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Eknath GHATE & Vinayak VATSAL

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ON THE LOCAL BEHAVIOUR OF ORDINARY Λ-ADIC REPRESENTATIONS

by Eknath GHATE and Vinayak VATSAL

1. Introduction.

In this paper we study the local behaviour of the Galois representations attached to ordinary Λ-adic forms and ordinary classical cusp forms. In both cases the splitting of the local representation is shown to be closely related to whether or not the corresponding form has complex multiplication.

Let us state our main result in the classical setting. Let $f = \sum_{n=1}^{\infty} a(n,f)q^n$ be an elliptic modular cusp form of weight $k \geqslant 2$, level $N \geqslant 1$ and nebentypus $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$. Assume that f is normalized to have first coefficient 1, and that it is a common eigenform of all the Hecke operators T_ℓ for primes ℓ with $(\ell, N) = 1$. Fix an embedding ι_∞ of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ and let K_f denote the number field generated by the Fourier coefficients $a(n,f)$ of f via the embedding ι_∞. If in addition f is a newform (or primitive form) in the sense of Atkin-Lehner, then f is also an eigenform for the operators U_ℓ for primes $\ell \mid N$.

Let p be a prime number. Fix an embedding ι_p of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ and let $\wp$ be the prime of $\bar{\mathbb{Q}}$ induced by this embedding. We continue to write $\wp$ for the restriction of $\wp$ to any subfield of $\bar{\mathbb{Q}}$. In particular $\wp$ is a prime of K_f. Let $K_{f,\wp}$ denote the completion of K_f at $\wp$. Following Eichler, Shimura, and Deligne, we can attach a Galois representation

$$\rho_f = \rho_{f,\wp} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\wp})$$

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to f (and $\wp$) which has the property that for all primes ℓ outside Np

$$\operatorname{tr}\rho_f(\mathrm{Frob}_\ell) = a(\ell, f) \quad \text{and} \quad \det\rho_f(\mathrm{Frob}_\ell) = \chi(\ell)\ell^{k-1}.$$

We say that f is ordinary at $\wp$ (or $\wp$-ordinary) if $a(p, f)$ is a $\wp$-adic unit. The notion of ordinariness depends on the embeddings ι_∞ and ι_p. If f is ordinary at $\wp$ then results of Mazur-Wiles [MW86] (for conductor p^n, and $p > 2$) and Wiles [Wil88] (in general) show that the restriction of ρ_f to the decomposition group G_p at $\wp$ is 'upper triangular'. More precisely if V is a two dimensional vector space over $K_{f,\wp}$ which affords the representation $\rho_f|_{G_p}$ then it is known that there is a basis of V in which $\rho_f|_{G_p}$ has the following shape:

$$\rho_f|_{G_p} \sim \begin{pmatrix} \delta & u \\ 0 & \epsilon \end{pmatrix} \tag{1.1}$$

where $\epsilon, \delta : G_p \to K_{f,\wp}^\times$ are characters with ϵ unramified and $u : G_p \to K_{f,\wp}$ is a continuous function.

The function u has a cohomological interpretation. Let $c : G_p \to K_{f,\wp}$ be the map defined by

$$c(g) = \epsilon^{-1}(g) \cdot u(g)$$

for all $g \in G_p$. Write $K_{f,\wp}(\delta\epsilon^{-1})$ for $K_{f,\wp}$ thought of as a G_p-module via the character $\delta\epsilon^{-1}$. Then it is easy to check that $c \in \mathrm{Z}^1(G_p, K_{f,\wp}(\delta\epsilon^{-1}))$ is a 1-cocycle. Thus c represents a cohomology class

$$[c] \in \mathrm{H}^1(G_p, K_{f,\wp}(\delta\epsilon^{-1})).$$

Let us say that the representation $\rho_f|_{G_p}$ *splits*, or that ρ_f is *split at* p, if $V = V_1 \oplus V_2$ where each V_i is a line stable under G_p. It is easy to see that the representation

$$\rho_f|_{G_p} \text{ splits if and only if } [c] = 0.$$

Ralph Greenberg has asked the following question concerning the splitting behaviour of the restriction of ρ_f to G_p.

Question 1. — Let f be a primitive $\wp$-ordinary cusp form of weight at least two. When is the representation $\rho_f|_{G_p}$ split? Equivalently when is the cocycle c that is defined in terms of the upper-shoulder map u in (1.1) a coboundary?

Let us say that f has complex multiplication (or CM for short) if there is a Dirichlet character θ such that $a(\ell, f) = a(\ell, f) \cdot \theta(\ell)$ for all but finitely many primes ℓ. It is well-known that if f has CM then ρ_f splits at

p (see, for instance, [Gha04], Prop. 3 and 5). A natural guess for an answer to Question 1 is therefore:

$$(1.2) \qquad (??) \quad \rho_f|_{G_p} \text{ splits if and only if } f \text{ has CM.}$$

Almost all evidence in support of (1.2), if it is indeed true, seems to be in weight two. Indeed, if p is prime to N, then it follows from a theorem due to Serre and Tate (see [Ser89], Chapter IV) that if the modular form f corresponds to an elliptic curve E defined over $\mathbb{Q}$, then $\rho_{f,\wp} = \rho_{E,p}$ splits at p if and only if E, and therefore f, has CM. This result can be extended partially to the cases where f corresponds to a modular abelian variety A of higher dimension and p is not necessarily prime to N. For precise statements and a general survey of what is known about (1.2) in weight two, see [Gha04].

In this paper we shall give further evidence in support of (1.2) for weights larger than two. To state our main result, we need some notation. Let $\bar{\rho}_f$ denote the reduction of ρ_f at the maximal ideal of the ring of integers of $K_{f,\wp}$. Say that f is *p-distinguished* if the reductions of the characters δ and ϵ in (1.1) are distinct. Let $\bar{\rho}_f$ denote the mod p reduction of ρ_f. Finally let $M = \mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$ when p is odd. Then we prove (see Theorem 18):

THEOREM 2. — *Let p be an odd prime and let N_0 be an integer which is relatively prime to p. Let S denote the set of primitive $\wp$-ordinary non-*CM *forms f of weight $k \geqslant 2$ and level $N = N_0p^r$, for $r \geqslant 0$, satisfying*

(1) *f is p-distinguished, and,*

(2) *$\bar{\rho}_f$ is absolutely irreducible when restricted to* $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

Then the representation $\rho_f|_{G_p}$ is non-split for all but finitely many forms $f \in S$.

In other words if p is an odd prime and one fixes the prime-to-p part of the level, then only at most finitely many primitive $\wp$-ordinary forms f fail to satisfy (1.2), at least under the technical conditions 1 and 2 above. In particular, the theorem shows that non-CM forms satisfying (1.2) are ubiquitous.

Theorem 2 follows from a similar result for Λ-adic forms. Since this result is of independent interest, we give a brief exposition of it now, leaving details to the main body of the paper.

