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ON THE FARAUT-KORANYI HYPERGEOMETRIC FUNCTIONS IN RANK TWO

by Miroslav ENGLIŠ & Genkai ZHANG

1. Introduction.

Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^d$ in its Harish-Chandra realization (a Cartan domain); i.e. Ω is circular, centered at the origin, and convex. We denote by r, a, b, d and p the rank, the characteristic multiplicities, the dimension and the genus of Ω, respectively; thus

$$d = \frac{r(r-1)}{2}a + rb + r, \qquad p = (r-1)a + b + 2.$$

Let further G stand for the identity connected component of the group of biholomorphic self-maps of Ω, and K for the stabilizer of the origin in G. The elements of K are precisely the unitary maps on $\mathbb{C}^d$ that preserve Ω, and G acts transitively on Ω (thus Ω may be identified with the coset space G/K).

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Under the action $f \mapsto f \circ k$ ($k \in K$) of K, the space $\mathcal{P}$ of holomorphic polynomials on $\mathbb{C}^d$ admits the Peter-Weyl decomposition into multiplicity-free direct sum of irreducible subspaces

$$\mathcal{P} = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}},$$

where the summation extends over all *signatures* (or *partitions*) $\mathbf{m}$, i.e. r-tuples $(m_1, \dots, m_r)$ of nonnegative integers such that $m_1 \geqslant m_2 \geqslant \dots \geqslant m_r \geqslant 0$. For each $\mathbf{m}$, the elements of $\mathcal{P}_{\mathbf{m}}$ are homogeneous polynomials of degree $|\mathbf{m}| := m_1 + \dots + m_r$. Equipped with the Fischer (or Fock) scalar product

$$\begin{aligned} \langle f, g \rangle_F : &= f(\partial)\overline{g(\overline{z})}\Big|_{z=0} \\ &= \pi^{-d} \int_{\mathbb{C}^d} f(x)\overline{g(x)} e^{-|x|^2}\, dm(x), \end{aligned}$$

each space $\mathcal{P}_{\mathbf{m}}$ becomes a finite-dimensional Hilbert space of functions on $\mathbb{C}^d$, and thus has a reproducing kernel $K_{\mathbf{m}}(x, \overline{y})$, holomorphic in x and conjugate-holomorphic in y.

The *Faraut-Koranyi hypergeometric functions* [FK], [Y] on Ω are defined by

$$(1.1) \qquad {}_2\mathcal{F}_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} \middle| \, x \right) := \sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}} (\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} K_{\mathbf{m}}(x, \overline{x}).$$

Here $(\cdot)_{\mathbf{m}}$ is the *generalized Pochhammer symbol*

$$(1.2) \quad (\alpha)_{\mathbf{m}} := \prod_{j=1}^{r} \left(\alpha - \frac{j-1}{2} a \right)_{m_j}, \quad \text{where } (\alpha)_k := \alpha(\alpha+1)\dots(\alpha+k-1).$$

The right-hand side of (1.1) converges uniformly for x, y in compact subsets of Ω, for any $\alpha, \beta, \gamma \in \mathbb{C}$ such that $(\gamma)_{\mathbf{m}} \neq 0$ $\forall \mathbf{m}$.

The series (1.1) arise in many problems in analysis on bounded symmetric domains, and often their asymptotic behaviour at the boundary of Ω is of importance. (The most notable examples are perhaps the generalizations of the Forelli-Rudin inequalities [FK] and their applications to Besov spaces and operator theory [Zh] [E] and boundedness of the Bergman projections [CR]; or Fourier analysis of spherical functions on symmetric domains [Sh] [H], which turn out to be of the form (1.1) for special values of the parameters α, β, γ.) It has been shown already in [FK] that for α, β, γ such that

$$\frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} > 0 \qquad \forall \mathbf{m}, \tag{1.3}$$

one has

$${}_2\mathcal{F}_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix} \middle| x\right) \asymp 1 \quad \text{if } \operatorname{Re}(\alpha+\beta-\gamma) < -\frac{r-1}{2}a, \tag{1.4}$$

and

$${}_2\mathcal{F}_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix} \middle| x\right) \asymp h(x, \overline{x})^{\operatorname{Re}(\gamma-\alpha-\beta)} \quad \text{if } \operatorname{Re}(\alpha+\beta-\gamma) > \frac{r-1}{2}a, \tag{1.5}$$

where $h(x, \overline{y})$ is the Jordan triple determinant of Ω (see Section 2 below for details) and, for any pair of nonnegative functions f, g on some set, the notation

$$f(x) \asymp g(x)$$

means that there exist constants $0 < C_1 < C_2 < \infty$ (independent of x) such that

$$C_1 f \leqslant g \leqslant C_2 f.$$

The authors of [FK] actually considered only real α, β, γ, but their argument works even if one assumes only (1.3). Also, instead of (1.4) they only proved that ${}_2\mathcal{F}_1$ is bounded; however, since (1.3) implies that all summands in (1.1) are nonnegative, and the term corresponding to $\mathbf{m} = (0, 0, \ldots, 0)$ equals 1, the lower bound is trivial.

Much less is known, however, about the behaviour of ${}_2\mathcal{F}_1$ in the critical strip

$$-\frac{r-1}{2}a \leqslant \operatorname{Re}(\alpha+\beta-\gamma) \leqslant \frac{r-1}{2}a. \tag{1.6}$$

For $x = te$, where $0 \leqslant t \leqslant 1$ and $e \in \overline{\Omega}$ is a maximal tripotent (i.e. an element of the Shilov boundary of Ω, or, equivalently, any element in $\overline{\Omega}$ having the maximal Euclidean distance from the origin), the asymptotics of ${}_2\mathcal{F}_1$ were obtained by Yan [Y]. He showed that if $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy (1.3), then

$${}_2\mathcal{F}_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix} \middle| te\right) \asymp (1-t)^{-j(2\xi-j+1)a/2} p_\xi(t), \tag{1.7}$$

where $\xi := \frac{\alpha+\beta-\gamma}{a} + \frac{r-1}{2} \in [0, r-1]$, j is the integer such that $j-1 \leqslant \xi < j$, and $p_\xi(t) = 1 - \log(1-t)$ if ξ is an integer, while $p_\xi(t) = 1$ otherwise. Thus the behaviour of ${}_2\mathcal{F}_1$ in the critical strip (1.6) seems to be fairly complicated.

In the present paper, we extend Yan's result by describing completely the boundary behaviour of ${}_2\mathcal{F}_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\middle| x\right)$, with α,β,γ subject to the condition (1.3), on all of Ω if Ω has rank 2.

