\lambda_1(\Omega_\alpha)$ and $\lambda_{21} \circ \lambda_{12} = \mathrm{id}$ on $\lambda_2(\Omega_\alpha)$. Therefore $\lambda_{12} \circ \lambda_{21} = \mathrm{id}_{\Omega_1}$ and $\lambda_{21} \circ \lambda_{12} = \mathrm{id}_{\Omega_2}$ holds. Thus the maps λ_{12} and λ_{21} are isomorphisms. □

DEFINITION 2.— *Let M be a complex manifold. If an extension $(\Omega, \pi, \lambda_\alpha; f)$ of a map element $(\Omega_\alpha, \pi_\alpha; f_\alpha)$ over M is maximal, the holomorphic map $f : \Omega \to X$ is called the maximal extension of f_α, and Ω is called the maximal domain of definition for f_α over M.*

Let $(\Omega_\alpha, \pi_\alpha)$ be an étale domain over a Stein manifold M. Let $F_\alpha = \{f_\alpha^i\}_{i \in I}$ be a set of *holomorphic functions* on Ω_α. We say that $(\Omega_\beta, \pi_\beta, \lambda_{\beta\alpha}; F_\beta)$ is an extension of $(\Omega_\alpha, \pi_\alpha; F_\alpha)$, if there exist an étale map $\lambda_{\beta\alpha} : \Omega_\alpha \to \Omega_\beta$ and holomorphic functions $F_\beta = \{f_\beta^i\}_{i \in I}$ on Ω_β such that for any $i \in I$, $f_\alpha^i = f_\beta^i \circ \lambda_{\beta\alpha}$, $\pi_\alpha = \pi_\beta \circ \lambda_{\beta\alpha}$. In this situation, we say that $(\Omega_\beta, \pi_\beta, \lambda_{\beta\alpha}; F_\beta)$ is a *simultaneous extension* of F_α from $(\Omega_\alpha, \pi_\alpha; F_\alpha)$ over M.

Let F_α be a set of holomorphic functions on an étale domain $(\Omega_\alpha, \pi_\alpha)$ on M. A *maximal simultaneous extension* $(\Omega, \pi, \lambda_\alpha; F)$ of F_α is a simultaneous extension of $(\Omega_\alpha, \pi_\alpha; F_\alpha)$ such that, if $(\Omega_\beta, \pi_\beta, \lambda_{\beta\alpha}; F_\beta)$ is any simultaneous extension of $(\Omega_\alpha, \pi_\alpha; F_\alpha)$, then there is an étale map $\lambda_\beta : \Omega_\beta \to \Omega$ such that, for all $i \in I$, $f^i_\beta = f^i \circ \lambda_\beta$ and $\lambda_\alpha = \lambda_\beta \circ \lambda_{\beta\alpha}$. It follows that $\pi_\beta = \pi \circ \lambda_\beta$. We can prove the existence and uniqueness up to isomorphisms of a maximal simultaneous extension of a given system $(\Omega_\alpha, \pi_\alpha; F_\alpha)$ over M in a manner similarly to the case a maximal extension. When we take as F_α the set of all holomorphic functions on Ω_α, we abbreviate $(\Omega_\alpha, \pi_\alpha; F_\alpha)$ to $(\Omega_\alpha, \pi_\alpha)$.

DEFINITION 3.— *Let (Ω, π) be an étale domain over a Stein manifold M. The maximal simultaneous extension $(\tilde\Omega, \tilde\pi, \lambda)$ $(\lambda : \Omega \to \tilde\Omega,\ \pi = \tilde\pi \circ \lambda)$ of all the holomorphic functions on (Ω, π) is called the envelope of holomorphy of Ω over M.*

Note here that the étale map $\lambda : \Omega \to \tilde\Omega$ is not necessary injective.

PROPOSITION 1.— *Let $(\Omega_\alpha, \pi_\alpha)$, $(\Omega_\beta, \pi_\beta)$ be étale domains over a Stein manifold. Let $(\tilde\Omega_\alpha, \tilde\pi_\alpha, \lambda_\alpha)$, $(\tilde\Omega_\beta, \tilde\pi_\beta, \lambda_\beta)$ be envelopes of holomorphy. Let $\mu_{\beta\alpha} : \Omega_\alpha \to \Omega_\beta$ be an étale map. Then there is an étale map $\tilde\mu_{\beta\alpha} : \tilde\Omega_\alpha \to \tilde\Omega_\beta$ such that $\lambda_\beta \circ \mu_{\beta\alpha} = \tilde\mu_{\beta\alpha} \circ \lambda_\alpha$.*

Proof.— Set $F_\alpha = \mu^*_{\beta\alpha}\mathcal{O}_\beta$, where $\mathcal{O}_\beta$ is the set of holomorphic functions on Ω_β. Then $(\tilde\Omega_\alpha, \tilde\pi_\alpha, \lambda_\alpha; \tilde F_\alpha)$ is a simultaneous extension of $(\Omega_\alpha, \pi_\alpha; F_\alpha)$, where $\tilde F_\alpha = \lambda^{*\,-1}_\alpha(F_\alpha)$. Note here that $\lambda^*_\alpha : \tilde F_\alpha \to F_\alpha$ and $\mu^*_{\beta\alpha} : \mathcal{O}_\beta \to F_\alpha$ are bijective. Since $(\tilde\Omega_\beta, \tilde\pi_\beta, \lambda_\beta)$ is a maximal extension of $(\Omega_\beta, \pi_\beta; \mathcal{O}_\beta)$, we see that the simultaneous extension $(\tilde\Omega_\beta, \tilde\pi_\beta, \lambda_\beta \circ \mu_{\beta\alpha})$ of $(\Omega_\alpha, \pi_\alpha; F_\alpha)$ is maximal. Therefore there is an étale map $\tilde\mu_{\beta\alpha} : \tilde\Omega_\alpha \to \tilde\Omega_\beta$ with the required property. $\square$

2. The holomorphic extension index.

In this section, we shall give the definition of the *holomorphic extension index* of a complex manifold and discuss on its properties. Fix $n \geqslant 2$. Let X be a complex manifold and $\sigma : H^n_{r\rho} \to X$ a holomorphic map. By the argument in section 1, we can consider the maximal extension $\hat\sigma : \Omega \to X$ of the map element $(H^n_{r\rho}, \iota; \sigma)$, where Ω is étale over the polydisk Δ^n,

$\pi : \Omega \to \Delta^n$ and $\iota : H^n_{r\rho} \to \Delta^n$ is the inclusion. As we shall see below, the maximal domain Ω of definition for σ is not always a subdomain of Δ^n.

DEFINITION 4. — *Let X be a complex manifold. A holomorphic map*

$$\sigma : H^n_{r\rho} \to X, \quad 0 < r, \rho < 1$$

is called an n-probe into X, if the maximal domain of definition for σ over Δ^n is a subdomain in Δ^n.

DEFINITION 5. — *A complex manifold is said to be n-probable, if every holomorphic map*

$$\sigma : H^n_{r\rho} \to X$$

is an n-probe.

Let $\sigma : H^n_{r\rho} \to X$ be an n-probe whose maximal domain is Ω. Put

$$A_\sigma = \Delta^n \setminus \Omega.$$

Then A_σ is a closed subset of Δ^n and is called the *singular set* of the probe. Considering the Hausdorff dimension $d(A_\sigma)$ of A_σ, we shall define an index of X as follows.

DEFINITION 6. — *The n-th holomorphic extension index* $\mathrm{Hex}_n(X)$ *of a complex manifold X is defined by*

$$\mathrm{Hex}_n(X) = 2n - \sup_\sigma d(A_\sigma),$$

where σ runs over all n-probes into X.

Here we put

$$d(\emptyset) = -\infty.$$

By definition, $\mathrm{Hex}_n(X)$ is a real number satisfying

$$0 \leqslant \mathrm{Hex}_n(X) \leqslant 2n, \quad \text{or} \quad \mathrm{Hex}_n(X) = +\infty.$$

If $\mathrm{Hex}_n(X) = 2n$, then every n-probe extends to a holomorphic map of Δ^n into X outside a closed set of Hausdorff dimension zero. For $m \geqslant n$, let $\eta : H^m_{r\rho} \to H^n_{r\rho}$ by

$$\eta(z_1, \ldots, z_{m-1}, z_m) = (z_{m-n+1}, \ldots, z_{m-1}, z_m).$$

Then for any n-probe σ, we have an m-probe $\sigma \circ \eta$ and the inclusion relation

$$\Delta^{m-n} \times A_\sigma \subset A_{\sigma\circ\eta}.$$

Therefore $d(A_{\sigma\circ\eta}) \geqslant d(A_\sigma) + 2(m-n)$. Hence

$$2n - \sup_\sigma d(A_\sigma) \geqslant 2m - \sup_\sigma d(A_{\sigma\circ\eta}) \geqslant 2m - \sup_\tau d(A_\tau),$$

where τ runs over all m-probes. Thus we have a relation

$$\mathrm{Hex}_n(X) \geqslant \mathrm{Hex}_m(X) \quad \text{for} \quad n \leqslant m,$$

provided that X is both n- and m-probable.