Let p be an odd prime and let $\Lambda = \mathbb{Z}_p[[X]]$ be the power series ring in one variable over $\mathbb{Z}_p$. Let $\mathcal{F}$ be a primitive $\wp$-ordinary Λ-adic form and let $\rho_{\mathcal{F}}$ be the associated Galois representation. As in the classical situation, the representation $\rho_{\mathcal{F}}|_{G_p}$ turns out to be 'upper triangular'. Hence there are analogous notions of what it means for the Λ-adic form $\mathcal{F}$ to be p-distinguished, and for the representation $\rho_{\mathcal{F}}|_{G_p}$ to split. One may also speak of the residual representation $\bar{\rho}_{\mathcal{F}}$ of $\rho_{\mathcal{F}}$. Finally, Λ-adic forms, like their classical counterparts, are of two types, CM and non-CM. We can now state the following result (see Theorem 13 below):

THEOREM 3. — *Let p be an odd prime and let $\mathcal{F}$ be a primitive $\wp$-ordinary Λ-adic form. Assume that*

(1) *$\mathcal{F}$ is p-distinguished, and,*

(2) *$\bar{\rho}_{\mathcal{F}}$ is absolutely irreducible when restricted to* $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

Then $\rho_{\mathcal{F}}|_{G_p}$ splits if and only if $\mathcal{F}$ has CM.

The proof of Theorem 3 forms the heart of this paper. The key issue is whether there are *infinitely* many weight one specializations of $\mathcal{F}$ that are classical weight one cusp forms, since as we show, under conditions 1 and 2 above, this is equivalent to both $\rho_{\mathcal{F}}|_{G_p}$ being split and $\mathcal{F}$ being of CM type. A recent result of Buzzard, which allows one to tell whether a p-adic Galois representation arises from a classical weight one cusp form, plays an important role here. In fact conditions 1 and 2 are needed in Theorem 3 in order to apply the main result of [Buz03].

Theorem 2 follows naturally from Theorem 3. Details may be found in Section 4, but the main point is that if $\rho_{\mathcal{F}}$ does not split at p then ρ_f does not split at p for all but finitely many specializations f of $\mathcal{F}$ of weight larger than two. The proof of this last statement is essentially an elaboration of the Weierstrass preparation theorem which implies that once a power series in Λ is non-zero, then it can specialize to zero only finitely often. Note that conditions 1 and 2 are needed in Theorem 2 because of the analogous conditions in Theorem 3.

Using deformation theory, it is possible to give examples of primitive $\wp$-ordinary Λ-adic forms $\mathcal{F}$ which do not have CM and for which *all* the classical weight $k \geqslant 2$ specializations have locally non-split Galois representation. This is shown to hold in [Va05] for the 23-adic family $\mathcal{F}$ containing the Ramanujan Δ function. Interestingly, the members of this family do not satisfy condition 2 in Theorem 2 above.

Finally, let us remark that Question 1 is related to work of Coleman on the image of the p-adic derivation θ on p-adic modular forms (see

[Col96], especially p. 232, Remark 2) and to the work of Buzzard and Taylor on the existence of pairs of characteristic zero companion forms [BT99], Theorem 2. This is explained in [Gha05].

2. Λ-adic forms.

To begin, we recall some of the theory of Λ-adic forms due to Hida. References for this material are the papers [Hid86a], [Hid86b] (for $p \geqslant 5$), and [Wil88] and Chapter 7 of [Hid93] (for general p). In the rest of this paper we will assume that $p \geqslant 3$.

Let $\Lambda = \mathbb{Z}_p[[X]]$, let K denote a finite extension of the quotient field of Λ and let L denote the integral closure of Λ in K. Let ζ denote a p-power root of unity in $\bar{\mathbb{Q}}_p$ and let $k \geqslant 1$ be a positive integer. The assignment $X \mapsto \zeta(1+p)^k - 1$ yields an algebra homomorphism

$$\varphi_{k,\zeta} : \Lambda \to \bar{\mathbb{Q}}_p.$$

We shall say that a height one prime $P \in \mathrm{Spec}(L)(\bar{\mathbb{Q}}_p)$ has *weight* k if the corresponding Λ-algebra homomorphism $P : L \to \bar{\mathbb{Q}}_p$ extends $\varphi_{k,\zeta}$ on Λ for some $k \geqslant 1$ and some ζ. In addition we say that P is *arithmetic* if P has weight $k \geqslant 2$. In the sequel we will need the notion of the *specialization* of an L-valued object B at a height one prime $P : L \to \bar{\mathbb{Q}}_p$. This object is $\bar{\mathbb{Q}}_p$-valued, and is obtained from B by composing B with the homomorphism P. It is denoted by $P(B)$.

Now fix an integer N_0 which is prime to p. We will need the following Dirichlet characters:

- A given character ψ modulo $N_0 p$.
- The Teichmüller character ω of conductor p.
- The character χ_ζ mod p^r for each root of unity ζ of order p^{r-1} with $r \geqslant 1$, defined by first decomposing
$$(\mathbb{Z}/p^r)^\times = (\mathbb{Z}/p)^\times \times \mathbb{Z}/p^{r-1}$$
where the second factor is generated by $1+p$, and then by setting
$$\chi_\zeta = 1 \text{ on } (\mathbb{Z}/p)^\times \quad \text{and} \quad \chi_\zeta(1+p) = \zeta.$$

Note that $\chi = \psi\omega^{-k}\chi_\zeta$ is a Dirichlet character modulo $N = N_0 p^r$ for each integer $k \geqslant 1$ and each root of unity ζ of exact order p^{r-1}, $r \geqslant 1$.

Now consider a point $P \in \mathrm{Spec}(L)(\bar{\mathbb{Q}}_p)$ as above, so that P extends the homomorphism $\varphi_{k,\zeta} : X \mapsto \zeta(1+p)^k - 1$ of Λ. If ζ has exact order p^{r-1}, $r \geqslant 1$, we put $N = N(P) = N_0 p^r$ and $\chi = \chi(P) = \psi\omega^{-k}\chi_\zeta$. We call N and χ the *level* and *character* of P respectively. Note that the level N is always divisible by p, even if the p-component of χ is trivial. Finally let $S_k(N, \chi)$ be the space of classical cusp forms of weight k, level N and character χ.

DEFINITION 4. — *Let $\mathcal{F} = \sum_{n=1}^{\infty} a(n, \mathcal{F})q^n \in L[[q]]$ denote a formal q-expansion with coefficients $a(n, \mathcal{F}) \in L$. Then $\mathcal{F}$ is said to be a Λ-adic cusp form of level N_0 and character ψ if for each arithmetic point $P \in \mathrm{Spec}(L)(\bar{\mathbb{Q}}_p)$ lying over $\varphi_{k,\zeta}$, with $k \geqslant 2$ and ζ of order p^{r-1}, $r \geqslant 1$, the specialization*

$$P(\mathcal{F}) \in \bar{\mathbb{Q}}_p[[q]]$$

of $\mathcal{F}$ at P is the q-expansion of a classical cusp form $f \in S_k(N, \chi)$ of level $N = N_0 p^r$ and character $\chi = \psi\omega^{-k}\chi_\zeta$.

We say that $\mathcal{F}$ is *normalized* if $a(1, \mathcal{F}) = 1$. We say that $\mathcal{F}$ is a Λ-adic *eigenform* of level N_0 if every arithmetic specialization $f = P(\mathcal{F})$ is an eigenvector for the classical Hecke operators T_ℓ for all primes ℓ with $\ell \nmid N_0 p$, and for U_ℓ if $\ell | N_0 p$. In particular, we require that each $f = P(\mathcal{F})$ is an eigenvector for U_p.

Let $f \in S_k(N, \chi)$ be any eigenvector for all the operators T_ℓ with $\ell \nmid N$. Atkin-Lehner theory implies that one can associate to f a unique primitive form f^* of minimal level dividing N which has the same eigenvalues as f for almost all primes ℓ. We say that a Λ-adic eigenform $\mathcal{F}$ of level N_0 is a *newform* of level N_0 if for every arithmetic specialization $f = P(\mathcal{F}) \in S_k(N, \chi)$, the associated primitive form f^* has level divisible by N_0. Finally, let us say that $\mathcal{F}$ is *primitive* of level N_0 if it is normalized and is a newform of level N_0.