THEOREM 1. — *Let Ω be a Cartan domain of rank 2, α,β,γ any complex numbers such that* (1.3) *is fulfilled, and x a point of Ω. Let $t \leqslant T$ be the singular values of x (i.e. $x = k(te_1 + Te_2)$ for some $k \in K$ and minimal orthogonal tripotents e_1, e_2) and denote $\nu = \mathrm{Re}\,(\alpha+\beta-\gamma)$. Then*
(1.8)
$$
{}_2\mathcal{F}_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\middle| x\right) \asymp \begin{cases}
(1-t)^{-\nu}(1-T)^{-\nu} & \text{if } \nu > \frac{a}{2}\,, \\
(1-t)^{-a/2}(1-T)^{-a/2}\big[1+\log\frac{1}{1-t}\big] & \text{if } \nu = \frac{a}{2}\,, \\
(1-t)^{-a/2}(1-T)^{-\nu} & \text{if } 0 < \nu < \frac{a}{2}\,, \\
(1-t)^{-a/2}\big[1+\log\frac{1-t}{1-T}\big] & \text{if } \nu = 0, \\
(1-t)^{-\nu-\frac{a}{2}} & \text{if } -\frac{a}{2} < \nu < 0, \\
1+\log\frac{1}{1-t} & \text{if } \nu = -\frac{a}{2}\,, \\
1 & \text{if } \nu < -\frac{a}{2}.
\end{cases}
$$

This also gives the behaviour of ${}_2\mathcal{F}_1$ for Ω of arbitrary rank and $x \in \Omega$ having only two nonzero singular values. The proofs rely on an interesting integral representation for $K_{\mathbf{m}}(x,\overline{x})$ on domains of rank 2: namely,

$$
K_{\mathbf{m}}(t_1,t_2) = \frac{d_{\mathbf{m}}}{(\frac{d}{2})_{\mathbf{m}}}\,\frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2}\int_0^1 [t_1-(t_1-t_2)y]^{m_1-m_2}\,(t_1t_2)^{m_2}\,[y(1-y)]^{\frac{a}{2}-1}\,dy.
$$

(See again Section 2 for the notation.) It would be nice to have an analogous representation for general $\mathbf{m}$ and Ω.

The proof of Theorem 1 is given in Section 4, after dealing with some preliminaries in Section 2 and establishing the above-mentioned integral representation in Section 3. The extension to points with at most two nonzero singular values in domains of arbitrary rank appears in Section 5.

Finally, our result remains in force also for noninteger $a > 0$; this is briefly explained in the last Section 6.

We remark that all our results extend, without change, also to the more general hypergeometric functions

$$ {}_{k+1}\mathcal{F}_k \left(\begin{array}{c} \alpha_1, \ldots, \alpha_{k+1} \\ \beta_1, \ldots, \beta_k \end{array} \middle| \, x \right) := \sum_{\mathbf{m}} \frac{(\alpha_1)_{\mathbf{m}} \ldots (\alpha_{k+1})_{\mathbf{m}}}{(\beta_1)_{\mathbf{m}} \ldots (\beta_k)_{\mathbf{m}}} K_{\mathbf{m}}(x, \overline{x}), $$

with $\alpha_1, \ldots, \alpha_{k+1}, \beta_1, \ldots, \beta_k$ such that $(\alpha_1)_{\mathbf{m}} \ldots (\alpha_{k+1})_{\mathbf{m}} / (\beta_1)_{\mathbf{m}} \ldots (\beta_k)_{\mathbf{m}} > 0 \ \forall \mathbf{m}$. On the other hand, we are unable to say anything about the behaviour of the "polarized" functions

$$ {}_2\mathcal{F}_1 \left(\begin{array}{c} \alpha, \beta \\ \gamma \end{array} \middle| \, x, \overline{y} \right) := \sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}} (\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} K_{\mathbf{m}}(x, \overline{y}) $$

for $x \neq y$ (more precisely – for x, y which cannot be diagonalized by the same Jordan frame).

2. Preliminaries.

The main purpose of this section is to set up all kinds of notations and collect various useful facts on Cartan domains; see e.g. [Ar], [FK], or [Lo]. In particular, we fix a *Jordan frame* $e_1, \ldots, e_r \in \mathbb{C}^d$, so that each $x \in \mathbb{C}^d$ has the *polar decomposition*

$$ x = k(t_1 e_1 + \ldots + t_r e_r), \qquad k \in K, \ t_1 \geqslant t_2 \geqslant \ldots \geqslant t_r \geqslant 0 \tag{2.1} $$

(the numbers $t_1, \ldots, t_r$ — the *singular values* of x — are determined uniquely, but k need not be). Further, x belongs to $\Omega, \partial\Omega$, or $\mathbb{C}^d \setminus \Omega$ if and only if $t_1 < 1$, $t_1 = 1$, or $t_1 \geqslant 1$, respectively. For x as in (2.1), it is known that ([FK], Lemma 3.2)

$$ K_{\mathbf{m}}(x, \overline{x}) = K_{\mathbf{m}}(\textstyle\sum_j t_j^2 e_j, \overline{e}), \tag{2.2} $$

where $e := \sum_j e_j$.

The *Jordan triple determinant* $h(x,\overline{y})$ is the (unique) K-invariant polynomial on $\Omega \times \Omega$ (i.e. $h(kx,\overline{ky}) = h(x,\overline{y})\ \forall k \in K$), holomorphic in x and conjugate-holomorphic in y, such that for x as in (2.1)

$$h(x,\overline{x}) = \prod_{j=1}^{r}(1-t_j^2). \tag{2.3}$$

The kernels $K_{\mathbf{m}}$ are related to h by the *Faraut-Koranyi formula* [FK]

$$h(x,\overline{y})^{-\nu} = \sum_{\mathbf{m}}(\nu)_{\mathbf{m}}\, K_{\mathbf{m}}(x,\overline{y}), \tag{2.4}$$

in which the right-hand side converges, uniformly for x, y in compact subsets of Ω, for any $\nu \in \mathbb{C}$.