Obviously, Stein manifolds are n-probable with $\mathrm{Hex}_n = +\infty$ for any $n \geqslant 2$. Any Moishezon manifolds are n-probable with $\mathrm{Hex}_n \geqslant 4$ for any $n \geqslant 2$. By Ivashkovich [I1], we know that compact Kähler manifolds and any compact complex surface are n-probable with $\mathrm{Hex}_n \geqslant 4$ for any $n \geqslant 2$. Any subdomain of an n-probable manifold is n-probable. The compact complex 3-manifold of Example 1 in section 5 is n-probable for any $n \geqslant 2$ with $\mathrm{Hex}_2 = 0$. Many flat twistor spaces over conformally flat real 4-dimensional manifolds have fractal Hex_2 and Hex_3, see Example 2. In general, however, there are non-probable compact complex manifolds, see Example 3 in section 5.

We have the following easy sufficient condition for a map to be a probe.

PROPOSITION 2.— *Let X be a complex manifold and $\sigma : H^n_{r\rho} \to X$ a holomorphic map. Suppose that σ extends holomorphically to $\Delta^n \setminus A \to X$, where A is a closed subset of Δ^n with $d(A) < 2n-1$. Then σ is an n-probe.*

Proof.— Put $\Omega = \Delta^n \setminus A$. Let Ω_σ be the maximal domain of definition over Δ^n. Then we have the following commutative diagram of the canonical étale maps:

$$\begin{array}{ccc} \Omega & \xrightarrow{\lambda} & \Omega_\sigma \\ \cap & & \mu \downarrow \\ \Delta^n & = & \Delta^n. \end{array}$$

Put $\Omega_0 = \lambda(\Omega)$. It is clear that λ is injective and $\mu|_{\Omega_0} : \Omega_0 \to \mu(\Omega_0) = \Omega$ is biholomorphic. Since μ is étale, we have $d(\mu^{-1}(A)) = d(A) < 2n-1$.

Therefore $\Omega_\sigma \setminus \mu^{-1}(A)$ is connected. Since $\Omega_0 \subset \Omega_\sigma \setminus \mu^{-1}(A)$, we see that Ω_0 coincides with $\Omega_\sigma \setminus \mu^{-1}(A)$. Hence μ is injective. $\square$

3. A characterization of the index.

For n-probable manifolds with $\mathrm{Hex}_n(X) > 2$, we have the following property.

THEOREM 2.— *Let X be a complex manifold which is n-probable with $\mathrm{Hex}_n(X) > 2$. Let (Ω, π) be an étale domain over a Stein manifold M with $\dim_{\mathbb{C}} M = n$. Then, for every holomorphic map $\sigma : \Omega \to X$, the maximal domain Ω_σ of definition for σ over M is a subdomain of $\tilde{\Omega}_\sigma$ satisfying*

$$d(\tilde{\Omega}_\sigma \setminus \Omega_\sigma) \leqslant 2n - \mathrm{Hex}_n(X),$$

where $\tilde{\Omega}_\sigma$ is the envelope of holomorphy for Ω_σ over M.

Proof.— We can clearly assume that $\Omega = \Omega_\sigma$ without loss of generality. Take a Hartogs domain $H = H^n_{r\rho}$ and consider a holomorphic open embedding $\varphi : H \to \Omega$ such that $\pi \circ \varphi$ is also injective. Put $\hat{T} = \varphi(H)$ and $T = \pi \circ \varphi(H)$. Note here that $\pi|\hat{T} : \hat{T} \to T$ is biholomorphic. Put $\Delta = \Delta^n$. Since M is of Stein, $\pi \circ \varphi$ extends to $\bar{\varphi} : \Delta \to M$ and $\bar{\varphi}$ is biholomorphic onto its image $\Delta_T := \bar{\varphi}(\Delta)$. Let $\hat{U}_T$ be the connected component of $\pi^{-1}(\Delta_T)$ which contains $\hat{T}$. Put $U_T = \pi(\hat{U}_T) \subset \Delta_T$.

Consider the holomorphic map $\sigma_T := \sigma \circ (\pi|\hat{T})^{-1} : T \to X$. Since X is n-probable, there are a closed set A_{σ_T} in Δ_T with $d(A_{\sigma_T}) \leqslant 2n - \mathrm{Hex}_n(X)$, and a holomorphic map $\hat{\sigma}_T : \Delta_T \setminus A_{\sigma_T} \to X$ which extends σ_T. Then, since $(\Omega, \pi; \sigma)$ is the maximal domain of extension for σ over M, there is an étale map $\mu : \Delta_T \setminus A_{\sigma_T} \to \Omega$ by which $(\Omega, \pi, \mu; \sigma)$ becomes an extension of $(\Delta_T \setminus A_{\sigma_T}, \iota; \hat{\sigma}_T)$ over M, where $\iota : \Delta_T \setminus A_{\sigma_T} \to M$ is the natural inclusion. Since $\iota = \pi \circ \mu$, μ is injective and we have $\pi = \mu^{-1}$ on $\mu(\Delta_T \setminus A_{\sigma_T})$. Note that $\mu(\Delta_T \setminus A_{\sigma_T})$ is a subdomain contained in $\pi^{-1}(U_T)$ and contains $\hat{T}$. Therefore $\mu(\Delta_T \setminus A_{\sigma_T})$ is contained in $\hat{U}_T$.

Suppose that there is a point $x \in \hat{U}_T \setminus \mu(\Delta_T \setminus A_{\sigma_T})$. Since $\hat{U}_T \subset \Omega$, there is a neighborhood $\hat{W} (\subset \hat{U}_T)$ of x where σ is defined. Since π is étale, we can assume that $\pi(W)$ is neighborhood of $\pi(x)$ which is biholomorphic to W. Therefore σ_T is defined on $\pi(W)$. Hence $\pi(x) \in \Delta_T \setminus A_{\sigma_T}$. This is absurd, since $\mu \circ \pi(x) = x \notin \mu(\Delta_T \setminus A_{\sigma_T})$. Hence we infer that $\mu(\Delta_T \setminus A_{\sigma_T})$ coincides with $\hat{U}_T$.

Therefore $\pi|\hat{U}_T : \hat{U}_T \to \Delta_T \setminus A_{\sigma_T}$ is a biholomorphic map. We glue Δ_T and Ω by identifying $\Delta_T \setminus A_{\sigma_T}$ and $\hat{U}_T$ by $\pi|\hat{U}_T$ to obtain a new domain $\Delta_T \cup \Omega$. It is clearly étale over $\tilde{\Omega}$.

Thus, taking holomorphic open embeddings $\varphi_\alpha : H \to \Omega$ such that $\pi \circ \varphi_\alpha$ are injective, we form the union

$$\hat{\Omega} = \bigcup_\alpha \Delta_{T_\alpha} \cup \Omega \tag{1}$$

by the method described above. Then, there are a closed subset A_α in $\Delta_\alpha = \Delta_{T_\alpha}$ with $d(A_\alpha) \leqslant 2n - \mathrm{Hex}_n(X)$ and a holomorphic map $\sigma_\alpha : \Delta_\alpha \setminus A_\alpha \to X$ such that $\sigma_\alpha|T_\alpha = \sigma|T_\alpha$. By the definition of $\hat{\Omega}$, $\pi : \Omega \to M$ extends to an étale map of $\hat{\pi} : \hat{\Omega} \to M$. Put

$$A = \bigcup_\alpha A_\alpha.$$

It is easy to check that $A_\alpha \cap \Delta_\beta = A_\beta \cap \Delta_\alpha$. Therefore A is a closed set in $\hat{\Omega}$. Note also that $d(A) \leqslant 2n - \mathrm{Hex}_n(X)$ in $\hat{\Omega}$.

Lemma 1. — *The domain $\hat{\Omega}$ is of Stein.*

Proof. — It is enough to show that $\hat{\Omega}$ is p_7-convex by [DG]. Take any holomorphic open embedding $\varphi : H \to \hat{\Omega}$ such that $\hat{\pi} \circ \varphi$ is injective. If $\varphi(H) \subset \Omega$, then φ extends to a holomorphic map $\hat{\varphi}$ of Δ into $\hat{\Omega}$ by the definition of $\hat{\Omega}$. Suppose that $\varphi(H) \not\subset \Omega$. We have to show that φ extends to a holomorphic map $\hat{\varphi}$ of Δ into $\hat{\Omega}$.

