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Jochen HEINLOTH

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COHERENT SHEAVES WITH PARABOLIC STRUCTURE AND CONSTRUCTION OF HECKE EIGENSHEAVES FOR SOME RAMIFIED LOCAL SYSTEMS

by Jochen HEINLOTH

Introduction.

Before explaining the main result (Theorem 2.5) of this article in more detail, I would like to recall the setting of the geometric Langlands correspondence as in [23].

Let C be a smooth projective curve over a finite field $\mathbb{F}_q$. (As pointed out in [11] and [23], a lot of the arguments carry over to the case when C is defined over the complex numbers.) Then the Langlands correspondence – as proven by Lafforgue [18] – provides a bijection between irreducible ℓ-adic local systems defined on some open subset $U \subset C$ and certain irreducible representations of $\mathrm{GL}_n(\mathbb{A})$ contained in the space $\mathcal{C}^\infty\big(\mathrm{GL}_n(k(C))\backslash \mathrm{GL}_n(\mathbb{A})\big)$ called the space of automorphic functions. Here we denoted by $\mathbb{A} := \prod'_{x\in C} K_x$ the ring of adeles of the function field $k(C)$ of C, and by $\mathcal{C}^\infty\big(\mathrm{GL}_n(k(C))\backslash \mathrm{GL}_n(\mathbb{A})\big)$ the space of functions (with values in $\overline{\mathbb{Q}}_\ell$) that are right invariant under some compact open subgroup of $\mathrm{GL}_n(\mathbb{A})$ (for notations see Section 0). More precisely it is known (see e.g. [22]) that for any representation π_{E} corresponding to some local system E there is a compact open subgroup K such that π_{E} contains a K-invariant function A_{E}, often unique up to scalar. Further, this compact subgroup is determined by the ramification of E. Finally, note that the group $\mathrm{GL}_n(\mathbb{A})$ does not act on the K-invariant functions, but the algebra

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of K-bi-invariant functions acts on these by convolution. This is the action of the K-Hecke algebra. The function A_{E} is an eigenvector for this action, and it is determined by this condition.

Drinfeld noted [8] that this correspondence should have a geometric interpretation. First consider the case $\mathsf{K} = \mathrm{GL}_n(\mathcal{O})$. Weil explained that the double quotient $\mathrm{GL}_n(k(C))\backslash \mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathcal{O})$ can be identified with the set of isomorphism classes of vector bundles on C (choose a trivialisation at all local rings of C and at the generic point of C, the transition functions give an adele):

$$\mathrm{Bun}_n(\mathbb{F}_q) = \mathrm{GL}_n(k(C))\backslash \mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathcal{O}).$$

Furthermore, Grothendieck explained that any complex A of ℓ-adic sheaves on a scheme $X_{/\mathbb{F}_q}$ gives rise to a function on the set of its points by

$$\begin{aligned} \mathrm{trace} : \mathbb{D}^b(X) &\longrightarrow \prod_{n\in\mathbb{N}} \mathrm{Funct}\big(X(\mathbb{F}_{q^n})\big), \\ \mathsf{A} &\longmapsto \mathrm{tr}_{\mathsf{A}}(x) := \mathrm{trace}(\mathrm{Frob}_{\mathbb{F}_{q^n}}, \mathsf{A}_{|x}) \end{aligned}$$

and an irreducible perverse complex is determined by this function (see [21]).

Thus, Drinfeld expected that the above A_{E} should be of the form $\mathrm{tr}_{\mathsf{A}_{\mathsf{E}}}$ for some irreducible perverse sheaf A_{E} on the moduli space of vector bundles on C. He proved this for unramified local systems of rank 2. Later Laumon [20] gave a conjectural construction of A_{E} for local systems of arbitrary rank, and recently Frenkel, Gaitsgory and Vilonen [11], [13], proved that by Laumon's construction one indeed obtains a sheaf A_{E}.

Moreover, the action of the Hecke algebra also has a geometric interpretation in this case. Consider for example the characteristic function of the double coset $\mathrm{GL}_n(\mathcal{O}_x)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_x \end{smallmatrix}\right)\mathrm{GL}_n(\mathcal{O}_x)$, where π_x is a local parameter at some point $x \in C$. For a vector bundle $\mathcal{E}$ the multiplication of the corresponding adele by an element of this set produces a subbundle $\mathcal{E}' \subset \mathcal{E}$ such that the cokernel is $k(x)$. Further, every such subbundle can be obtained in this way. Drinfeld therefore considered the stack Hecke^1 classifying pairs of bundles $\mathcal{E}' \subset \mathcal{E}$ such that the cokernel has length 1, i.e. $\deg(\mathcal{E}') = \deg(\mathcal{E}) - 1 =: d-1$. This has forgetful maps

$$\mathrm{Bun}^d \xleftarrow{\mathrm{pr}_{\mathrm{big}}} \mathrm{Hecke}^1 \xrightarrow{\mathrm{pr}_{\mathrm{small}} \times \mathrm{quot}} \mathrm{Bun}^{d-1} \times C$$

With this definition the sheaf A_{E} has the additional property that

$$\mathbf{R}(\mathrm{pr}_{\mathrm{small}} \times \mathrm{quot})_! \, \mathrm{pr}_{\mathrm{big}}^* \, \mathsf{A}_{\mathsf{E}} \cong \mathsf{A}_{\mathsf{E}} \boxtimes \mathsf{E}[-n+1](-n+1),$$

and a similar definition works for more general Hecke stacks. One says that A_{E} is a *Hecke eigensheaf*.

Drinfeld [9] also proved an analogous result for local systems of rank 2 with unipotent ramification at a finite set of points $S \subset C(\mathbb{F}_q)$, this time producing a complex A_E on the moduli space of vector bundles of rank 2 with parabolic structure at S. The purpose of this article is to generalize this result.

We will start with an irreducible local system E with unipotent ramification at a finite set of points $S \subset C(\mathbb{F}_q)$, and we further have to assume that the ramification group at these points acts indecomposably, i.e. that the sheaf $j_*\mathsf{E}$ (where $j : C - S \to C$) has 1 dimensional stalks at all points $p \in S$. This additional condition is the reason why for the moment we can only prove our main theorem for local systems of rank ≤ 3.

In this case the corresponding automorphic function should be defined on the space $\mathrm{GL}_n(k(C))\backslash \mathrm{GL}_n(\mathbb{A})/\mathsf{K}_S$, where

$$\mathsf{K}_S = \prod_{x \in C-S} \mathrm{GL}_n(\mathcal{O}_x) \times \prod_{x \in S} \mathrm{Iw}_x$$

and $\mathrm{Iw}_x \subset \mathrm{GL}_n(\mathcal{O}_x)$ is the subgroup of matrices which are upper triangular mod x. As before we can interpret this set as vector bundles with the additional structure of a complete flag of subspaces of the stalks at all points in S:

$$\begin{aligned}\mathrm{Bun}_{n,S}(T) := \big\langle (\mathcal{E}, (V_{i,p})_{i=1,\dots,n,\ p\in S}) \mid \mathcal{E} \in \mathrm{Bun}_n \,;\\ 0 \subset V_{1,p} \subset \cdots \subset V_{n,p} = \mathcal{E} \otimes k(p) \big\rangle.\end{aligned}$$

This is usually called the stack of vector bundles with (quasi-)parabolic structure. Note that this can also be described as:

$$\begin{aligned}\mathrm{Bun}_{n,S}(T) := \big\langle (\mathcal{E}, (\mathcal{E}^{(i,p)})_{i=1,\dots,n,\ p\in S}) \mid \mathcal{E} \in \mathrm{Bun}_n \,;\\ \mathcal{E} \subset \mathcal{E}^{(1,p)} \subset \cdots \subset \mathcal{E}^{(n,p)} = \mathcal{E}(p) \big\rangle\end{aligned}$$

which has a simple generalization to coherent sheaves: one only has to replace "$\subset$" by arbitrary maps "$\to$" and to add the condition that the induced maps $\mathcal{E}^{(i,p)} \to \mathcal{E}^{(i,p)}(p)$ are the natural ones. This reformulation made our construction possible.

The first step of our construction is to recall that in principle a candidate for the automorphic function A_E is known, but we do not know of an explicit calculation of this function. Therefore, we have to prove an explicit formula (Proposition 1.2). This motivates a generalization of Laumon's construction, and – as a by-product of the notion of parabolic torsion sheaf – we get a geometric interpretation of some Hecke operators for the group K, i.e. of the Iwahori–Hecke algebra. Our main result is then the following:

THEOREM 2.5. — *For any irreducible local system* E *of rank* $n \leq 3$ *on* $C - S$ *with indecomposable unipotent ramification at* S *there is an irreducible perverse sheaf* A_{E} *on* $\mathrm{Bun}_{n,S}$ *which is an eigensheaf for the Iwahori-Hecke algebra.*

The strategy of the proof is the same as in [11], using parabolic sheaves instead of coherent sheaves, but some additional problems arise from the ramification of E. We reduce the theorem to an analogue (Proposition 7.1) of the vanishing conjecture of loc. cit. In particular, we show that the above theorem would follow for local systems of general rank if this analogue held in general.

The structure of the article is as follows. We start with the calculation of the Whittaker function for the Steinberg representation given in first section. This is an elementary calculation which served as motivation for our construction.

In the second section we introduce the notion of a coherent sheaf with parabolic structure and prove the results needed to give an analogue of Laumon's "fundamental diagram" and of Laumon's Whittaker sheaf $\mathcal{L}^d_{\mathsf{E}}$. As in the unramified situation we then define two candidates for an automorphic sheaf. At the end of this section we define the geometric Hecke operators corresponding to operators of the Iwahori-Hecke algebra which are needed to give a precise formulation of our main Theorem 2.5.

After this short exposition of our results we try to clarify the notion of parabolic sheaves in Section 3. We explain the general structure of parabolic torsion sheaves. Further, we give an explicit description of the corresponding moduli stack, and finally we note some semicontinuity results. We then use these basic results to prove some properties of the Whittaker sheaf $\mathcal{L}^d_{\mathsf{E}}$ (Section 4). Here we give a substitute for the Springer resolution in the case of parabolic sheaves which can be used to calculate this sheaf, and we prove a Hecke property of $\mathcal{L}^d_{\mathsf{E}}$. The problem arising in the proof of these results is that in our situation the above resolution is not small and the ramification of E also generates additional cohomology. By simultaneously proving the Hecke property and the fact that $\mathcal{L}^d_{\mathsf{E}}$ can be calculated via the resolution we see that the two effects cancel out.

In the fifth section we then compare the geometric construction of Section 2 with the calculation of the Whittaker function. The key idea here is to define an analogue of Drinfeld's compactification as given in [11]. However, we can not copy the proofs of loc. cit., which use results on the affine Grassmannian for which we do not know the corresponding statements for the affine flag manifold. Instead, we give an elementary proof of a much weaker result, sufficient for our purpose.

With these results available we can follow the strategy of [11] again and apply Lafforgue's result to deduce the existence of a Hecke eigensheaf on the moduli space of parabolic vector bundles whenever we know that the two candidates constructed coincide. This is the content of Section 6.

In the last two sections we then prove a generalization of the vanishing theorem of [11] for local systems of rank ≤ 3 and deduce the assumption needed to prove our theorem in Section 6. This is again very similar to the arguments in loc.cit., however we have to take care of the Iwahori-Hecke operators, for example we have to prove that some of them are central elements of the algebra (see Lemma 7.6).

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0. Notations and preliminary remarks.

We want to fix some notations used throughout this article.

0.1. The curve and its rings.

We fix a smooth projective, geometrically irreducible curve C defined over a finite field $k = \mathbb{F}_q$ and a finite (non-empty) set of points $S \subset C$. For simplicity we will assume that these points are rational. This assumption is not essential, since we could extend k to make the points rational and then descend the final result back to k. We denote by:

- $k(C)$ the field of rational functions on C;
- $\mathcal{O}_p$ (resp. $\widehat{\mathcal{O}}_p$) the local ring (resp. the complete local ring) at a point $p \in C$;
- $K_p := \mathrm{Quot}(\widehat{\mathcal{O}}_p)$;

- $\mathbb{A} := \prod'_{p \in C} K_p$ the ring of adeles of $k(C)$;
- $\mathcal{O} := \prod_{p \in C} \widehat{\mathcal{O}}_p$;
- $\Omega := \Omega_{C/k}$ the sheaf of differentials on C.

0.2. Groups.

- We note by GL_n the algebraic group of invertible $n \times n$ matrices;
- $\mathsf{B}_n \subset \mathrm{GL}_n$ the group of upper-triangular matrices;
- $\mathsf{N}_n \subset \mathsf{B}_n$ the group of unipotent upper triangular matrices;
- $\mathsf{P}_1 \subset \mathrm{GL}_n$ the subgroup fixing the subspace spanned by the first $n-1$ base vectors and acting trivially on the quotient by this subspace, i.e.

$$\mathsf{P}_1(R) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in \mathrm{GL}_{n-1}(R),\ v \in R^{n-1} \right\}.$$

- $\mathrm{Iw}_p \subset \mathrm{GL}_n(\widehat{\mathcal{O}}_p)$ the group of matrices which are upper triangular mod p.

We will further fix a non-trivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^*$.

Choosing a meromorphic differential form ω this defines

$$\Psi : \mathsf{N}_n(k(C)) \backslash \mathsf{N}_n(\mathbb{A}) \longrightarrow \overline{\mathbb{Q}}_\ell^*,$$
$$\Psi\big((U_p)_{p \in C}\big) := \prod_{p \in C} \psi\Big(\mathrm{trace}_{k(p)/\mathbb{F}_q}\Big(\mathrm{Res}_p\Big(\sum_{i=1}^{n-1} u_{p,i,i+1}\Big)\omega\Big)\Big),$$

where $u_{p,i,i+1}$ is the i-th entry of the first upper diagonal of the matrix U_p.

To avoid the choice of a meromorphic differential form we will (as in [10]) often replace the group $\mathrm{GL}_n \times C/C$ by the group

$$\mathrm{GL}_n^\Omega := \mathrm{Aut}\Big(\bigoplus_{i=1}^{n} \Omega^{\otimes n-i}\Big).$$

More precisely, $\mathrm{GL}_n \times C = \mathrm{Aut}(\mathcal{O}^{\oplus n})$ is the automorphism group of the trivial vector bundle over C, since for any ring R the automorphisms of the trivial rank n-bundle over $\mathrm{Spec}(R)$ are the same as elements of $\mathrm{GL}_n(R)$. In the same way points of GL_n^Ω are invertible matrices in which the (i,j)-th entry is a section of $\mathrm{Hom}(\Omega^{\otimes i-1}, \Omega^{\otimes j-1}) \cong \Omega^{\otimes j-i}$. In particular, the choice of a meromorphic differential ω induces a group isomorphism $\mathrm{GL}_n(\mathbb{A}) \xrightarrow{\sim} \mathrm{GL}_n^\Omega(\mathbb{A})$.

Denote by $\mathsf{N}_n^\Omega \subset \mathrm{GL}_n^\Omega$ the upper triangular matrices, with diagonal entries $1 \in \mathrm{Hom}(\Omega^i, \Omega^i)$, then Ψ is given by the composition

$$\mathsf{N}_n(\mathbb{A}) \xrightarrow{\sim} \mathsf{N}_n^\Omega(\mathbb{A}) \xrightarrow{\Sigma \mathrm{Res}} \mathbb{F}_q \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^*,$$

where the first map is the restriction of the above isomorphism to unipotent matrices and $\sum \mathrm{Res}$ is the sum of the residues of the upper diagonal entries.

0.3. Fourier transform.

For the additive character $\psi\colon \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^*$ chosen above we denote by L_ψ the Artin-Schreier sheaf on $\mathbb{A}^1$. Let

$$\mathrm{AS} : \mathbb{A}^1 \longrightarrow \mathbb{A}^1, \quad x \longmapsto x^q - x$$

be the Artin-Schreier covering with structure group $\mathbb{F}_q$, then L_ψ is the ψ-isotypic component of $\mathrm{AS}_* \overline{\mathbb{Q}}_\ell$. This is additive in the sense that for the addition map $+ : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ we have $+^* \mathsf{L}_\psi \cong \mathsf{L}_\psi \boxtimes \mathsf{L}_\psi$.

For a vector bundle $E \xrightarrow{p} X$ of rank n on a scheme (or algebraic stack) denote by $p^\vee : E^\vee \to X$ the dual bundle and by $\langle .,. \rangle : E \times_X E^\vee \to \mathbb{A}^1$ the contraction. The Fourier transform defined in [21] is given by

$$\mathcal{F}our : D^b(E) \longrightarrow D^b(E^\vee), \quad \mathsf{K} \longmapsto \mathbf{R}p_{E^\vee,!}\big(p_E^* \mathsf{K} \otimes \langle .,. \rangle^* \mathsf{L}_\psi\big)[n].$$

0.4. The trace function of a complex.

For a complex K of $\overline{\mathbb{Q}}_\ell$-adic sheaves on a scheme (or algebraic stack) X we denote by tr_K the function:

$$\mathrm{tr}_\mathsf{K} : \coprod_{n>0} X(\mathbb{F}_{q^n}) \longrightarrow \overline{\mathbb{Q}}_\ell, \quad x \longmapsto \mathrm{tr}_\mathsf{K}(x) := \mathrm{trace}(\mathrm{Frob}_x, \mathsf{K}_{|x}).$$

0.5. Algebraic stacks.

For the general theory of algebraic stacks we refer to the book of Laumon and Moret-Bailly [24]. In particular, an algebraic stack will be a stack that admits a smooth representable covering by a scheme.

We will view stacks as sheaves of categories for the fppf-topology. Thus to define a stack $\mathcal{M}$ over k, we usually give the category of T-valued points of $\mathcal{M}$ for any scheme T over k and denote this as $\mathcal{M}(T) := \langle \text{objects} \rangle$, where we use the brackets $\langle\ \rangle$ instead of $\{\ \}$ to denote the category of *objects* in which the only morphisms are isomorphisms of the *objects*.

Sometimes it is easier to give the T-valued points of a stack only for affine schemes T over the given base, which is equivalent to the data for all schemes by the descent condition for stacks. This point of view is used as definition in loc. cit.

To use the usual operations on constructible sheaves and the corresponding derived categories given in loc. cit. we need that our stacks satisfy the Bernstein-Lunts condition, i.e. for every $n \in \mathbb{N}$ we can find n-acyclic presentations for these stacks.

In our case we will often know that our stacks have a presentation as quotients $[X/G]$, where G is a reductive algebraic group acting on a scheme X. Stacks of this form satisfy the Bernstein Lunts condition (see [24], 18.7.5). For the moduli stack of vector bundles over a curve this is not true, but we have an ascending open covering $U_1 \subset U_2 \subset \cdots \subset \mathrm{Bun}_n^d$ in which each of the $U_i \cong [X_i/G_i]$ is a Bernstein-Lunts stack. For us this will be sufficient, since our sheaves will be supported in such a subset.

0.6. Some remarks on generalized vector bundles.

Recall that for a flat algebraic group G acting on a scheme X there is a quotient stack $[X/G]$ classifying principal G-bundles together with a G-equivariant morphism to X. In this section we will be concerned with the particular case of two vector bundles E_0, E_1 over some base and a homomorphism of $\mathcal{O}_C$-modules $E_0 \xrightarrow{\phi} E_1$. We take $G := E_0$ acting via ϕ on $X := E_1$:

Definition 0.1 (see [3]). — Let $E_0 \xrightarrow{\phi} E_1$ be an $\mathcal{O}_C$-module homomorphism between two vector bundles on a scheme (or an algebraic stack) X. Then the quotient stack $[E_1/E_0]$ is called a *generalized vector bundle* over X.

Lemma 0.1. — *Let $E_0 \xrightarrow{\phi} E_1$ be an $\mathcal{O}_C$-module homomorphism between two vector bundles on some scheme (or algebraic stack) X. The stack $[E_1/E_0]$ can be described as follows. For any affine scheme $T = \mathrm{Spec}(A) \xrightarrow{f} X$ over X:*

$$[E_1/E_0](T) = \left\langle \begin{matrix} \text{objects} = \{s \in H^0(T,f^*E_1)\} \text{ and for } s,t \in H^0(T,f^*E_1) \\ \mathrm{Hom}(s,t) = \{h \in H^0(T,f^*E_0) \mid s + \phi(h) = t\} \end{matrix} \right\rangle.$$

Moreover, any quasi-isomorphism of such complexes gives rise to an equivalence of the corresponding stacks, thus the stack $[E_1/E_0]$ depends only on the class of the complex $E_0 \to E_1$ in the derived category of coherent sheaves on X.

Example. — Let $C \xrightarrow{p} X$ be a smooth projective curve over some noetherian base scheme X, and let $\mathcal{F}_1, \mathcal{F}_2$ be coherent sheaves on C, flat over X. By [16] the complex $\mathbf{R}p_*(\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2))$ can be represented by a homomorphism of vector bundles $\mathcal{E}_0 \to \mathcal{E}_1$ on X. By abuse of notation we denote by $\underline{\mathrm{Ext}}(\mathcal{F}_1, \mathcal{F}_2)$ the corresponding generalized vector bundle on X.

Note that this is well defined by the above lemma. The description of the categories of sections given in the lemma tells us that this stack classifies extensions $0 \to \mathcal{F}_2 \to \mathcal{F} \to \mathcal{F}_1 \to 0$, i.e. for any $T \xrightarrow{f} X$:

$$\underline{\mathrm{Ext}}(\mathcal{F}_1, \mathcal{F}_2)(T) = \big\langle 0 \to f^*\mathcal{F}_2 \to \mathcal{F} \to f^*\mathcal{F}_1 \to 0 \text{ exact sequence on } T \big\rangle.$$

Proof of Lemma 0.1. — First note that the claimed description of $[E_1/E_0]$ defines a stack:

1) We can glue morphisms, because sections of E_0 form a sheaf.

2) Any descent datum of objects is effective (i.e. we can glue objects): let U_i be an affine covering of the affine scheme T. A descent datum for this covering is a collection of objects $s_i \in \Gamma(U_i, f^*E_1)$ together with morphisms $h_{ij} \in \Gamma(U_{ij}, f^*E_0)$ such that $s_{i|U_{ij}} + \phi(h_{ij}) = s_{j|U_{ij}}$ and $h_{ik|U_{ijk}} = h_{jk|U_{ijk}} + h_{ij|U_{ijk}}$.

This implies that h_{ij} is a 1-cocycle, and since T is affine it must be a coboundary, i.e. we can find $h_i \in H^0(U_i, f^*E_0)$ with $h_i - h_j = h_{ij}$ on U_{ij}. Therefore we may define $s_i' := s_i - \phi(h_i)$, and this collection of sections glues to give $s \in H^0(T, f^*E_1)$ with $s_{|U_i} = s_i'$.

Thus we may define a morphism of stacks

$$\left\langle \begin{matrix} \text{objects} = \{s \in H^0(T, f^*E_1)\} \text{ and for } s, t \in H^0(T, f^*E_1) \\ \mathrm{Hom}(s,t) = \{h \in H^0(T, f^*E_0) \mid s + \phi(h) = t\} \end{matrix} \right\rangle \to [E_1/E_0](T)$$

mapping a section $T \to E_1$ to the composition $T \to E_1 \to [E_1/E_0]$.

Since $H^1(T, f^*E_0) = 0$ for any affine T, any $s \in [E_1/E_0](T)$ is isomorphic to some $s' \in H^0(T, E_1)$ and by definition any morphism between two elements s, t in the image of this functor is given by a section of $H^0(T, E_0)$. Thus the morphism is an equivalence of stacks.

The above description of the stack $[E_1/E_0]$ also shows that a quasi-isomorphism of complexes induces an equivalence of the categories of points of the corresponding stacks. □

LEMMA 0.2. — *Let*

$$\begin{array}{ccccccc} 0 \to & E_0' & \xhookrightarrow{i_0} & E_0 & \xrightarrow{p_0} & E_0'' & \to 0 \\ & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi''} & \\ 0 \to & E_1' & \xhookrightarrow{i_1} & E_1 & \xrightarrow{p_1} & E_1'' & \to 0 \end{array}$$

be an exact sequence of (2 term-)complexes of locally free sheaves on some (quasi-separated) scheme X. Denote by

$$[E_1'/E_0'] \xrightarrow{i} [E_1/E_0] \xrightarrow{p} [E_1''/E_0'']$$

the induced morphisms of the generalized bundles, and let $s'' : X \to [E_1''/E_0'']$ be a section. Then locally over X the stack

$$p^{-1}(s'') = [E_1/E_0] \times_{[E_1''/E_0'']} X$$

is isomorphic to $[E_1'/E_0']$. More precisely such an isomorphism exists over any $U \to X$ such that there is a lift $s_1 \in \Gamma(U, E_1)$ with $p(s_1) \cong s''_{|U}$.

Remark. — We might state the above as "$p^{-1}(s'')$ is a principal homogeneous space for $[E_1'/E_0']$". More generally, we will call a morphism of stacks a *generalized affine space bundle* if it can be factored into a sequence of maps each of them locally (over the target space) isomorphic to a generalized vector bundle.

Proof. — We may assume that $X = U$, such that there exists s_1 in $H^0(U, E_1)$ with $p(s_1) = s''$ (e.g. we can take U affine). Let $f : T \to U = X$ be an affine U-scheme. Using the previous lemma, we find that $p^{-1}(s'')(T)$ is the category with:

$$\begin{aligned} \text{objects} &= \big\{(s, h'') \in \Gamma(T, f^*E_1) \times \Gamma(T, f^*E_0'') \mid p_1(s) + \phi''(h'') = s_1\big\}, \\ \operatorname{Hom}\big((s,h''),(t,g'')\big) &= \big\{h \in \Gamma(T, f^*E_0) \mid s + \phi(h) = t \\ &\qquad\qquad \text{and } p_0(h) = h'' - g''\big\}. \end{aligned}$$

Thus we define $[E_1'/E_0'] \to p_1^{-1}(s^n)$ by

$$H^0(T, E_1') \ni s' \longmapsto (i_1(s') + s_1, 0) \quad \text{and} \quad H^0(T, E_0') \ni h' \longmapsto i_0(h').$$

This is essentially surjective, since for affine T and any $h'' \in H^0(T, f^*E_0'')$ there is an $h \in H^0(T, f^*E_0)$ with $p_0(h) = h''$, and therefore any $(s, h'') \cong (s - \phi(h), 0)$. Morphisms of two objects in the image of the above map are given by $H^0(T,E_0') = \ker(H^0(T,E_0) \to H^0(T,E_0''))$, therefore the above map is an equivalence of categories. $\square$

Application. — We will apply this lemma in the following situation: consider the morphism of stacks classifying diagrams (with exact lines and columns) of torsion sheaves on a curve C:

$$\left\langle \begin{matrix} \mathcal{T}_1' & & & \\ \downarrow & & & \\ \mathcal{T}_1 & \hookrightarrow \mathcal{T}_2 & \twoheadrightarrow \mathcal{T}_3'' \\ \downarrow & \downarrow & \| \\ \mathcal{T}_1'' & \hookrightarrow \mathcal{T}_2'' & \twoheadrightarrow \mathcal{T}_3'' \end{matrix} \right\rangle \xrightarrow{\text{forget}_{\mathcal{T}_2}} \left\langle \begin{matrix} \mathcal{T}_1' & & \\ \downarrow & & \\ \mathcal{T}_1 & & \\ \downarrow & & \\ \mathcal{T}_1'' & \hookrightarrow \mathcal{T}_2'' & \twoheadrightarrow \mathcal{T}_3'' \end{matrix} \right\rangle$$

where the degree of each torsion sheaf is fixed.