The point $e = e_1 + \ldots + e_r$ belongs to the Shilov boundary $\partial_0\Omega$ of Ω. The group K acts transitively on $\partial_0\Omega$, so that $\partial_0\Omega = \{ke, k \in K\} \simeq K/L$, where L is the stabilizer of e in K. Each Peter-Weyl space $\mathcal{P}_{\mathbf{m}}$ contains a unique L-invariant polynomial $\phi_{\mathbf{m}}$ satisfying the normalization condition $\phi_{\mathbf{m}}(e) = 1$. We will usually write $\phi_{\mathbf{m}}(t_1,\ldots,t_r)$ instead of $\phi_{\mathbf{m}}(t_1e_1 + \ldots + t_re_r)$. The polynomials $\phi_{\mathbf{m}}$ satisfy

$$\begin{gathered}\phi_{(0,0,\ldots,0)} = 1,\\ \phi_{(m_1+1,m_2+1,\ldots,m_r+1)}(t_1,\ldots,t_r) = t_1\cdots t_r\,\phi_{\mathbf{m}}(t_1,\ldots,t_r),\end{gathered} \tag{2.5}$$

and are related to the reproducing kernels $K_{\mathbf{m}}$ by the formula

$$K_{\mathbf{m}}(x,\overline{e}) = \frac{d_{\mathbf{m}}}{(d/r)_{\mathbf{m}}}\,\phi_{\mathbf{m}}(x), \tag{2.6}$$

where $d_{\mathbf{m}} := \dim \mathcal{P}_{\mathbf{m}}$. It is known that the last dimension is given by the formula ([Up], Lemmas 2.5 and 2.6)

$$d_{\mathbf{m}} = \frac{(d/r)_{\mathbf{m}}}{(q)_{\mathbf{m}}}\,\pi_{\mathbf{m}}$$

where

$$q := \frac{r-1}{2}a + 1$$

and

$$\pi_{\mathbf{m}} := \prod_{1\leqslant i<j\leqslant r} \frac{m_i - m_j + \frac{j-i}{2}a}{\frac{j-i}{2}a}\;\frac{(\frac{j-i+1}{2}a)_{m_i-m_j}}{(\frac{j-i-1}{2}a+1)_{m_i-m_j}}. \tag{2.7}$$

Thus we may rewrite (2.6) as

$$K_{\mathbf{m}}(x,\bar{e}) = \frac{\pi_{\mathbf{m}}}{(q)_{\mathbf{m}}}\,\phi_{\mathbf{m}}(x).$$

Combining the last formula with (2.2), we thus get

$$K_{\mathbf{m}}(x,\overline{x}) = \frac{\pi_{\mathbf{m}}}{(q)_{\mathbf{m}}}\,\phi_{\mathbf{m}}(t_1^2,\dots,t_r^2) \tag{2.8}$$

for x as in (2.1).

It is immediate from Stirling's formula that for any α,β such that $(\beta)_{\mathbf{m}} \neq 0\ \forall \mathbf{m}$,

$$\left|\frac{(\alpha)_{\mathbf{m}}}{(\beta)_{\mathbf{m}}}\right| \asymp \prod_{j=1}^{r}(1+m_j)^{\mathrm{Re}\,(\alpha-\beta)}. \tag{2.9}$$

Similarly, from (2.7) it follows that

$$\pi_{\mathbf{m}} \asymp \prod_{1\leqslant i<j\leqslant r}(1+m_i-m_j)^a.$$

Consequently, if α,β,γ satisfy (1.3), then

$${}_2\mathcal{F}_1\left(\begin{array}{c}\alpha,\beta\\ \gamma\end{array}\middle|\,x\right) \asymp \Lambda_{\mathrm{Re}\,(\alpha+\beta-\gamma)}(t_1^2,\dots,t_r^2) \tag{2.10}$$

for x as in (2.1), where

$$\Lambda_\nu(t_1,\dots,t_r) := \sum_{\mathbf{m}}\prod_{j=1}^{r}(1+m_j)^{\nu-q}\cdot\prod_{1\leqslant i<j\leqslant r}(1+m_i-m_j)^a\cdot\phi_{\mathbf{m}}(t_1,\dots,t_r). \tag{2.11}$$

Finally, we recall that the polynomials $\phi_{\mathbf{m}}$ have also a combinatorial interpretation in terms of *Jack symmetric polynomials* $J_{\mathbf{m}}^{(\lambda)}$ with parameter λ (cf. [MD], Section 10 of Chapter VI): namely,

$$\phi_{\mathbf{m}}(t_1,\dots,t_r) = j_{\mathbf{m}}^{-1}\,J_{\mathbf{m}}^{(2/a)}(t_1,\dots,t_r), \tag{2.12}$$

where

$$j_{\mathbf{m}} := J_{\mathbf{m}}^{(2/a)}(\underbrace{1,\dots,1}_{r}) = \left(\frac{2}{a}\right)^{|\mathbf{m}|}\left(\frac{ra}{2}\right)_{\mathbf{m}}. \tag{2.13}$$

We will usually suppress the superscripts $(2/a)$ in the sequel. For future use, we also note here explicitly that the Jack polynomials $J_{\mathbf{m}}$ are independent of r, in the sense that

$$J_{\mathbf{m}}(t_1,\dots,t_r) = J_{(m_1,\dots,m_r,\underbrace{0,0,\dots,0}_{k})}(t_1,\dots,t_r,\underbrace{0,0,\dots,0}_{k}), \tag{2.14}$$

for any $k \geqslant 0$.

3. An integral representation.

Let Δ_r be the $(r-1)$-dimensional simplex

$$\Delta_r := \{y \in \mathbb{R}^r : \ y_1,\dots,y_r \geqslant 0, \ y_1+\dots+y_r = 1\}$$

in $\mathbb{R}^r$, and let σ_{r-1} stand for the $(r-1)$-dimensional Lebesgue measure on Δ_r.

PROPOSITION 2. — *Let Ω be a Cartan domain of rank r. Then*

$$\phi_{(n,0,\dots,0)}(t_1,\dots,t_r) = \frac{\Gamma(\frac{ra}{2})}{\Gamma(\frac{a}{2})^r} \int_{\Delta_r} (y_1t_1+\dots+y_rt_r)^n \, (y_1\dots y_r)^{\frac{a}{2}-1} \, d\sigma_{r-1}(y).$$

Proof. — Observe, first of all, that

$$\begin{aligned}
\mathcal{I}_r(c_1,\dots,c_r) &:= \int_{\Delta_r} y_1^{c_1} y_2^{c_2} \dots y_r^{c_r} \, d\sigma_{r-1}(y) \\
&= \int_0^1 \int_0^{1-y_1} \dots \int_0^{1-y_1-\dots-y_{r-2}} y_1^{c_1} \dots y_{r-1}^{c_{r-1}} (1-y_1-\dots-y_{r-1})^{c_r} \\
&\qquad \times dy_{r-1} \dots dy_2 \, dy_1 \\
&= \int_0^1 (1-y_1)^{c_1+\dots+c_r} y_1^{c_1} \int_0^{1-y_1} \dots \int_0^{1-y_1-\dots-y_{r-2}} \\
&\qquad \times \left(\frac{y_2}{1-y_1}\right)^{c_2} \dots \left(\frac{y_{r-1}}{1-y_1}\right)^{c_{r-1}} \left(1 - \frac{y_2}{1-y_1} - \dots - \frac{y_{r-1}}{1-y_1}\right)^{c_r} \\
&\qquad \times dy_{r-1} \dots dy_2 \, dy_1
\end{aligned}$$