We consider a projection $\Phi : \Delta \to \Delta^{n-1} \times [0, 1)$ defined by

$$\Phi(z_1, \ldots, z_{n-1}, z_n) = ((z_1, \ldots, z_{n-1}), |z_n|),$$

and a projection $p : \Delta \to \Delta^{n-1}$ defined by

$$p(z_1, \ldots, z_{n-1}, z_n) = (z_1, \ldots, z_{n-1}).$$

For any set G, we put $\tilde{G} = \Phi^{-1}(\Phi(G))$. For a point $z \in \Delta$, we put $\ell_z = p^{-1}(p(z))$. Note that $d(p(G)) \leqslant d(G)$ holds for any set G in Δ. Put $K = \Delta \setminus H$ and $B = \varphi^{-1}(A)$. The set B is a closed subset of H with $d(\Phi(B)) \leqslant d(B) \leqslant 2n - \mathrm{Hex}_n(X) < 2n - 2$, $d(\tilde{B}) < 2n - 1$ and $d(p(B)) < 2n - 2$. To show that φ extends to a holomorphic map of Δ into

$\hat{\Omega}$, fix a point $o \in H$ satisfying $\ell_o \cap B = \emptyset$, and take an arbitrary point $a \in H$.

First consider the case where $\ell_a \cap H \not\subset \tilde{B}$. Replacing a with a point in $(\ell_a \cap H) \setminus B$, we can assume $a \notin \tilde{B}$ without loss of generality. Since $d(\tilde{B}) < 2n-1$, there is a continuous path $\omega : [0,1] \to H \setminus \tilde{B}$ with $\omega(0) = o$ and $\omega(1) = a$. Consider a number

$$t_0 = \sup\{t \in [0,1] : \varphi \text{ extends to a neighborhood of } \ell_{\omega(s)} \text{ for } \ 0 \leqslant s \leqslant t\}.$$

It is clear that $t_0 > 0$. Put $b = \omega(t_0)$. Let $U(c, \varepsilon)$ indicate the open polydisk in Δ^{n-1} with the center c and the common radius $\varepsilon > 0$. Let G be the domain defined by

$$G = p^{-1}(U(p(b), \varepsilon)) \cap \{z \in \Delta : |b_n| - \varepsilon < |z_n| < |b_n| + \varepsilon\},$$

where $\varepsilon > 0$ is such a small number that G is contained in $H \setminus \tilde{B}$. We can choose $\delta > 0$ so small that the point $\omega(t)$ is in G for any t with $t_0 - \delta < t < t_0$. Then there is a small connected open neighborhood V of $p(\omega(I))$, $I = (t_0 - \delta, t_0)$, such that φ extends to a biholomorphic map $\varphi' : G \cup p^{-1}(V) \to \hat{\Omega}$. Put $B' = \varphi'^{-1}(A) \subset G \cup V$. Since $p(B')$ is a closed subset with $d(p(B')) < 2n-2$, there is an open polydisk $V' \subset V \setminus p(B')$. Put

$$\tilde{V} = p^{-1}(V') \cap \{z \in \Delta : |z_n| < |b_n| + \varepsilon\}.$$

Then, the Hartogs domain

$$H' = G \cup \tilde{V}$$

is contained in $(G \cup \tilde{V}) \setminus B'$. Thus $\varphi' : H' \to \Omega$, and $\varphi'(H')$ is a member of T_α's in (1). Hence φ' embeds the polydisk

$$\hat{G} = p^{-1}(U(p(b), \varepsilon)) \cap \{z \in \Delta : |z_n| < |b_n| + \varepsilon\}$$

into $\hat{\Omega}$. This contradicts the definition of t_0. Hence t_0 cannot be less than 1 and we have $t_0 = 1$. The above argument shows that the set

$$\{t \in [0,1] : \varphi \text{ extends to a neighborhood of } \ell_{\omega(s)} \text{ for } \ 0 \leqslant s \leqslant t\}$$

is closed in $[0,1]$. Therefore φ extends also in a neighborhood of ℓ_a. Thus we are done.

Next consider the case where $\ell_a \cap H \subset \tilde{B}$. Letting $\varepsilon_1, \dots, \varepsilon_{n-1}$ be complex constants with small absolutes values, we introduce a new system of coordinates on Δ by

$$\begin{cases} w_j &= z_j + \varepsilon_j z_n, \quad j = 1, \dots, n-1 \\ w_n &= z_n, \end{cases} \tag{2}$$

and projections

$$\Phi(w_1, \dots, w_{n-1}, w_n) = ((w_1, \dots, w_{n-1}), |w_n|),$$
$$p(w_1, \dots, w_{n-1}, w_n) = (w_1, \dots, w_{n-1}).$$

with respect to the new system of coordinates. The complex line ℓ_a with respect $(w_1, \dots, w_n)$ is given in $(z_1, \dots, z_n)$ by

$$z_j = a_j + \varepsilon_j(a_n - z_n), \quad j = 1, \dots, n-1,$$

where $a = (a_1, \dots, a_{n-1}, a_n)$. Since $d(B) < 2n-2$, for almost all $(n-1)$-vectors $(\varepsilon_1, \dots, \varepsilon_{n-1}) \in \mathbb{C}^{n-1}$, the line ℓ_a does not intersects B except possibly the point a, by virtue of a result of Shiffman [Sh, Corollary p. 119]. Therefore, we can repeat the argument in the first case where $\ell_a \cap H \not\subset \tilde{B}$, and we see that φ extends to a neighborhood of ℓ_a. Hence we conclude that $\varphi : H \to \hat{\Omega}$ extends to a holomorphic map $\hat{\varphi} : \Delta \to \hat{\Omega}$. □

LEMMA 2. — *$\hat{\Omega}$ is the envelope of holomorphy for Ω over M.*

Proof. — Since $\Omega = \hat{\Omega} \setminus A$ with $d(A) < 2n-2$, $(\hat{\Omega}, \hat{\pi}, j)$, $j : \Omega \to \hat{\Omega}$, is the simultaneous extension of all the holomorphic functions on (Ω, π). Since $\hat{\Omega}$ is of Stein, $(\hat{\Omega}, \hat{\pi}, j)$ is maximal. Thus $(\hat{\Omega}, \hat{\pi}, j)$ is the envelope of holomorphy for (Ω, π) over M. □

Thus Theorem 2 is proved. □

As a corollary, we have the following

PROPOSITION 3. — *Let X be a complex manifold which is n-probable with $\mathrm{Hex}_n(X) > 2$. Let S an analytic subset in a Stein manifold W with codimension 1, and A a closed subset of $W \setminus S$ with $d(A) \leqslant 2n - \mathrm{Hex}_n(X)$. Let G be a domain in W which intersects every irreducible component of S. Then any holomorphic map $\tau : G \cup (W \setminus (S \cup A)) \to X$ extends holomorphically to $W \setminus B$, where B is a closed subset in $S \cup A$ with $d(B) \leqslant 2n - \mathrm{Hex}_n(X)$.*

Proof. — Put $W^* = G \cup (W \setminus (S \cup A))$. Let $\iota : W^* \to W$ be the natural inclusion and we regard W^* as an étale domain over the Stein manifold W by ι. Let $(W^*_\tau, \pi, \mu; \tilde{\tau})$, $\mu : W^* \to W^*_\tau$, be the maximal domain of definition for τ over W. Since $\iota = \pi \circ \mu$, we see that μ is injective. Let $\tilde{W}^*_\tau$, $j : W^*_\tau \to \tilde{W}^*_\tau$, be the envelope of holomorphy for W^*_τ, and $\tilde{\pi} : \tilde{W}^*_\tau \to W$ the natural étale map. Then we have $\tilde{\pi} \circ j = \pi$. By Theorem 2, the canonical étale map j is injective and the inequality $d(\tilde{W}^*_\tau \setminus j(W^*_\tau)) \leqslant 2n - \mathrm{Hex}_n(X)$ holds. Therefore, $\tilde{\pi}$ is also injective, since $j \circ \mu : W^* \to \tilde{W}^*_\tau$ is injective and $\tilde{\pi} \circ j \circ \mu = \iota$. Since G intersects every irreducible codimension 1 component of S, every holomorphic function on W^* extends to W. Hence it extends to $\tilde{W}^*_\tau$. Since $\tilde{W}^*_\tau$ and W are of Stein, this implies that $\tilde{\pi} : \tilde{W}^*_\tau \to W$ is biholomorphic. This proves the proposition. $\square$

4. Some properties of holomorphic extension indices.

If there is a surjective holomorphic map between n-probale complex manifolds, we have the following.