On the right hand stack the exact triangle of complexes

$$\mathbf{R}\operatorname{Hom}(\mathcal{T}_3'',\mathcal{T}_1') \longrightarrow \mathbf{R}\operatorname{Hom}(\mathcal{T}_3'',\mathcal{T}_1) \xrightarrow{p} \mathbf{R}\operatorname{Hom}(\mathcal{T}_3'',\mathcal{T}_1'')$$

can be represented by an exact sequence of 2-term complexes of vector bundles. There is a canonical s'' of $\mathbf{R}\operatorname{Hom}(\mathcal{T}_3'',\mathcal{T}_1'')$ given by the extension in the lower line, and the projection map from $p^{-1}(s'')$ to the base stack is the map $\text{forget}_{\mathcal{T}_2}$.

Thus, by the above lemma, we see that the fibres of this morphism are isomorphic to the stack $\underline{\operatorname{Ext}}(\mathcal{T}_3'',\mathcal{T}_1')$. These stacks are generalized affine spaces, in particular the étale cohomology of the fibres is one-dimensional.

0.7. A lemma used more than once...

The following general lemma is stated in [11], a similar calculation is done in [6]. I would like to thank Sergey Lysenko for explanations about this.

LEMMA 0.3. — *Let $\mathcal{E} \xrightarrow{p} X$ be a (generalized) vector bundle, and denote by $s_0 : X \to \mathcal{E}$ the zero-section of $\mathcal{E}$. Let further $\mathsf{K} \in D^b_{\text{ét}}(\mathcal{E})$ be a complex of étale sheaves on $\mathcal{E}$ such that the restriction of K to the complement of the zero-section descends to the projective bundle $\mathbb{P}(\mathcal{E})$ (e.g. a $\mathbb{G}_m$-invariant complex of sheaves on $\mathcal{E}$). Then*

$$\mathbf{R}p_*\mathsf{K} = s_0^*\mathsf{K}.$$

Proof. — We may assume that $\mathcal{E}$ is a vector bundle, since for a generalized vector bundle $[\mathcal{E}_1/\mathcal{E}_0]$ the functor $\mathbf{R}p_*$ is defined via an acyclic representable covering of the bundle, i.e. by definition we may replace $[\mathcal{E}_1/\mathcal{E}_0]$ by $\mathcal{E}_1$. Let $j : \mathcal{E}^0 := \mathcal{E} - s_0(X) \hookrightarrow \mathcal{E}$ be the inclusion. Then we have an exact triangle

$$\to j_! j^* \mathsf{K} \longrightarrow \mathsf{K} \longrightarrow s_{0,*} s_0^* \mathsf{K} \xrightarrow{[1]} .$$

For the first term $j_! j^* \mathsf{K}$ we have to prove that $\mathbf{R}p_* j_! j^* \mathsf{K} = 0$. If we can show this we are done, since the lemma is true for the last term, and the right hand map then gives the claimed isomorphism.

Write $\mathsf{K}^0 := j^* \mathsf{K}$. Then by assumption $\mathsf{K}^0 \cong \mathrm{proj}^*(\overline{\mathsf{K}})$, where $\mathrm{proj} : \mathcal{E}^0 \to \mathbb{P}(\mathcal{E})$ is the projection to the projectivized bundle and $\overline{\mathsf{K}}$ is a sheaf on $\mathbb{P}(\mathcal{E})$. To get a relation between $\mathcal{E}$ and $\mathbb{P}(\mathcal{E})$, blow-up the zero-section of $\mathcal{E}$, and denote the blow-up by $\mathrm{Bl}_{s_0}(\mathcal{E})$:

$$\begin{tikzcd}
\mathrm{Bl}_{s_0}(\mathcal{E}) \arrow[d, "\mathrm{bl}"'] \arrow[drr, "\mathrm{pr}_{\mathbb{P}(\mathcal{E})}"] & & \\
\mathcal{E} \arrow[d, "p"'] & \mathcal{E}^0 \arrow[l, hook', "j"] \arrow[ul, hook, "\tilde{j}"'] \arrow[r, "\mathrm{proj}"'] & \mathbb{P}(\mathcal{E}). \arrow[dll, "\bar{p}"] \\
X & &
\end{tikzcd}$$

Note that $\mathrm{pr}_{\mathbb{P}(\mathcal{E})} : \mathrm{Bl}_{s_0}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ is the line bundle $\mathcal{O}(-1)$ over $\mathbb{P}(\mathcal{E})$. Let $s_{\mathbb{P}(\mathcal{E})} : \mathbb{P}(\mathcal{E}) \to \mathrm{Bl}_{s_0}(\mathcal{E})$ be the zero-section (i.e. the inclusion of the special fibre of the blow-up). Since

$$j_! \,\mathrm{proj}^*(\overline{\mathsf{K}}) = \mathbf{R}\,\mathrm{bl}_! \tilde{j}_! \,\mathrm{proj}^*(\overline{\mathsf{K}}) \xlongequal{\text{bl.projective}} \mathbf{R}\,\mathrm{bl}_* \tilde{j}_! \,\mathrm{proj}^*(\overline{\mathsf{K}}),$$

we need to show that $\mathbf{R}(p \circ \mathrm{bl})_*(\tilde{j}_! \,\mathrm{proj}^*(\overline{\mathsf{K}})) = 0$. But this is easy, since – as before – there is an exact triangle on $\mathrm{Bl}_{s_0}(\mathcal{E})$

$$\to \tilde{j}_! \,\mathrm{proj}^* \overline{\mathsf{K}} \longrightarrow \mathrm{pr}^*_{\mathbb{P}(\mathcal{E})} \overline{\mathsf{K}} \longrightarrow s_{\mathbb{P}(\mathcal{E}),*} \overline{\mathsf{K}} \to,$$

and if we apply $\mathbf{R}\,\mathrm{pr}_{\mathbb{P}(\mathcal{E})}$ then the natural map induces

$$\mathbf{R}\,\mathrm{pr}_{\mathbb{P}(\mathcal{E}),*} \mathrm{pr}^*_{\mathbb{P}(\mathcal{E})} \overline{\mathsf{K}} \xlongequal{\text{proj. formula}} \overline{\mathsf{K}} \cong \mathbf{R}\,\mathrm{pr}_{\mathbb{P}(\mathcal{E}),*} s_{\mathbb{P}(\mathcal{E}),*} \overline{\mathsf{K}}.$$

Thus $\mathbf{R}\,\mathrm{pr}_{\mathbb{P}(\mathcal{E}),*} \tilde{j}_! \,\mathrm{proj}^* \overline{\mathsf{K}} = 0$, and therefore $\mathbf{R}(p \circ \mathrm{bl})_* \tilde{j}_! \,\mathrm{proj}^* \overline{\mathsf{K}} = 0$, since $p \circ \mathrm{bl}$ factors through $\mathbb{P}(\mathcal{E})$. □

1. The Whittaker function for the Steinberg representation.

As indicated in the introduction, for any local system E of rank n on $C - S$ with indecomposable unipotent ramification at points in S there is a particular function f_{E} on $\mathrm{GL}_n(\mathbb{A})$ which one expects to span the automorphic representation corresponding to E.

In this section we will give a formula for this function, more precisely we will give an explicit formula for a function W_{E} from which f_{E} may be obtained by some explicit transformation. This formula served as motivation for our construction, whereas it is not needed to define the geometric construction. The reader might want to skip the simple, but lengthy calculation.

1.1. The Whittaker space.

We will denote by $\mathcal{C}^\infty(\mathrm{GL}_n(\mathbb{A}))$ the space of functions f on $\mathrm{GL}_n(\mathbb{A})$ with values in $\overline{\mathbb{Q}}_\ell$ such that there exists a compact open subgroup $\mathsf{K} \subset \mathrm{GL}_n(\mathbb{A})$ (depending on f) such that $f(xk) = f(x)$ for all $x \in \mathrm{GL}(\mathbb{A}), k \in \mathsf{K}$. The same notation will be used for other locally compact groups.

Recall that we have chosen a particular additive character Ψ on $\mathsf{N}_n(\mathbb{A})$ (see 0.2). The space of functions

$$\mathcal{C}^\infty\big(\mathrm{GL}_n(\mathbb{A})\big)^{\mathsf{N}_n(\mathbb{A}),\Psi} := \left\{ f \in \mathcal{C}^\infty(\mathrm{GL}_n(\mathbb{A})) \;\middle|\; \begin{array}{l} f(ug) = \Psi(u)f(g) \\ \forall u \in \mathsf{N}_n(\mathbb{A}), g \in \mathrm{GL}_n(\mathbb{A}) \end{array} \right\}$$

is called *Whittaker representation* of $\mathrm{GL}_n(\mathbb{A})$. A subrepresentation π of this representation of $\mathrm{GL}_n(\mathbb{A})$ is called a *Whittaker model* for the isomorphism class of π.

Similarly let $\mathcal{C}^\infty_{\mathrm{cusp}}(\mathrm{GL}_n(\mathbb{A}))^{\mathsf{P}_1(k(C))}$ be the space of functions which are $\mathsf{P}_1(k(C))$-invariant and cuspidal[(1)] (see [10]). Recall the theorem of Shalika [25], 5.9, as stated in loc.cit.:

THEOREM 1.1 (Shalika). — *There is an isomorphism of representations of* $\mathrm{GL}_n(\mathbb{A})$

$$\Phi : \mathcal{C}^\infty\big(\mathrm{GL}_n(\mathbb{A})\big)^{\mathsf{N}_n(\mathbb{A}),\Psi} \longrightarrow \mathcal{C}^\infty_{\mathrm{cusp}}\big(\mathrm{GL}_n(\mathbb{A})\big)^{\mathsf{P}_1(k(C))}$$

given by $f \mapsto \Phi(f)(g) := \sum_{y \in \mathsf{N}_{n-1}(k(C))\backslash \mathrm{GL}_{n-1}(k(C))} f\big(\big(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\big) g\big)$.

(1) A reader unfamiliar with this notion may ignore it for the moment, it will be explained again.

Since the character Ψ is a product of characters of the groups $\mathsf{N}_n(K_p)$, we may construct functions in the Whittaker representation as products of functions on $\mathrm{GL}_n(K_p)$ which satisfy the analogous transformation condition for elements of $\mathsf{N}_n(K_p)$. Thus, using Shalika's theorem, the strategy to construct automorphic functions has been to construct functions in the Whittaker model, then to apply Φ and try to prove that the resulting function is not only invariant under the action of $\mathsf{P}_1(k(C))$ but really invariant under the action of $\mathrm{GL}_n(k(C))$.

In this chapter we will only be concerned with the local question, i.e. with representations of $\mathrm{GL}_n(K_p)$ for one fixed prime p. The global Whittaker function

$$W_{\mathsf{E}}(g) := \prod_{p\in C} W_{\mathsf{E},p}(g_p) \quad \text{for } g = (g_p)_{p\in C} \in \mathrm{GL}_n(\mathbb{A})$$

corresponding to our local system will be given as the product of the local functions $W_{\mathsf{E},p}$. For all $p \in C - S$ these are given by the formula of Shintani and Casselman, Shalika (see [10]), whereas for $p \in S$ the local factor is the Whittaker function of the Steinberg representation (twisted by the eigenvalue λ_p of Frob_p on the one-dimensional stalk $(j_*\mathsf{E})_p$) which is calculated below.

1.2. The Steinberg representation.

Fix a point $p \in S \subset C$ and choose a local parameter π at p. Let

$$\delta_\lambda : ({K^*}_p/\mathcal{O}_p^*)^n \longrightarrow \overline{\mathbb{Q}}_\ell^*, \quad (\pi^{d_i}) \longmapsto \lambda^{\sum_i d_i} \prod_{i<j} q^{-(d_i-d_j)}.$$

This may be viewed as a character of $\mathsf{B}_n(K_p)$, by applying δ_λ to the diagonal entries of an element of $\mathsf{B}_n(K_p)$. In this interpretation δ_λ is the modulus character multiplied by $\lambda^{\mathrm{valuation(det)}}$.

The (twisted) Steinberg representation St_λ of $\mathrm{GL}_n(K_p)$ is the unique irreducible subrepresentation of the induced representation

$$\mathrm{Ind}_{\mathsf{B}_n(K_p)}^{\mathrm{GL}_n(K_p)}\delta_\lambda := \left\{ f \in \mathcal{C}^\infty(\mathrm{GL}_n(K_p)) \;\middle|\; \begin{array}{l} f(bg) = \delta_\lambda(b)f(g) \\ \forall b \in \mathsf{B}_n(K_p),\, g \in \mathrm{GL}_n(K_p) \end{array} \right\}.$$

Here again $\mathcal{C}^\infty(\mathrm{GL}_n(K_p))$ denotes the $\overline{\mathbb{Q}}_\ell$-valued functions which are invariant under some compact open subgroup. For this representation there is a unique (up to scalar) nontrivial Iw-invariant vector, which is an eigenvector of the Iwahori-Hecke algebra [5]. We denote this vector by f_{Iw}. Furthermore we know that this representation has a Whittaker model, and we denote the Iw-invariant vector in the Whittaker model by W_λ and normalize it by the condition that $W_\lambda(1) = 1$.

1.3. The Whittaker function – statement of the formula.

For any $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ denote by

$$\operatorname{diag}(\pi^{\underline{d}}) := \begin{pmatrix} \pi^{d_1} & & \\ & \ddots & \\ & & \pi^{d_n} \end{pmatrix}$$

the diagonal matrix and by σ the permutation matrix corresponding to the permutation $\sigma(e_i) = e_{\sigma(i)}$ (where e_i are the standard basis of $\mathbb{Z}^n$).

Proposition 1.2. — *The unique* Iw*-invariant function* W_λ *in the Whittaker model of* St_λ, *normalized by* $W_\lambda(1) = 1$, *is given by*

$$W_\lambda\big(\operatorname{diag}(\pi^{\underline{d}})\cdot\sigma\big) = \frac{\operatorname{sign}(\sigma)\lambda^{\sum d_i}}{q^{\sum_{i<j} d_i - d_j}\operatorname{vol}(\mathrm{Iw}\,\sigma\,\mathrm{Iw})}$$

if $d_i \geq d_{i+1} - \delta_{\sigma^{-1}(i)>\sigma^{-1}(i+1)}$ *and* $W_\lambda(\operatorname{diag}(\pi^{\underline{d}})\cdot\sigma) = 0$ *otherwise.*

Here $\delta_{\sigma^{-1}(i)>\sigma^{-1}(i+1)} = 1$ *if* $\sigma^{-1}(i) > \sigma^{-1}(i+1)$, *i.e. the entry in line* i *of* σ *is right of the entry below and* $\delta_{\sigma^{-1}(i)>\sigma^{-1}(i+1)} = 0$ *otherwise; the volume is normalized by* $\operatorname{vol}(\mathrm{Iw}) = 1$.

Remark. — The proposition is sufficient to calculate W_λ since

$$\begin{aligned}\mathrm{GL}_n(K_p) = \mathsf{B}_n(K_p)\,\mathrm{GL}_n(\mathcal{O}_p) &= \bigcup_{\sigma\in S_n} \mathsf{B}_n(K_p)\,\mathrm{Iw}\,\sigma\,\mathrm{Iw} \\ &= \bigcup_{\sigma\in S_n} \mathsf{B}_n(K_p)\mathsf{N}_n(\mathcal{O}_p)\sigma\,\mathrm{Iw} = \bigcup_{\sigma\in S_n} \mathsf{B}_n(K_p)\sigma\,\mathrm{Iw}\,.\end{aligned}$$

Example. — For GL_2 we have

$$W_\lambda\left(\begin{pmatrix} \pi^{d_1} & \\ & \pi^{d_2}\end{pmatrix}\right) = \begin{cases} q^{d_2-d_1}\lambda^{d_1+d_2} & \text{if } d_1 \geq d_2, \\ 0 & \text{otherwise,}\end{cases}$$

$$W_\lambda\left(\begin{pmatrix} & \pi^{d_1} \\ \pi^{d_2} & \end{pmatrix}\right) = \begin{cases} -q^{d_2-d_1-1}\lambda^{d_1+d_2} & \text{if } d_1 \geq d_2 - 1, \\ 0 & \text{otherwise.}\end{cases}$$

This is the formula used in Drinfeld's article [9].

1.4. Eigenvalues of some Hecke operators on the Steinberg representation.

To calculate W_λ, we need to compute the eigenvalues of the Hecke operators on the Iw-invariant vector in St_λ. To this end we use the function f_{Iw}. For an element $g \in \mathrm{GL}_n(K_p)$ we denote by T_g the Hecke operator given by convolution with the characteristic function of the double coset $\mathrm{Iw}\, g\, \mathrm{Iw}$, i.e.

$$\begin{aligned} T_g : \mathcal{C}^\infty(\mathrm{GL}_n(K_p)/\,\mathrm{Iw}) &\longrightarrow \mathcal{C}^\infty(\mathrm{GL}_n(K_p)/\,\mathrm{Iw}), \\ f &\longmapsto (T_g f)(x) := \sum_{h \in \mathrm{Iw}\, g\, \mathrm{Iw}\,/\,\mathrm{Iw}} f(x \cdot h). \end{aligned}$$

The Hecke operators given by the following particular matrices $t_{\leq i}$ will be very useful[(2)]:

$$t_{\leq i}(e_j) = \begin{cases} \pi \cdot e_i & \text{if } j = 1, \\ e_{j-1} & \text{if } 1 < j \leq i, \\ e_j & \text{if } j > i \end{cases} \quad \text{i.e.} \quad t_{\leq i} = \left(\begin{array}{cccc|c} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \pi & & & & \\ \hline & & & & \mathbf{1}_{n-i} \end{array}\right).$$

The following – presumably well-known – lemma gives the eigenvalues of some of the operators T_g on f_{Iw}:

LEMMA 1.3. — 1) *For all* $\underline{d} \in \mathbb{Z}^n$ *with* $d_1 \geq \cdots \geq d_n$ *one has*

$$T_{(\mathrm{diag}(\pi^{\underline{d}}))} f_{\mathrm{Iw}} = \lambda^{\sum d_i} f_{\mathrm{Iw}}.$$

2) $T_\sigma f_{\mathrm{Iw}} = \mathrm{sign}(\sigma) f_{\mathrm{Iw}}$ *for all* $\sigma \in S_n$.

3) $T_{t_{\leq i}} f_{\mathrm{Iw}} = (-1)^{i-1} \lambda f_{\mathrm{Iw}}$.

Unfortunately, most of the results on Hecke algebras in the literature are formulated only for semi-simple groups. However, the Iwahori-Hecke algebra for GL_n differs from the one for SL_n only by the additional element $T_{t_{\leq n}}$.

Proof. — First we note that Borel shows in [5] that the eigenvalue of T_σ is

$$(T_\sigma) f_{\mathrm{Iw}} = \mathrm{sign}(\sigma) f_{\mathrm{Iw}} \quad \text{for } \sigma \in S_n.$$

Further, we may assume $f_{\mathrm{Iw}}(1) = 1$, since $f_{\mathrm{Iw}}(1) = 0$ would imply that f_{Iw} is identically 0 (see the calculations below), thus for $u \in \mathsf{N}_n(K)$ and $\underline{d} \in \mathbb{Z}^n$

$$f_{\mathrm{Iw}}(\mathrm{diag}(\pi^{\underline{d}}) u) = \lambda^{\sum d_i} q^{-\sum_{i<j} d_i - d_j} f_{\mathrm{Iw}}(1).$$

(2) In case $i = n$ the corresponding operation on parabolic bundles is the upper modification.

We apply this in the case $d_1 \geq \cdots \geq d_n$ to calculate

$$\begin{aligned}
T_{\operatorname{diag}(\pi^{\underline{d}})} f_{\mathrm{Iw}}(1) &= \sum_{g \in \mathrm{Iw}\operatorname{diag}(\pi^{\underline{d}})\,\mathrm{Iw}\,/\,\mathrm{Iw}} f_{\mathrm{Iw}}(g) \\
&= \sum_{g \in \mathsf{N}_n(\mathcal{O})\operatorname{diag}(\pi^{\underline{d}})\,\mathrm{Iw}\,/\,\mathrm{Iw}} f_{\mathrm{Iw}}(g) \\
&= \operatorname{vol}\big(\mathsf{N}_n(\mathcal{O})\operatorname{diag}(\pi^{\underline{d}})\,\mathrm{Iw}\,\big)\,\delta\big(\operatorname{diag}(\pi^{\underline{d}})\big)\, f_{\mathrm{Iw}}(1) \\
&= q^{(\sum_{i<j} d_i - d_j)}\delta\big(\operatorname{diag}(\pi^{\underline{d}})\big) f_{\mathrm{Iw}}(1) = \lambda^{(\sum d_i)} f_{\mathrm{Iw}}(1).
\end{aligned}$$

Here we used that an element of Iw is a product of an element in $\mathsf{N}_n(\mathcal{O})$ and a lower diagonal matrix contained in Iw, and that for $d_1 \geq \cdots \geq d_n$

$$\begin{pmatrix} 1 & & 0 \\ \mathfrak{p} & \ddots & \\ \mathfrak{p} & \mathfrak{p} & 1 \end{pmatrix} \operatorname{diag}(\pi^{\underline{d}}) = \operatorname{diag}(\pi^{\underline{d}}) \begin{pmatrix} 1 & & 0 \\ \pi^{d_1-d_2}\mathfrak{p} & \ddots & \\ \pi^{d_1-d_n}\mathfrak{p} & \pi^{d_2-d_n}\mathfrak{p} & 1 \end{pmatrix} \in \operatorname{diag}(\pi^{\underline{d}})\mathrm{Iw}.$$

Finally, to compute the eigenvalue of the operator $T_{t_{\leq i}}$ we first note that by 2):

$$\operatorname{sign}(\sigma) = T_\sigma f_{\mathrm{Iw}}(1) = \sum_{g \in \mathrm{Iw}\,\sigma\,\mathrm{Iw}\,/\,\mathrm{Iw}} f_{\mathrm{Iw}}(g) = \operatorname{vol}(\mathrm{Iw}\,\sigma\,\mathrm{Iw}) f_{\mathrm{Iw}}(\sigma).$$

Further we need a description of the corresponding double coset $\mathrm{Iw} \cdot t_{\leq i} \cdot \mathrm{Iw}\,/\,\mathrm{Iw}$. Take an element $k \in \mathrm{Iw}$ and look at $k \cdot t_{\leq i}$:

$$\begin{aligned}
\begin{pmatrix} \mathrm{Iw}_i & \mathcal{O} \\ \mathfrak{p} & \mathrm{Iw}_{n-i} \end{pmatrix} \cdot t_{\leq i} &= \left(\begin{array}{ccc} \multicolumn{2}{c}{\mathrm{Iw}_i \cdot t} & \mathcal{O} \\ \hline \underbrace{\pi\cdot\mathfrak{p}}_{\text{first column}} & \mathfrak{p} & \mathrm{Iw}_{n-i} \end{array}\right) \\
&= t_{\leq i} \cdot \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{Iw}_i} & \pi^{-1}\mathcal{O} \\ & & \mathcal{O} \\ \hline \underbrace{\pi\cdot\mathfrak{p}}_{\text{first column}} & \mathfrak{p} & \mathrm{Iw}_{n-i} \end{array}\right) \begin{array}{l} \leftarrow \text{first line} \\ \\ \\ \end{array}.
\end{aligned}$$

Thus, the matrices of the form

$$t_{\leq i} \left(\begin{array}{c|c} 1_i & \pi^{-1}v_1 \cdots \pi^{-1}v_{n-i} \\ & 0 \\ \hline & 1_{n-i} \end{array}\right) \begin{array}{l} \leftarrow \text{first line} \\ \\ \\ \end{array}, \qquad v_i \in \mathbb{F}_q,$$

form a set of representatives for $\mathrm{Iw}\, t_{\leq i}\, \mathrm{Iw}\,/\,\mathrm{Iw}$. We denote the right hand unipotent matrix by u_v with $v \in \mathbb{F}_q^{n-i}$ and the permutation $\sigma_{\leq i} := (t_{\leq i}\operatorname{diag}(\pi^{-1}, 1, \ldots, 1))$ then we have

$$\begin{aligned} T_{t_{\leq i}} f_{\mathrm{Iw}}(\sigma_{\leq i}^{-1}) &= \sum_{v \in \mathbb{F}_q^{n-i}} f_{\mathrm{Iw}}(\sigma_{\leq i}^{-1} t_{\leq i} u_v) = \sum_{v \in \mathbb{F}_q^{n-i}} f_{\mathrm{Iw}}\big(\operatorname{diag}(\pi, 1, \ldots, 1) u_v\big) \\ &= q^{n-i} \lambda q^{-(n-1)} = q^{-i+1}\lambda. \end{aligned}$$

□

1.5. The Whittaker function – proof of the formula.

First we show the vanishing assertion. For $u \in \mathsf{N}_n(K), \gamma \in \mathrm{Iw}$ and a permutation σ we know that

$$W_\lambda\big(u \operatorname{diag}(\pi^{\underline{d}})\sigma\gamma\big) = \psi(u) W_\lambda\big(\operatorname{diag}(\pi^{\underline{d}})\sigma\gamma\big).$$

Thus if $\sigma^{-1}\operatorname{diag}(\pi^{\underline{d}})^{-1} u \operatorname{diag}(\pi^{\underline{d}})\sigma \in \mathrm{Iw}$ then we must either have $\psi(u) = 1$ or $W_\lambda(\operatorname{diag}(\pi^{\underline{d}})\sigma) = 0$, i.e. if $\sigma^{-1}(i) > \sigma^{-1}(i+1)$ our function W_λ can be non-zero only if

$$(\pi^{d_{i+1}} - d_i) u_i \in \mathfrak{p} \implies \operatorname{Res}(u_i) = 0,$$

that is $d_i \geq d_{i+1} - 1$ and if $\sigma^{-1}(i) < \sigma^{-1}(i+1)$, we need $d_i \geq d_{i+1}$. This gives the necessary condition for $W_\lambda \neq 0$ claimed in the lemma.

Next, we note that our formula holds for diagonal matrices with $d_1 \geq d_2 \geq \cdots \geq d_n$, because (as in the proof of Lemma 1.3)

$$\begin{aligned} \lambda^{\sum d_i} \cdot W_\lambda(1) = T_{\operatorname{diag}(\pi^{\underline{d}})} W_\lambda(1) &= \sum_{g \in N(\mathcal{O})\operatorname{diag}(\pi^{\underline{d}})\cdot \mathrm{Iw}\,/\,\mathrm{Iw}} W_\lambda(g) \\ = W_\lambda\big(\operatorname{diag}(\pi^{\underline{d}})\big) \cdot \operatorname{vol}\big(\mathrm{Iw} \cdot \operatorname{diag}(\pi^{\underline{d}}) \cdot \mathrm{Iw}\big) &= W_\lambda\big(\operatorname{diag}(\pi^{\underline{d}})\big) \cdot q^{\sum_{i<j} d_i - d_j}. \end{aligned}$$

Now we proceed by descending induction on the number i such that $\sigma(j) = j$ for all $j > i$: assume that $\sigma(j) = j$ for all $j > i$ and $\sigma^{-1}(i) < i$.