There is an extensive theory of Λ-adic forms due to Hida [Hid86a], [Hid86b] under the additional assumption of ordinariness, which we describe now. Let f be a normalized eigenform for all the Hecke operators of level $N = N_0 p^r$, with $(N_0, p) = 1$ and $r \geqslant 1$, and weight $k \geqslant 2$. Assume that f^* has level divisible by N_0. We say that f is $\wp$-stabilized if either f is p-new and is $\wp$-ordinary, or f is p-old, and is obtained from a primitive $\wp$-ordinary form f_0 of level N_0 by the formula $f(z) = f_0(z) - \beta f_0(pz)$, where β is the non $\wp$-adic-unit root of

$$E_p(x) = x^2 - a(p, f_0)x + \chi(p)p^{k-1}.$$

One checks that f is actually an eigenform of all the Hecke operators of level N_0p, with U_p eigenvalue equal to α, the unique $\wp$-adic unit root of $E_p(x)$. We say that a primitive Λ-adic form of level N_0 is *$\wp$-ordinary* if each arithmetic specialization $P(\mathcal{F})$ is a $\wp$-stabilized form in $S_k(N,\chi)$, where k, N, χ are the weight, level, and character of P respectively.

THEOREM 5 ([Hid86b], Cor. 1.3). — *Let $p \geqslant 3$ denote a prime number. Let N_0 denote a fixed integer with $(N_0, p) = 1$, and let ψ denote a character modulo N_0p. Then the following statements hold.*

(1) *There are only finitely many primitive $\wp$-ordinary Λ-adic forms of level N_0 and character ψ.*

(2) *Any $\wp$-stabilized cusp form f of weight $k \geqslant 2$ and level $N = N_0p^r$ for $r \geqslant 1$ occurs as the arithmetic specialization of some primitive $\wp$-ordinary Λ-adic form $\mathcal{F}$ of level N_0, for suitable ψ.*

(3) *The form $\mathcal{F}$ in part (2) is unique up to Galois conjugacy.*

As we have already remarked, Hida states his theorems in [Hid86a], [Hid86b] for $p \geqslant 5$; the extension to $p = 3$ is sketched in [Hid93], via the method of [Wil88]. Let us also make a remark about terminology: since the primitive Λ-adic forms occurring in this paper are all $\wp$-ordinary, we will from this point on use the adjective 'primitive' to describe those Λ-adic forms that are primitive *and* $\wp$-ordinary.

3. Local Splitting of Λ-adic Representations.

Let $\mathcal{F} \in L[[q]]$ be a primitive Λ-adic form of level N_0. Write $K = K_{\mathcal{F}}$ for the quotient field of L. Then Hida attaches a Galois representation

$$\rho_{\mathcal{F}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(K_{\mathcal{F}})$$

to $\mathcal{F}$ such that for each arithmetic point P of L, $P(\rho_{\mathcal{F}})$, the specialization of $\rho_{\mathcal{F}}$ at P, is isomorphic to the representation ρ_f attached to $f = P(\mathcal{F})$ by Deligne.

One has to be a little careful about considering specializations of the $\mathrm{GL}_2(K_{\mathcal{F}})$-valued representation $\rho_{\mathcal{F}}$, since the entries of a matrix realization of $\rho_{\mathcal{F}}$ will in general have non-trivial denominators and the corresponding rational functions will have poles. Note however that if ℓ is a prime number such that $\ell \nmid N_0p$, then the trace $\mathrm{Tr}(\rho_{\mathcal{F}}(\mathrm{Frob}_\ell))$ is given by

$$\mathrm{Tr}(\rho_{\mathcal{F}}(\mathrm{Frob}_\ell)) = a(\ell, \mathcal{F}) \in L.$$

It can be shown that for any arithmetic point P, there exists a realization of $\rho_{\mathcal{F}}$ that is well-defined at P, and that the specialization of $\rho_{\mathcal{F}}$ at P is well-defined (because the traces are in L, and are independent of the particular realization by matrices). Under the hypotheses 1 and 2 of Theorem 3, it can be shown that $\rho_{\mathcal{F}}$ may even be realized by matrices in $\mathrm{GL}_2(L)$, but we will not need this in our arguments.

The restriction of $\rho_{\mathcal{F}}$ to G_p also turns out to be 'upper triangular'. More precisely the representation $\rho_{\mathcal{F}}|_{G_p}$ has the following shape:

$$\rho_{\mathcal{F}}|_{G_p} \sim \begin{pmatrix} \delta_{\mathcal{F}} & u_{\mathcal{F}} \\ 0 & \epsilon_{\mathcal{F}} \end{pmatrix} \tag{3.1}$$

where $\epsilon_{\mathcal{F}}, \delta_{\mathcal{F}} : G_p \to K_{\mathcal{F}}^{\times}$ are characters with $\epsilon_{\mathcal{F}}$ unramified, and $u_{\mathcal{F}} : G_p \to K_{\mathcal{F}}$ is a map that again is not necessarily a homomorphism. As in the classical case there is an obvious notion of what it means for $\rho_{\mathcal{F}}$ to be split at p. Similarly, let

$$c_{\mathcal{F}} = \epsilon_{\mathcal{F}}^{-1} \cdot u_{\mathcal{F}} \in \mathrm{Z}^1(G_p, K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}^{-1})) \tag{3.2}$$

be the associated cocycle (in this context see also the work of Mazur-Tilouine [MT90]). Then the representation

$$\rho_{\mathcal{F}}|_{G_p} \text{ splits if and only if } [c_{\mathcal{F}}] = 0 \text{ in } \mathrm{H}^1(G_p, K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}^{-1})).$$

The following question is the natural analogue of Greenberg's question in the Λ-adic setting.

Question 6. — Say $\mathcal{F}$ is a primitive Λ-adic form. When is the representation $\rho_{\mathcal{F}}|_{G_p}$ split? Equivalently when is the cocycle $c_{\mathcal{F}}$ a coboundary?

To answer this question, we introduce the notion of complex multiplication in this setting.

DEFINITION 7. — *A Λ-adic form $\mathcal{F}$ is said to have complex multiplication (or* CM*) if some arithmetic specialization $f = P(\mathcal{F})$ of $\mathcal{F}$ has complex multiplication.*

PROPOSITION 8. — *If $\mathcal{F}$ is a primitive Λ-adic form with* CM *then every arithmetic specialization of $\mathcal{F}$ has* CM.

Proof. — Let f be an arithmetic specialization of $\mathcal{F}$ of CM type. Since f is $\wp$-ordinary and $k \geqslant 2$, an easy check using Galois representations shows that p must split in K. When p is split, Theorem 7.1 in [Hid86a] implies that the CM form f is the specialization of an explicit primitive Λ-adic form $\mathcal{F}'$, all of whose specializations have CM. The proposition results now from the uniqueness assertion in part (3) of Theorem 5. □

Remark 9. — It follows from the proposition that if a primitive Λ-adic form $\mathcal{F}$ does not have CM then no arithmetic specialization of $\mathcal{F}$ has CM. In particular if f is a classical $\wp$-stabilized primitive form of weight $k \geqslant 2$ then the Λ-adic form $\mathcal{F}$ containing it is of CM type if and only if f has complex multiplication.