THEOREM 3. — *Let X and Y are n-probable complex manifolds and $f : X \to Y$ a surjective holomorphic map. Let $\{U_\alpha\}_\alpha$ be any open covering of Y. Then we have the following inequality*

$$\mathrm{Hex}_n(X) \geqslant \inf_\alpha \{\mathrm{Hex}_n(Y), \mathrm{Hex}_n(f^{-1}(U_\alpha))\}.$$

Proof. — Put $e = \inf_\alpha\{\mathrm{Hex}_n(Y), \mathrm{Hex}_n(f^{-1}(U_\alpha))\}$. Suppose that $\sigma : H \to X$ be any n-probe. Then $\tau = f \circ \sigma$ is an n-probe of Y. By the assumption, there is a closed subset $A_\tau \subset \Delta$ with $d(A_\tau) \leqslant 2n - \mathrm{Hex}_n(Y)$ such that $\hat{\tau} : \Delta \setminus A_\tau \to Y$ is the maximal extension of τ. On the other hand, there is a closed subset $A_\sigma \subset \Delta$ with $d(A_\sigma) \leqslant 2n - \mathrm{Hex}_n(X)$ such that $\hat{\sigma} : \Delta \setminus A_\sigma \to X$ is the maximal extension of σ. Since $f \circ \hat{\sigma} : \Delta \setminus A_\sigma \to X$ is an extension of σ, we have $A_\tau \subset A_\sigma$. Hence $d(A_\tau) \leqslant d(A_\sigma)$ holds. Suppose that $A_\tau \neq A_\sigma$ and take a point $a \in A_\sigma \setminus A_\tau$. Choose U_α such that $\hat{\tau}(a) \in U_\alpha$ and consider a neighborhood $V \subset \Delta \setminus A_\tau$ of a such that $\hat{\tau}(V) \subset U_\alpha$. Then $\hat{\sigma}(V \setminus A_\sigma) \subset f^{-1}(U_\alpha)$. Since $f^{-1}(U_\alpha)$ is n-probable and $\mathrm{Hex}_n(f^{-1}(U_\alpha)) \geqslant e$, we see that $d(V \cap A_\sigma) \leqslant 2n - e$. Hence $d(A_\sigma, a) \leqslant 2n - e$ (see Appendix) and consequently $d(A_\sigma \setminus A_\tau) \leqslant 2n - e$. Therefore we have $d(A_\sigma) \leqslant 2n - e$ for any σ. Hence we have $\mathrm{Hex}_n(X) \geqslant e$. $\square$

PROPOSITION 4. — *Let X be a complex manifold and $D = \cup_\lambda D_\lambda$ a finite union of non-singular hypersurfaces $D_\lambda \subset X$. Assume that $X \setminus D$ and any component D_λ are n-probable and that $\mathrm{Hex}_n(X \setminus D) > 2$. Then X is n-probable and $\mathrm{Hex}_n(X) \geqslant \min\{\min_\lambda \mathrm{Hex}_n(D_\lambda), 2\}$ holds.*

Proof. — Let $\sigma : H \to X$ be any n-probe. To show that X is n-probable, it is enough to consider the case where $\sigma(H) \not\subset D_\lambda$ for any λ. Put $S = \sigma^{-1}(D)$, $\Omega = H \setminus S$, and $\sigma_1 := \sigma|_\Omega : \Omega \to X \setminus D$. By Theorem 2, σ_1 extends to $\tilde{\sigma}_1 : \tilde{\Omega} \setminus A \to X \setminus D$, where $\tilde{\Omega}$ is an envelope of holomorphy of $H \setminus S$ and where A is a closed subset of $\tilde{\Omega}$ with $d(A) \leqslant 2n - \mathrm{Hex}_n(X \setminus D)$. By a theorem of Dloussky [D], there is an analytic subset S_1 of pure codimension 1 in Δ such that $\tilde{\Omega} = \Delta \setminus S_1$. Hence σ extends at least to $\tilde{\sigma} : (\Delta \setminus (A \cup S_1)) \cup H \to X$. Since $d(A \cup S_1) \leqslant \max\{2n - \mathrm{Hex}_n(X \setminus D), 2n-2\} = 2n-2 < 2n-1$ (see Proposition 9), we see that X is n-probable by Proposition 2 and that $\mathrm{Hex}_n(X) \geqslant \min\{\min_\lambda \mathrm{Hex}_n(D_\lambda), 2\}$. □

Let X, Y be complex manifolds of the same dimension and suppose that there is a holomorphic surjective map $f : X \to Y$ with discrete fibers. In the case of unramified even covering, we have the following proposition. This shows that n-probability of X implies that of Y, but the converse statement does not hold true, even if f is proper. See Example 3, where $\mathrm{Hex}_n(Y) = 2$.

PROPOSITION 5. — *Let X, Y be complex manifolds and $f : X \to Y$ a holomorphic unramified even covering.*

(a) *If X is n-probable, then Y is n-probable and $\mathrm{Hex}_n(X) = \mathrm{Hex}_n(Y)$ holds.*

(b) *If Y is n-probable with $\mathrm{Hex}_n(Y) > 2$, then X is n-probable and $\mathrm{Hex}_n(X) = \mathrm{Hex}_n(Y)$ holds.*

Proof. — (a) Let $\sigma : H \to Y$ be any holomorphic map. Since H is simply connected, σ lifts to $\tilde{\sigma} : H \to X$. Then by the assumption, $\tilde{\sigma}$ extends holomorphically to a map $\hat{\sigma} : \Delta \setminus A_{\tilde{\sigma}} \to X$, where $\hat{\sigma}$ is the maximal extension of $\tilde{\sigma}$ over Δ. The map $f \circ \hat{\sigma} : \Delta \setminus A_{\tilde{\sigma}} \to Y$ gives an extension of σ. Since f is an unramified even covering over Y and since $\hat{\sigma}$ is maximal, we see that $f \circ \hat{\sigma}$ does not extend to any neighborhood of a point in $A_{\tilde{\sigma}}$. Hence $f \circ \hat{\sigma}$ is the maximal extension of σ over Δ and $A_\sigma = A_{f \circ \hat{\sigma}}$. Therefore σ is an n-probe and consequently Y is n-probable. Here we have $\mathrm{Hex}_n(X) \leqslant 2n - \sup_{\sigma : H \to Y} d(A_{\tilde{\sigma}}) = 2n - \sup_{\sigma : H \to Y} d(A_\sigma) = \mathrm{Hex}_n(Y)$.

Hence $\mathrm{Hex}_n(X) \leqslant \mathrm{Hex}_n(Y)$ holds. To show the inequality $\mathrm{Hex}_n(X) \geqslant \mathrm{Hex}_n(Y)$, take any $\varepsilon > 0$. There is an n-probe $\tau_0 : H \to X$ such that $d(A_{\tau_0}) > \sup_{\tau: H\to X} d(A_\tau) - \varepsilon$. Since we now know that Y is n-probable, $f \circ \tau_0 : H \to Y$ extends to an n-probe into Y. Since $A_{f\circ\tau_0} = A_{\tau_0}$ as above, we have

$$\sup_{\sigma: H\to Y} d(A_\sigma) \geqslant d(A_{f\circ\tau_0}) = d(A_{\tau_0}) \geqslant \sup_{\tau: H\to X} d(A_\tau) - \varepsilon$$

Therefore we have $\sup_{\sigma: H\to Y} d(A_\sigma) \geqslant \sup_{\tau: H\to X} d(A_\tau)$. Hence $\mathrm{Hex}_n(Y) \leqslant \mathrm{Hex}_n(X)$.

(b) By (a), it is enough to show that X is n-probable. Let $\tilde{\tau} : H \to X$ be any holomorphic map. Since Y is n-probable, $f \circ \tilde{\tau}$ extends to $\tau : \Delta \setminus A_\tau \to Y$ with $d(A_\tau) \leqslant 2n - \mathrm{Hex}_n(Y) < 2n-2$. Hence $\Delta \setminus A_\tau$ is simply connected. Therefore τ lifts to $\Delta \setminus A_\tau \to X$, which gives an extension $\tilde{\tau}$. Thus $\tilde{\tau}$ is an n-probe by Proposition 2. Hence X is n-probable. □

For the case of branched coverings, we have the following. As the branched covering of P^1 by an elliptic curve shows, the indices Hex_n are no longer preserved. See section 5 for further examples.

THEOREM 4.— *Let X, Y be complex manifolds and $f : X \to Y$ a proper surjective holomorphic map with discrete fibers. Suppose that X is n-probable with* $\mathrm{Hex}_n(X) > 2$, *then Y is n-probable and the inequality*

$$\mathrm{Hex}_n(X) \geqslant \mathrm{Hex}_n(Y) \geqslant 2$$

holds true.