We apply the Hecke operator $T_{t_{\leq i}}$ to express the value of $W_\lambda(\operatorname{diag}(\pi^{\underline{d}}) \cdot \sigma)$ for elements σ with $\sigma^{-1}(i) = i - k$ in terms of the value of W_λ at points with $\sigma^{-1}(i) = i - k + 1$, which we know by induction. Since W_λ is an eigenfunction for $T_{t_{\leq i}}$, with eigenvalue $(-1)^{i-1}\lambda$, we get

$$\begin{aligned} (-1)^{i-1}\lambda \cdot W_\lambda\big(\operatorname{diag}(\pi^{\underline{d}})\sigma\big) &= (T_{\leq i} W_\lambda)\big(\operatorname{diag}(\pi^{\underline{d}})\sigma\big) \\ &= \sum_{k \in \mathrm{Iw}\, t_{\leq i}\, \mathrm{Iw}\,/\,\mathrm{Iw}} W_\lambda\big(\operatorname{diag}(\pi^{\underline{d}})\sigma \cdot k\big). \end{aligned}$$

In the proof of Lemma 1.3 we used representatives of $\mathrm{Iw}\, t_{\leq i}\, \mathrm{Iw} / \mathrm{Iw}$ given as

$$t_{\leq i}\begin{pmatrix} 1_i & \pi^{-1}v_1 \cdots \pi^{-1}v_{n-i} \\ & 0 \\ \hline & 1_{n-i} \end{pmatrix} \begin{array}{l} \leftarrow\text{first line} \\ \\ \\ \end{array} = t_{\leq i} u_v, \quad v \in \mathbb{F}_q^{n-i}.$$

We write $t_{\leq i} = \sigma_{\leq i} \operatorname{diag}(\pi, 1, \dots, 1)$. Then the above equals

$$\begin{aligned} T_{\leq i} W_\lambda\big(\operatorname{diag}(\pi^{\underline{d}})\sigma\big) &= \sum_{v \in \mathbb{F}_q^{n-i}} W_\lambda\big(\operatorname{diag}(\pi^{\underline{d}})\sigma\sigma_{\leq i} \operatorname{diag}(\pi, 1, \dots, 1) u_v\big) \\ &= \sum_{v \in \mathbb{F}_q^{n-i}} W_\lambda\big(\operatorname{diag}(\pi^{(d_1, \dots, d_{\sigma(i)}+1, \dots, d_n)})\sigma\sigma_{\leq i} u_v\big). \end{aligned}$$

Where we used $\sigma\sigma_{\leq i}(1) = \sigma(i)$. Now $\sigma\sigma_{\leq i} u_v (\sigma\sigma_{\leq i})^{-1}$ is a unipotent matrix in which all the entries of the first upper diagonal are zero (the non trivial entries are in line $\sigma(i)$ which is $< i$ and columns $> i$). The character ψ vanishes on such elements. Therefore

$$T_{\leq i} W_\lambda\big(\operatorname{diag}(\pi^{\underline{d}})\sigma\big) = q^{n-i} W_\lambda\big(\operatorname{diag}(\pi^{d_1}, \dots, \pi^{d_{\sigma(i)}+1}, \dots, \pi^{d_n})\sigma \circ \sigma_{\leq i}\big).$$

Note that $(\sigma\sigma_{\leq i})^{-1}(i) = \sigma^{-1}(i) + 1$ therefore by induction this last expression is non-zero if $d_{\sigma(i)-1} \geq d_{\sigma(i)} \geq d_{\sigma(i)+1} - 1$, or equivalently

$$d_{\sigma(i)-1} + 1 \leq d_{\sigma(i)} + 1 \leq d_{\sigma(i)+1}$$

which gives the sufficient condition for $W_\lambda(\operatorname{diag}(\pi^{\underline{d}})\sigma)$ to be non zero. To conclude we have to check that we get the right power of q in the induction step:

1) $\operatorname{vol}(\sigma\sigma_{\leq i} =: \sigma') = q^{\#\{k<j \mid \sigma'(k) > \sigma'(j)\}}$ and we have

$$\#\{k < j \mid \sigma'(k) > \sigma'(j)\} = \#\{k < j \mid \sigma(k) > \sigma(j)\} - \big(i - \sigma(i)\big) + \big(\sigma(i) - 1\big).$$

2) Write $d'_{\sigma(i)} := d_{\sigma(i)} + 1$ and $d'_j := d_j$ for $j \neq \sigma(i)$. Then

$$q^{\sum_{k<j} d'_k - d'_j} = q^{(\sum_{k<j} d_k - d_j) + (n - \sigma(i)) - (\sigma(i) - 1)}.$$

So these terms differ by a factor q^{n-i}, which is what we needed to show. $\square$

2. An analogue of Laumon's construction.

We fix an irreducible local system E of rank n on our curve $C - S$, ramified at a finite set of points $S \subset C(\mathbb{F}_q)$, such that the ramification group at any point $p \in S$ acts unipotently and indecomposably. We will state this condition as "E has indecomposable unipotent ramification at S".

We want to give a geometric construction for an irreducible perverse sheaf corresponding to the Fourier transform $\Phi(W_{\mathsf{E}})$ of the Whittaker function W_{E}, computed in the previous section. We will follow Laumon's construction closely, the only new ingredient needed for the construction being the notion of a coherent sheaf with parabolic structure. We will also need to prove generalizations of some results on vector bundles to the case of quasi-parabolic vector bundles.

2.1. Parabolic vector bundles.

Denote by $\mathrm{Bun}^d_{n,S}$ the moduli space (algebraic stack) of vector bundles of rank n and degree d on C with a full flag at the points of S, i.e. for any scheme $T_{/k}$:

$$\mathrm{Bun}^d_{n,S}(T) := \left\langle (\mathcal{E}, \mathcal{E}^{(i,p)})_{i=1,\dots,n-1, p\in S} \;\middle|\; \begin{array}{l} \mathcal{E}, \mathcal{E}^{(i,p)} \text{ vector bundles on } C \times T, \\ \mathcal{E} \subset \mathcal{E}^{(1,p)} \subset \cdots \subset \mathcal{E}^{(n-1,p)} \subset \mathcal{E}(p), \\ \mathcal{E}^{(i,p)}/\mathcal{E} \text{ flat over } T, \\ \operatorname{rank}\mathcal{E} = n, \deg(\mathcal{E}) = d, \deg(\mathcal{E}^{(i,p)}) = d + i \end{array} \right\rangle.$$

Remark. — Usually one defines a vector bundle with full (quasi-) parabolic structure to be a vector bundle $\mathcal{E}$ together with a full flag $V_{1,p} \subsetneq \cdots \subsetneq V_{n,p} = \mathcal{E} \otimes k(p)$ of subspaces of the stalk of $\mathcal{E}$ at p. This is equivalent to the above definition – set $V_{i,p} := \ker\big(\mathcal{E} \otimes k(p) \to \mathcal{E}^{(i,p)} \otimes k(p)\big)$ and conversely $\mathcal{E}^{i,p} := \big(\ker(\mathcal{E} \to \mathcal{E} \otimes k(p)/V_{i,p})\big)(p)$.

From this reformulation we get a description of the points of $\mathrm{Bun}^d_{n,S}$: Denote as before $\mathsf{K} := \prod_{p \in (C-S)} \mathrm{GL}_n(\mathcal{O}_p) \times \prod_{p\in S} \mathrm{Iw}_p$, then[3]

$$\mathrm{Bun}^d_{n,S}(\mathbb{F}_q) = \mathrm{GL}_n(k(C)) \backslash \mathrm{GL}_n(\mathbb{A})^{\mathrm{norm}(\det)=d} / \mathsf{K}.$$

[3] Recall that given a vector bundle $\mathcal{E}$ one can choose a trivialisation of $\mathcal{E}$ at the generic point and at all complete local rings of C. The transition functions then give an element of $\mathrm{GL}_n(\mathbb{A})$, the double quotient is obtained by forgetting the trivialisations, keeping the flags at S.

And the double quotient $\mathsf{P}_1(F)\backslash \mathrm{GL}_n(\mathbb{A})/\mathsf{K}$ contains the points of the bundle $\mathrm{Hom}^{\mathrm{inj}}(\mathcal{O},\mathcal{E}) \to \mathrm{Bun}^d_{n,S}$.

Notations.

1) We will write $\mathcal{E}^\bullet := (\mathcal{E}, \mathcal{E}^{(i,p)})_{i=1,\dots,n-1;p\in S}$.

2) Since $\mathcal{E} \subset \mathcal{E}^{(i,p)} \subset \mathcal{E}(p)$ we also get $\mathcal{E}(p) \subset \mathcal{E}^{(i,p)}(p)$, thus a parabolic bundle is a chain of vector bundles

$$\mathcal{E}^{(i,p)} \subset \mathcal{E}^{(i+1,p)} \subset \cdots \subset \mathcal{E}^{(n-1,p)} \subset \mathcal{E}(p) \subset \mathcal{E}^{(1,p)}(p) \subset \cdots,$$

where the cokernel of every inclusion is of length 1. For any integer $k \in \mathbb{Z}$ we denote by $\mathcal{E}^{(kn+i,p)} := \mathcal{E}^{(i,p)}(kp)$.

Note furthermore that since the map $\mathcal{E} \to \mathcal{E}(p)$ is an isomorphism on $C - \{p\}$, for two distinct points $p, q \in S$ the vector bundle $\mathcal{E}^{(i,p)} + \mathcal{E}^{(j,q)} \subset \mathcal{E}(p+q)$ is a vector bundle of degree $d+i+j$. We denote it by $\mathcal{E}^{(i,p)+(j,q)}$. Analogously we define $\mathcal{E}^{(i,S)} := \mathcal{E}^{\sum_{p\in S}(i,p)}$.

Thus we can shift the whole complex to obtain parabolic structures on the vector bundle $\mathcal{E}^{(i,p)}$ for all i. This is called the i-th *upper modification* of $\mathcal{E}^\bullet$.

3) $\mathcal{E}^\bullet(\frac{i}{n}p) := (\mathcal{E}^{(i,p)}, \mathcal{E}^{(j,q)+(i,p)})_{j=1,\dots,n-1,q\in S}$. This notation might be justified, because $\mathcal{E}^{(i,p)}$ is of degree $d+i = d+n(i/n)$ and for $i=n$ we get the canonical parabolic structure on the vector bundle $\mathcal{E}(p)$.

We now want to mimic Laumon's construction of automorphic sheaves for unramified local systems. Consider for example the case of bundles of rank 2. We will view $\Phi(W_{\mathsf{E}})$ as a function on vector bundles together with a meromorphic section of Ω. At a point $\Omega \hookrightarrow \mathcal{E}$ such that $\Omega \to \mathcal{E}$ and $\Omega \to \mathcal{E}^{(1,S)}$ are both maximal embeddings $\Phi(W_{\mathsf{E}})$ is defined as the sum over all sections of $\mathcal{E}/\Omega$ with at most simple poles at S. But the line bundle $(\mathcal{E}/\Omega)(S) \cong \mathcal{E}^{(1,S)}/\Omega$, thus we might equivalently sum over all holomorphic sections of $\mathcal{E}^{(1,S)}/\Omega$.

To apply a similar consideration to bundles of larger rank, our calculation of W_{E} suggests that we need to consider quotients of $\mathcal{E}^\bullet$ by subsheaves which are not maximal. We therefore look for a notion of coherent sheaves with parabolic structure[4] which allows the operation $\mathcal{F}^\bullet \mapsto \mathcal{F}^\bullet(\frac{i}{n}S)$. This is easy with the above definition of parabolic structure.

[4] While I was thinking about this, Norbert Hoffmann explained to me that one can formally adjoin quotients of vector bundles with parabolic structure to the category of such bundles to obtain an abelian category. The definition below may be viewed as a geometric interpretation of these quotients. I would like to thank him for the helpful discussion.

2.2. Parabolic coherent sheaves.

DEFINITION 2.1. — A *coherent sheaf on C with n-step parabolic structure at S* – also called *parabolic sheaf* for short – is a collection of coherent sheaves

$$\mathcal{F}^\bullet := \big(\mathcal{F} = \mathcal{F}^{(0,p)}, \mathcal{F}^{(i,p)}\big)_{i=1,\dots,n-1;\, p\in S}$$

together with morphisms $\phi^{(i,p)}: \mathcal{F}^{(i-1,p)} \to \mathcal{F}^{(i,p)}$ for $i = 1,\dots,n$ and $p \in S$ (where $\mathcal{F}^{(n,p)} := \mathcal{F}(p)$) such that in the resulting sequence

$$\cdots \xrightarrow{\phi^{(n,p)}(-p)} \mathcal{F} \xrightarrow{\phi^{(1,p)}} \mathcal{F}^{(1,p)} \xrightarrow{\phi^{(2,p)}}$$

$$\cdots \xrightarrow{\phi^{(n-1,p)}} \mathcal{F}^{(n-1,p)} \xrightarrow{\phi^{(n,p)}} \mathcal{F}(p) \xrightarrow{\phi^{(1,p)}(p)} \mathcal{F}^{(1,p)}(p) \to \cdots$$

the composition of n maps $\mathcal{F}^{(i,p)} \xrightarrow{\phi^{(i,p)}(p)\circ\cdots\circ\phi^{(i+1,p)}} \mathcal{F}^{(i,p)}(p)$ is the natural morphism.

Notes. — 1) If the sheaf $\mathcal{F}^{(i,p)}$ is not torsion free at p for some i, then the natural map $\mathcal{F}^{(i,p)} \to \mathcal{F}^{(i,p)}(p)$ is not injective, so at least one of the $\phi^\bullet$'s is not injective (see the examples below). However, by the same argument we see that all $\phi^{(i,p)}$ are isomorphisms on $C - S$, in particular all the $\mathcal{F}^{(i,p)}$ have the same generic rank, thus we define the *rank* of $\mathcal{F}^\bullet$ as $\mathrm{rank}(\mathcal{F}^{(0,p)})$.

2) The *degree* of $\mathcal{F}^\bullet$ is defined as the collection

$$\deg(\mathcal{F}^\bullet) := \big(\deg(\mathcal{F}^{(i,p)})\big)_{0\le i<n, p\in S}.$$

3) We denote by $\mathrm{Coh}^{\underline{d}}_{r,C,S}$ the *algebraic stack of coherent sheaves of rank r on C with n-step parabolic structure at S and (multi-)degree* $\underline{d} = (d^{(i,p)})_{0\le i<n,p\in S}$. Since we usually fix the curve C, we will omit it and write $\mathrm{Coh}^{\underline{d}}_{r,S}$ to shorten this lengthy notation.

4) We denote by $\mathrm{Bun}^{\underline{d}}_{r,S} \subset \mathrm{Coh}^{\underline{d}}_{r,S}$ the substack of torsion free sheaves, i.e. the substack where all $\mathcal{F}^{(i,p)}$ are vector bundles. Note that these stacks include the stacks of vector bundles with partial parabolic structure at S, in particular for constant degree $\underline{d} = (d,\dots,d)$ this substack is the moduli stack of vector bundles without additional structure.

Usually we will consider $\mathrm{Bun}^{\underline{d}}_{r,S}$ only in the case where $\underline{d}^{(i,p)} = d + i$ for some $d \in \mathbb{Z}$ and $r = n$, but the other stacks will arise in connection with Hecke operators.

5) As in the case of vector bundles we define

$$\mathcal{F}^{(i,p)+(j,q)} := (\mathcal{F}^{(i,p)} \oplus \mathcal{F}^{(j,q)})/\mathcal{F}$$

(for the diagonal embedding of $\mathcal{F}$). Note that this quotient is the sheaf isomorphic to $\mathcal{F}^{(i,p)}$ on $C-q$ and isomorphic to $\mathcal{F}^{(j,q)}$ on $C-p$. These sheaves glue, since both are canonically isomorphic to $\mathcal{F}$ on $C-\{p,q\}$. Analogously we define $\mathcal{F}^{(i,S)}$.

6) Again we define upper modifications as

$$\mathcal{F}^\bullet\left(\frac{i}{n}p\right) := (\mathcal{F}^{(i,p)}, \mathcal{F}^{(j,q)+(i,p)})_{0\le j<n, q\in S}.$$

Example. — In our case, given an injection $\Omega^{\otimes(n-1)} \hookrightarrow \mathcal{E}$, we get an induced parabolic structure on the quotient $\mathcal{E}/\Omega^{\otimes(n-1)}$. We only use that $\mathcal{E}(p)/\Omega^{\otimes(n-1)}(p) = (\mathcal{E}/\Omega^{\otimes(n-1)})(p)$ to get

$$\begin{tikzcd}
\Omega^{\otimes(n-1)} \arrow[r, "\mathrm{id}"] \arrow[d] & \Omega^{\otimes(n-1)} \arrow[r, "\mathrm{id}"] \arrow[d] & \cdots \arrow[r, "\mathrm{id}"] & \Omega^{\otimes(n-1)} \arrow[r, hook] \arrow[d] & \Omega^{\otimes(n-1)}(p) \arrow[d] \\
\mathcal{E} \arrow[r, "\phi^{(1,p)}"] \arrow[d] & \mathcal{E}^{(1,p)} \arrow[r, "\phi^{(2,p)}"] \arrow[d] & \cdots \arrow[r] & \mathcal{E}^{(n-1,p)} \arrow[r, "\phi^{(n,p)}"] \arrow[d] & \mathcal{E}(p) \arrow[d] \\
\mathcal{E}/\Omega^{\otimes(n-1)} \arrow[r] & \mathcal{E}^{(1,p)}/\Omega^{\otimes(n-1)} \arrow[r] & \cdots \arrow[r] & \mathcal{E}^{(n-1,p)}/\Omega^{\otimes(n-1)} \arrow[r] & (\mathcal{E}/\Omega^{\otimes(n-1)})(p).
\end{tikzcd} \tag{2.1}$$

Note that we can view $\Omega^{\otimes(n-1)}$ (or any coherent sheaf) as parabolic sheaf by defining

$$\Omega^{\otimes(n-1)(i,p)} := \Omega^{\otimes(n-1)}$$

for $i = 0, \ldots, n-1$. With this definition the above diagram is an extension of parabolic sheaves.

From this example we see that:

LEMMA-DEFINITION 2.1. — *The category of (quasi-)coherent sheaves with n-step parabolic structure is abelian.*

We denote homomorphisms of parabolic sheaves by $\mathrm{Hom}_{\mathrm{para}}(.,.)$, *and the same for* $\mathrm{Ext}^1_{\mathrm{para}}$, *etc.*

The category of quasi-coherent sheaves has enough injectives.

Proof. — The kernel and cokernel of a morphism can be defined componentwise. All compatibilities thus follow from the corresponding ones for coherent sheaves and we conclude that the category of sheaves with parabolic structure is abelian. Furthermore the above example shows that:

Remark 2.2. — The stack $\mathrm{Coh}^d_{k,C}$ classifying coherent sheaves of rank k and degree d on C can be embedded into the stack of parabolic sheaves:

$$j\colon \mathrm{Coh}^d_{k,C} \hookrightarrow \mathrm{Coh}^{(d,\dots,d)}_{k,S}, \quad \mathcal{F} \longmapsto \mathcal{F}^\bullet := (\mathcal{F}, \mathcal{F}^{(i,p)} := \mathcal{F}).$$

For a coherent sheaf $\mathcal{F}$ on C we will write $(\mathcal{F})^\bullet$ for its image $j(\mathcal{F})$. The functors $(.)^\bullet$ and $(.)^{(0,S)}$ are adjoint functors:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{para}}\big((\mathcal{F})^\bullet, \mathcal{G}^\bullet\big) &= \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \mathcal{G}^{(0,S)}),\\ \mathrm{Hom}_{\mathrm{para}}\big(\mathcal{G}^\bullet, (\mathcal{F})^\bullet\big) &= \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{G}^{(n-1,S)}, \mathcal{F}). \end{aligned}$$

For an injective sheaf $\mathcal{I}$ the adjunction property yields

$$\mathrm{Hom}_{\mathrm{para}}\big(\mathcal{F}^\bullet, (\mathcal{I})^\bullet\big) = \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}^{(n-1,S)}, \mathcal{I}).$$

Since the functor $(.)^{(n-1,S)}$ is exact we conclude that $\mathrm{Hom}(., (\mathcal{I})^\bullet)$ is exact. Thus choosing embeddings $\mathcal{G}^{i,p} \hookrightarrow \mathcal{I}_{i,p}$ into injective sheaves $\mathcal{I}_{i,p}$ we get an embedding $\mathcal{G}^\bullet \hookrightarrow \bigoplus(\mathcal{I}_{i,p})^\bullet(\frac{n-1}{n}S + \frac{n-i-1}{n}p)$ of $\mathcal{G}$ into an injective parabolic sheaf. □

By the above we also have:

LEMMA 2.3. — *The extensions of a parabolic sheaf $\mathcal{F}^\bullet$ by a line bundle $\mathcal{L}$ are classified by* $\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{F}^{(n-1,S)}, \mathcal{L})$, *i.e.*

$$\mathrm{Ext}^1_{\mathrm{para}}\big(\mathcal{F}^\bullet, (\mathcal{L})^\bullet\big) = \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{F}^{(n-1,S)}, \mathcal{L}).$$

Proof. — By the above remark any injective resolution of $\mathcal{L}$ defines an injective resolution of $(\mathcal{L})^\bullet$, and to such a resolution we may apply the adjunction formula. □

Note that we could give another proof of this lemma, calculating the Yoneda-Ext groups directly via the diagram (2.1). The only thing one has to check is that in this diagram we have

$$\mathcal{E}^{(i,p)} \cong \mathcal{E}^{(n-1,p)} \times_{(\mathcal{E}^{(n-1,p)}/\Omega^{\otimes n-1})} (\mathcal{E}^{(i,p)}/\Omega^{\otimes n-1}).$$

COROLLARY 2.4. — *Let $\mathcal{F}^\bullet$ be a parabolic sheaf and let $\mathcal{L}$ be a line bundle on C. Then we have by Serre duality*

$$\mathrm{Ext}^1_{\mathrm{para}}\big(\mathcal{F}^\bullet, (\mathcal{L})^\bullet\big) = \Big(\mathrm{Hom}_{\mathrm{para}}\big((\mathcal{L}\otimes\Omega^{-1})^\bullet(-\tfrac{n-1}{n}S), \mathcal{F}^\bullet\big)\Big)^\vee.$$

Proof. — This is just an application of the adjunction isomorphism to

$$\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{F}^{(n-1,S)},\mathcal{L}) = \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{L}\otimes\Omega^{-1},\mathcal{F}^{(n-1,S)})^\vee. \qquad \square$$

The above version of Serre duality (Corollary 2.4) suggests to denote

$$\mathcal{O}^\bullet := (\mathcal{O})^\bullet, \quad \Omega^\bullet := (\Omega)^\bullet(\tfrac{n-1}{n}S)$$

and analogously $\Omega^{\bullet,k} := (\Omega^{\otimes k})^\bullet(k\frac{n-1}{n}S)$. Then we can put $\mathcal{L} := \Omega^{\otimes k}$ to deduce from the corollary that

$$\mathrm{Ext}^1_{\mathrm{para}}(\mathcal{F}^\bullet,\Omega^{\bullet,k}) \cong \mathrm{Hom}_{\mathrm{para}}(\Omega^{\bullet,k-1},\mathcal{F}^\bullet)^\vee.$$

2.3. The fundamental diagram.

Reformulating the preceding calculations for families of parabolic sheaves allows us to construct a variant of Laumon's "fundamental diagram" as follows. We will call a coherent parabolic sheaf $\mathcal{F}^\bullet$ *good* if

$$\mathrm{Hom}_{\mathrm{para}}(\mathcal{F}^\bullet,\Omega^{\bullet,n-i+1}) = 0 \quad \text{for all } 1\le i\le n-1.$$

By Serre duality this condition guarantees that

$$\mathrm{Ext}^1_{\mathrm{para}}(\Omega^{\bullet,n-i},\mathcal{F}^\bullet) = \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^{n-i},\mathcal{F}^{(-(n-i)(n-1),S)}) = 0,$$

and moreover the same will be true for any quotient of $\mathcal{F}^\bullet$.

We denote by $\mathrm{Bun}^{d,\mathrm{good}}_{n,S}\subset\mathrm{Bun}^d_{n,S}$ and $\mathrm{Coh}^{d,\mathrm{good}}_{n,S}\subset\mathrm{Coh}^d_{n,S}$ the open substacks of good parabolic sheaves.

Denote by $\mathcal{E}^\bullet_{\mathrm{univ}}$ (resp. $\mathcal{F}^\bullet_{\mathrm{univ}}$) the universal parabolic sheaf on $\mathrm{Bun}^{d,\mathrm{good}}_{n,S}\times C$ (resp. on $\mathrm{Coh}^{d,\mathrm{good}}_{n,S}\times C$) and let p_i be the projection to the i-th factor.

We can view the sheaf $p_{1,*}(\mathcal{H}om(p_2^*\Omega^{\bullet,n-1},\mathcal{E}^\bullet_{\mathrm{univ}}))$ as the classifying stack for good parabolic vector bundles $\mathcal{E}^\bullet$ together with a homomorphism $\Omega^{\bullet,n-1}\to\mathcal{E}^\bullet$. Denote this stack by

$$\mathrm{Hom}_n := \big\langle(\mathcal{F}^\bullet,\mathrm{pr}_C^*\,\Omega^{\bullet,n-1}\xrightarrow{\phi}\mathcal{F}^\bullet)\mid\mathcal{F}^\bullet\in\mathrm{Coh}^{d,\mathrm{good}}_{n,S}\big\rangle.$$

Write $\mathrm{Hom}_n^{\mathrm{inj}}$ for the open substack of Hom_n where ϕ is injective.

Similarly write Ext^1_n for the stack classifying extensions of parabolic sheaves by $\Omega^{\bullet,n}$:

$$\mathrm{Ext}^1_n := \big\langle 0\to\mathrm{pr}_C^*\,\Omega^{\bullet,n}\to\mathcal{F}^\bullet_{n+1}\to\mathcal{F}^\bullet\to 0\mid\mathcal{F}^\bullet\in\mathrm{Coh}^{d,\mathrm{good}}_{n,S}\big\rangle.$$

Note that we defined the substacks of good bundles, to guarantee that Hom_n and Ext^1_n are vector bundles over $\mathrm{Coh}^{\underline{d},\mathrm{good}}_{n,S}$.

As in Laumon's construction we have:

1) To give a short exact sequence $0 \to \Omega^{\bullet,n-1} \to \mathcal{F}^\bullet_n \to \mathcal{F}^\bullet \to 0$ it is sufficient to specify the datum $0 \to \Omega^{\bullet,n-1} \to \mathcal{F}^\bullet_n$. Furthermore, if $\mathcal{F}^\bullet_n$ is good, all of its quotients are good as well.

Thus if we denote by $\mathrm{Ext}^{1,\mathrm{good}}_n \subset \mathrm{Ext}^1_n$ the substack consisting of extensions in which the middle term is a good parabolic sheaf, then have an isomorphism $I_n : \mathrm{Hom}^{\mathrm{inj}}_n \xrightarrow{\sim} \mathrm{Ext}^{1,\mathrm{good}}_{n-1}$.

2) Over $\mathrm{Coh}^{\underline{d},\mathrm{good}}_{n,S}$ the bundles Hom_n and Ext^1_n are dual vector bundles.