$$= \int_0^1 (1-y_1)^{c_1+\ldots+c_r+r-2} y_1^{c_1} \int_0^1 \int_0^{1-z_2} \cdots \int_0^{1-z_2-\ldots-z_{r-2}}$$

$$\times\, z_2^{c_2} \ldots z_{r-1}^{c_{r-1}} (1-z_2-\ldots-z_{r-1})^{c_r}\, dz_{r-1} \ldots dz_2\, dy_1$$

$$= \int_0^1 (1-y_1)^{c_1+\ldots+c_r+r-2} y_1^{c_1} \mathcal{I}_{r-1}(c_2,\ldots,c_r)\, dy_1$$

$$= \frac{\Gamma(c_1+1)\Gamma(c_2+\ldots+c_r+r-1)}{\Gamma(c_1+c_2+\ldots+c_r+r)}\, \mathcal{I}_{r-1}(c_2,\ldots,c_r).$$

An easy induction argument thus gives

$$\mathcal{I}_r(c_1,\ldots,c_r) = \frac{\Gamma(c_1+1)\ldots\Gamma(c_r+1)}{\Gamma(c_1+\ldots+c_r+r)}.$$

It follows that

$$p_n(t_1,\ldots,t_r) := \int_{\Delta_r} (y_1t_1+\ldots+y_rt_r)^n\, (y_1\ldots y_r)^{\frac{a}{2}-1}\, d\sigma_{r-1}(y)$$

$$= \sum_{n_1+\ldots+n_r=n} \frac{n!}{n_1!\ldots n_r!}\, t_1^{n_1}\ldots t_r^{n_r}\, \mathcal{I}_r(n_1+\frac{a}{2}-1,\ldots,n_r+\frac{a}{2}-1)$$

$$= \sum_{n_1+\ldots+n_r=n} \frac{n!}{n_1!\ldots n_r!}\, \frac{\Gamma(n_1+\frac{a}{2})\ldots\Gamma(n_r+\frac{a}{2})}{\Gamma(n+\frac{ra}{2})}\, t_1^{n_1}\ldots t_r^{n_r}$$

$$= \frac{n!\Gamma(\frac{a}{2})^r}{\Gamma(n+\frac{ra}{2})} \sum_{n_1+\ldots+n_r=n} \frac{(\frac{a}{2})_{n_1}\ldots(\frac{a}{2})_{n_r}}{n_1!\ldots n_r!}\, t_1^{n_1}\ldots t_r^{n_r}.$$

Consequently,

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{ra}{2})}{n!\Gamma(\frac{a}{2})^r}\, p_n(t_1,\ldots,t_r) = \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(\frac{a}{2})_{n_1}\ldots(\frac{a}{2})_{n_r}}{n_1!\ldots n_r!}\, t_1^{n_1}\ldots t_r^{n_r}$$

$$= \prod_{j=1}^{r} (1-t_j)^{-a/2}.$$

On the other hand, by the Faraut-Koranyi formula,

$$
\begin{aligned}
\prod_{j=1}^{r}(1-t_j)^{-a/2} &= \sum_{\mathbf{m}} (\tfrac{a}{2})_{\mathbf{m}} \, \frac{\pi_{\mathbf{m}}}{(q)_{\mathbf{m}}} \, \phi_{\mathbf{m}}(t_1,\dots,t_r) \\
&= \sum_{n=0}^{\infty} (\tfrac{a}{2})_n \, \frac{\pi_{(n,0,\dots,0)}}{(q)_n} \, \phi_{(n,0,\dots,0)}(t_1,\dots,t_r),
\end{aligned}
\tag{3.1}
$$

since $(\frac{a}{2})_{\mathbf{m}} = 0$ if $m_2 > 0$. Since both $\phi_{(n,0,\dots,0)}$ and p_n are homogeneous polynomials in $t_1,\dots,t_r$ of degree n, comparing the two expressions shows that

$$
\phi_{(n,0,\dots,0)}(t_1,\dots,t_r) = \frac{(q)_n}{\pi_{(n,0,\dots,0)}(\frac{a}{2})_n} \, \frac{\Gamma(n+\frac{ra}{2})}{n!\Gamma(\frac{a}{2})^r} \, p_n(t_1,\dots,t_r).
$$

Since $\pi_{(n,0,\dots,0)} = \dfrac{(q)_n(\frac{ra}{2})_n}{n!(\frac{a}{2})_n}$ by (2.7), the result follows. □

Taking in particular $r = 2$ in Proposition 2 we obtain the following corollaries.

Corollary 3. — *Let Ω be a Cartan domain of rank 2. Then*

$$
\phi_{(n,0)}(t_1,t_2) = \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2} \int_0^1 [t_1-(t_1-t_2)y]^n \, [y(1-y)]^{\frac{a}{2}-1} \, dy.
\tag{3.2}
$$

Corollary 4. — *Let Ω be a Cartan domain of rank 2. Then*

$$
\phi_{\mathbf{m}}(t_1,t_2) = \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2} \int_0^1 [t_1-(t_1-t_2)y]^{m_1-m_2} \, (t_1t_2)^{m_2} \, [y(1-y)]^{\frac{a}{2}-1} \, dy.
$$

Proof. — Immediate from the preceding corollary and (2.5). □

Remark 5. — A direct way of arrriving at Corollary 3 is as follows. Recall that the ordinary (Gauss) hypergeometric function, defined for $|z| < 1$ by the series

$$
{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,\middle|\, z\right) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \, z^n
$$

satisfies, for any $z \in \mathbb{C} \setminus [1, \infty)$,

$$(3.3)\quad {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,\middle|\, z\right) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 (1-zy)^{-\beta}y^{\alpha-1}(1-y)^{\gamma-\alpha-1}\,dy$$

whenever $\operatorname{Re}\gamma > \operatorname{Re}\alpha > 0$. Thus we have, for any $t_1, t_2 \in [0,1)$,

$$\begin{aligned}
(1-t_1)^{-a/2}(1-t_2)^{-a/2} &= (1-t_1)^{-a}\left(1-\frac{t_2-t_1}{1-t_1}\right)^{-a/2}\\
&= (1-t_1)^{-a}{}_2F_1\left(\begin{matrix}a/2,a\\ a\end{matrix}\,\middle|\,\frac{t_2-t_1}{1-t_1}\right)\\
&= \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2}(1-t_1)^{-a}\int_0^1\left[1-\frac{t_2-t_1}{1-t_1}y\right]^{-a}[y(1-y)]^{\frac{a}{2}-1}\,dy\\
&= \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2}\int_0^1 [1-t_1+(t_1-t_2)y]^{-a}\,[y(1-y)]^{\frac{a}{2}-1}\,dy\\
&= \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\int_0^1 [t_1-(t_1-t_2)y]^n\,[y(1-y)]^{\frac{a}{2}-1}\,dy.
\end{aligned}$$