Remark 10. — The construction of ordinary Λ-adic CM forms in [Hid86a] is based on the construction of a certain Λ-adic Hecke character $\Phi : \mathrm{Gal}(\bar{\mathbb{Q}}/K) \to L^\times$. We will give a variant of this construction in the course of proving proposition 14. See also the discussion in [Hid93], pp. 235–236. Now, if P is an arithmetic point of weight k in $\mathrm{Spec}(L)(\bar{\mathbb{Q}}_p)$, then the specialization $\phi = \Phi(P)$ is a $\overline{\mathbb{Q}}_p$-valued character of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ with infinity type $(k-1, 0)$ and $f = P(\mathcal{F})$ is the CM theta series associated by Hecke to ϕ. This holds also when P has weight one, but if ϕ is stable under the action of $\mathrm{Gal}(K/\mathbb{Q})$, then the form f degenerates to an Eisenstein series. On the other hand, there are Λ-adic forms without CM which have weight one specializations of CM type (see [Hid93], p. 237). However, we shall prove (see Proposition 14) that if a primitive Λ-adic form $\mathcal{F}$ has infinitely many weight one specializations that are classical weight one forms with CM then $\mathcal{F}$ is necessarily of CM type.

Remark 11. — There is an analogue of part (3) of Theorem 5 for $\wp$-stabilized forms of weight one. Namely, it can be shown (under a hypothesis similar to condition 1 in Theorem 2) that any such weight one form is contained in a unique Λ-adic form, up to conjugacy. This does not follow from Hida's theory. Rather, it follows from an analysis of deformations of weight one forms. For a detailed study of this kind of deformation theory, the reader is referred to forthcoming work of the second-named author and Ralph Greenberg [GV03].

PROPOSITION 12. — *Let $\mathcal{F}$ be a primitive Λ-adic form with* CM. *Then $\rho_{\mathcal{F}}$ splits on G_p.*

Proof. — It follows from the fact that $\mathcal{F}$ admits an arithmetic specialization f that is ordinary and of CM type that the prime p is split in K. Following Hida's construction of CM forms in Theorem 7.1

of [Hid86a], one checks that $\rho_{\mathcal{F}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{\mathcal{F}})$ is given explicitly by $\rho_{\mathcal{F}} = \mathrm{Ind}_{\mathbb{Q}}^{K}(\Phi)$ where Φ is the Λ-adic Hecke character mentioned in Remark 10. It follows that the restriction of $\rho_{\mathcal{F}}$ to $\mathrm{Gal}(K/\mathbb{Q})$ decomposes as $\Phi \oplus \bar{\Phi}$, where $\bar{\Phi}$ is the conjugate of Φ under $\mathrm{Gal}(K/\mathbb{Q})$. Since p is split, we have $G_p \subset \mathrm{Gal}(\bar{\mathbb{Q}}/K)$, and $\rho_{\mathcal{F}}$ splits on G_p as well. □

To state the next theorem fix a primitive Λ-adic form $\mathcal{F}$. Let

$$\bar{\rho}_{\mathcal{F}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k)$$

denote the reduction of $\rho_{\mathcal{F}}$ corresponding to the maximal ideal $\mathfrak{m}_L$ of L. Here $k = L/\mathfrak{m}_L$ is a finite field of characteristic p. Equivalently $\bar{\rho}_{\mathcal{F}}$ is the reduced representation $\bar{\rho}_f$ associated to one (hence all) of the Deligne representations ρ_f as f varies through the classical arithmetic specializations of $\mathcal{F}$. We say that $\mathcal{F}$ is *p-distinguished* if the characters $\delta_{\mathcal{F}}$ and $\epsilon_{\mathcal{F}}$ appearing in (3.1) have distinct reductions modulo $\mathfrak{m}_L$. It follows that $\mathcal{F}$ is p-distinguished if and only if f is p-distinguished for one (therefore every) arithmetic specialization f of $\mathcal{F}$.

The following theorem (which is a restatement of Theorem 3 above) is the Λ-adic analogue of Theorem 2 in the introduction. It says that the answer to the Λ-adic analogue of Greenberg's question (Question 6) is what one expects, at least under certain technical conditions on the representation $\bar{\rho}_{\mathcal{F}}$.

THEOREM 13. — *Let p be an odd prime. Let $\mathcal{F}$ be a primitive Λ-adic form such that*

(1) *$\mathcal{F}$ is p-distinguished, and,*

(2) *$\bar{\rho}_{\mathcal{F}}$ is absolutely irreducible when restricted to* $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

Then $\rho_{\mathcal{F}}|_{G_p}$ is split if and only if $\mathcal{F}$ has CM.

We will in fact prove a more informative result:

PROPOSITION 14. — *Let p be an odd prime and let $\mathcal{F}$ be a primitive Λ-adic form satisfying the conditions 1 and 2 of the theorem above. Then the following statements are equivalent.*

(i) *$\rho_{\mathcal{F}}|_{G_p}$ splits.*

(ii) *$\mathcal{F}$ has infinitely many weight one specializations that are classical.*

(iii) *$\mathcal{F}$ has infinitely many weight one specializations that are classical* CM *forms.*

(iv) $\mathcal{F}$ *is of* CM *type.*

The theorem simply states that (i) $\iff$ (iv). Observe that (iv) $\Longrightarrow$ (i) is just Proposition 12 above. We will show (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) under conditions 1 and 2 above. In fact these conditions are only needed to show that (i) $\Longrightarrow$ (ii), although condition 2 is used in a weak way in the proof of (ii) $\Longrightarrow$ (iii).

Proof of (i) $\Longrightarrow$ (ii). — We want to study the weight one specializations of $\mathcal{F}$. So say $\mathcal{F}$ has q-expansion in $L[[q]]$. Let P be a weight one point of L. It is known (see, for instance, [MW86]) that the specialization of $\mathcal{F}$ at P may not be a classical form of weight one; equivalently the specialization of $\rho_{\mathcal{F}}$ at P may not be the representation attached by Deligne and Serre to a classical weight one form. However, we claim that under the hypothesis that $\rho_{\mathcal{F}}|_{G_p}$ is split, and conditions 1 and 2 above, infinitely many weight one specializations of $\mathcal{F}$ are classical weight one forms. To prove this we use a result of Buzzard [Buz03], to be stated shortly, which gives conditions under which a p-adic Galois representation arises from a classical weight one cusp form. The main result of [Buz03] extends earlier work of Buzzard and Taylor [BT99], but it is really the new result of Buzzard that we need here since this result allows the p-adic Galois representation to be ramified at p.

Let N_0 denote the level of $\mathcal{F}$ and let $\psi : G_{\mathbb{Q}} \to \bar{\mathbb{Q}}_p^\times$ be the character modulo N_0p which is the character of $\mathcal{F}$. Let $\kappa : G_{\mathbb{Q}} \to \Lambda^\times$ be the character obtained by composing the projection

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) = 1 + p\mathbb{Z}_p$$

where $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, with the character

$$1 + p\mathbb{Z}_p \to \Lambda^\times$$

which takes $(1+p)^s$ for $s \in \mathbb{Z}_p$ to $(1+X)^s \in \Lambda^\times$. Finally let $\nu : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ denote the p-adic cyclotomic character. Then it is known that

$$\det(\rho_{\mathcal{F}}) = \psi \cdot \kappa \cdot \nu^{-1}.$$

Since the specialization of κ at $\varphi_{k,\zeta}$ for $k \geqslant 1$ is easily checked to be $\omega^{-k}\nu^k\chi_\zeta$ we see that the specialization of $\det(\rho_{\mathcal{F}})$ at $\varphi_{k,\zeta}$ is $\chi\nu^{k-1}$ with $\chi = \psi\omega^{-k}\chi_\zeta$. This is of course compatible with the well known formula for the determinant of a Galois representation attached to a classical cusp form of weight $k \geqslant 1$ and nebentypus χ.