Since we want to construct a sheaf on the moduli stack of vector bundles with full parabolic structure at S, we fix a parabolic degree $\underline{d}$ given by $d^{(i,p)} = d + i$ for some fixed $d \in \mathbb{Z}$ and define a fundamental diagram (which we split into several diagrams):

$$\begin{tikzcd}
\mathrm{Hom}_n \arrow[d] & \mathrm{Hom}^{\mathrm{inj}}_n \arrow[l, hook', "j_{\mathrm{Hom}}"'] \arrow[r, hook, "I_n", "\sim"'] & \mathrm{Ext}^{1,\mathrm{good}}_{n-1} \arrow[r, hook, "j_{\mathrm{Ext}}", "\sim"'] & \mathrm{Ext}^1_{n-1} \arrow[dr] & \cdots \\
\mathrm{Coh}^{\underline{d},\mathrm{good}}_{n,S} & & & & \mathrm{Coh}^{\underline{d}_{n-1,S},\mathrm{good}}_{n-1,S}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Ext}^1_{n-1} \arrow[dr] \arrow[rr, dashed, bend left, leftrightarrow, "\text{dual bundles}"] & & \mathrm{Hom}_{n-1} \arrow[dl] & \cdots \\
& \mathrm{Coh}^{\underline{d}_{n-1},\mathrm{good}}_{n-1,S} & &
\end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd}
\cdots & \mathrm{Hom}_{n-1} \arrow[dl] & \mathrm{Hom}^{\mathrm{inj}}_{n-1} \arrow[l, hook', "j_{\mathrm{Hom}}"'] \arrow[r, hook, "I_{n-1}", "\sim"'] & \mathrm{Ext}^{1,\mathrm{good}}_{n-2} \arrow[r, hook, "j_{\mathrm{Ext}}", "\sim"'] & \mathrm{Ext}^1_{n-2} \arrow[d] & \cdots \\
\mathrm{Coh}^{\underline{d}_{n-1},\mathrm{good}}_{n-1,S} & & & & \mathrm{Coh}^{\underline{d}_{n-2},\mathrm{good}}_{n-2,S} &
\end{tikzcd}$$

Here the last line is the same as the first one with n replaced by $n-1$. Thus we can continue this to end up with $\mathrm{Coh}^{\underline{d}_0}_{0,S}$ (we drop the superscript "good", since all torsion sheaves are good). We have to keep track of the degrees of the parabolic sheaves $\underline{d}_{n-i} := \underline{d} - \sum_{j=1}^{i} \deg(\Omega^{\bullet,n-j})$, thus

$$\underline{d}^{(*,p)}_{n-i} = \big(d_{n-i} - (n-i-1), d_{n-i} - (n-i-2), \ldots, \underbrace{d_{n-i}, \ldots, d_{n-i}}_{i+1 \text{ times}}\big)$$

with $d_{n-i} := d - \sum_{j=1}^{i}((n-j)(2g-2)+(n-j+1))$. In particular, continuing the above diagram to the right, the last term will be $\mathrm{Coh}_{0,S}^{(d_0,\ldots,d_0)}$.

Laumon's construction started with a sheaf on $\mathrm{Coh}_0^{d_0}$ which corresponds to the Whittaker function for unramified local systems. This sheaf is pulled back to $\mathrm{Ext}_0^{1,\mathrm{good}}$, then one applies $j_{\mathrm{Hom},!*}I_1^*$ to the resulting sheaf, after that one applies the Fourier transform for the bundles in (2.2) and then continues with pull backs and intermediate extensions for the maps j_{Hom} and j_{Ext} in the upper line of the diagram until one ends up with a sheaf on Hom_n.

To do the same in our situation we need to find a sheaf on $\mathrm{Coh}_{0,S}^{d_0}$ that corresponds to the Whittaker function as calculated in Section 1.

2.4. The Whittaker sheaf $\mathcal{L}_{\mathsf{E}}^{d}$.

As noted in Section 2.2, there is an open embedding of torsion sheaves of degree d_0 on $C-S$ to parabolic torsion sheaves:

$$j\colon \mathrm{Coh}_{0,C-S}^{d_0} \hookrightarrow \mathrm{Coh}_{0,S}^{d_0}, \quad \mathcal{T} \longmapsto \mathcal{T}^{\bullet} = (\mathcal{T}, \mathcal{T}^{(i,p)} := \mathcal{T}).$$

The map j is open, since the condition that $\mathrm{supp}(\mathcal{T}^{(0,S)}) \subset C-S$ is open.

Furthermore we have Laumon's Whittaker sheaf $\mathcal{L}_{\mathsf{E}|C-S}^{d_0}$ on $\mathrm{Coh}_{0,C-S}^{d_0}$. Recall the definition of $\mathcal{L}_{\mathsf{E}|C-S}^{d_0}$: let $(C-S)^{(d_0)}$ be the d_0-th symmetric product of $C-S$ and denote by

$$j_{C-S}\colon (C-S)^{(d_0)} \longrightarrow \mathrm{Coh}_{0,C-S}^{d_0}, \quad D \longmapsto \mathcal{O}/\mathcal{O}(-D),$$

which is almost an embedding (see [20]). Let ${\mathsf{E}_{|C-S}}^{(d_0)}$ be the symmetric power of E restricted to the symmetric product $(C-S)^{(d_0)}$ of the curve $C-S$, then $\mathcal{L}_{\mathsf{E}|C-S}^{d_0} := j_{C-S,!*}{\mathsf{E}_{|C-S}}^{(d_0)}$.

Definition 2.2. — We define the *Whittaker sheaf* corresponding to E to be

$$\mathcal{L}_{\mathsf{E}}^{d} := j_{!*}\mathcal{L}_{\mathsf{E}|C-S}^{d} = (j \circ j_{C-S})_{!*}(\mathsf{E}_{|C-S})^{(d)}.$$

We will prove some properties of the Whittaker sheaf justifying its name in Section 4.

2.5. Putting everything together: the Fourier transform of $\mathcal{L}_{\mathsf{E}}^{d}$.

Now let $\mathrm{quot} : \mathrm{Ext}_0^1 \to \mathrm{Coh}_{0,S}^{\underline{d}_0}$, $(\mathcal{O}^\bullet \hookrightarrow \mathcal{F}_1^\bullet \twoheadrightarrow \mathcal{T}^\bullet) \mapsto \mathcal{T}^\bullet$ be the quotient map and denote the Fourier transform (recalled in 0.3) by $\mathcal{F}our : D^b(\mathrm{Hom}_k) \to D^b(\mathrm{Ext}_k^1)$. Following the fundamental diagram (2.2) from right to left we define:

Definition 2.3. — We inductively define the sheaves $\mathsf{F}_{\mathsf{E}}^k$ and $\mathsf{F}_{\mathsf{E}!}^k$ on the connected components of $\mathrm{Hom}_k^{\mathrm{inj}}$ which occur in the fundamental diagram as

$$\begin{aligned}\mathsf{F}_{\mathsf{E}}^1 &:= I_1^* j_{\mathrm{Ext}}^* \mathrm{quot}^* \mathcal{L}_{\mathsf{E}}^{d_0}[d_0] =: \mathsf{F}_{\mathsf{E},!}^1,\\ \mathsf{F}_{\mathsf{E}}^{k+1} &:= I_{k+1}^* j_{\mathrm{Ext}}^* \mathcal{F}our(j_{\mathrm{Hom}\,,!*} \mathsf{F}_{\mathsf{E}}^k),\\ \mathsf{F}_{\mathsf{E},!}^{k+1} &:= I_{k+1}^* j_{\mathrm{Ext}}^* \mathcal{F}our(j_{\mathrm{Hom}\,,!} \mathsf{F}_{\mathsf{E},!}^k).\end{aligned}$$

We define $\mathsf{F}_{\mathsf{E}}^k$ and $\mathsf{F}_{\mathsf{E},!}^k$ to be zero on all other connected components of $\mathrm{Hom}_k^{\mathrm{inj}}$.

Note that to keep track of the parabolic degrees we formulated the construction of $\mathsf{F}_{\mathsf{E}}^k$ on $\mathrm{Hom}_k^{\mathrm{inj}}$ above a fixed connected component $\mathrm{Coh}_{k,S}^{\underline{d}_k,\mathrm{good}} \subset \mathrm{Coh}_{k,S}^{\mathrm{good}}$. However, we will consider $\mathsf{F}_{\mathsf{E}}^k$ and $\mathrm{Hom}_k^{\mathrm{inj}}$ as defined above all the connected components corresponding to the special degrees $\underline{d}_k$ that occur in the definition of the fundamental diagram together and for convenience we defined $\mathsf{F}_{\mathsf{E}}^k$ and $\mathsf{F}_{\mathsf{E},!}^k$ to be zero on the other components.

The restriction of the sheaf $\mathsf{F}_{\mathsf{E}}^n$ to the stack of vector bundles with a section of $\Omega^{\bullet,n-1}$ will be our candidate to descend to an automorphic sheaf on $\mathrm{Bun}_{n,S}$. By construction this is a perverse sheaf, which is irreducible on all connected components of $\mathrm{Hom}_k^{\mathrm{inj}}$ (because we assumed that E is irreducible, therefore $\mathcal{L}_{\mathsf{E}}^{d}$ is an irreducible perverse sheaf and this property is preserved by $\mathcal{F}our$, $j_{\mathrm{Hom},!*}$ and j_{Ext}^*).

As in [19] we also define the sheaves $\mathsf{F}_{\mathsf{E},!}^k$, because it will be easy to prove that these have a Hecke eigensheaf property, and finally (in Section 8) we will show that they are isomorphic to $\mathsf{F}_{\mathsf{E}}^k$ for $k \leq n \leq 3$.

To end this section we want to state our main theorem. To do this we need to define geometric Hecke operators for parabolic sheaves. We first give an example indicating the relation between parabolic torsion sheaves and the Iwahori-Hecke algebra:

2.6. Parabolic torsion sheaves and Hecke operators.

Assume for the moment that $n = 2$, $S = \{p\}$, and consider the stack $\mathrm{Coh}^{1,1}_{0,p}$. Take any $\mathcal{T}^\bullet \in \mathrm{Coh}^{1,1}_{0,p}$. If $\mathrm{supp}(\mathcal{T}) = q \neq p$, then $\mathcal{T}^\bullet \cong (k_q)^\bullet$ where k_q is the residue field at q. But if $\mathrm{supp}(\mathcal{T}) = p$, then $\mathcal{T}^\bullet$ is isomorphic to exactly one of the following sheaves:

1) $\mathcal{T}_0 = k_p \xrightarrow{\mathrm{id}} \mathcal{T}_1 = k_p \xrightarrow{0} \mathcal{T}_0(p) \cong k_p \xrightarrow{\mathrm{id}} \cdots,$

2) $\mathcal{T}_0 = k_p \xrightarrow{0} \mathcal{T}_1 = k_p \xrightarrow{\mathrm{id}} \mathcal{T}_0(p) \cong k_p \xrightarrow{0} \cdots,$

3) $\mathcal{T}_0 = k_p \xrightarrow{0} \mathcal{T}_1 = k_p \xrightarrow{0} \mathcal{T}_0(p) \cong k_p \xrightarrow{0} \cdots.$

We want to relate these sheaves to some Hecke operators of the Iwahori-Hecke algebra at p, acting on parabolic vector bundles of rank 2. To do this, we consider torsion free extensions of a parabolic vector bundle $\mathcal{E}'^\bullet$ by the first complex:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{E}'^{(0,p)} \arrow[r] \arrow[d] & \mathcal{E}'^{(1,p)} \arrow[r] \arrow[d] & \mathcal{E}'^{(0,p)}(p) \arrow[r] \arrow[d] & \mathcal{E}'^{(1,p)}(p) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{E}^{(0,p)} \arrow[r, "\phi^1"] \arrow[d] & \mathcal{E}^{(1,p)} \arrow[r, "\phi^2"] \arrow[d] \arrow[ur, dashed] & \mathcal{E}^{(0,p)}(p) \arrow[r, "\phi^1(p)"] \arrow[d] & \mathcal{E}^{(1,p)}(p) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & k_p \arrow[r, "\mathrm{id}"] & k_p \arrow[r, "0"] & k_p \arrow[r, "\mathrm{id}"] & k_p \arrow[r] & \cdots
\end{tikzcd}$$

The middle map in the lower sequence is 0, therefore ϕ^2 factors through $\mathcal{E}^{(1,p)} \to \mathcal{E}'^{(0,p)}(p)$. Since all the bundles $\mathcal{E}^{(i,p)}$ are locally free this map is injective, and since the two bundles have the same degree it is an isomorphism.

The same argument shows that ϕ^1 does not factor through $\mathcal{E}'^{(1,p)}$, so the upper line is given by a parabolic structure on the vector bundle $\mathcal{E}^{(1,p)} \cong \mathcal{E}'^{(0,p)}(p)$, different from the canonical structure $\mathcal{E}'^\bullet(p)$. Thus extensions of this type are the set indexing the summation of the Hecke-Operator $T_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \circ T_{\left(\begin{smallmatrix}0&1\\\pi&0\end{smallmatrix}\right)}$. According to Lemma 1.3 this operator acts with eigenvalue $\mathrm{trace}(\mathrm{Frob}_p, (j_*\mathsf{E})_\mathsf{p})$ on the Whittaker function. Analogously we find that summing over extensions of parabolic bundles by the second torsion sheaf calculates $T_{\left(\begin{smallmatrix}0&1\\\pi&0\end{smallmatrix}\right)} \circ T_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$. Finally the third torsion sheaf gives the Hecke operator $T_{\left(\begin{smallmatrix}0&1\\\pi&0\end{smallmatrix}\right)}$ which acts with eigenvalue $-\,\mathrm{trace}(\mathrm{Frob}_p, (j_*\mathsf{E})_\mathsf{p})$ on the Whittaker function. Note that this torsion sheaf is a point of codimension 2 in $\mathrm{Coh}^{1,1}_{0,p}$, and thus the perverse sheaf $\mathcal{L}^1_\mathsf{E}$ will have some H^1 at this point. The minus sign of the eigenvalue will come from taking the trace of Frob on this cohomology group of odd degree (see Corollary 4.5). Therefore we define *generalized Hecke operators* as follows.

Fix non negative degrees $\underline{d} = \underline{d}_1 + \underline{d}_2$, and let $\mathrm{Hecke}_n^{\underline{d}_1,\underline{d}_2}$ be the stack classifying extensions of parabolic sheaves of degree $\underline{d}_2$ by torsion sheaves of degree $\underline{d}_1$, i.e.

$$\mathrm{Hecke}_n^{\underline{d}_1,\underline{d}_2} := \big\langle (0 \to \mathcal{F}'^\bullet \to \mathcal{F}^\bullet \to \mathcal{T}^\bullet \to 0) \\ \mid \mathcal{F}'^\bullet \in \mathrm{Coh}_{n,S}^{\underline{d}_2}, \mathcal{T}'^\bullet \in \mathrm{Coh}_{0,S}^{\underline{d}_1} \big\rangle.$$

The forgetful maps give rise to a correspondence

$$\mathrm{Coh}_{n,S}^{\underline{d}_1+\underline{d}_2} \xleftarrow{\mathrm{pr}_{\mathrm{big}}} \mathrm{Hecke}_n^{\underline{d}_1,\underline{d}_2} \xrightarrow{\mathrm{pr}_{\mathrm{small}} \times \mathrm{quot}} \mathrm{Coh}_{n,S}^{\underline{d}_2} \times \mathrm{Coh}_{0,S}^{\underline{d}_1}.$$

DEFINITION 2.4. — The *generalized Hecke operator* $H_n^{\underline{d}_1,\underline{d}_2}$ is defined as

$$H_n^{\underline{d}_1,\underline{d}_2} : D^b(\mathrm{Coh}_{n,S}^{\underline{d}}) \longrightarrow D^b(\mathrm{Coh}_{n,S}^{\underline{d}_2} \times \mathrm{Coh}_{0,S}^{\underline{d}_1}),$$

$$\mathsf{K} \longmapsto H_n^{\underline{d}_1,\underline{d}_2}\mathsf{K} := \mathbf{R}(pr_{\mathrm{small}} \times \mathrm{quot})_! \circ \mathrm{pr}_{\mathrm{big}}^* \mathsf{K}.$$

To define operators on parabolic vector bundles which correspond to the action of the Iwahori-Hecke algebra on functions on $\mathrm{Bun}_{n,S}^{\underline{d}}(\mathbb{F}_q)$ we have to forget the scalar automorphisms of our sheaves as follows:

Let $\underline{\epsilon}$ be a parabolic degree satisfying $(0,\dots,0) < \underline{\epsilon} \leq (1,\dots,1)$. (Here $\underline{\epsilon} \leq (1,\dots,1)$ is a short hand for the condition that for any torsion sheaf $\mathcal{T}^\bullet$ of degree $\underline{\epsilon}$ we have $\deg(\mathcal{T}^{(i,p)+(j,q)}) \leq 1$ for all $i,j \in \mathbb{Z}$, $p,q \in S$. In case that $\epsilon^{(0,p)} = 1$ this is equivalent to $\epsilon^{(i,p)} \leq 1$ for all i,p, but if $\epsilon^{(0,p)} = 0$ we add the condition $\epsilon^{(i,p)} + \epsilon^{(j,q)} \leq 1$ for $p \neq q$.)

Note that on every non-trivial parabolic sheaf we have a free action of scalar automorphisms. In the language of stacks this means that $\mathbb{G}_m$ acts freely by 2-automorphisms on $\mathrm{Coh}_{0,S}^{\underline{\epsilon}}$. We can quotient out these automorphisms (see [1]) and we denote the quotient stack by

$$\overline{\mathrm{Coh}}_{0,S}^{\underline{\epsilon}} := \mathrm{Coh}_{0,S}^{\underline{\epsilon}} / \text{diagonal } \mathbb{G}_m\text{-automorphisms}.$$

In our situation this stack can also be defined as follows: choose (i_0,p_0) with $\epsilon_{i_0,p_0} = 1$ then we have

$$\overline{\mathrm{Coh}}_{0,S}^{\underline{\epsilon}} = \big\langle \mathcal{T}^\bullet, \phi : k \xrightarrow{\sim} H^0(C, \mathcal{T}^{(i_0,p_0)}) \mid \mathcal{T}^\bullet \in \mathrm{Coh}_{0,S}^{\underline{\epsilon}} \big\rangle.$$

The morphism $\mathrm{Coh}_{0,S}^{\underline{\epsilon}} \to \overline{\mathrm{Coh}}_{0,S}^{\underline{\epsilon}}$ is given by

$$\mathcal{T}^\bullet \longmapsto \big(\mathcal{T}^\bullet \otimes (\mathcal{T}^{(i_0,p_0)})^{-1}, \phi : k \xrightarrow{\sim} H^0(C, \mathcal{T}^{(i_0,p_0)} \otimes (\mathcal{T}^{(i_0,p_0)})^{-1})\big).$$

For different choices of (i_0,p_0) the resulting stacks are canonically isomorphic (tensor with $(\mathcal{T}^{(i_1,p_1)})^{-1}$).

The morphism supp : $\mathrm{Coh}_{0,S}^{\underline{\epsilon}} \to C$ which maps a sheaf $\mathcal{T}^\bullet$ to its support (for our choice of $\underline{\epsilon}$ all $\mathcal{T}^{(i,p)}$ are either zero or concentrated in a single point which does not depend on (i,p)), factors through a morphism supp : $\overline{\mathrm{Coh}}_{0,S}^{\underline{\epsilon}} \to C$.

We define more Hecke correspondences:

$$\begin{tikzcd}[column sep=small]
 & & \left\langle \mathcal{E}'^\bullet \subset \mathcal{E}^\bullet \;\middle|\; \begin{array}{l} \mathcal{E}^\bullet \in \mathrm{Bun}_{n,S}^{\underline{d}} \\ \mathcal{E}'^\bullet \in \mathrm{Bun}_{n,S}^{\underline{d}-\underline{\epsilon}} \end{array} \right\rangle \arrow[dll, "\mathrm{pr}_{\mathrm{big}}"'] \arrow[d, "\mathrm{pr}_{\mathrm{small}} \times \mathrm{quot}"'] \arrow[drr, "\mathrm{pr}_{\mathrm{small}} \times \mathrm{pr}_C"] & & \\
\mathrm{Bun}_{n,S}^{\underline{d}} & & \mathrm{Bun}_{n,S}^{\underline{d}-\underline{\epsilon}} \times \overline{\mathrm{Coh}}_{0,S}^{\underline{\epsilon}} \arrow[rr, "\mathrm{id} \times \mathrm{supp}"] & & \mathrm{Bun}_{n,S}^{\underline{d}-\underline{\epsilon}} \times C.
\end{tikzcd}$$

DEFINITION 2.5. — The *Hecke operator* $H^{\underline{\epsilon}}$ is defined by

$$H^{\underline{\epsilon}} \colon D^b(\mathrm{Bun}_{n,S}^{\underline{d}}) \longrightarrow D^b(\mathrm{Bun}_{n,S}^{\underline{d}-\underline{\epsilon}} \times \overline{\mathrm{Coh}}_{0,S}^{\underline{\epsilon}}),$$

$$\mathsf{K} \longmapsto H^{\underline{\epsilon}}\mathsf{K} := \mathbf{R}(\mathrm{pr}_{\mathrm{small}} \times \mathrm{quot})_! \circ \mathrm{pr}_{\mathrm{big}}^* \mathsf{K}.$$

For $\underline{\epsilon} = (1,\dots,1) =: \underline{1}$ we set

$$H_C^{\underline{1}} \colon D^b(\mathrm{Bun}_{n,S}^{\underline{d}}) \longrightarrow D^b(\mathrm{Bun}_{n,S}^{\underline{d}-\underline{1}} \times C),$$

$$\mathsf{K} \longmapsto H_C^{\underline{1}}\mathsf{K} := \mathbf{R}(\mathrm{pr}_{\mathrm{small}} \times \mathrm{pr}_C)_! \circ \mathrm{pr}_{\mathrm{big}}^* \mathsf{K}.$$

Finally note that the sheaf $\mathcal{L}_{\mathsf{E}}^{\underline{1}}$ descends to a sheaf $\overline{\mathcal{L}}_{\mathsf{E}}^{\underline{1}}$ on $\overline{\mathrm{Coh}}_{0,S}^{\underline{1}}$. Denote by $j_C \colon C - S \hookrightarrow C$ the inclusion. We will prove the following:

THEOREM 2.5. — *Let* E *be an irreducible local system on the curve* $C - S$ *with indecomposable unipotent ramification at* S *and assume* $n = \mathrm{rank}(\mathsf{E}) \leq 3$. *Then*

1) $\mathsf{F}_{\mathsf{E}}^{\underline{n}} \cong \mathsf{F}_{\mathsf{E},!}^{\underline{n}}$.

2) $\mathsf{F}_{\mathsf{E}}^{\underline{n}}$ *descends to a nonzero perverse sheaf* $\mathsf{A}_{\mathsf{E}}^{\mathrm{good}}$ *on* $\mathrm{Bun}_{n,S}^{\mathrm{good}}$.

3) $\mathsf{A}_{\mathsf{E}}^{\mathrm{good}}$ *extends to a* Hecke eigensheaf A_{E} *on* $\mathrm{Bun}_{n,S}$, *i.e. there is a unique extension* A_{E} *of* $\mathsf{A}_{\mathsf{E}}^{\mathrm{good}}$ *to* $\mathrm{Bun}_{n,S}$ *such that*

$$H^{\underline{1}}\mathsf{A}_{\mathsf{E}} \cong \mathsf{A}_{\mathsf{E}} \boxtimes \overline{\mathcal{L}}_{\mathsf{E}}^{\underline{1}}[-n+1](-n+1),$$

$$H^{\underline{\epsilon}}\mathsf{A}_{\mathsf{E}} = 0 \quad \textit{for } \underline{0} < \underline{\epsilon} < \underline{1},$$

$$H_C^{\underline{1}}\mathsf{A}_{\mathsf{E}} \cong \mathsf{A}_{\mathsf{E}} \boxtimes j_{C,*}\mathsf{E}[-n+1](-n+1)$$

and the isomorphism

$$H_C^{\underline{1}} \circ H_C^{\underline{1}}\mathsf{A}_{\mathsf{E}} \cong \mathsf{A}_{\mathsf{E}} \boxtimes j_{C,*}\mathsf{E} \boxtimes j_{C,*}\mathsf{E}[-2n+2](-2n+2)$$

is S_2*-equivariant.*

Note that we have chosen to define the operators $H^{\mathfrak{E}}$ in such a way that they correspond to the usual Hecke operators for functions whereas in [11] the operators are normalized such that they preserve perversity. Furthermore we defined non zero $\mathsf{F}^n_{\mathsf{E}}$ and thus A_{E} only on the connected components of $\mathrm{Bun}^{\underline{d}}_{n,S}$ satisfying $d^{(i,p)} = d^{(0,p)} + i$ for all i,p, i.e. which parameterize vector bundles with full parabolic structure at S. The Hecke property for the operators $H^{\mathfrak{E}}$ justifies the definition of $\mathsf{F}^n_{\mathsf{E}}$ and A_{E} to be zero on the other components.

We will show (Corollary 4.5) that the theorem implies that the function $\mathrm{tr}_{\mathsf{A}_{\mathsf{E}}}$ is an eigenfunction for the action of the Iwahori-Hecke algebra. Indeed, by the example given above we have already seen that the points of $\overline{\mathrm{Coh}}^{\frac{1}{n}}_{0,S}$ give a set of generators for the Iwahori-Hecke algebra (the invertible element corresponding to $\otimes\mathcal{O}(\frac{1}{n}p)$ and the operators corresponding to the transpositions in S_n generate the algebra).

To emphasize which part of the proof depends on the assumption $n = \mathrm{rank}(\mathsf{E}) \leq 3$, we divide it into several parts. Proposition 6.7 proves the above theorem under the assumption that $\mathsf{F}^n_{\mathsf{E}} = \mathsf{F}^n_{\mathsf{E},!}$. In Section 8 we deduce this assumption from a vanishing theorem 7.1. All this works for general n, but the proof of this vanishing theorem given in Section 7 relies on the assumption $n \leq 3$.

3. Some properties of parabolic sheaves.

This section is an attempt to clarify the notion of parabolic sheaves. First we give a description of the isomorphism classes of parabolic torsion sheaves, then we prove some lemmata concerning homological algebra of parabolic sheaves. At the end of this section we give an explicit description of the moduli space of torsion sheaves on $\mathbb{A}^1$ with parabolic structure at 0. All these results are simple, but for completeness they are collected in this paragraph.

3.1. The structure of parabolic torsion sheaves.

The structure theorem for modules over principal ideal domains shows that any torsion sheaf on a curve C/k is a direct sum of sheaves of the form $\mathcal{O}/(\mathfrak{p}^d) =: \mathcal{O}_{dp}$ for some prime ideals $\mathfrak{p}$. A similar result holds for parabolic torsion sheaves. The constituents of a sheaf $\mathcal{T}^\bullet$ supported in $p \in S$ will be of the form (we only give the sequence $\cdots \to \mathcal{T}^{(i-1,p)} \xrightarrow{\phi^{(i,p)}} \mathcal{T}^{(i,p)} \to \cdots$)

$$\cdots \to \mathcal{O}/\mathfrak{p}^d \to \cdots \to \mathcal{O}/\mathfrak{p}^d \twoheadrightarrow \mathcal{O}/\mathfrak{p}^{(d-1)} \to \cdots \to \mathcal{O}/\mathfrak{p}^{(d-1)} \hookrightarrow \mathcal{O}/\mathfrak{p}^d \to \cdots.$$

More precisely these are isomorphic to $\mathcal{O}^\bullet_{kp/n}(\frac{i}{n}p) := \mathcal{O}^\bullet(\frac{i}{n}p)/\mathcal{O}^\bullet(\frac{i-k}{n}p)$ for some $0 \le k < i \in \mathbb{N}$ (in the sequence above $d = \lceil k/n \rceil$ is the smallest integer bigger than k/n). We call

$$\left|\deg(\mathcal{T}^\bullet)\right| := \sum_{i=0,\dots,n-1;\, p\in S} \deg(\mathcal{T}^{(i,p)})$$

the *total degree* of a parabolic torsion sheaf.