The n-th term in the last infinite sum is plainly a homogeneous polynomial in t_1, t_2 of degree n. On the other hand, (2.3), the Faraut-Koranyi formula, (2.8), and the fact that $(\frac{a}{2})_{\mathbf{m}} = 0$ if $m_2 > 0$ imply that

$$\begin{aligned}
(1-t_1)^{-a/2}(1-t_2)^{-a/2} &= \sum_{\mathbf{m}:\, m_2=0}\left(\frac{a}{2}\right)_{\mathbf{m}}\frac{\pi_{\mathbf{m}}}{(q)_{\mathbf{m}}}\,\phi_{\mathbf{m}}(t_1,t_2)\\
&= \sum_{n=0}^{\infty}\left(\frac{a}{2}\right)_n\frac{\pi_{(n,0)}}{(q)_n}\,\phi_{(n,0)}(t_1,t_2),
\end{aligned}$$

and $\phi_{(n,0)}$ is again a homogeneous polynomial of degree n. The uniqueness of the homogeneous expansion of a function therefore implies that

$$\begin{aligned}
\phi_{(n,0)}(t_1,t_2) &= \frac{(q)_n}{\pi_{(n,0)}(\frac{a}{2})_n}\frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2}\frac{(a)_n}{n!}\int_0^1 [t_1-(t_1-t_2)y]^n\,[y(1-y)]^{\frac{a}{2}-1}\,dy\\
&= \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2}\int_0^1 [t_1-(t_1-t_2)y]^n\,[y(1-y)]^{\frac{a}{2}-1}\,dy,
\end{aligned}$$

since $\pi_{(n,0)} = \dfrac{(q)_n(a)_n}{(\frac{a}{2})_n n!}$ for $r=2$, by (2.7). This completes the proof.

Remark 6.— In a similar way one can obtain the more general formula for domains of arbitrary rank

$$\phi_{(n,0,\dots,0)}(t_1,t_2,0,\dots,0)=\frac{\Gamma(a+n)}{\Gamma(\frac{a}{2})^2(\frac{ra}{2})_n}\int_0^1[t_1-(t_1-t_2)y]^n\,[y(1-y)]^{\frac{a}{2}-1}\,dy.$$

Remark 7.— Another way of arriving at Corollary 3 is as follows. Since $\phi_{(j,0)}(t_1,t_2)$ is a symmetric polynomial in t_1,t_2 and is homogeneous of degree j, it must be a linear combination of $(t_1+t_2)^k(t_1t_2)^{\frac{j-k}{2}}$, $k=0,\dots,j$. It follows that $\phi_{(j,0)}(t_1,t_2)=(t_1t_2)^{j/2}G_j(\frac{t_1+t_2}{2\sqrt{t_1t_2}})$, for some polynomial G_j of degree at most j. Substituting this into (2.4) we thus get

$$(1-t_1)^{-\nu}(1-t_2)^{-\nu}=\sum_{\mathbf{m}}(\nu)_{\mathbf{m}}\frac{\pi_{\mathbf{m}}}{(q)_{\mathbf{m}}}(t_1t_2)^{\frac{m_1+m_2}{2}}G_{m_1-m_2}\left(\frac{t_1+t_2}{2\sqrt{t_1t_2}}\right).$$

Recalling the generating function for Gegenbauer polynomials $C_n^{(\lambda)}(x)$ ([BE], §3.15)

$$(1-2xt+t^2)^{-\lambda}=\sum_{n=0}^{\infty}C_n^{(\lambda)}(x)t^n,$$

setting $t=\sqrt{t_1t_2}$, $x=\frac{t_1+t_2}{2\sqrt{t_1t_2}}$, and comparing with the preceding formula, we obtain

$$\sum_{\mathbf{m}}(\nu)_{\mathbf{m}}\frac{\pi_{\mathbf{m}}}{(q)_{\mathbf{m}}}t^{m_1+m_2}G_{m_1-m_2}(x)=\sum_{n=0}^{\infty}C_n^{(\nu)}(x)t^n.$$

Taking $\nu=\frac{a}{2}$, so that $(\nu)_{\mathbf{m}}=0$ if $m_2>0$, and looking at the coefficients at like powers of t on both sides, we thus see that

$$G_m(x)=\frac{(q)_m}{\pi_{(m,0)}(\frac{a}{2})_m}C_m^{(a/2)}(x)=\frac{m!}{(a)_m}C_m^{(a/2)}(x),$$

whence

$$\phi_{(m,0)}(t_1,t_2)=(t_1t_2)^{m/2}\frac{m!}{(a)_m}C_m^{(a/2)}\left(\frac{t_1+t_2}{2\sqrt{t_1t_2}}\right).$$

Using the integral formula for Gegenbauer polynomials ([BE], §3.15, formula (22))

$$C_m^\lambda(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(m+2\lambda)\Gamma(\lambda+\frac{1}{2})}{m!\Gamma(\lambda)\Gamma(2\lambda)} \int_0^\pi [x+\sqrt{x^2-1}\cos\phi]^m (\sin\phi)^{2\lambda-1}\, d\phi,$$

and the formula for $\Gamma(2\lambda)$ ([BE], §1.2, formula (15)), we get

$$C_m^{(a/2)}(x) = \frac{2^{1-a}\Gamma(a+m)}{m!\Gamma(\frac{a}{2})^2} \int_0^\pi [x+\sqrt{x^2-1}\cos\phi]^m (\sin\phi)^{a-1}\, d\phi,$$

and a change of variable to $y = \frac{1-\cos\phi}{2}$ eventually leads to (3.2). □

Corollaries 3 and 4 are also a special case $r = 2$ of a result of P. Sawyer, see [Sa], Theorem 5.3 (but Proposition 2 is not). Further, our proof of Proposition 2 is completely elementary (or, in any case, much simpler than that of Theorem 5.3 in [Sa]). Plainly, it would be of interest to have a representation analogous to Proposition 2 for general signatures **m**.

4. Domains of rank 2.

LEMMA 8.— *For each $\nu > 1$, there exists a constant $C_\nu < \infty$ (depending on ν) such that*

$$(4.1) \qquad (1+k)^{-\nu} \leqslant \sum_{m=0}^{\infty} (1+m)^{-\nu}(1+m+k)^{-\nu} z^m \leqslant C_\nu (1+k)^{-\nu},$$

for all $z \in [0,1]$ and all $k = 0, 1, 2, \ldots$.