By assumption $\rho_{\mathcal{F}}$ is split on G_p. In view of (3.1) we have

$$\rho_{\mathcal{F}}|_{I_p} \sim \begin{pmatrix} \psi\kappa\nu^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

since $\epsilon_{\mathcal{F}}$ is trivial on I_p and therefore $\delta_{\mathcal{F}} = \det(\rho_{\mathcal{F}})$ on I_p. Let P be a weight one point of L extending $\varphi_{1,\zeta} : \Lambda \to \bar{\mathbb{Q}}_p$. It follows that $P(\rho_{\mathcal{F}}) = \rho_{P(\mathcal{F})}$ has the following shape on I_p:

$$\rho_{P(\mathcal{F})}|_{I_p} \sim \begin{pmatrix} \psi\omega^{-1}\chi_\zeta & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.3}$$

Note that the character $\psi\omega^{-1}\chi_\zeta$ has finite order.

We now state the above mentioned result of Buzzard. To do this we need some notation. Let $\mathcal{O}$ denote the ring of integers in a finite extension of $\mathbb{Q}_p$. Let $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation. Let λ denote the maximal ideal of $\mathcal{O}$ and let $\bar{\rho}$ denote the mod λ reduction of ρ.

Theorem 15 (Buzzard [Buz03]). — *Assume that*

- *ρ is ramified at finitely many primes,*
- *$\bar{\rho}$ is modular,*
- *$\bar{\rho}$ is absolutely irreducible when restricted to* $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$,
- *$\bar{\rho}|_{G_p}$ is the direct sum of two distinct character α and $\beta : G_p \to \mathcal{O}^\times$ such that the reductions of α and β mod λ are distinct characters, and $\alpha(I_p)$ and $\beta(I_p)$ are finite.*

Then ρ is modular, in the sense that there is a primitive cusp form f of weight one and an embedding $\iota : \mathcal{O} \hookrightarrow \mathbb{C}$ such that $\iota \circ \rho$ is isomorphic to the classical representation attached to f by Deligne and Serre.

We have already verified all the hypotheses of the theorem for the representation $\rho = \rho_{P(\mathcal{F})}$. Indeed, the first two hypotheses are automatically satisfied in our situation. The latter two hypotheses follow from the conditions 1 and 2 imposed on $\rho_{\mathcal{F}}$, and from (3.3) which shows $\rho = \rho_{P(\mathcal{F})}$ acts on I_p by the finite order character $\chi = \psi\omega^{-1}\chi_\zeta$ and the trivial character. Thus we deduce that $\rho = \rho_{P(\mathcal{F})}$ is isomorphic to the Deligne-Serre representation ρ_f where f is a primitive weight 1 cusp form, necessarily of level $N = N_0 p^r$ and character $\psi\omega^{-1}\chi_\zeta$, where ζ has exact order p^{r-1} for $r \geqslant 1$. Here we have suppressed mentioning the embedding ι in the discussion. As we vary the point P, and therefore $r \geqslant 1$, we obtain infinitely many classical weight 1 specializations of $\mathcal{F}$ as required.

(ii) $\Longrightarrow$ (iii). — We now show that if a primitive Λ-adic $\mathcal{F}$ form has infinitely many classical weight one specializations then it must have infinitely many weight one specializations of CM type. Once and for all we

pick an embedding of $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ so that all the weight one specializations ρ_f of $\rho_{\mathcal{F}}$ take values in $\mathrm{GL}_2(\mathbb{C})$. Let

$$\tilde{\rho}_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{PGL}_2(\mathbb{C})$$

denote the projectivization of ρ_f. It is well known that the image of $\tilde{\rho}_f$ is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$ which, according to a now standard classification, is either

- cyclic,
- dihedral, or,
- A_4, A_5 or S_4.

Since there are infinitely many classical weight one specializations f of $\mathcal{F}$, one of the three possibilities has to occur infinitely often.

The first possibility is automatically excluded since f is a cusp form and ρ_f is irreducible (by condition 1 even the reduction $\bar{\rho}_f = \bar{\rho}_{\mathcal{F}}$ is absolutely irreducible on $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$) whereas if the image of $\tilde{\rho}_g$ for a weight one modular form g is cyclic then g is an Eisenstein series and ρ_g is reducible.

On the other hand by (3.3) the image of I_p in $\mathrm{GL}_2(\mathbb{C})$ has trivial intersection with the scalar matrices, so that the image of I_p injects into $\mathrm{PGL}_2(\mathbb{C})$. Since the order of $\psi\omega^{-1}\chi_\zeta$ is at least the order of ζ which is p^{r-1} say, for $r \geqslant 1$, the order of the image of inertia in $\mathrm{PGL}_2(\mathbb{C})$ increases without bound as r increases. But the groups A_4, A_5 and S_4 have bounded order. It follows that only finitely many weight one specializations f of $\mathcal{F}$ are of the third kind.

Thus, the only possibility is that the image of $\tilde{\rho}_f$ is dihedral for infinitely many weight one specializations f of $\mathcal{F}$. In this case each ρ_f is induced from a finite order Hecke character ϕ of some quadratic field K of $\mathbb{Q}$. Since the conductor of ρ_f is $N_0 p^r$ and the discriminant of K must divide the conductor we see that there are only finitely many choices for the field K. So one of these quadratic fields K has to occur infinitely often. We fix one such and call it K again. Now let σ be a generator of $\mathrm{Gal}(K/\mathbb{Q})$ and let $H = \mathrm{Gal}(\bar{\mathbb{Q}}/K)$. Then $\rho_f|_H = \phi \oplus \phi^\sigma$, and the nontrivial element σ of $\mathrm{Gal}(K/\mathbb{Q})$ interchanges ϕ and ϕ^σ.

Since f is $\wp$-ordinary clearly p cannot be inert in K, since in this case ρ_f is irreducible on the decomposition group at p. If $p = \mathfrak{p}^2$ is ramified in K then the $\wp$-ordinariness of f forces ϕ to be unramified at $\mathfrak{p}$. It follows that in this case at most one power of p can divide the level of f. But the forms f have level N_0p^r for r arbitrarily large. So p does not ramify in K either. It follows that $p = \mathfrak{p}\mathfrak{p}^\sigma$ must split in K.

The $\wp$-ordinariness of f then shows that ϕ must be ramified at exactly one of the two primes $\mathfrak{p}$ and $\mathfrak{p}^\sigma$ for r sufficiently large. Indeed (3.3) shows that exactly one of the two characters through which inertia acts is non-trivial when $r \geqslant 2$. On the other hand since $I_p = I_{\mathfrak{p}} \subset H$ we see that ρ_f acts by the two characters ϕ and ϕ^σ on H. So exactly one of ϕ and ϕ^σ is ramified at $\mathfrak{p}$, or equivalently, ϕ is ramified at exactly one of $\mathfrak{p}$ and $\mathfrak{p}^\sigma$. Without loss of generality we may assume that infinitely many of the ϕ are ramified at $\mathfrak{p}$ (and not at $\mathfrak{p}^\sigma$).

We now claim that K must be an imaginary quadratic field. To see this we need the following lemma.