First we consider parabolic torsion sheaves supported at a single point $p \in S$ and we choose a local parameter π at p. Then the complete local ring at p is $\mathcal{O}^\wedge_{C,p} \cong k[[\pi]]$. To simplify the original argument G. Faltings remarked that in this situation a parabolic torsion sheaf $\mathcal{T}^\bullet$ supported at p is the same as the $\mathbb{Z}/n\mathbb{Z}$-graded $k[[T = \pi^{1/n}]]$ module $M := \bigoplus_{i=0}^{n-1} \mathcal{T}^{(i,p)}$ with multiplication by T given by $(\phi^{(i,p)})_{i=0,\dots,n}$. Here the structure of $k[[T]]$-modules implies that every cyclic submodule $k[[T]]/(T^\ell) \subset M$ of maximal length is a direct summand. Further any such submodule may be used to define a graded inclusion $k[[T]]/(T^\ell) \hookrightarrow M$ and any splitting of this inclusion as $k[[T]]$-module also gives rise to a graded splitting. Translating this back into a statement of parabolic sheaves we get:

LEMMA 3.1. — *Let $\mathcal{T}^\bullet$ be a parabolic torsion sheaf supported in $p \in S$, and let further $\mathcal{O}^\bullet_{kp/n}(\frac{i}{n}p) \xhookrightarrow{\psi} \mathcal{T}^\bullet$ be an inclusion such that the total degree $|\deg(\mathcal{O}^\bullet_{kp/n}(\frac{i}{n}p))|$ is maximal. Then there is a splitting of ψ.* □

From this lemma we get:

PROPOSITION 3.2 (Structure of parabolic torsion sheaves). — 1) *Any parabolic torsion sheaf is a direct sum of sheaves of the form*

$$\mathcal{O}_{jp/n}(\tfrac{i}{n}p)^\bullet = \mathcal{O}^\bullet(\tfrac{i}{n}p)/\mathcal{O}^\bullet(\tfrac{i-j}{n}p), \quad i,j \in \mathbb{N},\, p \in S$$

and sheaves supported outside S.

2) *Any parabolic torsion sheaf $\mathcal{T}^\bullet$ has a filtration $\mathcal{T}^\bullet_j \subset \mathcal{T}^\bullet_{j+1} \subset \cdots \subset \mathcal{T}^\bullet$ such that the filtration quotients $\mathcal{T}^\bullet_{j+1}/\mathcal{T}^\bullet_j$ are isomorphic to one of the following:*

(a) $\mathcal{T}^\bullet_{j+1}/\mathcal{T}^\bullet_j \cong (k(q))^\bullet$ *and* $q \notin S$;

(b) *there is a* $p_0 \in S$ *and* $0 \le i_0 < n$ *such that*

$$\mathcal{T}^{(i,p)}_{j+1}/\mathcal{T}^{(i,p)}_j = \begin{cases} k(p_0) & i \equiv i_0 \bmod n,\ p = p_0 \in S, \\ 0 & \textit{else}; \end{cases}$$

3) *any parabolic torsion sheaf* $\mathcal{T}^\bullet$ *of constant degree* $\deg \mathcal{T}^\bullet = (d, \dots, d)$ *has a filtration* $\mathcal{T}_1^\bullet \subset \cdots \subset \mathcal{T}_i^\bullet \subset \cdots \subset \mathcal{T}^\bullet$ *such that* $\deg(\mathcal{T}_i^\bullet) = (i, \dots, i)$.

Proof. — Since for any torsion sheaf $\mathcal{T}$ we have a canonical decomposition $\mathcal{T} = \bigoplus_{q \in \mathrm{supp}(\mathcal{T})} \mathcal{T} \otimes \mathcal{O}_{C,p}$, we may assume that $\mathcal{T}^\bullet$ is a parabolic torsion sheaf concentrated in a single point q, i.e. $\mathrm{supp}(\mathcal{T}^{(i,p)}) = q$ for any (i,p).

If $q \notin S$, we know that all the $\mathcal{T}^{(i,p)}$ are isomorphic because the functor $\otimes \mathcal{O}_C(S)$ is the identity functor on sheaves supported in $C - S$. Hence $\mathcal{T}^\bullet = (\mathcal{T}^{(0,p)})^\bullet$ and for torsion sheaves without extra structure the lemma holds.

For torsion sheaves supported in S the previous lemma implies 1) and the sheaves $\mathcal{O}_{kp/n}(\frac{j}{n}p)$ have a filtration satisfying 2).

To prove 3) by induction on d pick a summand $\mathcal{O}^\bullet_{jp/n}(\frac{i}{n}p)$ of $\mathcal{T}^\bullet$. Shifting $\mathcal{T}^\bullet$ we may assume that $i = 0$. If $j/n \geq 1$ then this has a submodule of degree $(1, \dots, 1)$ and we are done. Otherwise a complement $\mathcal{T}'^\bullet$ to this summand will have degree $\underline{d}'$ with

$$d'^{(k,q)} = \begin{cases} d - 1 & \text{for } 0 \leq k < j,\ q \in S, \\ d & \text{for } j \leq k < n,\ q \in S. \end{cases}$$

But then the map $\mathcal{T}'^{(n-1,p)} \to \mathcal{T}'^{(0,p)}(p)$ must have a non zero kernel. Take a summand $\mathcal{O}^\bullet_{j'p/n}(\frac{i'}{n}p)$ of $\mathcal{T}'^\bullet$ contributing to this kernel. Again unless $j' < n - j$ this contains a subsheaf of degree $(0, \dots, 0, 1, \dots, 1)$ where 1 is repeated $n-j$ times. Thus, doing one more induction we find a subsheaf $\mathcal{T}_1^\bullet$ of $\mathcal{T}^\bullet$ of degree $(1, \dots, 1)$. □

Finally note that for an arbitrary parabolic sheaf the torsion subsheaves are always a direct summand:

Remark 3.3. — Let $\mathcal{F}^\bullet$ be a parabolic sheaf on C/k. Then $\mathcal{F}^\bullet = \mathcal{E}^\bullet \oplus \mathcal{T}^\bullet$, where $\mathcal{T}^\bullet$ is a parabolic torsion sheaf and all $\mathcal{E}^{(i,p)}$ are torsion free.

Proof. — We know that $\mathcal{T}^\bullet := \mathrm{torsion}(\mathcal{F}^\bullet) \subset \mathcal{F}^\bullet$ is a parabolic torsion sheaf and $\mathcal{F}^{(0,S)} \cong \mathcal{T}^{(0,S)} \oplus \mathcal{E}^{(0,S)}$. And since the $\phi^{(i,p)}$ are isomorphisms over the generic fibre of C the images $\phi^{i,p}(\mathcal{E}^{(0,p)})$ can be used to define maximal torsion free subsheaves of $\mathcal{F}^{(i,p)}$, these define the desired decomposition. □

3.2. Homological algebra of parabolic sheaves.

LEMMA 3.4. — *For coherent parabolic sheaves on $C_{/k}$ the functors $\mathrm{Ext}^i_{\mathrm{para}}$ vanish for $i > 1$.*

Proof. — Let $\mathcal{F}^\bullet$ be a parabolic sheaf. We prove that $\mathrm{Ext}^i_{\mathrm{para}}(\,.\,,\mathcal{F}^\bullet) = 0$ for $i > 1$ by descending induction on the rank and degree of $\mathcal{F}^\bullet$.

For a line bundle $\mathcal{L}$ on C the functor $\mathrm{Hom}_{\mathrm{para}}(\,.\,,(\mathcal{L})^\bullet(\frac{i}{n}S))$ coincides with a Hom-functor on coherent sheaves, and for $\mathrm{Ext}^i_{\mathcal{O}_C}$ the lemma holds. By induction, we may therefore assume that $\mathcal{F}^\bullet$ is a parabolic torsion sheaf. By Proposition 3.2 giving the structure of parabolic torsion sheaves, we may further assume that $\mathcal{F}^\bullet$ is a quotient of a line bundle by a subsheaf, both of arbitrarily high degree, which establishes the claim. □

LEMMA 3.5. — *Let $\mathcal{T}^\bullet$ be a parabolic torsion sheaf and $\mathcal{E}^\bullet$ a parabolic vector bundle. Then:*

1) $\dim(\mathrm{Ext}^1_{\mathrm{para}}(\mathcal{T}^\bullet,\mathcal{E}^\bullet))$ *and* $\dim(\mathrm{Hom}_{\mathrm{para}}(\mathcal{E}^\bullet,\mathcal{T}^\bullet))$ *only depend on* $\mathrm{rank}(\mathcal{E}^\bullet)$, $\deg(\mathcal{E}^\bullet)$ *and* $\deg(\mathcal{T}^\bullet)$.

2) *More precisely, for* $\mathcal{T}^{(i,p)} = k(p_0)$ *if* $(i,p) = (i_0,p_0)$ *and* $\mathcal{T}^{(i,p)} = 0$ *otherwise, we have*

$$\dim\big(\mathrm{Ext}^1_{\mathrm{para}}(\mathcal{T}^\bullet,\mathcal{E}^\bullet)\big) = \deg\big(\mathcal{E}^{(i+1,p)}\big) - \deg\big(\mathcal{E}^{(i,p)}\big),$$
$$\dim\big(\mathrm{Hom}_{\mathrm{para}}(\mathcal{E}^\bullet,\mathcal{T}^\bullet)\big) = \deg\big(\mathcal{E}^{(i,p)}\big) - \deg\big(\mathcal{E}^{(i-1,p)}\big).$$

3) *If* $\deg(\mathcal{T}^\bullet) = (d)$ *is constant, we get*

$$\dim\big(\mathrm{Ext}^1_{\mathrm{para}}(\mathcal{T}^\bullet,\mathcal{E}^\bullet)\big) = d\cdot \mathrm{rank}(\mathcal{E}^\bullet) = \dim\big(\mathrm{Hom}_{\mathrm{para}}(\mathcal{E}^\bullet,\mathcal{T}^\bullet)\big).$$

Proof. — We give a proof of the statements on $\mathrm{Ext}^1_{\mathrm{para}}$, the case of homomorphisms is even simpler. Since $\mathcal{E}^\bullet$ is torsion free, $\mathrm{Hom}_{\mathrm{para}}(\mathcal{T}^\bullet,\mathcal{E}^\bullet)$ is zero. Thus for any exact sequence $0 \to \mathcal{T}'^\bullet \to \mathcal{T}^\bullet \to \mathcal{T}''^\bullet \to 0$ the sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{para}}(\mathcal{T}''^\bullet,\mathcal{E}^\bullet) \longrightarrow \mathrm{Ext}^1_{\mathrm{para}}(\mathcal{T}^\bullet,\mathcal{E}^\bullet) \longrightarrow \mathrm{Ext}^1_{\mathrm{para}}(\mathcal{T}'^\bullet,\mathcal{E}^\bullet) \to 0$$

is exact as well.

To prove the lemma, apply this remark to the filtration $\mathcal{T}^\bullet_i \subset \mathcal{T}^\bullet$ constructed in Proposition 3.2 1) and reduce to the case $\mathcal{T}^{(i,p)} = k(p_0)$ if $(i,p) = (i_0,p_0)$ and $\mathcal{T}^{(i,p)} = 0$ otherwise. We may shift $\mathcal{E}^\bullet,\mathcal{T}^\bullet$ and assume that $i_0 = 0$. Write $\mathcal{T} = (\mathcal{L})^\bullet/(\mathcal{L})^\bullet(-\frac{1}{n}p_0)$ for some line bundle $\mathcal{L}$ and for simplicity choose $\deg(\mathcal{L}) \ll 0$ such that $\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{L}, \mathcal{E}^{(i,p)}) = 0$ for all $p \in S$, $-1 \leq i \leq n$. Then

$$\begin{aligned}
\dim\big(\mathrm{Ext}^1_{\mathrm{para}}(\mathcal{T}^\bullet, \mathcal{E}^\bullet)\big) &= \chi\big(\mathbf{R}\,\mathrm{Hom}_{\mathrm{para}}(\mathcal{L}^\bullet(-\tfrac{1}{n}p_0), \mathcal{E}^\bullet)\big) \\
&\qquad\qquad - \chi\big(\mathbf{R}\,\mathrm{Hom}_{\mathrm{para}}(\mathcal{L}^\bullet, \mathcal{E}^\bullet)\big) \\
&= \chi\big(\mathbf{R}\,\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{L}, \mathcal{E}^{(1,p_0)})\big) \\
&\qquad\qquad - \chi\big(\mathbf{R}\,\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{L}, \mathcal{E}^{(0,p_0)})\big) \\
&= \dim\big(\mathrm{Hom}_{\mathcal{O}}(\mathcal{L}, \mathcal{E}^{(1,p_0)})\big) - \dim\big(\mathrm{Hom}_{\mathcal{O}}(\mathcal{L}, \mathcal{E}^{(0,p_0)})\big) \\
&= \deg(\mathcal{E}^{(1,p_0)}) - \deg(\mathcal{E}^{(0,p_0)}) \qquad \square
\end{aligned}$$

3.3. The moduli stack of parabolic torsion sheaves.

First let us consider the moduli stack of torsion sheaves on $\mathbb{A}^1$ with parabolic structure at $p = 0$ as an example:

This stack classifies sequences of torsion sheaves[5]:

$$\cdots \to \mathcal{T}^0 \xrightarrow{\phi^1} \mathcal{T}^1 \xrightarrow{\phi^2} \mathcal{T}^2 \xrightarrow{\phi^3} \cdots \xrightarrow{\phi^{n-1}} \mathcal{T}^{n-1} \xrightarrow{\phi^n} \mathcal{T}^0(p) \xrightarrow{\phi^1(p)} \cdots$$

with the property that the induced maps $\mathcal{T}_i \to \mathcal{T}_i(p)$ are the natural ones.

Recall that a single torsion sheaf $\mathcal{T}$ on $\mathbb{A}^1$ can be described by giving its vector space of global sections $H^0(\mathbb{A}^1, \mathcal{T})$ together with the endomorphism given by multiplication by the coordinate t of $\mathbb{A}^1 = \mathrm{Spec}(k[t])$. Hence we get a presentation of the moduli space of torsion sheaves of degree d on $\mathbb{A}^1$:

$$\mathrm{Coh}^d_{0,\mathbb{A}^1} \cong [\mathrm{Mat}_{d,d} \,/\, \mathrm{GL}_d],$$

where GL_d acts on $\mathrm{Mat}_{d,d}$ by conjugation. (Under this identification the support of a sheaf is given by the eigenvalues of the corresponding matrix, and the length of the indecomposable summands is given by the Jordan decomposition.)

For torsion sheaves with parabolic structure we can define a similar presentation as follows: The coordinate t induces isomorphisms $\mathcal{T}(p) \cong \mathcal{T}$ and under this identification the natural map $\mathcal{T}^i \to \mathcal{T}^i(p) \xrightarrow{\sim} \mathcal{T}^i$ is given by the multiplication by t. Thus for any collection $(\phi^i : k^{\oplus d_{i-1}} \to k^{\oplus d_i})_{i=1}^n$ we may define $\mathcal{T}^j$ by $(k^{\oplus d_j}, \phi^j \circ \phi^{j-1} \cdots \phi^1 \circ \phi^n \circ \cdots \circ \phi^{j+1})$ and with this definition the ϕ^i automatically define homomorphisms $\mathcal{T}^{i-1} \to \mathcal{T}^i$ of $\mathcal{O}_{\mathbb{A}^1}$-modules. This proves:

(5) I drop the upper index p since we have assumed that $S = \{p\} = \{0\}$.

LEMMA 3.6. — *There is an isomorphism*

$$\mathrm{Coh}_{0,\{p\}}^{d_0,\dots,d_{n-1}} \cong \big[\mathrm{Mat}_{d_1,d_0} \times \mathrm{Mat}_{d_2,d_1} \times \cdots \times \mathrm{Mat}_{d_0,d_{n-1}} \\ /(\mathrm{GL}_{d_0} \times \cdots \times \mathrm{GL}_{d_{n-1}})\big],$$

where an element $(g_0,\dots,g_{n-1}) \in \mathrm{GL}_{d_0} \times \cdots \times \mathrm{GL}_{d_{n-1}}$ *operates on* $(\phi^1,\dots,\phi^n) \in \mathrm{Mat}_{d_1,d_0} \times \mathrm{Mat}_{d_2,d_1} \times \cdots \times \mathrm{Mat}_{d_0,d_{n-1}}$ *as*

$$(g_0,\dots,g_{n-1}) \cdot (\phi^1,\dots,\phi^n) := (g_1\phi^1 g_0^{-1}, g_2\phi^2 g_1^{-1},\dots, g_0\phi^n g_{n-1}^{-1}). \qquad \square$$

COROLLARY 3.7. — *For any smooth curve* C *and any finite set* $S \subset C(k)$ *the stack* $\mathrm{Coh}_{0,S}^{\underline{d}}$ *is smooth. In case* $\underline{d}$ *is constant it is of dimension* 0.

Proof. — To show the lifting property for smoothness at a point $\mathcal{T}^\bullet \in \mathrm{Coh}_{0,S}^{\underline{d}}$, we only need to consider sheaves on $\mathrm{Spec}(\Pi_{q\in \mathrm{supp}(\mathcal{T})}\widehat{\mathcal{O}}_{C,q})$. But for a smooth curve we know that $\widehat{\mathcal{O}}_{C,q} \cong k[[t]] \cong \widehat{\mathcal{O}}_{\mathbb{A}^1,0}$, and therefore it is sufficient to prove the corollary in case $C = \mathbb{A}^1$ and $S = \{0\}$, which is proven in the previous lemma. $\square$

In case one does not want to consider deformations of parabolic sheaves one could use the above lemma and the fundamental diagram to get smooth presentations of the stacks $\mathrm{Coh}_{n,S}^{\underline{d}}$:

COROLLARY 3.8. — *For any smooth curve* C *and any finite set* $S \subset C(k)$ *the stacks* $\mathrm{Coh}_{n,S}^{\underline{d}}$ *are smooth algebraic stacks.* $\square$

4. Properties of the Whittaker sheaf $\mathcal{L}_{\mathsf{E}}^{\underline{d}}$.

Our main goal in this section is to prove a Hecke property of the sheaf $\mathcal{L}_{\mathsf{E}}^{\underline{d}}$ defined in 2.2 (Proposition 4.8). In the case of unramified local systems Laumon [19] proved this in two steps: first he introduced a small resolution of the stack of torsion sheaves, defined as the stack classifying torsion sheaves, together with a full flag of subsheaves. Thereby he obtained a geometric description of the Whittaker sheaf, which he then used to prove the Hecke property.

Translating this into our situation we encounter two problems. The first one is that $\mathcal{L}_{\mathsf{E}}^{\underline{1}}$ is already a complex of sheaves. The second problem is that the analogue of Laumon's resolution is not small in the case of parabolic torsion sheaves.

Since $\mathcal{L}^d_{\mathsf{E}}$ is a perverse sheaf on the moduli stack of parabolic torsion sheaves and most of the questions are local in the étale topology we will often be able to reduce to the case that our curve is $\mathbb{A}^1$ and our local system is ramified only at the point 0. Therefore our first aim is to calculate $\mathcal{L}^1_{\mathsf{E}}$ in this case. After translating these results into the general situation we then proceed with Laumon's strategy as described above. Here we simultaneously prove that the Hecke property of $\mathcal{L}^d_{\mathsf{E}}$ holds and that we can give a geometric description (Lemma 4.10) of $\mathcal{L}^d_{\mathsf{E}}$.

4.1. Calculation of the sheaf $j_{!*}\mathsf{E}$ on $\mathrm{Coh}^1_{0,\mathbb{A}^1,0}$.

Consider the case $C = \mathbb{A}^1$ and $S = \{0\}$. Let E_n be the n-dimensional local system on $\mathbb{G}_m$, ramified at 0, such that the ramification group acts unipotently and indecomposably – i.e. the invariants under the ramification group are 1-dimensional – constructed as follows: we have

$$\mathrm{Ext}^1_{\mathbb{G}_m}\big(\mathbb{Q}_\ell(-1),\mathbb{Q}_\ell\big) = H^1\big(\mathbb{G}_m,\mathbb{Q}_\ell(1)\big) = H^1(\mathbb{G}_m,\mathbb{Q}_\ell)(1) = \mathbb{Q}_\ell$$

and therefore there is a canonical nontrivial extension E_2 of the sheaf $\mathbb{Q}_\ell(-1)$ by the constant sheaf $\mathbb{Q}_\ell$. The long exact cohomology sequence corresponding to this extension gives $H^1(\mathbb{G}_m,\mathsf{E}_2) = \mathbb{Q}_\ell(-2)$, thus we can repeat this argument to define E_n, filtered by $\mathbb{Q}_\ell = \mathsf{E}_1 \subset \mathsf{E}_2 \subset \cdots \subset \mathsf{E}_{n-1} \subset \mathsf{E}_n$ with subquotients $\mathsf{E}_i/\mathsf{E}_{i-1} \cong \mathbb{Q}_\ell(-i+1)$. Alternatively we could describe this as $\mathrm{Sym}^{n-1}(\mathsf{E}_2)$.

Since $\mathrm{Coh}^1_{0,\mathbb{G}_m}$ – the stack of torsion sheaves of length 1 on $\mathbb{G}_m$ – is isomorphic to $[\mathbb{G}_m/\mathbb{G}_m]$ for the trivial action of $\mathbb{G}_m$ on $\mathbb{G}_m$, the sheaf E_n descends to a sheaf on $\mathrm{Coh}^1_{0,\mathbb{G}_m}$ which we denote again by E_n.

We want to calculate the middle extension $j_{!*}\mathsf{E}_n$ with respect to the inclusion $j\colon \mathrm{Coh}^1_{0,\mathbb{G}_m} \to \mathrm{Coh}^{1,\dots,1}_{0,\mathbb{A}^1,\{0\}}$ (where $\mathrm{Coh}^{1,\dots,1}_{0,\mathbb{A}^1,\{0\}}$ is the stack of torsion sheaves with k-step parabolic structure – in this section we allow $n \neq k$). Because of the theorem on smooth base change it is sufficient to do this on a smooth representation of these stacks. We will use the presentation $\mathbb{A}^k \xrightarrow{\pi} \mathrm{Coh}_{0,\mathbb{A}^1,\{0\}}$ given by the quotient construction in Lemma 3.6. This fits into a cartesian square

$$\begin{tikzcd}
 & \mathbb{G}_m^k \arrow[dl, "m"'] \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^k \arrow[d] \\
\mathbb{G}_m \arrow[r, "/\mathbb{G}_m\text{-trivial}"'] & \mathrm{Coh}^1_{0,\mathbb{G}_m} \arrow[r, "j"] & \mathrm{Coh}^{1,\dots,1}_{0,\mathbb{A}^1,0}
\end{tikzcd}$$

where m is the multiplication map. So we are left calculating $j_{!*}m^*\mathsf{E}_n$ on $\mathbb{A}^k$ and to simplify notations we will often denote $m^*\mathsf{E}_n$ again by E_n.

We use the standard notations $D_i := \{x_i = 0\} \subset \mathbb{A}^k$ and for a subset $I \subset \{1, \dots, k\}$ define $D_I := \bigcap_{i\in I} D_i$. Finally denote by $U_i := \mathbb{A}^k - \bigcup_{\#I=i} D_I$. This stratification of the complement of $\mathbb{G}_m^k$ gives rise to open immersions $j_i : U_i \to U_{i+1}$:

$$\mathbb{G}_m{}^k = U_1 \xhookrightarrow{j_1} U_2 \xhookrightarrow{j_2} \cdots \xhookrightarrow{j_{k-1}} U_k = \mathbb{A}^k - (0, \dots, 0) \xhookrightarrow{j_k} \mathbb{A}^k$$

And $j_{!*}m^*\mathsf{E}_n = \tau^{<k}\mathbf{R}j_{k,*}\tau^{<k-1}\mathbf{R}j_{k-1}\cdots\tau^{<1}\mathbf{R}j_{1,*}m^*\mathsf{E}_n$ (this is a definition in Intersection Homology II [14] and a proposition (2.1.11) in Faisceaux Pervers [4]).

For $k = 1$ we have $R^pj_*\mathsf{E}_{n|0} = \mathbb{Q}_\ell$ if $p = 0$ and $R^pj_*\mathsf{E}_{n|0} = \mathbb{Q}_\ell(-n)$ if $p = 1$ on $\mathbb{A}^1$.

Therefore on $\mathbb{A}^k$ the stalk at 0 is

$$R^pj_*\mathsf{E}_{n|0} \xrightarrow[\sim]{(*)} H^p(\mathbb{A}^k_{\overline{\mathbb{F}}_q}, \mathbf{R}j_*\mathsf{E}_n) = H^p(\mathbb{G}^k_{m,\overline{\mathbb{F}}_q}, \mathsf{E}_n)$$

and this isomorphism is compatible with the action of the Galois group. The isomorphism (*) holds, because E_n is an extension of constant sheaves, for which the two cohomology groups are canonically isomorphic.