Proof.— Since the term $m = 0$ in the sum is precisely $(1+k)^{-\nu}$, the lower bound is trivial. For the upper bound, observe that

$$\sum_{m=1}^{\infty} (1+m)^{-\nu}(1+m+k)^{-\nu} z^m \leqslant \sum_{m=1}^{\infty} (1+m)^{-\nu}(1+m+k)^{-\nu}$$
$$\leqslant \int_1^\infty x^{-\nu}(x+k)^{-\nu}\, dx.$$

For $k=0$, the last integral equals $\frac{1}{2\nu-1}$. For $k\geqslant 1$, making the change of variable $x=ks$ transforms the integral into

$$\begin{aligned}
k^{1-2\nu}\int_{1/k}^{\infty} s^{-\nu}(s+1)^{-\nu}\,ds &\leqslant c_\nu k^{1-2\nu}\left[\int_{1/k}^{1} s^{-\nu}\,ds+\int_{1}^{\infty} s^{-2\nu}\,ds\right]\\
&= c_\nu k^{1-2\nu}\left[\frac{k^{\nu-1}-1}{\nu-1}+\frac{1}{2\nu-1}\right]\\
&\leqslant c_\nu k^{1-2\nu}\left[\frac{k^{\nu-1}-1}{\nu-1}+\frac{1}{\nu-1}\right]=\frac{c_\nu}{\nu-1}\,k^{-\nu}
\end{aligned}$$

for some finite constant c_ν, and the desired assertion follows. □

Proof of Theorem 1. — The cases $\nu<-\frac{a}{2}$ and $\nu>\frac{a}{2}$ are covered by (1.4) and (1.5), thus we may assume that $-\frac{a}{2}\leqslant\nu\leqslant\frac{a}{2}$. In view of (2.10), it is enough to prove the assertion for

$$\Lambda_\nu(t,T)=\sum_{\mathbf{m}}[(1+m_1)(1+m_2)]^{\nu-\frac{a}{2}-1}(1+m_1-m_2)^a\,(tT)^{m_2}\phi_{(m_1-m_2,0)}(t,T)$$

in the place of ${}_2\mathcal{F}_1$. Setting $m_2=m$ and $m_1=m+k$, the sum becomes

$$\sum_{m,k=0}^{\infty}(1+m+k)^{\nu-\frac{a}{2}-1}(1+m)^{\nu-\frac{a}{2}-1}(1+k)^a\,(tT)^m\,\phi_{(k,0)}(t,T). \tag{4.2}$$

Consider first the case of

$$-\frac{a}{2}<\nu<\frac{a}{2}.$$

Then $\nu-\frac{a}{2}-1<-1$, so we may apply Lemma 8 to conclude that

$$\begin{aligned}
(4.2)&\asymp\sum_{k=0}^{\infty}(1+k)^{\nu+\frac{a}{2}-1}\,\phi_{(k,0)}(t,T)\\
&\asymp\sum_{k=0}^{\infty}\frac{(\nu+\frac{a}{2})_k}{k!}\,\phi_{(k,0)}(t,T),
\end{aligned}\tag{4.3}$$

by (2.9). By Corollary 3, the last sum equals (note that $0 \leqslant t-(t-T)y < 1$)

$$\sum_{k=0}^{\infty} \frac{(\nu+\frac{a}{2})_k}{k!} \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2} \int_0^1 [t-(t-T)y]^k \, [y(1-y)]^{\frac{a}{2}-1} \, dy$$

$$= \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2} \int_0^1 [1-t+(t-T)y]^{-\nu-\frac{a}{2}} \, [y(1-y)]^{\frac{a}{2}-1} \, dy$$

$$= (1-t)^{-\nu-\frac{a}{2}} \, {}_2F_1\left(\begin{matrix} a/2, a/2+\nu \\ a \end{matrix} \middle| \, \frac{T-t}{1-t} \right), \qquad \text{by (3.3).}$$

As $0 \leqslant t \leqslant T < 1$, we have $\frac{T-t}{1-t} \in [0,1)$. Since for any $A, B, C > 0$,

$$\text{(4.4)} \qquad {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \middle| \, z \right) \asymp \begin{cases} (1-z)^{C-A-B} & \text{if } C < A+B \\ 1-\log(1-z) & \text{if } C = A+B \\ 1 & \text{if } C > A+B \end{cases}$$

on the interval $z \in [0,1)$, it follows that

$$\Lambda_\nu(t,T) \asymp \begin{cases} (1-t)^{-a/2}(1-T)^{-\nu} & \text{if } \nu > 0\,, \\ (1-t)^{-\nu-\frac{a}{2}} \left[1+\log \frac{1-t}{1-T}\right] & \text{if } \nu = 0\,, \\ (1-t)^{-\nu-\frac{a}{2}} & \text{if } \nu < 0\,, \end{cases}$$

which settles the three middle lines in (1.8).

It remains to deal with the cases $\nu = \pm\frac{a}{2}$. For $\nu = -\frac{a}{2}$, we get instead of (4.3)

$$(4.2) \asymp \sum_{k=0}^{\infty} \frac{\phi_{(k,0)}(t,T)}{k+1}.$$

Since

$$\sum_{k=0}^{\infty} \frac{z^k}{k+1} = \frac{1}{z} \log \frac{1}{1-z} \asymp 1 + \log \frac{1}{1-z}$$

for $z \in [0,1)$, the same argument as above shows that

$$(4.2) \asymp \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2} \int_0^1 \left[1+\log \frac{1}{1-t+(t-T)y}\right] [y(1-y)]^{\frac{a}{2}-1} \, dy.$$