LEMMA 16. — *Let F be a real quadratic field. Let $\mathfrak{p}$ be a split prime of F, let $\mathfrak{n}_0$ be any ideal of F prime to $\mathfrak{p}$ and let ν be the formal product of a subset of the two infinite places of F. Then the ray class field modulo $\mathfrak{n}_0\mathfrak{p}^r\nu$ has bounded order as r tends to ∞.*

Proof. — This follows from class field theory. Assume $\mathfrak{n}_0 = 1$ and $\nu = 1$. Let $\mathrm{Cl}_F(\mathfrak{p}^r)$ denote the ray class group of F modulo $\mathfrak{p}^r$, and let Cl_F denote the class group of F. Then the exact sequence

$$1 \to \mathcal{O}_F^\times \to (\mathcal{O}_F/\mathfrak{p}^r)^\times \to \mathrm{Cl}_F(\mathfrak{p}^r) \to \mathrm{Cl}_F \to 1 \tag{3.4}$$

shows that the ray class field modulo $\mathfrak{p}^r$ has order $n(\mathfrak{p}^r) \cdot h_F$ where h_F is the class number of F and $n(\mathfrak{p}^r)$ is the index in $(\mathcal{O}_F/\mathfrak{p}^r)^\times$ of the subgroup generated by the image of $\mathcal{O}_F^\times$ under the natural map $\mathcal{O}_F^\times \to (\mathcal{O}_F/\mathfrak{p}^r)^\times$. So it is enough to show that the index $n(\mathfrak{p}^r)$ is bounded independently of r. Let $\mathcal{O}_F^1$ denote the subgroup of $\mathcal{O}_F^\times$ consisting of those global units that map to the principal units $\mathcal{O}_{F,\mathfrak{p}}^1$ under the natural inclusion $\mathcal{O}_F^\times \hookrightarrow \mathcal{O}_{F,\mathfrak{p}}^\times$. Then it is an immediate consequence of Leopoldt's conjecture (which is trivial in this case since $F/\mathbb{Q}$ is quadratic, and the unit group has rank 1) that the closure of $\mathcal{O}_F^1$ in $\mathcal{O}_{F,\mathfrak{p}}^1$ has finite index in $\mathcal{O}_{F,\mathfrak{p}}^1$. It follows that the closure of $\mathcal{O}_F^\times$ in $\mathcal{O}_{F,\mathfrak{p}}^\times$ is also of finite index in $\mathcal{O}_{F,\mathfrak{p}}^\times$, say this index is M. Since $\mathcal{O}_{F,\mathfrak{p}}^\times = \varprojlim(\mathcal{O}_F/\mathfrak{p}^r)^\times$ one see that $n(\mathfrak{p}^r) \leqslant M$, for all r, proving the lemma in the case $\mathfrak{n}_0 = 1$ and $\nu = 1$. The proof of the general case is no more difficult. □

Suppose, towards a contradiction, that the quadratic field $K = F$ above is real. Then by the lemma there are only finitely many characters ϕ of conductor dividing $N_0\mathfrak{p}^\infty\nu$ where ν is the formal product of the infinite places of K. But we have just produced infinitely many characters ϕ of this type. So K is an imaginary quadratic field, as claimed. In particular infinitely many of the classical weight one specializations f of $\mathcal{F}$ are of CM type, as desired.

(iii) $\Longrightarrow$ (iv). — Now consider a primitive Λ-adic form $\mathcal{F}$ of level N_0 which admits infinitely many weight one specializations $f = \sum \phi(\mathfrak{a})q^{\mathrm{N}(\mathfrak{a})}$ of CM type, coming from finite order characters ϕ on the fixed imaginary quadratic field K. By ordinariness, each ϕ can be taken to be ramified at $\mathfrak{p}$ and unramified at $\mathfrak{p}^\sigma$. We now show that there is a form $\mathcal{F}'$ of CM type which simultaneously interpolates infinitely many of these CM forms.

To do this note that for each ϕ above there is a decomposition $\phi = \phi_t \cdot \phi_w$ where ϕ_t has order prime to p and ϕ_w has p-power order. Since the reductions of the ϕ must all be the same (the forms f are specializations of the same Λ-adic form) the ϕ_t must in fact all be equal. On the other hand if $\mathfrak{q}$ is a prime of K with $\mathfrak{q}|N_0$ then the restriction, say $\phi_\mathfrak{q}$, of ϕ_w to $I_\mathfrak{q}$ is tamely ramified and of p-power order, so there are only finitely many possibilities for $\phi_\mathfrak{q}$. In what follows we may assume that the infinite set of characters ϕ above all have the the same ϕ_t and the same $\phi_\mathfrak{q}$ for each $\mathfrak{q}|N_0$. Fix any one such ϕ and call it ϕ_0.

Set $\beta_\phi = \phi\phi_0^{-1}$. Then β_ϕ has p-power order and is unramified outside $\mathfrak{p}$ for each ϕ. In particular β_ϕ is a character of $\mathrm{Cl}_K(\mathfrak{p}^m)$, the ray class group modulo $\mathfrak{p}^m$, for some m sufficiently large depending on ϕ. The exact sequence (3.4) above, for K instead of F, shows that $\mathrm{Cl}_K(\mathfrak{p}^\infty) = \varprojlim \mathrm{Cl}_K(\mathfrak{p}^m)$ has the following structure as an abelian group:

$$\mathrm{Cl}(\mathfrak{p}^\infty) = T \oplus \Gamma \tag{3.5}$$

where T is finite and $\Gamma \cong \mathbb{Z}_p$. Each β_ϕ may be viewed as a character of $\mathrm{Cl}_K(\mathfrak{p}^\infty)$.

Since the character group of T is finite, there is a character β_T of T such that $\beta_\phi = \beta_T$ on T for infinitely many ϕ. We restrict our discussion below to such ϕ.

Now let L_0 be the completed group algebra $\mathbb{Z}_p[[\Gamma]]$. Then the projection

$$\mathrm{Gal}(\bar{\mathbb{Q}}/K) \twoheadrightarrow \mathrm{Cl}_K(\mathfrak{p}^\infty) \to \Gamma$$

gives a character $\Phi_0 : \mathrm{Gal}(\bar{\mathbb{Q}}/K) \to \Gamma \subset L_0^\times$. Let

$$\Phi_\mathbb{Q} = \det(\mathrm{Ind}_\mathbb{Q}^K(\Phi_0)) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to L_0^\times .$$

Then $\Phi_{\mathbb{Q}}$ factors through $\mathrm{Gal}(K\mathbb{Q}_\infty/\mathbb{Q})$, and identifies $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ with a finite index subgroup $\Gamma_{\mathbb{Q}}$ of $\Gamma \subset L_0^\times$. In fact, $\Gamma_{\mathbb{Q}}$ is the image under Φ_0 of the inertia group $I_{\mathfrak{p}}$. Thus we may view L_0 as a Λ_0-algebra, where $\Lambda_0 = \mathbb{Z}_p[\Gamma_{\mathbb{Q}}]$. If we fix generators w and u of Γ and $\Gamma_{\mathbb{Q}}$ respectively, with $w^{p^\gamma} = u$ for $p^\gamma = [\Gamma : \Gamma_{\mathbb{Q}}]$, then $L_0 \cong \mathbb{Z}_p[[Y]]$ with $w \mapsto 1+Y$ and $\Lambda_0 \cong \Lambda = \mathbb{Z}_p[[X]]$ with $u \mapsto 1+X$, and $(1+Y)^{p^\gamma} = 1+X$.