To calculate the latter cohomology group, we can factor m into an isomorphism $\mathbb{G}_m^k \to \mathbb{G}_m^k$, $(a_i) \mapsto (\Pi a_i, a_2, \dots, a_n)$, followed by the projection onto the first factor, to obtain:

$$\begin{aligned} H^*(\mathbb{G}_m^k, m^*\mathsf{E}_n) &\cong H^*(\mathbb{G}_m^k, \mathrm{pr}_1^*\,\mathsf{E}_n) \cong H^*(\mathbb{G}_m, \mathsf{E}_n) \otimes H^*(\mathbb{G}_m^{k-1}, \mathbb{Q}_\ell) \\ &= \begin{cases} \mathbb{Q}_\ell & \text{if } * = 0, \\ \mathbb{Q}_\ell(-1)^{\oplus k-1} \oplus \mathbb{Q}_\ell(-n) & \text{if } * = 1, \\ \text{etc.} \end{cases} \end{aligned}$$

Analogously we get a formula for the stalk of $R^pj_*\mathsf{E}_n$ at a point lying on exactly r of the divisors:

$$R^*j_*\mathsf{E}_{n|D_{(i_1,\dots,i_r)} - \bigcup_{j\notin \underline{i}} D_{(i_1,\dots,i_r,j)}} \cong H^*(\mathbb{G}_m, \mathsf{E}_n) \otimes H^*(\mathbb{G}_m^{r-1} \times \mathbb{A}^{k-r}, \mathbb{Q}_\ell).$$

If the terms of weight $\geq 2n$ did not appear, then the truncation functors $\tau^{<i}$ used in the definition of $j_{!*}\mathsf{E}_n$ would be trivial and $\mathbf{R}j_*\mathsf{E}_n$ would "be an irreducible perverse sheaf". But these terms do disappear if we pass to the inductive limit of all the $\mathsf{E}_n \hookrightarrow \mathsf{E}_{n+1} \hookrightarrow \cdots$. Therefore define

$$\mathsf{E}_\infty := \varinjlim \mathsf{E}_n.$$

PROPOSITION 4.1. — *For $n \geq k$ there is an exact triangle of complexes on $\mathbb{A}^k$:*

$$\to j_{!*}\mathsf{E}_n \longrightarrow \mathbf{R}j_*\mathsf{E}_\infty \longrightarrow j_!\mathsf{E}_\infty(-n) \xrightarrow{[1]} .$$

Proof (inductively calculating $\tau^{<i}\mathbf{R}j_{i,*}$). — We use the shorthand $j_{i\dots 1} := j_i \circ \cdots \circ j_1$. We start with the exact sequence of sheaves on $\mathbb{G}_m^k$:

$$0 \to \mathsf{E}_n \longrightarrow \mathsf{E}_\infty \longrightarrow \mathsf{E}_\infty(-n) \to 0.$$

Applying $j_{1,!*} = \tau^{<1}\mathbf{R}j_{1,*} = j_{1,*}$ we get on U_2

$$0 \to j_{1,*}\mathsf{E}_n \longrightarrow j_{1,*}\mathsf{E}_\infty \longrightarrow j_{1,!}\mathsf{E}_\infty(-n) \to 0.$$

Using the previous calculation $j_{1,*}\mathsf{E}_\infty = \mathbf{R}j_{1,*}\mathsf{E}_\infty$ and $j_{1,*} = j_{1,!*}$ we get

$$\to j_{1,!*}\mathsf{E}_n \longrightarrow \mathbf{R}j_{1,*}\mathsf{E}_\infty \longrightarrow j_{1,!}\mathsf{E}_\infty(-n) \xrightarrow{[1]} .$$

Now $j_{i\dots 1,!}\mathsf{E}_\infty(-n)$ is an extension of $j_{i\dots 1,!}\mathbb{Q}_\ell(-n-r)$ with $r \geq 0$, thus to do the induction we will need to calculate $\mathbf{R}j_{i+1,*}j_{i\dots 1,!}\mathbb{Q}_\ell$. We use the resolution of $j_{1\dots k,!}\mathbb{Q}_\ell$ on $\mathbb{A}^k$:

$$\begin{aligned} 0 \to j_{k\dots 1,!}\mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow \bigoplus_{j=1}^{k} \mathbb{Q}_{\ell|D_j} \\ \longrightarrow \bigoplus_{\substack{I\subset\{1,\dots,k\}\\ |I|=2}} \mathbb{Q}_{\ell|D_I} \to \cdots \to \mathbb{Q}_{\ell|D_{1,\dots,k}} \to 0 \end{aligned}$$

where $\mathbb{Q}_{\ell|D_I}$ is the constant sheaf on D_I. (We will often use this shorthand: for a closed subscheme $Z \overset{i}{\hookrightarrow} X$ and a sheaf K on X we write $\mathsf{K}_{|Z} := i_*i^*\mathsf{K}$.)

Restricting this resolution to U_{i+1} all terms $\mathbb{Q}_{\ell|D_I}$ with $|I| > i$ disappear and thus on U_{i+1} we have a resolution

$$0 \to j_{i\dots 1,!}\mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow \bigoplus_{j=1}^{k} \mathbb{Q}_{\ell|D_j} \to \cdots \to \bigoplus_{\substack{I\subset\{1,\dots,k\}\\ |I|=i}} \mathbb{Q}_{\ell|D_I} \to 0.$$

LEMMA 4.2. — *For any $m > i \geq 0$ the complex $j_{m\dots i+1,!*}j_{i\dots 1,!}\mathbb{Q}_\ell$ is quasi-isomorphic to*

$$j_{m\dots i+1,*}\Big(0 \to \mathbb{Q}_\ell \to \bigoplus_{j=1}^{k} \mathbb{Q}_{\ell|D_j} \to \cdots \to \bigoplus_{\substack{I\subset\{1,\dots,k\}\\ |I|=i}} \mathbb{Q}_{\ell|D_I} \to 0\Big).$$

Proof. — For $i = 0$ there is nothing to prove, so we may assume $i > 0$. Note that

$$R^p j_{i+1,*}(\mathbb{Q}_{\ell|D_I}) = \begin{cases} \mathbb{Q}_{\ell|D_I} & \text{if } p = 0, \\ \bigoplus_{I' \supset I,\, |I'|=k+1} \mathbb{Q}_\ell(k-i-1)_{|D_{I'}} & \text{if } p = 2(i+1-|I|)-1, \\ 0 & \text{otherwise.} \end{cases}$$

because j_{i+1} adds a smooth boundary of codimension $i+1-|I|$ to D_I. Therefore looking at the spectral sequence calculating $\mathbf{R}j_{i+1,*}j_{i\dots 1,!}\mathbb{Q}_\ell$ via our resolution of $j_{i\dots 1,!}\mathbb{Q}$ we see that the only terms appearing in cohomological dimension $< i+1$ are as claimed. This proves the lemma for $m = i+1$. Inductively we may apply the same argument for m to see that in our spectral sequence the cohomology in degrees p with $i+1 \leq p < 2(m-i)-1+i = 2m-(i+1)$ vanishes. □

We will use the last statement of the above proof again:

LEMMA 4.3. — *For $m > i+1$ we have*

$$j_{m,!*}(j_{m-1\dots i+1,!*}j_{i\dots 1,!}\mathbb{Q}_\ell) \cong \tau^{<2m-(i+1)}\mathbf{R}j_{m,*}(j_{m-1\dots i+1,!*}j_{i\dots 1,!}\mathbb{Q}_\ell) \quad \square$$

To finish the proof of Proposition 4.1, we still have to calculate

$$\tau^{<i+1}\mathbf{R}j_{i+1,*}\Big(\to j_{i\dots 1,!*}\mathsf{E}_n \to \mathbf{R}j_{i\dots 1,*}\mathsf{E}_\infty \to j_{i\dots 1,!}\mathsf{E}_\infty(-n) \xrightarrow{[1]} \Big).$$

By our calculation (Lemma 4.2) of $j_{i+1,!*}j_{i\dots 1,!}\mathbb{Q}_\ell = \tau^{<i+1}\mathbf{R}j_{i+1,*}j_{i\dots 1,!}\mathbb{Q}_\ell$ we know that

$$R^p j_{i+1,*}j_{i\dots 1,!}\mathsf{E}_\infty(-n) = \begin{cases} j_{i+1\dots 1,!}\mathsf{E}_\infty(-n) & \text{if } p = 0, \\ 0 & \text{if } 0 < p < i, \\ \text{of weight } \geq 2n & \text{if } p = i. \end{cases}$$

Considering the long exact cohomology sequence for

$$\mathbf{R}j_{i+1,*}j_{i\dots 1,!*}\mathsf{E}_n \longrightarrow \mathbf{R}j_{i+1\dots 1,*}\mathsf{E}_\infty \longrightarrow \mathbf{R}j_{i+1,*}j_{i\dots 1,!}\mathsf{E}_\infty(-n) \to$$

this calculation implies that the map

$$R^i j_{i+1\dots 1,*}\mathsf{E}_\infty \longrightarrow R^i j_{i+1,*}j_{i\dots 1,!}\mathsf{E}_\infty(-n)$$

must be zero because the weights of the two sheaves are distinct (here we use $n \geq k \geq i+1$). Thus we have proven the proposition. □

Later we will need the following description of $(j_{!*}\mathsf{E}_\infty)_{|D_I}$, which is implicit in the above:

LEMMA 4.4. — *On $U_2 = \mathbb{A}^k - (\bigcup_{i\neq j} D_{ij})$ the morphism*

$$j_{1,*}\mathsf{E}_\infty \longrightarrow i_{D_1,*}\, i^*_{D_1} j_{1,*}\mathsf{E}_\infty = i_{D_1,*}(\mathbb{Q}_{\ell|D_1\cap U_2})$$

induces an isomorphism $(\mathbf{R}j_*\mathsf{E}_\infty)_{|D_1} \xrightarrow{\sim} \mathbf{R}j_{k\dots 2,*}\mathbb{Q}_{\ell|D_1\cap U_2}$. *Therefore if $n \geq k$ we have*

$$(j_{!*}\mathsf{E}_n)_{|D_1} \xrightarrow{\sim} \mathbf{R}j_{k\dots 2,*}\mathbb{Q}_{\ell|D_1\cap U_2},$$

and more generally

$$(j_{!*}\mathsf{E}_n)_{|D_{1\dots\ell}} \xrightarrow{\sim} \mathbf{R}j_{k\dots\ell+1,*}\big((\mathbf{R}j_{\ell\dots 2,*}\mathbb{Q}_{\ell|D_1\cap U_2})_{|D_{1\dots\ell}\cap U_{\ell+1}}\big).$$

Proof. — For the first statement consider $j_{D_1} : U_1 \hookrightarrow \mathbb{A}^k - \bigcup_{i>1} D_i$ and $j'_1 : \mathbb{A}^k - \bigcup_{i>1} D_i \hookrightarrow U_2$. This induces an exact sequence

$$0 \to j_{D_1,!}\mathsf{E}_\infty \longrightarrow j_{D_1,*}\mathsf{E}_\infty \longrightarrow i_{D_1,*}\mathbb{Q}_{\ell|D_1} \to 0.$$

We therefore have to show that $\big(\mathbf{R}(j_{k\dots 2}\circ j'_1)_* j_{D_1,!}\mathsf{E}_\infty\big)_{|D_1} = 0$. Again we first show that the stalk at 0 vanishes. We know that

$$R^p(j_{k\dots 2}\circ j'_1)_* j_{D_1,!}\mathsf{E}_{\infty|0} = H^p\Big(\mathbb{A}^k - \bigcup_{i>1} D_i, j_{D_1,!}\mathsf{E}_\infty\Big),$$

because this is true for the other two sheaves in the sequence above (for the middle term we proved this to calculate $\mathbf{R}j_*\mathsf{E}_\infty$). The cartesian diagram

$$\begin{tikzcd}
\mathbb{G}_m^k \arrow[r, hook, "j_{D_1}"] \arrow[d, "(a_i)\mapsto(\Pi a_i, a_2,\dots a_k)"', "\cong"] & \mathbb{A}^1\times\mathbb{G}_m^{k-1} \arrow[d, "\cong"', "(a_i)\mapsto(\Pi a_i, a_2,\dots a_k)"] \\
\mathbb{G}_m^k \arrow[r, hook, "j_{D_1}"] \arrow[d, "\mathrm{pr}_1"'] & \mathbb{A}^1\times\mathbb{G}_m^{k-1} \arrow[d, "\mathrm{pr}_1"] \\
\mathbb{G}_m \arrow[r, hook, "j_{\mathbb{G}_m}"] & \mathbb{A}^1
\end{tikzcd}$$

shows that

$$\begin{aligned} H^*(\mathbb{A}^1\times\mathbb{G}_m^{k-1}, j_{D_1,!}\mathsf{E}_\infty) &= H^*(\mathbb{A}^1\times\mathbb{G}_m^{k-1}, \mathrm{pr}_1^* j_{\mathbb{G}_m,!}\mathsf{E}_\infty) \\ &= H^*(\mathbb{A}^1, j_{\mathbb{G}_m,!}\mathsf{E}_\infty)\otimes H^*(\mathbb{G}_m^{k-1}, \overline{\mathbb{Q}}_\ell) = 0, \end{aligned}$$

because $H^*(\mathbb{A}^1, j_{\mathbb{G}_m,!}\mathsf{E}_\infty) = 0$ (we know that $H^*(\mathbb{A}^1, j_{\mathbb{G}_m,!}\overline{\mathbb{Q}}_\ell) = 0$ and that E_∞ is an extension of constant sheaves).

Analogously we get that the fibre of the above complex at a point lying on D_1 and exactly c other divisors is isomorphic to $H^*(\mathbb{A}^1, j_{\mathbb{G}_m,!}\mathsf{E}_\infty)\otimes H^*(\mathbb{G}_m^{k-c-1}, \mathbb{Q}_\ell) = 0$. So we have proven, the first part of the lemma. The second part is then immediate using Proposition 4.1.

The description of $(j_{!*}\mathsf{E}_n)_{|D_I}$ follows, because by the second part we only have to prove that

$$\mathbf{R}j_{k\dots 2,*}(\mathbb{Q}_{\ell|D_1\cap U_2})_{|D_{1\dots\ell}} \cong \mathbf{R}j_{k\dots\ell+1,*}\big(\mathbf{R}j_{\ell\dots 2_*}(\mathbb{Q}_{\ell|D_1\cap U_2})_{|D_{1\dots\ell}\cap U_{\ell+1}}\big).$$

Now $D_1 \cap U_2 \cong \mathbb{G}_m^{k-1}$ and thus this is an easy statement for the direct image the constant sheaf on $\mathbb{G}_m^{k-1}$ under $j: \mathbb{G}_m^{k-1} \hookrightarrow \mathbb{A}^{k-1}$. □

COROLLARY 4.5. — *For an arbitrary curve C, let E be a rank n local system with indecomposable unipotent ramification at a finite set of points $S \subset C$. Let $I \subset \{1,\dots,k\}$ and let $D_{I,p} \subset \mathrm{Coh}^{\frac{1}{0},S}$ be the substack defined by $(\phi^{i,p}=0)_{i\in I}$ (i.e. for $C=\mathbb{A}^1$ this is the substack defined by $D_I \subset \mathbb{A}^k$) and denote by $D^0_{I,p}$ the substack defined by $\phi^{(j,p)} \neq 0$ for $j \notin I$. Finally let* $\mathrm{pr}: \mathrm{Coh}^{\frac{1}{0},S} \to \mathrm{Coh}^{\frac{1}{0}}$, $\mathcal{T}^\bullet \mapsto \mathcal{T}^{(0,S)}$, *be the projection. Then the following holds:*

1) *For $0 < |I| < k$ we have $\mathbf{R}\,\mathrm{pr}_!(j_{!*}\mathsf{E}_{|D_{I,p}}) = 0$.*
2) *There is a canonical isomorphism $\mathbf{R}\,\mathrm{pr}_!\, j_{!*}\mathsf{E} \cong \bar{\jmath}_{!*}\mathsf{E}$, where $\bar{\jmath}: C - S \to \mathrm{Coh}^{\frac{1}{0}}$ is the inclusion.*
3) *$\mathbf{R}\,\mathrm{pr}_!(j_{!*}\mathsf{E}_{|D^0_{I,p}}) \cong (\bar{\jmath}_{!*}\mathsf{E})_{|\mathrm{pr}(D_{I,p})}[|I|-1]$ for any I and again the isomorphism is canonical.*

Proof. — In the special case $(C,S) = (\mathbb{A}^1,\{0\})$ the corollary follows from Lemma 4.4 which shows that

$$j_{!*}\mathsf{E}_{n|D_I^0} \cong (j_*\mathsf{E}_n)_{|p} \otimes H^*(\mathbb{G}_m^{\times|I|-1}, \overline{\mathbb{Q}}_\ell) \text{ is constant and} \tag{4.1}$$

$$j_{!*}\mathsf{E}_{n|D_I} = \mathbf{R}j_{D^0\hookrightarrow D,*}(j_{!*}\mathsf{E}_{n|D_I^0}). \tag{4.2}$$

Combining these formulas we see that on D_I, we have

$$H^*_c\big(D_I, (j_{!*}\mathsf{E}_n)_{|D_I}\big) = 0$$

for $0 < |I| < k$. This follows using the Künneth formula and the fact that for $j: \mathbb{G}_m \to \mathbb{A}^1$ we have $H^*_c(\mathbb{A}^1, \mathbf{R}j_*\overline{\mathbb{Q}}_\ell) = 0$.

Now to prove the first part of the corollary we can base change the map pr by the map $\mathbb{A}^1 \to \mathrm{Coh}^{\frac{1}{0},\mathbb{A}^1}$ and restrict this to the fibre above 0:

$$\begin{tikzcd} & D_I \arrow[d] \\ \mathrm{Coh}^{\frac{1}{0},\{0\},\mathbb{A}^1} \arrow[d, "\mathrm{pr}'"'] & {[D_I/(\mathbb{G}_m^{k-1})]} \arrow[l, "\pi_0"'] \arrow[d, "\mathrm{pr}'"] \\ \mathrm{Coh}^{\frac{1}{0},\mathbb{A}^1} & \mathrm{Spec}(\mathbb{F}_q). \arrow[l, "0"'] \end{tikzcd}$$

We have to show that $H_c^*([D_I/(\mathbb{G}_m^{k-1})], \pi_0^* j_{!*}\mathsf{E}_n) = 0$. Since

$$H_c^*\big(D_I, (j_{!*}\mathsf{E}_n)_{|D_I}\big) = 0$$

the spectral sequence calculating the cohomology of a stack from the cohomology of a presentation gives this result. The second assertion follows because by 1) the cohomology of $j_{!*}\mathsf{E}_n$ restricted to the complement of the section $\mathcal{T} \mapsto (\mathcal{T})^\bullet$ vanishes. This follows because there is a resolution

$$\mathbb{Q}_{\ell|\cup_{i=2}^k D_i} \hookrightarrow \bigoplus_{2\le i\le k} \mathbb{Q}_{\ell|D_i} \longrightarrow \bigoplus_{\substack{2\le i,j\le k\\ i\neq j}} \mathbb{Q}_{\ell|D_{ij}} \to \cdots \to \mathbb{Q}_{\ell|D_{2\ldots k}} \to 0$$

and we just saw that $\mathbf{R}\,\mathrm{pr}_!(j_{!*}\mathsf{E}_{n|D_I}) = 0$ for all D_I occurring in this resolution.

Moreover this proves (still assuming $C = \mathbb{A}^1$) that the canonical morphism $\bar{j}_{!*}\mathsf{E}_n \to \mathbf{R}\,\mathrm{pr}_!\, j_{!*}\mathsf{E}_n$ given by the section $\mathrm{Coh}_0^1 \to \mathrm{Coh}_{0,S}^1$ is an isomorphism.

To prove 3) we note that $H^*(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell) \cong H_c^*(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)[1]$ and compare (4.1) with the Leray spectral sequence for

$$\begin{tikzcd}
{[\mathrm{pt}/\mathbb{G}_m]} \arrow[r, "\pi"] \arrow[dr, "\mathrm{id}"'] & {[\mathrm{pt}/\mathbb{G}_m^{|I|}] \cong D_{I,p}^0} \arrow[d, "\mathrm{pr}\,|D_{I,p}^0"] \\
 & {[\mathrm{pt}/\mathbb{G}_m] = \mathrm{pr}(D_{I,p}^0).}
\end{tikzcd}$$

Thus $j_{!*}\mathsf{E}_{n|D_{I,p}^0} \cong (\mathbf{R}\pi_!\mathbb{Q}_\ell) \otimes (j_*\mathsf{E}_n)_{|p}[I-1]$ and therefore

$$\mathbf{R}\,\mathrm{pr}_!(j_{!*}\mathsf{E}_{n|D_{I,p}^0}) \cong \mathbf{R}\,\mathrm{pr}_!\,\mathbf{R}\pi_!(j_*\mathsf{E}_n)_{|p}\big[|I|-1\big] = (\bar{j}_*\mathsf{E}_n)_{\mathrm{pr}(D_{I,p})}\big[|I|-1\big].$$

Again this isomorphism is induced from the canonical map $\mathbf{R}\,\mathrm{pr}_!(j_*\mathsf{E}_{n|D_{I,p}}) \to \mathbf{R}\,\mathrm{pr}_!(j_{!*}\mathsf{E}_{|D_{I,p}^0})$ restricted to the $-(|I|-1)$-th cohomology.

The general case follows from these calculations, because the statements are local in the étale topology on $\mathrm{Coh}_{0,C}^1$ $(= [C/\mathbb{G}_m])$. Therefore it is a problem which is local in the étale topology on C, thus to check that the morphisms given above are isomorphisms we may assume that $C = C_p^{\mathrm{sh}}$ is strictly henselian and $S = \{p\}$, i.e. $(C,p) \cong (\mathbb{A}^{1,h}_{\overline{\mathbb{F}}_q}, 0)$. In this case any irreducibly ramified sheaf on $\mathbb{A}^{1,h}_{\overline{\mathbb{F}}_q}$ is isomorphic to our sheaf E_n. $\square$

Remark. — Part 3) of this corollary implies that for any parabolic torsion sheaf $\mathcal{T}^\bullet$ of degree one $\mathrm{tr}(\mathrm{Frob}_{\mathrm{pr}(\mathcal{T}^\bullet)}, \mathbf{R}\,\mathrm{pr}_!(j_{!*}\mathsf{E}_{|\mathcal{T}^\bullet}))$ is the eigenvalue of the Hecke operator corresponding to $\mathcal{T}^\bullet$ applied to the Whittaker function W_{E}, i.e. writing $p = \mathrm{supp}(\mathcal{T}^\bullet)$ this is

$$(-1)^{\dim(\mathrm{Aut}(\mathcal{T}^\bullet))-1}\,\mathrm{tr}(\mathrm{Frob}_p, j_* E_{|p}).$$

To end this section we will prove two more corollaries to the above calculations. First we take up the situation of Proposition 4.1, i.e. $(C,S) = (\mathbb{A}^1, \{0\})$, and we keep the notations $j_i \colon \mathbb{A}^k - \bigcup_{|I|=i} D_I \hookrightarrow \mathbb{A}^k - \bigcup_{|I|=i+1} D_I$ and $j_{k\ldots i} := j_k \circ \cdots j_{i+1} \circ j_i$.

We have the following description of $j_{!*}\mathsf{E}_m$ for $m \leq k$:

Corollary 4.6. — *For any $0 < m \leq k$:*

1) *On $\mathbb{A}^k - \bigcup_{\substack{I\subset\{1,\ldots,k\}\\ |I|=m+1}} D_I$ there is a distinguished triangle of complexes*

$$\to j_{m\ldots 1,!*}\mathsf{E}_m \longrightarrow j_{m\ldots 1,!*}\mathsf{E}_k \longrightarrow j_{m\ldots 1,!}\mathsf{E}_{k-m}(-m) \xrightarrow{[1]} .$$

2) *For all $m + r \leq k$ we have*

$$\begin{aligned} j_{m+r\ldots 1,!*}\mathsf{E}_m &\cong \tau^{<m}\mathbf{R}j_{m+r,*}j_{m+r-1\ldots 1,!*}\mathsf{E}_m \\ &\cong \tau^{\leq m+2r-1}\mathbf{R}j_{m+r,*}j_{m+r-1\ldots 1,!*}\mathsf{E}_m. \end{aligned}$$

Moreover, there is an exact triangle

$$\to j_{!*}\mathsf{E}_{m-1} \longrightarrow j_{!*}\mathsf{E}_m \longrightarrow j_{k\ldots m,!*}j_{m-1\ldots 1,!}\mathbb{Q}_\ell \xrightarrow{[1]} .$$

Proof. — The first part of the corollary has been proven above. We may also recover it by comparing the triangle from Proposition 4.1 for E_k with the one for E_m.

To prove the second part, recall that by Lemma 4.3

$$j_{m+r+1,\ldots,m+1,!*}j_{m,\ldots,1,!}\mathbb{Q}_\ell \cong \tau^{\leq m+2r+1}\mathbf{R}j_{m+r+1,*}j_{m+r,\ldots,m+1,!*}j_{m\ldots 1,!}\mathbb{Q}_\ell.$$

Combined with Lemma 4.2 this implies that this complex has no cohomology in degrees $m+1, \ldots m+2r+1$.

Now we use induction on m: for $m = 1$ the sheaf E_1 is constant, thus the claim is true. By 1) we have an exact triangle on U_{m+1}

$$\to j_{m\ldots 1,!*}\mathsf{E}_m \longrightarrow j_{m\ldots 1,!*}\mathsf{E}_{m+1} \longrightarrow j_{m\ldots 1,!}\mathbb{Q}_\ell(-m) \xrightarrow{[1]} .$$

Apply $\mathbf{R}j_{m+1,*}$ to this complex. Then by induction we know that the left hand term has no cohomology in degrees $m, m+1$ and the right hand term has no cohomology in degree $m+1$. Thus – since $j_{m+1,!*} = \tau^{<m+1}\mathbf{R}j_{m+1,*}$ – we get that the following sequence is still exact:

$$\to j_{m+1\ldots 1,!*}\mathsf{E}_m \longrightarrow j_{m+1\ldots 1,!*}\mathsf{E}_{m+1} \longrightarrow j_{m+1,!*}j_{m\ldots 1}\mathbb{Q}_\ell \xrightarrow{[1]} .$$

Furthermore by induction the three functors $\tau^{<m+1}\mathbf{R}j_{m+2,*}$, $\tau^{\leq m+2}\mathbf{R}j_{m+2,*}$ and $j_{m+2,!*}$ give the the same result if we apply them to the left or right hand term of the triangle, thus the same is true for the middle term and again by induction we are done. □

Finally we note that there is a – perhaps surprising – analogue of Corollary 4.5 for the tensor product $j_{!*}\mathsf{E}_n \otimes j_{!*}\mathsf{E}_{n+k}$ which will be needed later on:

Corollary 4.7. — *Let (C,S) be a curve together with a finite set of points, and let E_m, E_{k+n} be local systems of rank $m \leq k$ and $k+n$ on $C-S$ with indecomposable unipotent ramification at all points in S. Let $\mathrm{pr} : \mathrm{Coh}^1_{0,S} \to \mathrm{Coh}^1_0$ be the map forgetting the k-step parabolic structure of the torsion sheaves, and denote by $\bar{j}$: $\mathrm{Coh}^1_{0,C-S} \to \mathrm{Coh}^1_{0,C}$ the inclusion. Then*

$$\mathbf{R}\,\mathrm{pr}_!(\mathcal{L}^1_{\mathsf{E}_m} \otimes \mathcal{L}^1_{\mathsf{E}_{k+n}}) = \bar{j}_*(\mathsf{E}_m \otimes \mathsf{E}_{k+n}).$$

Proof. — We have to show that $\mathbf{R}\,\mathrm{pr}_!(\mathcal{L}^1_{\mathsf{E}_m} \otimes \mathcal{L}^1_{\mathsf{E}_{k+n}})$ is the (middle) extension of its restriction to $\mathrm{Coh}^1_{0,C-S}$ to prove the corollary. This is a local problem on C, thus we may assume as before that $(C,S) = (\mathbb{A}^1, \{0\})$ and that E_m and E_{k+n} are the unipotently ramified sheaves on $\mathbb{G}_m$ defined at the beginning of this section.(6) We use the filtration

$$j_{!*}\mathsf{E}_{m-1} \longrightarrow j_{!*}\mathsf{E}_m \longrightarrow j_{k\ldots m,!*}j_{m-1\ldots 1,!}\mathbb{Q}_\ell(-m+1)$$

given by the previous corollary. We tensor this with $j_{!*}\mathsf{E}_{n+k}$ and apply $\mathbf{R}\,\mathrm{pr}_!$ to prove the corollary by induction on m.