Proof.— Put $\tilde{R} = \{x \in X : \mathrm{rank}_x f < \dim X\}$, and $R_0 = f(\tilde{R})$. Let $\sigma : H \to Y$ be any non-constant holomorphic map. If $\sigma(H) \subset R_0$, take an irreducible component Y_1 of R_0 such that $\sigma(H) \subset Y_1$. Put $X_1 = f^{-1}(Y_1)$ and $f_1 = f|_{X_1}$. Then $f_1 : X_1 \to Y_1$ is a proper surjective map with discrete fibres. Then there is a proper analytic subset R_1 of Y_1 such that $Y_1 \setminus R_1$ is non-singular and that

$$f_1|_{X_1\setminus f^{-1}(R_1)} : X_1 \setminus f^{-1}(R_1) \to Y_1 \setminus R_1$$

is proper and étale. If $\sigma(H) \subset R_1$, take an irreducible component Y_2 of R_1 such that $\sigma(H) \subset Y_2$. Put $X_2 = f^{-1}(Y_2)$ and $f_2 = f|_{X_2}$. Continue this process, until we obtain an irreducible analytic subsets Y_k in Y, X_k in X, and a proper analytic subset R_k in Y_k, with the following properties:

(i) $\sigma(H) \subset Y_k$ and $\sigma(H) \not\subset R_k$.

(ii) $Y_k \setminus R_k$ is non-singular.

(iii) The map $f_k = f|_{X_k} : X_k \to Y_k$ is proper and surjective with discrete fibres.

(iv) The map $f|_{X_k \setminus f^{-1}(R_k)} : X_k \setminus f^{-1}(R_k) \to Y_k \setminus R_k$ is proper and étale.

LEMMA 3.— *Let S be a proper analytic subset of an n-probable complex manifold X with* $\mathrm{Hex}_n(X) > 2$. *Then $X \setminus S$ is an n-probable manifold with*

$$\mathrm{Hex}_n(X \setminus S) \geqslant \min\{\mathrm{Hex}_n(X), 4\}.$$

Proof.— Since any subdomain of an n-probable manifold is n-probable, $X \setminus S$ is n-probable. Let $\sigma : H \to X \setminus S$ be any holomorphic map of a Hartogs domain H. Since X is n-probable, there is a closed set A_σ in the associated polydisk Δ with $d(A_\sigma) \leqslant 2n - \mathrm{Hex}_n(X) < 2n-2$ such that σ extends to $\hat{\sigma}_1 : \Delta \setminus A_\sigma \to X$. Put $V = \hat{\sigma}_1^{-1}(S)$, which is a proper analytic subset in $\Delta \setminus A_\sigma$. Suppose that V contains an $(n-1)$-dimensional component V_1. Since $d(A_\sigma) < 2n-2$, V_1 extends to an analytic subset of Δ. This is absurd, since $\emptyset \neq V_1 \cap H \subset V \cap H = \emptyset$. Further σ extends to $\hat{\sigma} : \Delta \setminus (A_\sigma \cup V) \to X \setminus S$, where $d(A_\sigma \cup V) \leqslant \max\{d(A_\sigma), 2n-4\}$. Hence we have the desired inequality. □

By Proposition 5(a), $Y_k \setminus R_k$ is n-probable, and we have

$$\mathrm{Hex}_n(Y_k \setminus R_k) = \mathrm{Hex}_n(X_k \setminus f^{-1}(R_k)) \geqslant \mathrm{Hex}_n(X \setminus f^{-1}(R_k)) > 2, \tag{3}$$

where the equality follows from Proposition 5(a), the first inequality is an easy consequence of the definition of Hex_n and the last inequality follows from Lemma 3. Put

$$\sigma_k = \sigma|_{H \setminus \sigma^{-1}(R_k)} : H \setminus \sigma^{-1}(R_k) \to Y_k \setminus R_k.$$

Let Ω_{σ_k} be the maximal domain of definition for σ_k over Δ and $\tilde{\Omega}_{\sigma_k}$ its envelope of holomorphy. Then, we see by (3) and Theorem 2 that

$$\Omega_{\sigma_k} \subset \tilde{\Omega}_{\sigma_k} \subset \Delta \quad \text{and} \quad d(\tilde{\Omega}_{\sigma_k} \setminus \Omega_{\sigma_k}) < 2n-2. \tag{4}$$

We decompose the analytic subset $\sigma^{-1}(R_k)$ in H into three parts,

$$\sigma^{-1}(R_k) = D + E + F,$$

where D, E, F are analytic subsets in H defined by

(i) $D = H \cap S$ for some pure $(n-1)$-dimensional analytic subset S in Δ,

(ii) $E =$ the union of $(n-1)$-dimensional components other than D,

(iii) $F =$ the union of components with dimensions less than $n-1$.

By a theorem of Dloussky [D], the envelope of holomorphy for $H \setminus \sigma^{-1}(R_k)$ over Δ is given by $\Delta \setminus S$. Therefore, since $H \setminus \sigma^{-1}(R_k) \subset \tilde{\Omega}_{\sigma_k} \subset \Delta$, we see that $\Delta \setminus S \subset \tilde{\Omega}_{\sigma_k} \subset \Delta$. It is clear that $\Omega_{\sigma_k} \cap (E \setminus D) = \emptyset$. Hence we have $E \setminus D \subset \tilde{\Omega}_{\sigma_k} \setminus \Omega_{\sigma_k}$. On the other hand, the inequality of (4) implies that $\tilde{\Omega}_{\sigma_k} \setminus \Omega_{\sigma_k}$ cannot contain non-empty open subset of an $(n-1)$-dimensional analytic subset. Hence we infer that $E \setminus D = \emptyset$, and consequently we have $E = \emptyset$. Therefore σ extends at least to $\Delta \setminus (S \setminus H)$ with $d(S \setminus H) \leqslant 2n-2$. Thus by Proposition 2, Y is n-probable with $\mathrm{Hex}_n(Y) \geqslant 2$. □

5. Examples.

Many of the examples listed below are flat twistor spaces over conformally flat real 4-dimensional manifolds.

Example 1. — The compact complex 3-manifold U/Γ constructed in [Kt2] has $\mathrm{Hex}_2(U/\Gamma) = 0$. Namely, there is a 2-probe which cannot extend across a closed ball with positive radius in Δ^2.

Example 2. — Flat twistor spaces over conformally flat real 4-dimensional manifolds of Schottky type often have fractal 2nd holomorphic extension indices. Explicit calculations and estimations of the indices are carried out in [O]. See also [I2].

Example 3. — We shall give an example of a compact complex manifolds X of dimension 3 for which *not* every holomorphic map

$$\sigma : H^n_{r\rho} \to X$$

is an n-probe into X for $n = 2, 3$. That is, X is *neither* 2-probable *nor* 3-probable. The product manifold of X with any compact complex manifold will give us an example of non n-probable higher dimensional complex manifolds for $n = 2, 3$.

Consider a real 4-dimensional submanifold L defined by

$$L = \{z = [z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3 : z_0\bar{z}_3 - \bar{z}_1 z_2 = 0\}$$

in the 3-dimensional projective space $\mathbb{P}^3$, and its complement

$$W = \mathbb{P}^3 \setminus L.$$

Let

$$\rho : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}(4)$$

be the group homomorphism defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \ B = \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix}, \ C = \begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix}, \ D = \begin{pmatrix} d & 0 \\ 0 & \bar{d} \end{pmatrix}.$$

Then $\mathrm{SL}_2(\mathbb{C})$ acts on W through ρ. Let

$$\mathbb{H}^+ = \{q = w + jx : w \in \mathbb{C},\ x \in \mathbb{R}^+\}$$

be the upper-half space of the quaternions. We consider a diffeomorphism

$$\phi : \mathbb{P}^3 \setminus L \to S^1 \times \mathbb{H}^+ \times \mathbb{P}^1$$

by

$$z = [z_0 : z_1 : z_2 : z_3] \mapsto (\theta(z), q(z), \zeta(z))$$

where

$$\theta(z) = \frac{v}{|v|}, \quad q(z) = u + j|v|, \quad \zeta(z) = [z_2 : z_3]$$

and

$$Z := \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} = \begin{pmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{pmatrix} \begin{pmatrix} z_2 & -\bar{z}_3 \\ z_3 & \bar{z}_2 \end{pmatrix}^{-1}.$$

Remark here that $(z_2, z_3) \neq (0,0)$ and $z \in L$ if and only if $v = 0$. By an easy calculation, we have

$$\theta(\rho(g)z) = \theta(z), \quad q(\rho(g)z) = (aq(z) + b)(cq(z) + d)^{-1},$$
$$\zeta(\rho(z)) = [cz_0 + dz_2 : \bar{c}z_1 + \bar{d}z_3]$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. Note here that $(cz_0 + dz_2, \bar{c}z_1 + \bar{d}z_3) \neq (0,0)$ follows from $z \notin L$.