Now, any character $\beta_\phi = \phi\phi_0^{-1}$ satisfying $\beta_\phi = \beta_T$ on T is uniquely determined by the number $\zeta = \beta_\phi(w) \in \mu_{p^\infty}$. Thus the specialization of Φ_0 under $Y \mapsto \zeta - 1$ is the character β_ϕ/β_T, and Φ_0 interpolates the characters $\phi/\phi_0\beta_T$. Thus to get a Λ-adic interpolation of the characters ϕ, we have to twist by the fixed character $\phi_0\beta_T$, and normalize correctly.

Recall that the character ϕ_0 is such that the CM form f_0 associated to ϕ_0 occurs as the specialization of the Λ-adic form $\mathcal{F}$ at some prime ideal of weight one. According to our normalizations, the character of f_0 has the form $\psi\omega^{-1}\chi_{\eta_0}$ for some $\eta_0 \in \mu_{p^\infty}$. Let ζ_0 be such that $\zeta_0^{p^r} = \eta_0$.

Now let $L = \mathcal{O}[[Y]]$, where $(1+Y)^{p^\gamma} = (1+X)$ and $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbb{Q}_p$ containing the values of ϕ_0, β_T, together with ζ_0 and $t = (1+p)^{1/p^\gamma}$. Let τ denote the automorphism of L defined by the change of variables $1+Y \mapsto \zeta_0^{-1}t^{-1}(1+Y)$. We think of Φ_0 as taking values in $L^\times$. Consider the character $\Phi : \mathrm{Gal}(\bar{\mathbb{Q}}/K) \to L^\times$ defined by

$$\Phi = \phi_0 \cdot \beta_T \cdot \tau(\Phi_0).$$

Then one checks readily that if ϕ is one of the infinitely many Hecke characters as above, corresponding to a CM specialization f of $\mathcal{F}$ with character given by $\psi\omega^{-1}\chi_\eta$, then the specialization of Φ under $Y \mapsto \zeta t - 1$ coincides with ϕ. Here ζ is determined by the requirement that $\zeta^{p^r} = \eta$. Since clearly the point $Y \mapsto \zeta t - 1$ lies above the point $X \mapsto \eta(1+p) - 1$, we find that the formal q-expansion $\mathcal{F}' = \sum_{\mathfrak{a}} \Phi(\mathfrak{a}) q^{\mathrm{N}(\mathfrak{a})}$ is a Λ-adic CM form interpolating infinitely many CM specializations of $\mathcal{F}$. One may further check that $\mathcal{F}'$ is a primitive form of level N_0.

Let G_Λ denote the absolute Galois group of the quotient field of Λ. We now show $\mathcal{F}$ and $\mathcal{F}'$ are Galois conjugates, that is $\mathcal{F} = (\mathcal{F}')^\tau$ for some $\tau \in G_\Lambda$.

LEMMA 17. — *Let $\mathcal{F}$ and $\mathcal{F}'$ be two primitive Λ-adic forms of level N_0 which have infinitely many specializations in common. Then $\mathcal{F}$ and $\mathcal{F}'$ are Galois conjugates.*

Proof. — Say $\mathcal{F} = \sum a(n,\mathcal{F})q^n \in L[[q]]$ and $\mathcal{F}' = \sum a(n,\mathcal{F}')q^n \in$

$L'[[q]]$. Let H be the Λ-adic Hecke algebra of Hida of level N_0. Then $\mathcal{F}$ and $\mathcal{F}'$ define algebra homomorphisms $\lambda_{\mathcal{F}} : H \to L$ and $\lambda_{\mathcal{F}'} : H \to L'$ respectively. Let Q and Q' denote the minimal prime ideals of H which are the respective kernels of these homomorphisms. Since $\mathcal{F}$ and $\mathcal{F}'$ have infinitely many specializations in common there are infinitely many pairs of algebra homomorphisms $P : L \to \bar{\mathbb{Q}}_p$ and $P' : L' \to \bar{\mathbb{Q}}_p$ such that $P \circ \lambda_{\mathcal{F}} = P' \circ \lambda_{\mathcal{F}'} = \lambda_{P,P'}$, say. Then the kernel of $\lambda_{P,P'}$ is a height one prime of H containing both Q and Q' and there are infinitely many such primes by hypothesis. We claim that this forces $Q = Q'$. Indeed if $Q \neq Q'$ then H/I, for $I = Q + Q'$, has only finitely many minimal prime ideals (it is Noetherian). This set of prime ideals is in bijection with the height 1 prime ideals of H containing both Q and Q'. It follows that $Q = Q'$. We conclude that $\mathcal{F}$ and $\mathcal{F}'$ are Galois conjugates. □

Returning to our situation, the infinite set of CM forms corresponding to the characters ϕ are specializations of both $\mathcal{F}$ and $\mathcal{F}'$. After the lemma we have $\mathcal{F} = (\mathcal{F}')^{\tau}$ for some $\tau \in G_{\Lambda}$. Since $\mathcal{F}'$ is a CM form, so is $(\mathcal{F}')^{\tau}$. We conclude that $\mathcal{F}$ is of CM type, completing the proof that (iii) $\Longrightarrow$ (iv). This also completes the proof of Theorem 13.

4. Descending to the Classical Situation.

In this section we deduce our main result, Theorem 2, from its Λ-adic analogue, Theorem 13. Let p be an odd prime and N_0 an integer that is relatively prime to p. The following theorem is a restatement of Theorem 2.

THEOREM 18. — *Let S denote the set of primitive $\wp$-ordinary forms f of weight $k \geqslant 2$ and level $N = N_0 p^r$, for $r \geqslant 0$, satisfying*

(1) *f is p-distinguished, and,*

(2) *$\bar{\rho}_f$ is absolutely irreducible when restricted to* $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

Then for all but finitely many $f \in S$, the representation $\rho_f|_{G_p}$ splits if and only if f has CM.

Proof. — By part (3) of Theorem 5, each of the forms $f \in S$ is the arithmetic specialization of a primitive Λ-adic form of level N_0, which is unique up to conjugacy. By part (1) of the same theorem, there are only finitely many such Λ-adic forms. So it suffices to prove the theorem for the subset $\mathcal{S}_{\mathcal{F}} \subset \mathcal{F}$ consisting of those $f \in S$ which are arithmetic specializations of the same primitive Λ-adic form $\mathcal{F}$ of level N_0. The conditions 1 and 2 above then imply the analogous conditions for $\mathcal{F}$, namely $\mathcal{F}$ is p-distinguished and $\bar{\rho}_{\mathcal{F}}$ is absolutely irreducible when restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

As always if $f \in S_{\mathcal{F}}$ has CM then $\rho_f|_{G_p}$ splits. So we may as well assume that no specialization of $\mathcal{F}$ is of CM type, in other words that $\mathcal{F}$ does not have CM. It follows from Theorem 13 that $\rho_{\mathcal{F}}|_{G_p}$ is not split. We now show that this forces $\rho_f|_{G_p}$ to be non-split for all but finitely many $f \in S_{\mathcal{F}}$. To see this suppose that $\mathcal{F} \in L[[q]]$. Then

$$L^\times = \mu_{q-1} \times (1 + \mathfrak{m}_L)$$

where q denotes the order of the residue field of L. Since both $\delta_{\mathcal{F}}$ and $\epsilon_{\mathcal{F}}$ are $L^\times$ valued we may decompose these characters as $\delta_{\mathcal{F}} = \delta_t \delta_w$ and $\epsilon_{\mathcal{F}} = \epsilon_t \epsilon_w$ according to the decomposition of L above. Let E_t denote the union of the finitely many tamely ramified abelian extensions of $\mathbb{Q}_p$ of order dividing $q-1$, let E_w denote the maximal abelian pro-p extension of $\mathbb{Q}_p$, and let $E = E_t \cdot E_w$ denote the compositum. Set $H = \mathrm{Gal}(\bar{\mathbb{Q}}_p/E)$. Then δ_t and ϵ_t are trivial on $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E_t)$ whereas δ_w and ϵ_w are trivial on $\mathrm{Gal}(\bar{\mathbb{Q}}_p/E_w)$. It follows that $\delta_{\mathcal{F}}$ and $\epsilon_{\mathcal{F}}$ are trivial on H and hence:

$$\rho_{\mathcal{F}}|_H \sim \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

for some homomorphism $\lambda : H \to L$.