For the right hand term we use Lemma 4.2 to replace

$$j_{k\ldots m,!*}\,j_{m-1\ldots 1,!}\,\mathbb{Q}_\ell(-m+1)$$

(6) In this case note that $\bar{j}_*(\mathsf{E}_m \otimes \mathsf{E}_{k+n}) \cong \bigoplus_{i=0}^{m-1} \bar{j}_*\mathsf{E}_{m+k+n-2i}(-i)$. This is just the Jordan decomposition for a tensor product (see for example [12], Exercise 11.11).

by the complex

$$\Big(\mathbb{Q}_\ell \to \bigoplus \mathbb{Q}_{\ell|D_i} \to \cdots \to \bigoplus_{|I|=m-1} \mathbb{Q}_{\ell|D_I}\Big)(-m+1).$$

By Corollary 4.5 we know that $\mathbf{R}\,\mathrm{pr}_!((j_{!*}\mathsf{E}_{n+k}) \otimes \mathbb{Q}_{\ell|D_I}) = 0$ for $0 < |I| < m \leq k$. Therefore

$$\begin{aligned}\mathbf{R}\,\mathrm{pr}_!\big(j_{!*}\,\mathsf{E}_{n+k} \otimes j_{k\ldots m,!*} j_{m-1\ldots 1,!}\mathbb{Q}_\ell(-m+1)\big)\\ = \mathbf{R}\,\mathrm{pr}_!\big(j_{!*}\,\mathsf{E}_{n+k} \otimes \mathbb{Q}_\ell(-m+1)\big) = \bar{\jmath}_*\,\mathsf{E}_{n+k}(-m+1).\end{aligned}$$

Now we apply the induction hypothesis to the right hand term of the filtration of $j_{!*}\,\mathsf{E}_m$ to get an exact triangle

$$\to \bar{\jmath}_*(\mathsf{E}_{m-1} \otimes \mathsf{E}_{n+k}) \longrightarrow \mathbf{R}\,\mathrm{pr}_!(\mathcal{L}^1_{\mathsf{E}_m} \otimes \mathcal{L}^1_{\mathsf{E}_{n+k}}) \longrightarrow \bar{\jmath}_*\,\mathsf{E}_{n+k}(-m+1) \xrightarrow{[1]} .$$

This proves that the middle term is a sheaf and that its dual its dual is a sheaf as well, thus it is a perverse sheaf which is the middle extension of its restriction to $\mathrm{Coh}^1_{0,C-S}$. □

4.2. A Hecke property on $\mathrm{Coh}^d_{0,S}$.

Consider as before $S \subset C$ and a rank n local system E on $C - S$ with indecomposable unipotent ramification at S. To reduce the number of constants we will assume that we are looking at n-step parabolic sheaves (it would be sufficient to assume that $\mathrm{rank}(\mathsf{E}) \geq$ length of structure).

Using the Definition 2.4 of the generalized *Hecke operators* the aim of this section is to prove:

PROPOSITION 4.8. — *$\mathcal{L}^d_{\mathsf{E}}$ is a Hecke eigensheaf on $\mathrm{Coh}^{d,\ldots,d}_{0,S}$, i.e. for all non-negative degrees $\underline{d} = \underline{d}' + \underline{d}''$ with $\underline{d} = (d,\ldots,d)$ we have*

$$H_0^{\underline{d}'',\underline{d}'}\mathcal{L}^d_{\mathsf{E}} = \begin{cases}\mathcal{L}^{d_1}_{\mathsf{E}} \boxtimes \mathcal{L}^{d_2}_{\mathsf{E}} & \text{if } \underline{d}', \underline{d}'' \text{ are constant}\\ 0 & \text{otherwise.}\end{cases}$$

To prove this, we need an analogue of Laumon's description of the Whittaker sheaf $\mathcal{L}^d_{\mathsf{E}}$. Let $\widetilde{\mathrm{Coh}}^d_{0,S}$ be the stack classifying parabolic torsion sheaves on (C,S) together with a complete flag of subsheaves:

$$\widetilde{\mathrm{Coh}}^d_{0,S} := \big\langle \mathcal{T}^\bullet_d \supset \mathcal{T}^\bullet_{d-1} \supset \cdots \supset \mathcal{T}^\bullet_1 \mid \mathcal{T}^\bullet_i \in \mathrm{Coh}^{(i,\ldots,i)}_{0,S}\big\rangle.$$

We have maps

$$\begin{tikzcd} \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S} \arrow[r, "\mathrm{gr}"] \arrow[d, "\mathrm{forget}_{\mathrm{flag}}"'] & \prod_{i=1}^{d} \mathrm{Coh}^{\underline{1}}_{0,S} \\ \mathrm{Coh}^{\underline{d}}_{0,S} & \end{tikzcd}$$

where

- $\mathrm{forget}_{\mathrm{flag}}((\mathcal{T}_i^\bullet)_{i=1,\dots,d}) := \mathcal{T}_d^\bullet$ is the forgetful map and
- $\mathrm{gr}((\mathcal{T}_i^\bullet)_{i=1,\dots,d}) = (\mathcal{T}_i^\bullet/\mathcal{T}_{i-1}^\bullet)_{i=1,\dots,d}$ with $\mathcal{T}_0^\bullet = 0$.

And define the sheaf

$$\widetilde{\mathcal{L}}^{\underline{d}}_{\mathsf{E}} := \mathbf{R}\,\mathrm{forget}_{\mathrm{flag},*}\,\mathrm{gr}^*(\mathcal{L}^{\underline{1}}_{\mathsf{E}})^{\boxtimes d}$$

on $\mathrm{Coh}^{\underline{d}}_{0,S}$. Note that the map $\mathrm{forget}_{\mathrm{flag}}$ is projective but not small (nor semi-small) in general.

Proposition 4.6. — *For any decomposition $\underline{d} = \underline{d}' + \underline{d}''$ we have*

$$H_0^{\underline{d}'',\underline{d}'}\widetilde{\mathcal{L}}^{\underline{d}}_{\mathsf{E}} = \begin{cases} \bigoplus_{S_d/(S_{d'}\times S_{d''})} \widetilde{\mathcal{L}}^{\underline{d}'}_{\mathsf{E}} \boxtimes \widetilde{\mathcal{L}}^{\underline{d}''}_{\mathsf{E}} & \text{if } \underline{d}' = (d',\dots,d') \text{ is constant,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — Extend the diagram used to define the Hecke-operators as follows:

$$\begin{tikzcd}[column sep=small] & \mathrm{Hecke}^{\underline{d}',\underline{d}''} \times_{\mathrm{Coh}^{\underline{d}}_{0,S}} \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S} \arrow[dl, "\widetilde{\mathrm{gr}}_{\mathrm{flag}}"'] \arrow[d] \arrow[rrr, "\widetilde{\mathrm{gr}}_{\mathrm{Ext}}" near end] \arrow[rr] & & \mathrm{Hecke}^{\underline{d}',\underline{d}''} \arrow[d, "\pi_{\mathrm{big}}"] \arrow[dr, "\pi_{\mathrm{small}}\times\mathrm{quot}"] & \\ (\mathrm{Coh}^{\underline{1}}_{0,S})^d & \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S} \arrow[l, "\mathrm{gr}"'] \arrow[rr, "\mathrm{forget}_{\mathrm{flag}}"] & & \mathrm{Coh}^{\underline{d}}_{0,S} & \mathrm{Coh}^{\underline{d}'}_{0,S} \times \mathrm{Coh}^{\underline{d}''}_{0,S} \end{tikzcd}$$

Using the base change theorem for the proper map $\mathrm{forget}_{\mathrm{flag}}$, we see that

$$H_0^{\underline{d}',\underline{d}''}\widetilde{\mathcal{L}}^{\underline{d}}_{\mathsf{E}} = \mathbf{R}\widetilde{\mathrm{gr}}_{\mathrm{Ext},!}\,\widetilde{\mathrm{gr}}^*_{\mathrm{flag}}\mathcal{L}^{1,\boxtimes d}_{\mathsf{E}}.$$

The fibre product $\mathrm{Hecke}^{\underline{d}',\underline{d}''} \times_{\mathrm{Coh}^{\underline{d}}_{0,S}} \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S}$ classifies

$$\langle \mathcal{T}'^\bullet \subset \mathcal{T}^\bullet \twoheadrightarrow \mathcal{T}''^\bullet,\ \mathcal{T}_1^\bullet \subset \cdots \subset \mathcal{T}_{d-1}^\bullet \subset \mathcal{T}^\bullet \rangle.$$

For every such collection of torsion sheaves we can pull back the filtration of $\mathcal{T}^\bullet$ to $\mathcal{T}'^\bullet$, and by fixing the degrees $\underline{d}'_i$ of the resulting torsion sheaves we obtain a stratification of the above stack

$$\mathrm{Hecke}^{\underline{d}',\underline{d}''} \times_{\mathrm{Coh}^{\underline{d}}_{0,S}} \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S} = \bigcup_{\underline{d}'_i} (\mathrm{Hecke}^{\underline{d}',\underline{d}''} \times_{\mathrm{Coh}^{\underline{d}}_{0,S}} \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S})^{\underline{d}'_i},$$

where the substacks of $\mathrm{Hecke}^{\underline{d}',\underline{d}''} \times_{\mathrm{Coh}^{\underline{d}}_{0,S}} \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S}$ are defined as

$$(\mathrm{Hecke}^{\underline{d}',\underline{d}''} \times_{\mathrm{Coh}^{\underline{d}}_{0,S}} \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S})^{\underline{d}'_i} := \left\langle \begin{pmatrix} \mathcal{T}'^\bullet \to \mathcal{T}^\bullet \to \mathcal{T}''^\bullet \\ \mathcal{T}^\bullet_i \subset \mathcal{T}^\bullet \end{pmatrix} \;\middle|\; \deg(\mathcal{T}'^\bullet \cap \mathcal{T}^\bullet_i) = \underline{d}'_i \right\rangle.$$

- *Fisrt case*: $\underline{d}'_i = (d'_i, \dots, d'_i)$ *is constant for all* i. — We have a commutative diagram

$$\begin{tikzcd}
(\mathrm{Hecke}^{\underline{d}',\underline{d}''} \times_{\mathrm{Coh}^{\underline{d}}_{0},S} \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S})^{\underline{d}'_i} \arrow[r, "\widetilde{\mathrm{forget}}_{\mathrm{Ext}}"] \arrow[d] & \widetilde{\mathrm{Coh}}^{\underline{d}'}_{0,S} \times \widetilde{\mathrm{Coh}}^{\underline{d}''}_{0,S} \arrow[d] \arrow[dr, "\mathrm{forget}'_{\mathrm{flag}} \times \mathrm{forget}''_{\mathrm{flag}}"] & \\
(\mathrm{Coh}^{1}_{0,S})^{\times d} \arrow[r, "\sim"] & (\mathrm{Coh}^{1}_{0,S})^{\times d'} \times (\mathrm{Coh}^{1}_{0,S})^{\times d''} & \mathrm{Coh}^{\underline{d}'}_{0,S} \times \mathrm{Coh}^{\underline{d}''}_{0,S}.
\end{tikzcd}$$

Where $\widetilde{\mathrm{forget}}_{\mathrm{Ext}}$ maps $(\mathcal{T}^\bullet_i \subset \mathcal{T}^\bullet, \mathcal{T}'^\bullet, \mathcal{T}''^\bullet)$ to the induced filtrations on $\mathcal{T}'^\bullet$ and $\mathcal{T}''^\bullet$. By Lemma 0.2 the map $\widetilde{\mathrm{forget}}_{\mathrm{Ext}}$ is smooth, the fibres being generalized affine spaces. These are of dimension 0, since both stacks are smooth of dimension 0, thus

$$\mathbf{R}\widetilde{\mathrm{gr}}_{\mathrm{Ext},!}(\widetilde{\mathrm{gr}}^*_{\mathrm{flag}}(\mathcal{L}^1_{\mathsf{E}})^{\boxtimes d})_{|(\mathrm{Hecke}^{\underline{d}',\underline{d}''} \times_{\mathrm{Coh}^{\underline{d}}_{0,S}} \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S})^{\underline{d}'_i}} = \widetilde{\mathcal{L}}^{\underline{d}'}_{\mathsf{E}} \boxtimes \widetilde{\mathcal{L}}^{\underline{d}''}_{\mathsf{E}}.$$

- *Second case*: $\underline{d}'_i$ *not a constant sequence for some* i. — Let $\mathrm{Flag}^{(\underline{d}'_i)}$ be the stack, classifying torsion sheaves $\mathcal{T}'^\bullet$ of degree $\underline{d}'$ together with a flag of subsheaves $\mathcal{T}'^\bullet_i$ of degree $(\underline{d}'_i)_{i=1,\dots,d}$. Then we can still factor the restriction of $\widetilde{\mathrm{gr}}_{\mathrm{Ext}}$ to the corresponding stratum into

$$(\mathrm{Hecke}^{\underline{d}',\underline{d}''} \times_{\mathrm{Coh}^{\underline{d}}_{0,S}} \widetilde{\mathrm{Coh}}^{\underline{d}}_{0,S})^{\underline{d}'_i} \xrightarrow{\widetilde{\mathrm{forget}}_{\mathrm{Ext}}} \mathrm{Flag}^{(\underline{d}'_i)} \times \mathrm{Flag}^{(\underline{d}''_i)} \longrightarrow \mathrm{Coh}^{\underline{d}'}_{0,S} \times \mathrm{Coh}^{\underline{d}''}_{0,S}.$$

Claim: $\mathbf{R}\widetilde{\mathrm{forget}}_{\mathrm{Ext},!}\widetilde{\mathrm{gr}}^*_{\mathrm{flag}}\mathcal{L}_{\mathsf{E}}^{1,\boxtimes d} = 0.$

As in the first case the map $\widetilde{\mathrm{forget}}_{\mathrm{Ext}}$ is smooth, and the fibres are generalized affine spaces: for a fixed point $(\mathcal{T}_i'^{\bullet}, \mathcal{T}_i''^{\bullet}) \in \mathrm{Flag}^{(\underline{d}_i')} \times \mathrm{Flag}^{(\underline{d}_i'')}$ the fibre of $\widetilde{\mathrm{forget}}_{\mathrm{Ext}}$ over this point consists of extensions

$$\begin{tikzcd}
\mathcal{T}_1'^{\bullet} \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \mathcal{T}_{d-1}'^{\bullet} \arrow[r, hook] \arrow[d, hook] & \mathcal{T}'^{\bullet} \arrow[d, hook] \\
\mathcal{T}_1^{\bullet} \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & \mathcal{T}_{d-1}^{\bullet} \arrow[r, hook] \arrow[d, two heads] & \mathcal{T}^{\bullet} \arrow[d, two heads] \\
\mathcal{T}_1''^{\bullet} \arrow[r, hook] & \cdots \arrow[r, hook] & \mathcal{T}_{d-1}''^{\bullet} \arrow[r, hook] & \mathcal{T}''^{\bullet}
\end{tikzcd}$$

Let $\mathrm{gr}_i \mathcal{T}'^{\bullet} := \mathcal{T}_i'^{\bullet}/\mathcal{T}_{i-1}'^{\bullet}$. Then we may factor $\widetilde{\mathrm{forget}}_{\mathrm{Ext}}$ into

$$\begin{aligned}
&(\mathrm{Hecke}^{\underline{d}',\underline{d}''} \times_{\mathrm{Coh}^{\underline{d}}_{0,S}} \widetilde{\mathrm{Coh}}^{d}_{0,S})^{\underline{d}_i'} \\
&\qquad \xrightarrow{\mathrm{gr}_{\mathrm{Ext}}} \prod_{i=1}^{d} \mathrm{Ext}(\mathrm{gr}_i \mathcal{T}''^{\bullet}, \mathrm{gr}_i \mathcal{T}'^{\bullet}) \longrightarrow \mathrm{Flag}^{(\underline{d}_i')} \times \mathrm{Flag}^{\underline{d}_i''},
\end{aligned}$$

where $\mathrm{Ext}(\mathrm{gr}_i \mathcal{T}''^{\bullet}, \mathrm{gr}_i \mathcal{T}'^{\bullet})$ is the generalized vector bundle over $\mathrm{Flag}^{(\underline{d}_i')} \times \mathrm{Flag}^{(\underline{d}_i'')}$ classifying extensions of the filtration quotients. Furthermore Lemma 0.2 shows that $\mathrm{gr}_{\mathrm{Ext}}$ is a generalized affine space bundle, which can be factored into maps with fibres $\mathrm{Ext}(\mathrm{gr}_i \mathcal{T}''^{\bullet}, \mathcal{T}_{i-1}'^{\bullet})$.

Since $\widetilde{\mathrm{gr}}_{\mathrm{flag}}$ also factors through $\mathrm{gr}_{\mathrm{Ext}}$, the sheaf $\widetilde{\mathrm{gr}}^*_{\mathrm{flag}}\widetilde{\mathcal{L}}^{\underline{d}}_{\mathsf{E}}$ is constant on the fibres of $\mathrm{gr}_{\mathrm{Ext}}$ and thus by the Künneth formula it is sufficient to prove that for $d = 1$ and any non-trivial decomposition

$$\underline{d} = (1,\dots,1) = \underbrace{(\epsilon_1,\dots,\epsilon_n)}_{=:\underline{d}'} + \underbrace{(1-\epsilon_1,\dots,1-\epsilon_n)}_{=:\underline{d}''}$$

we have $H_0^{\underline{d}',\underline{d}''}\mathcal{L}^{\underline{1}}_{\mathsf{E}} = 0$. But here we can apply the calculation of $\mathcal{L}^{\underline{1}}_{\mathsf{E}|D_I}$ given in Corollary 4.4 to establish the claim.

Now we have shown that $H_0^{\underline{d}',\underline{d}''}\widetilde{\mathcal{L}}^{\underline{d}}_{\mathsf{E}}$ has a filtration such that the subquotients are isomorphic to the sheaves $\widetilde{\mathcal{L}}^{\underline{d}'}_{\mathsf{E}} \boxtimes \widetilde{\mathcal{L}}^{\underline{d}''}_{\mathsf{E}}$. Furthermore we know that over the substack where $\mathrm{supp}(\mathcal{T}'^{\bullet}) \cup \mathrm{supp}(\mathcal{T}''^{\bullet})$ consists of d distinct points, this extension splits. The proof of the following lemma will only use this fact to show that all these sheaves are perverse sheaves which are the middle extension of their restrictions to any open subset. Therefore the filtration splits globally. □

LEMMA 4.10. — *The complex* $\widetilde{\mathcal{L}}^d_{\mathsf{E}} = \mathbf{R}\,\mathrm{forget}_!\,\mathrm{gr}^*((\mathcal{L}^1_{\mathsf{E}})^{\boxtimes d})$ *is a perverse sheaf which is the intermediate extension of its restriction to* $\mathrm{Coh}^d_{0,C-S}$:

$$\widetilde{\mathcal{L}}^d_{\mathsf{E}} = \mathbf{R}\,\mathrm{forget}_!\,\mathrm{gr}^*\big((\mathcal{L}^1_{\mathsf{E}})^{\boxtimes d} = j_{!*}(\widetilde{\mathcal{L}}^d_{\mathsf{E}|\mathrm{Coh}^d_{0,C-S}})\big).$$

In particular, it carries a natural action of the symmetric group S_d *and*

$$\mathcal{L}^d_{\mathsf{E}} = (\widetilde{\mathcal{L}}^d_{\mathsf{E}})^{S_d}.$$

Again we denoted by $j: \mathrm{Coh}^d_{0,C-S} \hookrightarrow \mathrm{Coh}^d_{0,S}$ the inclusion.

Proof of Lemma 4.10. — By Laumon's results [19] we know that the restriction of $\widetilde{\mathcal{L}}_{\mathsf{E}}$ to $\mathrm{Coh}^d_{0,C-S}$ is indeed a perverse sheaf which is the middle extension of its restriction to every open subset.

Since the question is local on $\mathrm{Coh}^d_{0,S}$ we may assume that our local system E is pure. Then $\mathcal{L}^1_{\mathsf{E}}$ is pure (it is irreducible and perverse) and thus, by Deligne's theorem (see [7], 6.2.6) $\widetilde{\mathcal{L}}^d_{\mathsf{E}}$ is also pure. Therefore we may apply the Decomposition Theorem (see [4], 5.4.6) to decompose $\widetilde{\mathcal{L}}^d_{\mathsf{E}} = j_{!*}j^*\widetilde{\mathcal{L}}^d_{\mathsf{E}} \oplus \mathrm{rest}^d$.

We prove the lemma by induction on d. Assume that $\mathrm{rest}^k = 0$ for all $k < d$. (By definition of $\mathcal{L}^1_{\mathsf{E}}$ the statement is true for $d = 1$.) By the induction hypothesis and the fact that the restriction of $\widetilde{\mathcal{L}}^d_{\mathsf{E}}$ to $\mathrm{Coh}^d_{0,C-S}$ is perverse we furthermore know that $\mathrm{supp}(\mathrm{rest}^d) \subset \langle \mathcal{T}^\bullet \,|\, \mathrm{supp}(\mathcal{T}^\bullet) = p \in S\rangle$. The preceding proposition shows a Hecke property of $\widetilde{\mathcal{L}}^d_{\mathsf{E}}$ and this implies in particular that $H_0^{\underline{i},(d)-\underline{i}}\mathrm{rest}^d = 0$ for all $\underline{i} > 0$.

Choose $\mathcal{T}^\bullet \in \mathrm{supp}(\mathrm{rest}^d)$ such that the degree of a maximal indecomposable summand of $\mathcal{T}^\bullet$ is maximal. Write $\mathcal{T}^\bullet = \mathcal{O}^\bullet_{ip/n}(\frac{j}{n}p) \oplus \mathcal{T}'^\bullet$, such that $\mathcal{O}^\bullet_{ip/n}(\frac{j}{n}p)$ is a direct summand of maximal degree (this is possible by Lemma 3.1). Note that $\mathcal{T}^\bullet \not\cong \mathcal{O}^\bullet_{dp}$ since the latter sheaf has a unique filtration with subquotients of degree $(1,\dots,1)$. Now define $\underline{d}' := \deg(\mathcal{T}'^\bullet)$ and look at the fibre F of the Hecke-correspondence $\mathrm{Hecke}_0^{\underline{d}',\underline{d}-\underline{d}'}$ over the point $(\mathcal{T}'^\bullet, \mathcal{O}_{ip/n}(\frac{j}{n}p)) \in \mathrm{Coh}^{\underline{d}'}_{0,S} \times \mathrm{Coh}^{\underline{d}-\underline{d}'}_{0,S}$. Then $\mathcal{T}^\bullet$ is the only sheaf contained in $\mathrm{supp}(\mathrm{rest}^d) \cap F$, because every non-trivial extension of the two sheaves contradicts our maximality assumption (again by Lemma 3.1).

Therefore if $\mathrm{rest}^d|_{\mathcal{T}^\bullet} \neq 0$ then $H_0^{\underline{d}',\underline{d}-\underline{d}'}\mathrm{rest}^d \neq 0$, contradicting our assumption that all the $\widetilde{\mathcal{L}}^k_{\mathsf{E}}$ are irreducible perverse sheaves for $k < d$. □

Proof of Proposition 4.8. — This now follows from the above lemma by taking S_d-invariants in the Hecke property of $\widetilde{\mathcal{L}}^d_{\mathsf{E}}$. □

5. The sheaf $\mathsf{F}^n_{\mathsf{E},!}$ corresponds to the function $\Phi(W_{\mathsf{E}})$.

The aim of this section is to explain the relation between the function $\mathrm{tr}_{\mathsf{F}^n_{\mathsf{E},!}}$ and Shalika's definition of $\Phi(W_{\mathsf{E}})$. As in the case of unramified local systems, the problem to compare the two functions stems from the fact that the interpretation of Laumon's diagram in terms of adeles does not immediately correspond to the definition of Φ. The main ingredient needed to solve this problem is an analogue of Drinfeld's compactification as defined in [11]. This moduli space is on the one hand related to the fundamental diagram and on the other hand its points have a simple adelic description. All this follows easily from [11].

However, to prove that the function $\mathrm{tr}_{\mathsf{F}^n_{\mathsf{E},!}}$ is indeed a non-zero multiple of the function $\Phi(W_{\mathsf{E}})$, we cannot copy the proof of [10], since this argument uses results on the affine Grassmannian for which we do not know analogous statements for the affine flag manifold. We will use an elementary approach instead to obtain an inductive argument to calculate the function $\mathrm{tr}_{\mathsf{F}^n_{\mathsf{E},!}}$ on a subset of its support which is sufficiently big to conclude the proof of our main theorem once we have calculated the whole function for $n-1$. We will then give a calculation for $n \leq 2$.

5.1. An analogue of Drinfeld's compactification.

First we rewrite the inductive definition of $\mathsf{F}^n_{\mathsf{E},!}$ as in the appendix of [19] and [11]:

Denote by $\langle \Omega\text{-}\operatorname{Ext} \subset \mathcal{E}^\bullet\rangle$ the stack classifying

$$\langle \Omega\text{-}\operatorname{Ext} \subset \mathcal{E}^\bullet\rangle := \left\langle \begin{array}{c} \mathcal{E}^\bullet \in \operatorname{Bun}_{n,S}^{d,\mathrm{good}},\ \mathcal{J}_i^\bullet \in \operatorname{Bun}_{i,S} \\ \mathcal{J}_1^\bullet \subset \mathcal{J}_2^\bullet \subset \cdots \subset \mathcal{J}_n^\bullet \subset \mathcal{E} \\ \alpha_i : \mathcal{J}_i^\bullet/\mathcal{J}_{i-1}^\bullet \xrightarrow{\sim} \Omega^{\bullet,n-i} \text{ a fixed isomorphism} \end{array} \right\rangle$$

We may define maps

$$\mathrm{quot} : \langle \Omega\text{-}\operatorname{Ext} \subset \mathcal{E}^\bullet\rangle \longrightarrow \operatorname{Coh}^d_{0,S}, \quad \left(\mathcal{E}^\bullet, (\mathcal{J}_i^\bullet)_{i=1,\dots,n}\right) \longmapsto \mathcal{E}^\bullet/\mathcal{J}_n^\bullet,$$

and

$$\mathrm{ext} : \langle \Omega\text{-}\operatorname{Ext} \subset \mathcal{E}^\bullet\rangle \longrightarrow \prod_{i=1}^{n-1} \operatorname{Ext}^1_{\mathrm{para}}(\Omega^{\bullet,n-1-i}, \Omega^{\bullet,n-i}) \xrightarrow{\Sigma\mathrm{Res}} \mathbb{A}^1,$$

$$\left(\mathcal{E}^\bullet, (\mathcal{J}_i^\bullet)_{i=1,\dots,n}\right) \longmapsto \sum_{i=1}^{n-1} \left(\Omega^{\bullet,n-i} \hookrightarrow \mathcal{J}_{i+1}^\bullet/\mathcal{J}_{i-1}^\bullet \twoheadrightarrow \Omega^{\bullet,n-i-1}\right)$$

(here we used Lemma 2.3 to identify the $\operatorname{Ext}^1_{\mathrm{para}}$-group with $H^1(C,\Omega) \cong \mathbb{A}^1$ and denote this residue map by Res);

$$\begin{gathered}\text{forget} : \langle \Omega\text{- Ext} \subset \mathcal{E}^\bullet\rangle \longrightarrow \mathrm{Hom}_n^{\mathrm{inj}},\\ \big(\mathcal{E}^\bullet, (\mathcal{J}_i^\bullet)_{i=1,\dots,n}\big) \longmapsto (\Omega^{\bullet,n-1} \cong \mathcal{J}_1^\bullet \hookrightarrow \mathcal{E}^\bullet).\end{gathered}$$

Then by definition of $\mathsf{F}^n_{\mathsf{E},!}$ we have

$$\mathsf{F}^n_{\mathsf{E},!} = \mathbf{R}\,\mathrm{forget}_!(\mathrm{quot}^*\mathcal{L}_{\mathsf{E}} \otimes \mathrm{ext}^*\, \mathsf{L}_\psi)[c], \tag{5.1}$$

where c is the dimension of the fibres of forget.

Remark. — We have an adelic description of the points of the stack $\langle \Omega\text{- Ext} \subset \mathcal{E}^\bullet\rangle$:

$$\langle \Omega\text{- Ext} \subset \mathcal{E}^\bullet\rangle(\mathbb{F}_q) \subset \mathsf{N}_n(k(C))\backslash \mathsf{N}_n(\mathbb{A}) \times_{\mathsf{N}(\mathcal{O})} \mathrm{GL}_n(\mathbb{A})/(\mathrm{GL}_n(\mathcal{O}_{C-S}) \times \mathrm{Iw}_S).$$

We will not need this (it is the same as in [10], Section 3), but note that this is not the set which is used in the definition of the function $\Phi(W_{\mathsf{E}})$.