Now

$$\begin{aligned}
&\frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2}\int_0^1 \log\frac{1}{1-t+(t-T)y}\,[y(1-y)]^{\frac{a}{2}-1}\,dy \\
&\qquad= \frac{d}{d\xi}\frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2}\int_0^1 [1-t+(t-T)y]^{-\xi}\,[y(1-y)]^{\frac{a}{2}-1}\,dy\,\Big|_{\xi=0} \\
&\qquad= \frac{d}{d\xi}(1-t)^{-\xi}\,{}_2F_1\left(\begin{matrix}a/2,\xi\\ a\end{matrix}\,\Big|\,\frac{T-t}{1-t}\right)\Big|_{\xi=0} \\
&\qquad= \log\frac{1}{1-t}+\frac{d}{d\xi}\,{}_2F_1\left(\begin{matrix}a/2,\xi\\ a\end{matrix}\,\Big|\,\frac{T-t}{1-t}\right)\Big|_{\xi=0},
\end{aligned}$$

and, for $0\leqslant z<1$,

$$\begin{aligned}
\frac{d}{d\xi}\,{}_2F_1\left(\begin{matrix}a/2,\xi\\ a\end{matrix}\,\Big|\,z\right)\Big|_{\xi=0} &= \sum_{j=0}^{\infty}\frac{(\frac{a}{2})_j}{(a)_j}\,\frac{z^j}{j!}\,\frac{d}{d\xi}(\xi)_j\Big|_{\xi=0} \\
&= \sum_{j=1}^{\infty}\frac{(\frac{a}{2})_j}{(a)_j}\,\frac{z^j}{j!}\,(\xi+1)_{j-1}\Big|_{\xi=0} \\
&= \sum_{j=1}^{\infty}\frac{(\frac{a}{2})_j}{(a)_j}\,\frac{z^j}{j}\asymp\sum_{j=1}^{\infty}(1+j)^{-a/2}\,\frac{z^j}{j} \\
&\asymp\sum_{j=1}^{\infty}(1+j)^{-\frac{a}{2}-1}\,z^j\asymp\sum_{j=1}^{\infty}\frac{(\frac{a}{4})_j(\frac{a}{4})_j}{(a)_j j!}\,z^j \\
&= {}_2F_1\left(\begin{matrix}a/4,a/4\\ a\end{matrix}\,\Big|\,z\right)-1\asymp z.
\end{aligned}$$

Since again $\frac{T-t}{1-t}\in[0,1)$, combining everything together we see that

$$\Lambda_{-a/2}(t,T)\asymp 1+\log\frac{1}{1-t}+\frac{T-t}{1-t}\asymp 1+\log\frac{1}{1-t}, \tag{4.5}$$

as asserted.

To deal with the last case $\nu = \frac{a}{2}$, we use the following transformation property of the Faraut-Koranyi hypergeometric functions (Kummer's relations; see [Y], Proposition 3.2)

$$(4.6)\qquad {}_2\mathcal{F}_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,\middle|\, x\right) = h(x,\overline{x})^{\gamma-\alpha-\beta}\,{}_2\mathcal{F}_1\left(\begin{matrix}\gamma-\alpha,\gamma-\beta\\ \gamma\end{matrix}\,\middle|\, x\right).$$

Applying it with $\alpha = \beta = \frac{3a}{2}$, $\gamma = \frac{5a}{2}$, we obtain

$$\begin{aligned}\Lambda_{a/2}(t,T) &\asymp {}_2\mathcal{F}_1\left(\begin{matrix}3a/2,3a/2\\ 5a/2\end{matrix}\,\middle|\, x\right) = h(x,x)^{-a/2}\,{}_2\mathcal{F}_1\left(\begin{matrix}a,a\\ 5a/2\end{matrix}\,\middle|\, x\right)\\ &\asymp h(x,x)^{-a/2}\Lambda_{-a/2}(t,T)\\ &\asymp (1-t)^{-a/2}(1-T)^{-a/2}\left[1+\log\frac{1}{1-t}\right],\end{aligned}$$

by (4.5). This completes the proof. □

Remark 9. — More generally, it follows from (4.6) that

$$\Lambda_\nu(t,T) \asymp (1-t)^{-\nu}(1-T)^{-\nu}\Lambda_{-\nu}(t,T), \qquad \forall\nu\in\mathbb{R}.$$

Thus, in principle, we could get the upper half of (1.8) from the lower half, or vice versa.

Remark 10. — For $t = T$, (1.8) reduces to

$${}_2\mathcal{F}_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,\middle|\, te\right) \asymp \begin{cases} h^{-\nu} & \text{if } \nu > \frac{a}{2},\\ h^{-a/2}\cdot\left(1+\log\frac{1}{1-t}\right) & \text{if } \nu = \frac{a}{2},\\ h^{-\frac{\nu}{2}-\frac{a}{4}} & \text{if } -\frac{a}{2} < \nu < \frac{a}{2},\\ 1+\log\frac{1}{1-t} & \text{if } \nu = -\frac{a}{2},\\ 1 & \text{if } \nu < -\frac{a}{2},\end{cases}$$

in complete agreement with Yan's result (1.7).

5. Partial extension to higher rank.

In this section, we deal with the case of a Cartan domain Ω of arbitrary rank and elements $x \in \Omega$ with polar decomposition $x = k(t_1 e_1 +$

t_2e_2), i.e. having at most only two nonzero eigenvalues. Our idea is to reduce this to the case of rank 2. To avoid confusion, we temporarily include the rank into the notation, writing $\phi_{\mathbf{m}}^{(r)}$, $j_{\mathbf{m}}^{(r)}$, $\Lambda_\nu^{(r)}$, etc.

Using the fact (2.14) that the Jack polynomials do not depend on r, we get from (2.12)

$$\phi_{\mathbf{m}}^{(r)}(t_1,t_2,0,\dots,0) = \frac{j_{\mathbf{m}}^{(2)}}{j_{\mathbf{m}}^{(r)}}\,\phi_{\mathbf{m}}^{(2)}(t_1,t_2).$$

(In particular, both sides vanish if $m_3 > 0$.) Since, by (2.13) and (2.9),

$$\frac{j_{\mathbf{m}}^{(2)}}{j_{\mathbf{m}}^{(r)}} = \frac{(a)_{\mathbf{m}}}{(\frac{ra}{2})_{\mathbf{m}}} \asymp [(1+m_1)(1+m_2)]^{-\frac{r-2}{2}a},$$

we conclude that

$$\begin{aligned}
&\Lambda_\nu^{(r)}(t_1,t_2,0,\dots,0)\\
&= \sum_{\mathbf{m}}\prod_{j=1}^{r}(1+m_j)^{\nu-\frac{r-1}{2}a-1}\cdot\prod_{1\leqslant i<j\leqslant r}(1+m_i-m_j)^a\cdot\,\phi_{\mathbf{m}}^{(r)}(t_1,t_2,0,\dots,0)\\
&= \sum_{m_1\geqslant m_2\geqslant 0}[(1+m_1)(1+m_2)]^{\nu-\frac{r-1}{2}a-1}\cdot[(1+m_1)(1+m_2)]^{(r-2)a}\\
&\qquad\qquad\times(1+m_1-m_2)^a\cdot\phi_{\mathbf{m}}^{(r)}(t_1,t_2,0,\dots,0)\\
&= \sum_{m_1\geqslant m_2\geqslant 0}[(1+m_1)(1+m_2)]^{\nu+\frac{r-3}{2}a-1}(1+m_1-m_2)^a\,\frac{j_{\mathbf{m}}^{(2)}}{j_{\mathbf{m}}^{(r)}}\,\phi_{\mathbf{m}}^{(2)}(t_1,t_2)\\
&\asymp \sum_{m_1\geqslant m_2\geqslant 0}[(1+m_1)(1+m_2)]^{\nu-\frac{a}{2}-1}(1+m_1-m_2)^a\,\phi_{\mathbf{m}}^{(2)}(t_1,t_2)\\
&= \Lambda_\nu^{(2)}(t_1,t_2).
\end{aligned}$$

We have thus arrived at the following theorem.