Now take a Fuchsian subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$ such that the quotient space $\mathbb{H}^+/\Gamma$ is a compact real C^∞ 3-manifold. We define a compact complex 3-manifold X by

$$X = (\mathbb{P}^3 \setminus L)/\Gamma,$$

which is a flat twistor space over $S^1 \times (\mathbb{H}^+/\Gamma)$. Let $\varpi : \mathbb{P}^3 \setminus L \to X$ be the natural projection. Define a holomorphic map $h : \mathbb{C}^3 \to \mathbb{P}^3$ by

$$(u_1, u_2, u_3) \mapsto [2u_1 - 1 : 2u_2 + 5u_3 : 5u_3 : 1].$$

Let $H^3_{r\rho} = G_1 \cup G_2$ $(r = \rho = 1/50)$ be the Hartogs domain defined by

$$G_1 = \left\{(u_1, u_2, u_3) \in \mathbb{C}^3 : |u_1| < 1/50,\ |u_2| < 1/50,\ |u_3| < 1 \right\}$$
$$G_2 = \left\{(u_1, u_2, u_3) \in \mathbb{C}^3 : |u_1| < 1,\ |u_2| < 1,\ 1 - 1/50 < |u_3| < 1 \right\}.$$

Lemma 4. — *$\Delta^3 \cap h^{-1}(L) \neq \emptyset$ but $H^3_{r\rho} \cap h^{-1}(L) = \emptyset$.*

Proof. — Note that

$$h^{-1}(L) = \{(u_1, u_2, u_3) \in \mathbb{C}^3 : 2u_1 - 10\bar{u}_2 u_3 = 25|u_3|^2 + 1\}. \tag{5}$$

Since $a = (\frac{1}{2}, 0, 0) \in \Delta^3 \cap h^{-1}(L)$, we see that $\Delta^3 \cap h^{-1}(L) \neq \emptyset$, On the other hand, for $(u_1, u_2, u_3) \in G_1$, we have

$$|2u_1 - 10\bar{u}_2 u_3| \leqslant \frac{2}{50} + \frac{10}{50} \times 1 = \frac{6}{25} < 1 \leqslant 25|u_3|^2 + 1.$$

Hence, by (5), $G_1 \cap h^{-1}(L) = \emptyset$. Similarly, for $(u_1, u_2, u_3) \in G_2$, we have

$$|2u_1 - 10\bar{u}_2 u_3| \leqslant 2 + 10 \times 1 = 12 < \frac{2501}{100} = 25\left(1 - \frac{1}{50}\right)^2 + 1 \leqslant 25|u_3|^2 + 1.$$

Hence, by (5), $G_2 \cap h^{-1}(L) = \emptyset$. Thus $H^3_{r\rho} \cap h^{-1}(L) = \emptyset$. $\square$

We consider a holomorphic map

$$\sigma : H^3_{r\rho} \to X, \qquad \sigma = \varpi \circ h | H^3_{r\rho}.$$

Obviously, the maximal domain of extension for f over Δ^3 is equal to $\Delta^3 \setminus h^{-1}(L)$ and hence is a 3-probe for X. Corresponding to the natural m-fold unramified covering of S^1, we construct an m-fold unramified covering X_m of X. Since $H^3_{r\rho}$ is simply connected, σ lifts to a holomorphic map

$$\sigma_m : H^3_{r\rho} \to X_m.$$

Note that a closed path in $\Delta^3 \setminus h^{-1}(L)$ around $h^{-1}(L)$ does not lift to a closed path in X_m, and we see that the maximal domain of extension for σ_m over Δ^3 is an m-fold unramified covering of $\Delta^3 \setminus h^{-1}(L)$. This implies that σ_m is *not* a 3-probe of X_m and that X_m is not 3-probable for $m \geqslant 2$. Restricting h to the 2-dimensional Hartogs domain

$$H^2_{r\rho} = H^3_{r\rho} \cap \{u_2 = 0\}$$

we can also show that X_m is *not* 2-probable for $m \geqslant 2$.

Example 4. — Every compact complex surface S of Class VI_o is a finite branched covering of a primary Hopf surface S_1. It is well-known that the universal covering S is either $\mathbb{C}^2$ of $\mathbb{C} \times \mathbf{D}$, where $\mathbf{D}$ is the unit disk in $\mathbb{C}$. Hence $+\infty = \mathrm{Hex}_2(S) > \mathrm{Hex}_2(S_1) = 4$.

Example 5. — Let X be a Blanchard manifold of type (B) (Kt1, p. 375]). By the construction, X is a non-singular quotient of $\mathbb{P}^3 \setminus \{\text{a projective line}\}$ by a free Abelian group of rank $= 4$, and is torus fiber space over $\mathbb{P}^1$. Let T be an elliptic curve and $\mu : T \to \mathbb{P}^1$ be the double branched covering. Take the normalization Y of the fiber product $X \times_\mu T$. Then Y is a finite branched covering of X, and the universal covering of Y is biholomorphic to $\mathbb{C}^3$. Hence we have $+\infty = \mathrm{Hex}_2(Y) > \mathrm{Hex}_2(X) = 4$ and $+\infty = \mathrm{Hex}_3(Y) > \mathrm{Hex}_3(X) = 4$.

A. Appendix : Local Hausdorff dimensions.

We shall give here the definition of *local Hausdorff dimensions* of sets and prove that they are left invariant by holomorphic maps with discrete fibers. Let $X = (X, d_X)$ be a metric space with the metric d_X. For a non-empty subset $A \subset X$, a *ball covering* of A is a countable family $\mathcal{B} = \{B_i\}$ of closed balls $B_i \subset X$ such that $A \subset \bigcup_i B_i$. By $I(A, \delta)$, we indicate the set of all ball coverings $\mathcal{B}$ of A such that $\ell(B_i) < \delta$ for all $B_i \in \mathcal{B}$. We define the following quantities,

$$v(s, \mathcal{B}) = \sum_{B_i \in \mathcal{B}} \ell(B_i)^s \tag{6}$$

$$h^s_\delta(A) = \inf_{\mathcal{B} \in I(A,\delta)} v(s, \mathcal{B}), \tag{7}$$

$$h^s(A) = \lim_{\delta \to 0} h^s_\delta(A). \tag{8}$$

Note that $h^s_\delta(A)$ is a decreasing function of $\delta > 0$.

DEFINITION 7. — *$h^s(A)$ is called the s-dimensional Hausdorff measure of A.*

DEFINITION 8. — *The Hausdorff dimension $d(A)$ of a non-empty set A is defined by*

$$d(A) = \sup\{\ s \in \mathbb{R} : h^s(A) = +\infty\ \},$$

which is equivalent to

$$d(A) = \inf\{\ s \in \mathbb{R} : h^s(A) = 0\ \}.$$

The Hausdorff dimension of an empty set is defined to be $-\infty$.

DEFINITION 9. — *For a point $x \in X$, we define the local Hausdorff dimension $d(A, x)$ by*

$$d(A, x) = \inf_{x \in U} d(A \cap U),$$

where we take inf *for all neighborhood U of x in X.*

Note that the set

$$E_c = \{x \in X : d(A, x) \geqslant c\}$$

is a closed set for every $c \in \mathbb{R}$, and that, if $x \notin A$ and A is a closed set, then $d(A, x) = -\infty$.

PROPOSITION 6.— *Suppose that X is σ-compact (i.e., a countable union of compact sets). Then, for any closed subset A in X, the equality*

$$d(A) = \sup_{x \in X} d(A, x) = \sup_{x \in A} d(A, x)$$

holds.

Proof. — The second equality is clear from the definition. The inequality

$$d(A) \geqslant \sup_{x \in A} d(A, x)$$

is clear, since $d(A) \geqslant d(A, x)$ holds for any $x \in X$.

To prove the inequality of the other direction, first we consider the case where A is compact. Take any $\varepsilon_0 > 0$ and fix r, s such that $\sup_{x \in A} d(A, x) < r < s < \sup_{x \in A} d(A, x) + \varepsilon_0$. By the choice of r, for any point $x \in A$, there is a open neighborhood $U_x \subset X$ of x such that $d(A \cap U_x) < r$. Since A is compact, there are finitely many points $x_1, \ldots, x_n$ in A such that $A \subset U_{x_1} \cup \ldots \cup U_{x_n}$. Take any $\varepsilon > 0$ and any k with $1 \leqslant k \leqslant n$. By the choice of s, there is $\delta_k > 0$ such that, for any δ with $0 < \delta < \delta_k$, the inequality

$$v(s, \mathcal{B}_k) < \frac{\varepsilon}{n}$$

holds for some $\mathcal{B}_k \in I(A \cap U_k, \delta)$. Put $\delta_0 = \min_k \{\delta_k\}$. Then for any δ with $0 < \delta < \delta_0$, there is $\mathcal{B} \in I(A, \delta)$ such that

$$v(s, \mathcal{B}) \leqslant \frac{\varepsilon}{n} \times n = \varepsilon,$$

and we see that $h^s_\delta(A) \leqslant \varepsilon$. Hence we have $h^s(A) = 0$. This implies that $d(A) < \sup_{x \in A} d(A, x) + \varepsilon_0$. Since $\varepsilon_0 > 0$ is arbitrary, we have $d(A) \leqslant \sup_{x \in A} d(A, x)$. Hence we have $d(A) = \sup_{x \in A} d(A, x)$ under the assumption that A is compact.