We first show that the homomorphism λ is non-zero. Let $c_{\mathcal{F}}$ denote the cocycle defined in (3.2). Since $\rho_{\mathcal{F}}|_{G_p}$ is non-split, the corresponding cohomology class $[c_{\mathcal{F}}] \neq 0$.

Lemma 19. — *The restriction map*

$$\mathrm{H}^1(G_p, K_{\mathcal{F}}(\delta_{\mathcal{F}}{\epsilon_{\mathcal{F}}}^{-1})) \to \mathrm{H}^1(H, K_{\mathcal{F}}(\delta_{\mathcal{F}}{\epsilon_{\mathcal{F}}}^{-1}))$$

is injective.

Proof. — It suffices to show that

$$\mathrm{H}^1(\mathrm{Gal}(E/\mathbb{Q}_p), K_{\mathcal{F}}(\delta_{\mathcal{F}}{\epsilon_{\mathcal{F}}}^{-1})) = 0$$

since the above group is the kernel of the restriction map by the inflation-restriction sequence. But $\Delta = \mathrm{Gal}(E_t/\mathbb{Q}_p)$ is a finite group and $\Gamma = \mathrm{Gal}(E_w/\mathbb{Q}_p)$ is well known to be isomorphic to $\Gamma_1 \times \Gamma_2$ where each $\Gamma_i = \mathbb{Z}_p$. Hence

$$\mathrm{H}^1(\mathrm{Gal}(E/\mathbb{Q}_p), K_{\mathcal{F}}(\delta_{\mathcal{F}}{\epsilon_{\mathcal{F}}}^{-1})) = \mathrm{H}^1(\Delta, K_{\mathcal{F}}(\delta_{\mathcal{F}}{\epsilon_{\mathcal{F}}}^{-1})^\Gamma) \oplus \mathrm{H}^1(\Gamma, K_{\mathcal{F}}(\delta_{\mathcal{F}}{\epsilon_{\mathcal{F}}}^{-1})^\Delta).$$

The first cohomology group on the right vanishes since Δ is a finite group. On the other hand if Δ acts non-trivially on $K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}{}^{-1})$ then the second group also vanishes. So we may assume that $K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}{}^{-1})^{\Delta} = K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}{}^{-1})$. But

$$\mathrm{H}^1(\Gamma, K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}{}^{-1})) = \mathrm{H}^1(\Gamma_1, K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}{}^{-1})^{\Gamma_2}) \oplus \mathrm{H}^1(\Gamma_2, K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}{}^{-1})^{\Gamma_1}).$$

One may check that $\delta_w\epsilon_w{}^{-1}$ is a non-trivial character of $\Gamma_1 \times \Gamma_2$. Without loss of generality assume that $\delta_w\epsilon_w{}^{-1}$ is non-trivial on Γ_2. Then again the first cohomology group on the right vanishes. As for the second we may assume that $\delta_w\epsilon_w{}^{-1}$ acts trivially on Γ_1 in which case this group is

$$\mathrm{H}^1(\Gamma_2, K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}{}^{-1})) = \frac{K_{\mathcal{F}}}{((\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}{}^{-1})(\gamma_2) - 1)K_{\mathcal{F}}} = 0$$

where γ_2 is a topological generator of Γ_2. This proves the lemma. □

By the lemma the restriction to H of the cohomology class $[c_{\mathcal{F}}]$ is still non-zero. It follows $\rho_{\mathcal{F}}|_H$ is non-split, and that the homomorphism $\lambda : H \to L$ is non-zero.

Let I denote the non-zero ideal of L generated by the image of λ. Since the intersection of infinitely many height one primes of L is the zero ideal, I is contained in only finitely many height one primes of L. Let $f = P(\mathcal{F}) \in S_{\mathcal{F}}$ be an arithmetic specialization of $\mathcal{F}$. Then $\rho_f|_H$ splits if and only if $I \subset P$. If follows that $\rho_f|_H$ does not split for all but finitely many specializations $f \in S_{\mathcal{F}}$. In particular $\rho_f|_{G_p}$ is non-split for all but finitely many $f \in S_{\mathcal{F}}$. This proves the theorem. □

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BIBLIOGRAPHY

[Buz03] K. BUZZARD, Analytic continuation of overconvergent eigenforms, J. Amer. Math. Soc., 16(1) (2003), 29–55.

[BT99] K. BUZZARD and R. TAYLOR, Companion forms and weight one forms, Ann. of Math., 149(3) (1999), 905–919.

[Col96] R. COLEMAN, Classical and overconvergent modular forms, Invent. Math., 124 (1996), 215–241.

[Gha04] E. GHATE, On the local behaviour of ordinary modular Galois representations, In Modular curves and abelian varieties, volume 224 of Progress in Mathematics, 105–124, Birkhäuser (2004).

[Gha05] E. Ghate, Ordinary forms and their local Galois representations, To appear.

[GV03] R. Greenberg and V. Vatsal, Iwasawa theory for Artin representations, To appear.

[Hid86a] H. Hida, Iwasawa modules attached to congruences of cusp forms, Ann. Sci. École Norm. Sup., 19(2) (1986), 231–273.

[Hid86b] H. Hida, Galois representations into $\mathrm{GL}_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms, Invent. Math., 85 (1986), 545–613.

[Hid93] H. Hida, Elementary Theory of L-functions and Eisenstein Series, LMSST 26, Cambridge University Press, Cambridge, (1993).

[MT90] B. Mazur and J. Tilouine, Représentations galoisiennes, différentielles de Kähler et “conjectures principales”, Inst. Hautes Études Sci. Publ. Math., 71 (1990), 65–103.

[MW86] B. Mazur and A. Wiles, On p-adic analytic families of Galois representations, Compositio Math., 59 (1986), 231–264.

[Miy89] T. Miyake, Modular forms, Springer Verlag, (1989).

[Ser89] J.-P. Serre, Abelian l-adic representations and elliptic curves. Second edition, Advanced Book Classics, Addison-Wesley Publishing Company, Redwood City, CA, (1989).

[Vat05] V. Vatsal, A remark on the 23-adic representation associated to the Ramanujan Delta function, Preprint.

[Wil88] A. Wiles, On ordinary λ-adic representations associated to modular forms, Invent. Math., 94 (1988), 529–573.

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Eknath GHATE,
Tata Institute of Fundamental Research
School of Mathematics
Homi Bhabha Road, Mumbai 400005 (India)
eghate@math.tifr.res.in

Vinayak VATSAL,
University of British Columbia
Department of Mathematics
Vancouver (Canada)
vatsal@math.ubc.ca