THEOREM 11. — *Let Ω be a Cartan domain of rank $r\geqslant 2$, e_1,e_2 a pair of orthogonal minimal tripotents, and α,β,γ complex numbers such that* (1.3) *is fulfilled. Denote $\nu=\operatorname{Re}(\alpha+\beta-\gamma)$, $t=\min(t_1,t_2)$, $T=\max(t_1,t_2)$. Then* (1.8) *holds true as x ranges in the set*

$$\{k(t_1e_1+t_2e_2);\ 0\leqslant t_1,t_2<1,\ k\in K\}.$$

Remark 12. — Another way of seeing that even

$$\Lambda_\nu^{(r)}(t_1,\dots,t_r) \asymp \Lambda_\nu^{(r+k)}(t_1,\dots,t_r,0,\dots,0), \tag{5.1}$$

for any $k \geqslant 0$, is the following. From (2.7) it can be checked that the quantity

$$\tilde{\pi}_{\mathbf{m}} := \frac{\pi_{\mathbf{m}}^{(r)}}{(\frac{ra}{2})_{\mathbf{m}}(q)_{\mathbf{m}}}$$

is independent of r as soon as r is greater than or equal to the length of $\mathbf{m}$; i.e. $\tilde{\pi}_{(\mathbf{m},0)} = \tilde{\pi}_{\mathbf{m}}$. Thus for x as in (2.1),

$${}_2\mathcal{F}_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,\middle|\, x\right) = \sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} \, \frac{\tilde{\pi}_{\mathbf{m}}}{(2/a)^{|\mathbf{m}|}} \, J_{\mathbf{m}}(t_1^2,\dots,t_r^2)$$

and the right-hand side is <u>independent of r</u>. (This is essentially the calculation in [Y], Proposition 4.1.) Consequently, if α,β,γ satisfy (1.3) and $\nu = \operatorname{Re}(\alpha+\beta-\gamma)$, then

$$\Lambda_\nu^{(r)}(t_1,t_2,\dots) \asymp \sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} \, \frac{\tilde{\pi}_{\mathbf{m}}}{(2/a)^{|\mathbf{m}|}} \, J_{\mathbf{m}}(t_1,t_2,\dots)$$

with the right-hand side independent of r. Hence (5.1) follows.

It would be interesting to know whether the definition (2.11) of $\Lambda_\nu^{(r)}$ can be modified in such a way that the estimate (5.1) become actually an equality.

6. The case of arbitrary a.

Note that the formulas (1.2) and (2.7) for the generalized Pochhammer symbol and $\pi_{\mathbf{m}}$ make sense even for any positive value of the number a (i.e. even if a is not the characteristic multiplicity of any Cartan domain). Similarly, the Jack polynomials $J_{\mathbf{m}}^{(2/a)}$ are defined for any $a > 0$. Using the relations (2.8) and (2.12), we can therefore define the hypergeometric functions ${}_2\mathcal{F}_1$ for any $a > 0$ by

$${}_2\mathcal{F}_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,\middle|\, x\right) := \sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} \, \frac{\pi_{\mathbf{m}}}{(q)_{\mathbf{m}}} \, j_{\mathbf{m}}^{-1} J_{\mathbf{m}}^{(2/a)}(t_1^2,\dots,t_r^2)$$

for x as in (2.1). (Actually, this is the definition used in [Y].)

It turns out that all our results remain in force also for these positive noninteger values of a. Indeed, looking at the proof of Proposition 2 we see that all arguments prior to (3.1) make sense even for any $a \in \mathbb{C}$ with positive real part (so that the integrals there exist), while the Faraut-Koranyi formula (2.4) (used in (3.1) with $\nu = \frac{a}{2}$) remains in force for all $a > 0$ ([Y], formula (34) on p. 1328). Thus Proposition 2 remains in force for any positive a too. Similarly, the only place in the proof of Theorem 1 in Section 4 where the fact that a be the characteristic multiplicity was needed were the Kummer relations (4.6), which likewise remain in force for any $a > 0$ by Proposition 3.2 in [Y].

BIBLIOGRAPHY

[Ar] J. Arazy, A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains, Multivariable operator theory, R.E. Curto, R.G. Douglas, J.D. Pincus, N. Salinas, Contemporary Mathematics, vol. 185, Amer. Math. Soc., Providence, p. 7–65 (1995).

[BE] H. Bateman, A.Erdélyi, Higher transcendental functions, vol. I, McGraw-Hill, New York – Toronto – London, (1953).

[CR] R.R. Coifman, R. Rochberg, Representation theorems for Hardy spaces, Asterisque, vol. 77, (1980), 11–66.

[E] M. Engliš, Compact Toeplitz operators via the Berezin transform on bounded symmetric domains, Integral Eq. Oper. Theory, vol. 33, (1999), 426-455, Erratum, ibid., vol. 34, (1999), 500-501.

[FK] J. Faraut, A.Koranyi, Function spaces and reproducing kernels on bounded symmetric domains, J. Funct. Anal., vol. 88, (1990), 64–89.

[H] S. Helgason, Groups and geometric analysis, Academic Press, Orlando, (1984).

[Lo] O. Loos, Bounded symmetric domains and Jordan pairs, Irvine, University of California, (1977).

[MD] I.G. Macdonald, Symmetric functions and Hall polynomials, 2nd edition, Clarendon Press, Oxford, (1995).

[Sa] P. Sawyer, Spherical functions on symmetric cones, Trans. Amer. Math. Soc., vol. 349, (1997), 3569–3584.

[Sh] N. Shimeno, Boundary value problems for the Shilov boundary of a bounded symmetric domain of tube type, J. Funct. Anal., vol. 140, (1996), 124-141.

[Up] H. Upmeier, Toeplitz operators on bounded symmetric domains, Trans. Amer. Math. Soc., vol. 280, (1983), 221–237.

[Y] Z. Yan, A class of generalized hypergeometric functions in several variables, Canad. J. Math., vol. 44, (1992), 1317–1338.

[Zh] K. Zhu, Holomorphic Besov spaces on bounded symmetric domains, Quart. J. Math. Oxford, vol. 46, (1995), 239–256.

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