Next we consider the case where A is non-compact. Since X is σ-compact and since A is closed, there is a increasing sequence $\{A_n\}_n$ of compact subsets in A such that $\cup_{k=1}^\infty A_k = A$. Since $d(A_n) = \sup_{x \in A_n} d(A_n, x)$ by the argument above, we have

$$\lim_{n \to \infty} d(A_n) \leqslant \sup_n \sup_{x \in A_n} d(A_n, x) = \sup_{x \in A} d(A, x)$$

We shall show the equality

$$d(A) = \lim_n d(A_n). \tag{9}$$

That $d(A) \geqslant \lim_n d(A_n)$ is clear. Suppose that there is an r such that $d(A) > r > \lim_n d(A_n)$. Fix $\varepsilon > 0$. For any $\delta > 0$, there is $\mathcal{B}_n \in I(A_n, \delta)$ such that $v(r, \mathcal{B}_n) < \frac{\varepsilon}{2^n}$. Since $\mathcal{B} = \bigcup_n \mathcal{B}_n \in I(A, \delta)$, we have

$$v(r, \mathcal{B}) \leqslant \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

This implies that $h^r(A) = 0$. Hence we get $d(A) \leqslant r$, which is a contradiction. Hence we have (9) and, consequently,

$$d(A) = \sup_{x \in A} d(A, x)$$

in the general case. □

Lemma 5. — *Let $X = (X, \rho_X)$ and $Y = (Y, \rho_Y)$ be σ-compact metric spaces and A a closed subset of Y. Suppose that f is a continuous map of X into Y such that for a positive constant K, the Lipschitz condition*

$$\rho_Y(f(x_1), f(x_2)) \leqslant K \rho_X(x_1, x_2)$$

holds for any points $x_1, x_2 \in X$. Then, the inequality

$$d(A, f(p)) \leqslant d(f^{-1}(A), p)$$

holds true for any point $p \in X$.

Proof. — Take any $\varepsilon > 0$. Then there is an open neighborhood U of p in X such that $d(f^{-1}(A) \cap U) < d(f^{-1}(A), p) + \varepsilon$. Put $\alpha = d(f^{-1}(A) \cap U)$. Take any $s > \alpha$. By definition, for any $\varepsilon_0 > 0$, we can find $\delta_0 > 0$ such that, for any $0 < \delta \leqslant \delta_0$, there is a ball covering $\mathcal{B} \in I(f^{-1}(A) \cap U, \delta)$ satisfying

$$\sum_{B_i \in \mathcal{B}} \ell(B_i)^s < \varepsilon_0.$$

By the Lipschitz condition, the inequality

$$\ell(f(B_i)) \leqslant K \ell(B_i)$$

holds for all i. Replacing each $f(B_i)$ with a ball C_i with $f(B_i) \subset C_i$ such that $\ell(f(B_i)) = \ell(C_i)$, we have a ball covering $\mathcal{B} \in I(A \cap f(U), K\delta)$ such that

$$\sum_{C_i \in \mathcal{B}(A \cap f(U), K\delta)} \ell(C_i)^s < K^s \varepsilon_0.$$

Thus we see that $h^s(A \cap f(U)) = 0$, and consequently, we have $d(A \cap f(U)) \leqslant d(f^{-1}(A) \cap U)$. This implies $d(A \cap f(U), f(p)) \leqslant d(f^{-1}(A) \cap U)$ by Proposition 6. Hence $d(A, f(p)) \leqslant \alpha + \varepsilon$ follows, and consequently we have $d(A, f(p)) \leqslant d(f^{-1}(A), p)$. □

The following lemma due to [Si, (2.A.13)] is useful.

LEMMA 6. — *Let $X = (X, \rho_X)$ and $Y = (Y, \rho_Y)$ be metric spaces. Assume that X is compact and that $f : X \to Y$ is a continuous map such that the fibers are finite. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if A is a set with $\ell(A) < \delta$ in Y, then every connected component B of $f^{-1}(A)$ satisfies $\ell(B) < \varepsilon$.*

Proof. — Here we give a copy of Siu's proof for reader's convenience. Suppose the lemma is false. Then there is $\varepsilon > 0$ with the following property: for every $n \in \mathbb{N}$, there exists some $A_n \subset Y$ with the diameter $\ell(A_n) < \frac{1}{n}$ such that some connected component B_n of $f^{-1}(A_n)$ has the diameter $\ell(B_n) \geqslant \varepsilon$.

Now take $a_n, b_n \in B_n$ such that $\rho_Y(a_n, b_n) \geqslant \frac{\varepsilon}{2}$. Since X is compact, there exists a subsequence $\{a_{n_k}\} \subset \{a_n\}$ converging to some point $a \in X$. Let

$$f^{-1}f(a) = \{c_1, \ldots, c_s\}.$$

Choose $0 < \eta < \frac{\varepsilon}{4}$ such that

$$U(c_1, \eta), \ldots, U(c_s, \eta)$$

are all mutually disjoint, where $U(c_j, \eta)$ are open balls of radius η centered at c_j. Since X is compact,

$$\bigcap_{j=1}^{s} f\left(X \setminus U(c_j, \eta)\right)$$

is closed in Y. Since

$$f(a) \notin \bigcap_{j=1}^{s} f\left(X \setminus U(c_j, \eta)\right),$$

there exists $r > 0$ such that

$$f^{-1}\left(U(f(a), r)\right) \subset \bigcup_{j=1}^{s} U(c_j, \eta).$$

There exists k_0 such that, for $k \geqslant k_0$, we have $\frac{1}{n_k} < \frac{r}{2}$ and

$$\rho_Y\left(f(a_{n_k}), f(a)\right) < \frac{r}{2}.$$

Fix $k \geqslant k_0$. For any $x \in A_{n_k}$, we have

$$\rho_Y(x, f(a)) \leqslant \rho_Y\left(x, f(a_{n_k})\right) + \rho_Y\left(f(a_{n_k}), f(a)\right) < \frac{1}{n_k} + \frac{r}{2} < r.$$

Hence $A_{n_k} \subset U(f(a), r)$. Hence $B_{n_k} \subset U(c_{i_k}, \eta)$ for some i_k. Since $a_{n_k}, b_{n_k} \in U(c_{i_k}, \eta)$, we have

$$\rho_X(a_{n_k}, b_{n_k}) < 2\eta < \frac{\varepsilon}{2},$$

which is a contradiction. □

For a ball covering $\mathcal{B} \in I(A, \delta)$, we define a ball covering $\tilde{\mathcal{B}}$ of $f^{-1}(A)$ as follows. For any $B_i \in \mathcal{B}$, consider connected components C_{i_λ} $(\lambda \in \Lambda_i)$ of $f^{-1}(B_i)$. Take a closed ball $B_{i_\lambda} \subset X$ such that $C_{i_\lambda} \subset B_{i_\lambda}$ with $\ell(C_{i_\lambda}) = \ell(B_{i_\lambda})$. Define $\tilde{\mathcal{B}}$ by

$$\tilde{\mathcal{B}} = \{B_{i_\lambda} \subset X : B_i \in \mathcal{B},\ \lambda \in \Lambda_i\}.$$

As a consequence of Lemma 6, we have

Lemma 7. — *Let $f : X \to Y$ be a continuous map between metric spaces. Suppose that the fibers of f are finite. If A is a subset of Y such that $f^{-1}(A)$ is contained in a compact subset of X. Then, for any $\varepsilon > 0$, there is $\delta_0 > 0$ such that, for any $0 < \delta \leqslant \delta_0$ and any $\mathcal{B} \in I(A, \delta)$, we have $\tilde{\mathcal{B}} \in I(f^{-1}(A), \varepsilon)$.*

Lemma 8. — *Let $f : X \to Y$ be a locally proper continuous map between locally compact metric spaces X and Y. Suppose that f satisfies the Lipschitz condition and that, for any point $q \in Y$, there is a neighborhood U of q and a positive integer r such that the number of points in $f^{-1}(q)$ is* *not more than r. Then, for a closed subset A of Y and for any point $p \in X$, we have*

$$d(f^{-1}(A), p) = d(A, f(p)).$$

Proof. — The inequality

$$d(f^{-1}(A), p) \geqslant d(A, f(p))$$

follows from Lemma 5. Thus it is sufficient to show the inequality

$$d(f^{-1}(A), p) \leqslant d(A, f(p)).$$

By the definition of local Hausdorff dimension, for any $\varepsilon > 0$, there is a neighborhood W of $f(p)$ such that $d(A \cap W) < d(A, f(p)) + \varepsilon$. Since f is locally proper, there are a neighborhood U of p in X, a neighborhood V of $f(p)$ with $[V] \subset W$ such that $f|[U] : [U] \to [V]$ is proper. Further, by the assumption on the fibers of f, we can assume that there is a positive integer r such that $(f|[U])^{-1}(x)$ consists of not more than r points for any $x \in [V]$.

Now put $g = f|[U]$ and $\alpha = d(A \cap V) \leqslant d(A \cap W)$. Then, $h^{\alpha+\varepsilon_1}(A \cap V) = 0$ for any $\varepsilon_1 > 0$. Suppose that $\varepsilon_2 > 0$ is given. We can find $\delta_1 > 0$ such that for any $0 < \delta \leqslant \delta_1$, there is a ball covering $\mathcal{B} \in I(A \cap V, \delta)$ with $v(\alpha + \varepsilon_1, \mathcal{B}) < \varepsilon_2$. Then by Lemma 7, there is $0 < \delta_2 < \delta_1$ such that, for any $0 < \delta \leqslant \delta_2$ and for any $\mathcal{B} \in I(A \cap V, \delta)$, we have $\tilde{\mathcal{B}} \in I(g^{-1}(A \cap V), \varepsilon_2)$ and $v(\alpha + \varepsilon_1, \tilde{\mathcal{B}}) \leqslant rv(\alpha + \varepsilon_1, \mathcal{B}) < r\varepsilon_2$. Hence $h_\delta^{\alpha+\varepsilon_1}(g^{-1}(A \cap V)) < r\varepsilon_2$ for any $0 < \delta \leqslant \delta_2$. Since $\varepsilon_2 > 0$ is arbitrary, this implies $h^{\alpha+\varepsilon_1}(g^{-1}(A \cap V)) = 0$. Hence $d(g^{-1}(A \cap V)) \leqslant \alpha + \varepsilon_1$. This shows $d(g^{-1}(A \cap V)) \leqslant \alpha = d(A \cap V)$ and consequently we have $d(g^{-1}(A), p) \leqslant d(A, g(p))$. Since $d(f^{-1}(A), p) = d(g^{-1}(A), p)$ and $f(p) = g(p)$, we have $d(f^{-1}(A), p) \leqslant d(A, f(p))$. □

We say that a continuous map $f : X \to Y$ is *locally proper*, if for any point $p \in X$ and any neighborhood W of $f(p)$, there are open neighborhood U of p in X and a neighborhood V of $f(p)$ in Y with $V \subset W$ such that

$$f|U : U \to V$$

is proper. Now we can show the following proposition.

Proposition 7. — *Let $f : X \to Y$ be a holomorphic map between (reduced) complex spaces X and Y with* $\dim X \leqslant \dim Y$. *Let A be a closed* *subset in Y. If every fiber of f consists of discrete points, then we have*

$$d(f^{-1}(A), p) = d(A, f(p))$$

for any point $p \in X$.

Proof. — Put $n = \dim X$. Since any holomorphic map satisfies the Lipschitz condition, the inequality $d(f^{-1}(A), p) \geqslant d(A, f(p))$ follows from Lemma 5. It is well-known that f is locally proper if every fiber of f consists of discrete points. We take open neighborhoods $U \ni p$ and $V \ni f(p)$ such that $f|U : U \to V$ is proper. Here V can be chosen arbitrary small and $Z = f(U)$ is an analytic subset of V by the proper mapping theorem of Remmert. Hence Z can be represented as an analytic cover $\pi : Z \to \Omega$ over a domain Ω in $\mathbb{C}^n$. Put $f_U = f|U$. Then, since $\pi \circ f_U : U \to \Omega$ is also an analytic cover, the conditions of Lemma 8 are satisfied. Hence we have

$$d((\pi \circ f_U)^{-1}(\pi(A)), p) = d(\pi(A), \pi \circ f_U(p)) \tag{10}$$

by Lemma 8. Since $f_U^{-1}(A) \subset (\pi \circ f_U)^{-1}(\pi(A))$, we have

$$d(f^{-1}(A), p) \leqslant d((\pi \circ f_U)^{-1}(\pi(A)), p). \tag{11}$$

On the other hand, by Lemma 5, we have

$$d(\pi(A), \pi \circ f_U(p)) \leqslant d(A, f(p)). \tag{12}$$

Combining (10), (11) and (12), we have

$$d(f^{-1}(A), p) \leqslant d(A, f(p)).$$ □

PROPOSITION 8. — *Let $f : X \to Y$ be a holomorphic map between (reduced) complex spaces X and Y with $\dim X \leqslant \dim Y$. Let A be a closed subset in Y. If every fiber of f consists of discrete points, then we have*

$$d(f^{-1}(A)) = d(A).$$

Proof. — Clear by Propositions 6 and 7. □

The following fact is used in proving Proposition 4.

Proposition 9. — *Let X be as in Proposition 6. Let A be a closed subset of X, and B a closed subset of $X \setminus A$. Then the inequality*

$$d(A \cup B) \leqslant \max\{d(A),\ d(B)\}$$

holds true.

Proof. — Suppose that $d(B) \leqslant d(A)$. Since $A \cup B$ is a closed subset of X, it is enough to check

$$d(A \cup B, x) \leqslant d(A) \tag{13}$$

for any $x \in A$ by the assumption $d(B) \leqslant d(A)$ and Proposition 6. Take any point $x \in A$ and a compact neighborhood K of x in X. Take any $d > d(A) = \max\{d(A),\ d(B)\}$. It is enough to show that, for any $\varepsilon > 0$, $h^d_\delta(K \cap (A \cup B)) < \varepsilon$ holds for any sufficiently small $\delta > 0$. Let $U(x,r)$ denote open ball with center x and radius r. Denote by $B(x,r)$ the closure of $U(x,r)$. By the choice of d, there is $\delta_1 > 0$ such that, for any $0 < \delta < \delta_1$, there is a closed ball covering $\mathcal{B} = \{B_\alpha\}_\alpha$, $B_\alpha = B(x_\alpha, r_\alpha)$, of $K \cap A$ with $2r_\alpha = \ell(B_\alpha) \leqslant \delta$ such that

$$\sum_\alpha \ell(B_\alpha)^d < \frac{\varepsilon}{2}.$$

Since $K \cap A$ is compact, there are finite open balls $U(x_{\alpha_1}, 2r_{\alpha_1}), \dots, U(x_{\alpha_s}, 2r_{\alpha_s})$ such that

$$K \cap A \subset \cup_{j=1}^s U(x_{\alpha_j}, 2r_{\alpha_j}).$$

Put $W = \cup_{j=1}^s U(x_{\alpha_j}, 2r_{\alpha_j})$. Since $(K \cap B) \setminus W$ is closed and since $d > d(B)$, there is an closed ball covering $\mathcal{B}' = \{B'_\beta\}_\beta$, $B'_\alpha = B(x'_\beta, r'_\beta)$, of $(K \cap B) \setminus W$ with $2r'_\beta = \ell(B'_\beta) \leqslant \delta$ such that

$$\sum_\beta \ell(B'_\beta)^d < \frac{\varepsilon}{2}.$$

Therefore, for any $0 < \delta < \delta_1$, there is an open ball covering $\mathcal{B} \cup \mathcal{B}' = \{B_\alpha, B'_\beta\}_{\alpha,\beta}$ of $K \cap (A \cup B)$ such that

$$\sum_\alpha \ell(B_\alpha)^d + \sum_\beta \ell(B'_\beta)^d < \varepsilon.$$

This proves the proposition. The case $d(B) \geqslant d(A)$ can be proved similarly. □

BIBLIOGRAPHY

[D] Dloussky G., Envelopes d'holomorphie et prolongements d'hypersurfaces, Séminaire Pierre Lelong p. 215-235, 1975-76, Lec. Notes in Math. 578, Springer 1977.

[DG] Docquier F., Grauert H., Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten, Math. Ann. 140, (1960) 94-123.

[I1] Ivashkovich S. M., The Hartogs-type extension theorem for the meromorphic maps into compact Kähler manifolds, Invent. math. 109, (1992) 47-54.

[I2] Ivashkovich S. M., Extension properties of meromorphic mappings with values in non-Kähler complex manifolds, preprint 2003.

[Kt1] Kato Ma., Factorization of compact complex 3-folds which admit certain projective structures, Tohoku Math. J. 41 (1989), 359-397.

[Kt2] Kato Ma., Examples on an Extension Problem of Holomorphic Maps and a Holomorphic 1-Dimensional Foliation, Tokyo J. Math. 13, (1990) 139-146.

[Kr] Krachni M., Prolongement d'applications holomorphes, Bull. Soc. math. France, 118(1990), 229-240.

[M] Malgrange B., Lectures on the theory of functions of several complex variables, Tata Inst. Fund. Research, Bombay, 1958.

[O] Okada N., An example of holomorphic maps which cannot be extended meromorphically across a closed fractal subset, Mini-Conference on Algebraic Geometry 2000, 42-53, Saitama University, Urawa.

[Sh] Shiffman B., On the removal of singularities of analytic sets, Michigan Math. J. 15 (1968), 111-120.

[Si] Siu Y. T., Techniques of extension of analytic objects, Dekker, New York, 1974.

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