To define a moduli space whose points will be a subset of

$$\mathsf{N}_n(k(C))\backslash \mathrm{GL}_n(\mathbb{A})/ \mathrm{GL}_n(\mathcal{O}_{C-S} \times \mathrm{Iw}_S)$$

we argue as in [11] and define a moduli space classifying parabolic vector bundles together with a full flag of subspaces of the generic fibre of the bundle, satisfying some regularity condition:

For a parabolic vector bundle $\mathcal{E}^\bullet$ we denote by $\bigwedge^k \mathcal{E}^\bullet$ its k-th exterior power, which is defined as the collection of the sequences of vector bundles

$$\cdots \longrightarrow \bigwedge^k \mathcal{E}^{(i,p)} \longrightarrow \bigwedge^k \mathcal{E}^{(i+1,p)} \longrightarrow \cdots \quad \text{for all } p \in S.$$

Analogously, denote for parabolic bundles $\mathcal{E}_1^\bullet \otimes \mathcal{E}_2^\bullet$ the tensor product taken componentwise, together with the natural maps.

DEFINITION 5.1 (Drinfeld's compactification). — The stack Ω-Plücker classifies

$$\Omega\text{-Plücker} := \left\langle \begin{gathered} \mathcal{E}^\bullet \in \mathrm{Bun}^d_{n,S} \\ s_1 : \Omega^{\bullet,n-1} \hookrightarrow \mathcal{E}^\bullet, \dots \\ s_i : \Omega^{\bullet,n-1} \otimes \dots \otimes \Omega^{\bullet,n-i} \hookrightarrow \wedge^i \mathcal{E}^\bullet, \dots \\ s_n : \Omega^{\bullet,n-1} \otimes \dots \otimes \Omega^\bullet \otimes \mathcal{O}^\bullet \hookrightarrow \wedge^n \mathcal{E}^\bullet \end{gathered} \right\rangle$$

such that the s_i satisfy the Plücker relations.

Recall that the Plücker relations are given by the condition that over the generic point of C the maps s_i define a full flag of subspaces of one (or equivalently all) $\mathcal{E}^{(j,p)}$. In particular we have a map

$$\mathrm{forget}_{\mathrm{Tor}} : \langle \Omega - \mathrm{Ext} \subset \mathcal{E}^\bullet \rangle \longrightarrow \Omega\text{-Plücker},$$

$$\left(\mathcal{E}^\bullet, (\mathcal{J}_i^\bullet, \alpha_i)_{i=1,\dots,n}\right) \longmapsto \left(\mathcal{E}^\bullet, s_i : \otimes_{j=1}^i \Omega^{\bullet,n-j} \xrightarrow[\sim]{\otimes_{j=1}^i \alpha_j} \wedge^i \mathcal{J}_i^\bullet \hookrightarrow \bigwedge^i \mathcal{E}^\bullet\right).$$

Furthermore if all the s_i are maximal embeddings (i.e. if the cokernel of s_i is torsion free in every degree (j,p)), then the s_i define a full flag of $\mathcal{E}^\bullet$ at every point of the curve, i.e. the s_i define a full flag of subbundles of $\mathcal{E}^\bullet$.

Therefore the points of this stack have a simple description in terms of the zero divisors of the maps s_i: we call a formal sum

$$D = \sum_{p \in C-S} n_p p + \sum_{p \in S} \frac{i_p}{n} p$$

(only finitely many $n_p \neq 0$) a *parabolic divisor*, i.e. it is a divisor, but the coefficients of points in S are allowed to lie in $\frac{1}{n}\mathbb{Z}$. For a parabolic divisor D we call $\deg(\mathcal{O}^\bullet(D))$ its degree. In the same way as usual divisors, parabolic divisors of a fixed degree $\underline{d}$ form a sheaf $\mathrm{Div}_{C,S}^{\underline{d}}$, and the subsheaf of effective parabolic divisors is represented by a symmetric product of the curve.

LEMMA-DEFINITION 5.2. — *The stack* Ω-Plücker *has a stratification by locally closed substacks indexed by degrees of parabolic divisors* $\underline{d}_1, \dots, \underline{d}_n$. *The strata are given by*

$$(\Omega\text{-Plücker})_{(\underline{d}_1,\dots,\underline{d}_n)} := \left\langle \begin{array}{c} \mathcal{E}^\bullet \in \mathrm{Bun}_{n,S}^{\underline{d}}, \ D_i \in \mathrm{Div}_{C,S}^{\underline{d}_i} \\ s_1 : \Omega^{\bullet,n-1}(D_1) \hookrightarrow \mathcal{E}^\bullet \\ s_i : \Omega^{\bullet,n-1}(D_1) \otimes \cdots \otimes \Omega^{\bullet,n-i}(D_{n-i}) \hookrightarrow \wedge^i \mathcal{E}^\bullet \\ s_n : \Omega^{\bullet,n-1}(D_1) \otimes \cdots \otimes \mathcal{O}^\bullet(D_n) \hookrightarrow \wedge^n \mathcal{E}^\bullet \end{array} \right\rangle$$

such that the s_i are maximal embeddings and satisfy the Plücker relations, and $\sum_{i=1}^k D_i$ is effective for all $1 \leq k \leq n$,

$$\cong \left\langle \begin{array}{c} \mathcal{E}^\bullet \in \mathrm{Bun}_{n,S}^{\underline{d}}, \ \mathcal{J}_i^\bullet \in \mathrm{Bun}_{i,S}, D_i \in \mathrm{Div}_{C,S}^{\underline{d}_i} \\ \mathcal{J}_1^\bullet \subset \mathcal{J}_2^\bullet \subset \cdots \subset \mathcal{J}_n^\bullet = \mathcal{E} \\ \alpha_i : \mathcal{J}_i^\bullet / \mathcal{J}_{i-1}^\bullet \xrightarrow{\sim} (\Omega^{n-i})^\bullet(D_i) \\ \sum_{i=1}^k D_i \text{ is effective for all } 1 \leq k \leq n. \end{array} \right\rangle.$$

For fixed parabolic divisors $D_1, \dots, D_n$ denote by Ω-Plücker$_{D_1,\dots,D_n}$ the corresponding substack of the above stack. □

Note that the above description of the strata of Ω-Plücker can also be used to describe the map $\mathrm{forget}_{\mathrm{Tor}}$. Namely, for a point $(\mathcal{E}^\bullet, (\mathcal{J}_i^\bullet)_{i=1,\dots,n})$ in $\langle \Omega - \mathrm{Ext} \subset \mathcal{E}^\bullet \rangle$ its image under $\mathrm{forget}_{\mathrm{Tor}}$ is $(\mathcal{E}^\bullet, (\mathcal{J}_i^{\bullet,\max}))_{i=1,\dots,n})$ where $\mathcal{J}_i^{\bullet,\max} \subset \mathcal{E}^\bullet$ is the subbundle defined by $\mathcal{J}_i^\bullet$. But this is only a pointwise description.

Remark 5.2. — The points of the stack Ω-Plücker can be described as a subset:

$$\Omega\text{-Plücker}(\mathbb{F}_q) \subset \mathsf{N}_n(k(C)) \backslash \mathrm{GL}_n^\Omega(\mathbb{A})/(\mathrm{GL}_n^\Omega(\mathcal{O}_{C-S}) \times \mathrm{Iw}_S).$$

Proof of Remark 5.2. — This is the same as Weil's description of vector bundles (see also [10]). However to compare the function W_{E} with a sheaf on Ω-Plücker we will need a precise form of the inclusion, therefore we will recall the construction of the map.

Given a point $(\mathcal{E}^\bullet, s_i, D_i) \in \Omega\text{-Plücker}_{D_1,\dots,D_n}$ we define an element of $\mathrm{GL}_n^\Omega(\mathbb{A})$ as follows: let $N := -(n-1)^2$ be the shift in the definition of $\Omega^{\bullet,n-1}$. Thus if all $D_i = 0$ then the bundle $\mathcal{E}^{(N,S)}$ is equipped with a filtration with subquotients $\Omega^{\otimes n-i}(-(i-1)S)$. Recall that in 0.2 we have chosen an identification of $\mathrm{GL}_n(\mathbb{A})$ with $\mathrm{GL}_n^\Omega(\mathbb{A})$, i.e. we decided to use $\bigoplus_{i=0}^{n-1} \Omega^{\otimes i}$ as standard bundle instead of the trivial one. Thus denote by η the generic point of C and choose a trivialization

$$f_\eta \colon \bigoplus_{i=0}^{n-1} \Omega_\eta^{\otimes i} \xrightarrow{\sim} \mathcal{E}_\eta^{(n-1,S)}$$

such that the image of $\bigoplus_{i=1}^{j} \Omega_\eta^{\otimes n-i}$ is the subspace defined by $(s_i)_{i\le j}$.

Further, for $p \in C - S$ choose a trivialization

$$f_p \colon \bigoplus_{i=0}^{n-1} \Omega^{\otimes i} \otimes \widehat{\mathcal{O}}_p \xrightarrow{\sim} \mathcal{E}^{N,S} \otimes \widehat{\mathcal{O}}_p$$

again compatible with the filtration induced by the s_i. Then $f_p^{-1} \circ f_\eta \in \mathrm{GL}_n^\Omega(K_p)$ will be an element of the form $N_p \cdot \mathrm{diag}(d_{n,p}, \dots, d_{1,p})$, where N_p is a unipotent upper triangular matrix and the second term is a diagonal matrix such that the valuations of the entries are given by the p-part of the divisors D_i.

Caution: in the definition of the adelic double quotient we divided by $\mathsf{N}_n(k(C))$ from the left and we want this to correspond to the flag at the generic point given by the s_i. Thus the transition function of $f_p^{-1} f_\eta \in \mathrm{GL}_n(K_p)$ is also given by multiplication from the left, i.e. an element of $\bigoplus_{i=0}^{n-1} \Omega_\eta^{\otimes i}$ is represented by a line-vector; the n-th component given by the coordinate of $\Omega^{\otimes n-1}$. Therefore the last entry of the diagonal matrix given above is indeed $d_{1,p}$.

For $p \in S$ we choose an isomorphism

$$f_p\colon \bigoplus_{i=0}^{n-1} \Omega^{\otimes i} \otimes \widehat{\mathcal{O}}_p \xrightarrow{\sim} \mathcal{E}^{(N,S)} \otimes \widehat{\mathcal{O}}_p$$

compatible with the filtration of the stalk $\mathcal{E}^{(N,S)} \otimes k(p)$, i.e. we choose f_p such that the induced map $\bigoplus_{i=1}^{j} \Omega^{\otimes n-i} \otimes k(p) \to \mathcal{E}^{(N,S)} \otimes k(p)$ factors through $\ker\big(\mathcal{E}^{(N,S)} \otimes k(p) \to (\mathcal{E}^{(N+j,S)} \otimes k(p))\big)$.

Again define $f_p^{-1} f_\eta \in \mathrm{GL}_n(K_p)$. To describe this element, let $D_i = (d_i + k_i/n)p + D_i'$ with $p \notin \mathrm{supp}(D_i')$ and $0 \le k_i < n$, and choose a local parameter π_p at p. Then $f_p^{-1} f_\eta(\Omega^{\otimes n-1})$ is contained in the $\widehat{\mathcal{O}}_p$-submodule

$$\pi_p^{d_1}\Big(\bigoplus_{j=k_1}^{n-1} \Omega^{\otimes j}\Big) \oplus \pi_p^{d_1+1}\Big(\bigoplus_{j=0}^{k_1-1} \Omega^{\otimes j}\Big),$$

equivalently in the last line of the matrix $f_p^{-1} f_\eta$ the first k_1 entries are of the form $\pi^{d_1+1}u$ with $u \in \widehat{\mathcal{O}}^*$, the k_1+1th is $\pi^{d_1}u$ for some $u \in \widehat{\mathcal{O}}^*$ and the others lie in (π^d). To apply a similar consideration to $\Omega^{\bullet,n-i}$ recall that $\Omega^{n-i,(N,S)} = \Omega^{\otimes n-i}(-(i-1)S)$. Thus $f_p^{-1} f_\eta(\Omega^{\otimes n-i})$ is contained in the subspace generated by $f_p^{-1} f_\eta(\bigoplus_{j=1}^{i-1} \Omega^{n-1})$ and either

$$\pi_p^{d_i-(i-1)}\Big(\bigoplus_{j=k_i+(i-1)}^{n-1} \Omega^{\otimes j}\Big) \oplus \pi_p^{d_i-(i-2)}\Big(\bigoplus_{j=0}^{k_i+i-2} \Omega^{\otimes j}\Big)$$

for $k_i \le n-i$ or otherwise

$$\pi^{d_i-(i-2)}\Big(\bigoplus_{j=k_i-(n-i)}^{n-1} \Omega^{\otimes j}\Big) \oplus \pi^{d_i-(i-3)}\Big(\bigoplus_{j=0}^{k_i-(n-i+1)} \Omega^{\otimes j}\Big).$$

Note that in this way we get an element of $\mathrm{GL}_n(K_p)$ for which we have calculated the value of the Whittaker function in Proposition 1.2. And the shift in the definition of $\Omega^{i,\bullet}$ assures that the support of the Whittaker function is the subset of Ω-Plücker where $D_1 \le D_2 \le \cdots \le D_{n-1}$. □

The map forget factors through Ω-Plücker:

$$\text{forget} : \langle \Omega\text{-}\operatorname{Ext} \subset \mathcal{E}^\bullet \rangle \xrightarrow{\text{forget}_{\mathrm{Tor}}} \Omega\text{-Plücker} \xrightarrow{\text{forget}'} \operatorname{Hom}_n^{\mathrm{inj}}.$$

By Proposition 1.2 the intersection of the support of the Whittaker function W_{E} with the points where $D_1 \ge 0$ lies in Ω-Plücker$(\mathbb{F}_q)$, and therefore the summation in the definition of $\Phi(W_{\mathsf{E}})$ is the same as the summation over the points in the fibres of forget$'$. Thus, to prove that $\mathrm{tr}_{\mathcal{F}^n_{\mathsf{E},!}}$ equals $\Phi(W_{\mathsf{E}})$ up to a scalar, it is sufficient to prove that (up to a scalar)

$$\mathrm{tr}_{\mathbf{R}\,\mathrm{forget}_{\mathrm{Tor},!}(\mathrm{quot}^*\mathcal{L}_{\mathsf{E}}\otimes\mathrm{ext}^*\,\mathsf{L}_\psi)} = W_{\mathsf{E}}.$$

Our first aim is to show that the left hand side of the last equation defines an element of the space of Whittaker functions (Proposition 5.3).

We denote by Ω- $\mathrm{Ext}_{D_1,\dots,D_n}$ the preimage $\mathrm{forget}_{\mathrm{Tor}}^{-1}(\Omega\text{-Plücker}_{D_1,\dots,D_n})$.

Note that whenever we have $0 \le D_1 \le D_2 \le \cdots \le D_n$, we can define a sheaf $\Psi_{D_1,\dots,D_n} := \mathrm{ext}^*_{D_1,\dots,D_n}\,\mathsf{L}_\psi$ via

$$\begin{aligned}\mathrm{ext}_{D_1,\dots,D_n} : \Omega\text{-Plücker}_{D_1,\dots,D_n}&\\ \longrightarrow \prod_{i=1}^{n-1} \mathrm{Ext}^1_{\mathrm{para}}\big(\Omega^{\bullet,n-i-1}(D_{i+1}), \Omega^{\bullet,n-i}(D_i)\big)&\\ \longrightarrow \prod_{i=1}^{n-1} \mathrm{Ext}^1_{\mathrm{para}}(\Omega^{\bullet,n-i-1}, \Omega^{\bullet,n-i}) \xrightarrow{\sum \mathrm{Res}} \mathbb{A}^1.&\end{aligned}$$

Let $\lfloor D_j \rfloor$ be the biggest divisor smaller than the parabolic divisor D_j and denote by d_j its degree. Then we also have a map

$$\mathrm{div} : \Omega\text{-Plücker}_{\underline{d}_1,\dots,\underline{d}_n} \longrightarrow C^{(d_1)} \times C^{(d_2-d_1)} \times \cdots \times C^{(d_n-d_{n-1})},$$

sending $(D_1,\dots,D_n)$ to $(\lfloor D_1\rfloor, \lfloor D_2\rfloor - \lfloor D_1\rfloor, \dots, \lfloor D_n\rfloor - \lfloor D_{n-1}\rfloor)$. To simplify notations we will denote the restriction of div to $\Omega\text{-Plücker}_{D_1,\dots,D_n}$ by the same symbol.

The aim of this section is to prove:

PROPOSITION 5.3. — *Let $D_1,\dots,D_n$ be parabolic divisors and assume $0 \le D_1$. Then:*

1) *If $D_i \not\le D_{i+1}$ for some i, then*

$$\mathbf{R}\,\mathrm{forget}_{\mathrm{Tor},!}\big(\mathrm{quot}^*\mathcal{L}^d_{\mathsf{E}} \otimes \mathrm{ext}^*\,\mathsf{L}_\psi)\big)_{|\Omega\text{-Plücker}_{D_1,\dots,D_n}} = 0.$$

2) *If $0 \le D_1 \le D_2 \le \cdots \le D_n$, then there is a sheaf W_{E} on $C^{(d_1)} \times \cdots \times C^{(d_n-d_1)}$ and a constant c such that*

$$\begin{aligned}\mathbf{R}\,\mathrm{forget}_{\mathrm{Tor},!}(\mathrm{quot}^*\mathcal{L}^d_{\mathsf{E}} \otimes \mathrm{ext}^*\,\mathsf{L}_\psi))_{|\Omega\text{-Plücker}_{D_1,\dots,D_n}}&\\ = \Psi_{D_1,\dots,D_n} \otimes \mathrm{div}^*\,\mathsf{W}_{\mathsf{E}}[-2c](-c).&\end{aligned}$$

The sheaf W_{E} and the constant c depend on the parabolic degrees of the $(D_i)_{i=1,\dots,n}$ and will be defined explicitly in the proof.

Note that the first assertion is the geometric reformulation of the support condition for the Whittaker function given in 1.2 and the second implies that the function $\mathrm{tr}_{\mathbf{R}\,\mathrm{forget}_{\mathrm{Tor},!}(\mathrm{quot}^*\mathcal{L}_{\mathsf{E}}^d\otimes\mathrm{ext}^*\,\mathsf{L}_\psi)}$ defines an element in $\mathcal{C}^\infty(\mathrm{GL}_n(\mathbb{A}))^{\mathsf{N}_n(\mathbb{A}),\Psi}$.

Proof. — We may assume that all D_i's are effective, since otherwise the fibres of $\mathrm{forget}_{\mathrm{Tor}}$ above $\Omega\text{-Plücker}_{D_1,\dots,D_n}$ are empty.

To study the fibres of the map $\mathrm{forget}_{\mathrm{Tor}}$, we note that this map factors through the stack $\Omega_{D_1,\dots,D_k}\,\mathrm{Ext}_{D_{k+1},\dots,D_n}$, which we define as the stack classifying

$$\left\langle \begin{array}{c} \mathcal{J}_1^\bullet \subset \cdots \subset \mathcal{J}_n^\bullet \subset \mathcal{E}^\bullet \\ \mathcal{J}_i^\bullet/\mathcal{J}_{i-1}^\bullet \xrightarrow{\sim} \Omega^{\bullet,n-i}(D_i) \text{ for } i \leq k \\ \mathcal{J}_i^\bullet/\mathcal{J}_{i-1}^\bullet \xrightarrow{\sim} \Omega^{\bullet,n-i} \text{ for } i > k \end{array} \right\rangle$$

such that $\mathcal{J}_k^\bullet \subset \mathcal{E}^\bullet$ is a maximal embedding and $\mathcal{J}_i^\bullet \subset \mathcal{E}^\bullet$ lies above $\Omega\text{-Plücker}_{D_1,\dots,D_n}$.

Consider the case $k = 1$ and denote by

$$\mathrm{forget}_{D_1} : \Omega\text{-}\,\mathrm{Ext}_{D_1,\dots,D_n} \longrightarrow \Omega_{D_1}\,\mathrm{Ext}_{D_2,\dots,D_n}$$

the forgetful map, which maps

$$\big(\mathcal{E}^\bullet, (\mathcal{J}_i^\bullet)_{i=1,\dots,n}\big) \longmapsto \big(\mathcal{E}^\bullet, (\mathcal{J}_i^\bullet + \mathcal{J}_1^{\bullet,\max})_{i=1,\dots,n}\big),$$

$\mathcal{J}_1^{\bullet,\max} = \Omega^{\bullet,n-1}(D_1)$ being the subbundle defined by $\mathcal{J}_1^\bullet$.

A point in the latter stack can alternatively be described as a maximal embedding $\Omega^{\bullet,n-1}(D_1) \hookrightarrow \mathcal{E}^\bullet$ together with a filtration

$$\bar{\mathcal{J}}_2^\bullet \subset \cdots \subset \bar{\mathcal{J}}_n^\bullet \subset \mathcal{E}^\bullet/(\Omega^{\bullet,n-1}(D_1))$$

and identifications $\bar{\mathcal{J}}_i^\bullet/\bar{\mathcal{J}}_{i-1}^\bullet \xrightarrow{\sim} \Omega^{\bullet,n-i}$. In this description the fibres of forget_{D_1} consist of the liftings of the inclusion

$$\begin{tikzcd} & \bar{\mathcal{J}}_n^\bullet \arrow[dl, dashed] \arrow[d, hook] \\ \mathcal{E}^\bullet/\Omega^{\bullet,n-1} \arrow[r, two heads] & \mathcal{E}^\bullet/(\Omega^{\bullet,n-1}(D_1)). \end{tikzcd}$$

And $\mathcal{E}^\bullet/\Omega^{\bullet,n-1} \cong \Omega_{D_1}^{\bullet,n-1}(D_1) \oplus \mathcal{E}^\bullet/(\Omega^{\bullet,n-1}(D_1))$, thus forget_{D_1} is a torsor for the group $\mathrm{Hom}(\bar{\mathcal{J}}_n^\bullet, \Omega_{D_1}^{\bullet,n-1}(D_1))$.

To describe such liftings we first lift the inclusion $\Omega^{\bullet,n-2} \cong \bar{\mathcal{J}}_2^\bullet \subset \mathcal{E}^\bullet/(\Omega^{\bullet,n-1}(D_1))$ to $\Omega^{\bullet,n-2} \xhookrightarrow{\tilde{\jmath}} \mathcal{E}^\bullet/\Omega^{\bullet,n-1}$ and then lift $\bar{\mathcal{J}}_n^\bullet/\Omega^{\bullet,n-2}$ to the cokernel of $\tilde{\jmath}$. Note that for a point in a fixed fibre of forget_{D_1}, its image under the map $\mathrm{ext} : \langle \Omega\text{-}\,\mathrm{Ext} \subset \mathcal{E}^\bullet\rangle \to \mathbb{A}^1$ depends only on the choice of $\tilde{\jmath}$ but not on the lift of $\bar{\mathcal{J}}^\bullet_n/\Omega^{\bullet,n-2}$, i.e. ext factors through the stack classifying points of $\Omega_{D_1}\operatorname{Ext}_{D_2,\dots,D_n}$ together with a lift $\tilde{\jmath}$. This is because the extension of $\Omega^{\bullet,n-2}$ by $\Omega^{\bullet,n-1}$ is given by the connecting homomorphism:

$$\operatorname{Hom}(\Omega^{\bullet,n-2},\mathcal{E}^\bullet/\Omega^{\bullet,n-1}) \longrightarrow \operatorname{Ext}^1(\Omega^{\bullet,n-1},\Omega^{\bullet,n-2}).$$

- *Assume that* $D_1 \not\leq D_2$. We claim that in this case

$$\mathbf{R}\operatorname{forget}_{D_1,!}(\operatorname{quot}^*\mathcal{L}^d_{\mathsf{E}}\otimes \operatorname{ext}^*\mathsf{L}_\psi)=0. \tag{5.2}$$

Write D for the effective part of D_1-D_2. Then the group

$$\operatorname{Hom}\big(\Omega^{\bullet,n-2},\Omega^{\bullet,n-1}_D(D)\big)\subset \operatorname{Hom}\big(\Omega^{\bullet,n-2},\Omega^{\bullet,n-1}_{D_1}(D_1)\big)$$

acts on the choices of $\tilde{\jmath}$. Note that this action changes the image under the map ext by the residue of the element in $\operatorname{Hom}(\Omega^{\bullet,n-2},\Omega^{\bullet,n-1}_D(D))$. However the cokernel of $\tilde{\jmath}$ is not affected by this action, because by construction we have a surjective map

$$\operatorname{Hom}\big(\Omega^{\bullet,n-2}(D_2),\Omega^{\bullet,n-1}_{D_1}(D_1)\big)\twoheadrightarrow \operatorname{Hom}\big(\Omega^{\bullet,n-2},\Omega^{\bullet,n-1}_D(D)\big)$$

and thus given $\tilde{\jmath}$ and $s\in\operatorname{Hom}(\Omega^{\bullet,n-2},\Omega^{\bullet,n-1}_D(D))$ we can find an isomorphism

$$\begin{array}{ccccccc}
\Omega^{\bullet,n-2} & \xrightarrow{\tilde{\jmath}} & \mathcal{E}^\bullet/(\Omega^{\bullet,n-1}(D_1))\oplus\Omega^{\bullet,n-1}_{D_1}(D_1) & \longrightarrow & \mathcal{F}^\bullet_{n-2} & \to & 0\\
\mathrm{id}\downarrow = & & \downarrow\cong & & & & \\
\Omega^{\bullet,n-2} & \xrightarrow{\tilde{\jmath}+s} & \mathcal{E}^\bullet/(\Omega^{\bullet,n-1}(D_1))\oplus\Omega^{\bullet,n-1}_{D_1}(D_1) & \longrightarrow & \mathcal{F}'^\bullet_{n-2} & \to & 0
\end{array}$$

simply by choosing a splitting of $\Omega^{\bullet,n-2}(D_2)\xhookrightarrow{\text{maximal}}\mathcal{E}^\bullet/(\Omega^{\bullet,n-1}(D_1))$ locally at D.

To see that this implies Formula (5.2), fix a lifting $\tilde{\jmath}$, denote by $\mathcal{F}^\bullet_{n-2}$ the cokernel of $\tilde{\jmath}$ and let $\operatorname{Lift}_{/n-2}$ be the space of liftings of $\bar{\mathcal{J}}^\bullet_n/\Omega^{\bullet,n-2}$ to $\mathcal{F}^\bullet_{n-2}$.

Consider the preimage of the $\operatorname{Hom}(\Omega^{\bullet,n-2},\Omega^{\bullet,n-1}_D(D))$-orbit of $\tilde{\jmath}$ in $\Omega\text{-}\operatorname{Ext}_{D_1,\dots,D_n}$. Then the above tells us that this preimage is isomorphic to the product

$$\operatorname{Hom}(\Omega^{\bullet,n-2},\Omega^{\bullet,n-1}_D(D))\times\operatorname{Lift}_{/n-2},$$

and furthermore the restriction of $\mathrm{quot}^*\mathcal{L}_{\mathsf{E}}^d \otimes \mathrm{ext}^* \mathsf{L}_\psi$ to this space is an exterior product, i.e. ext factors through the projection to the first factor and quot factors through the projection to $\mathrm{Lift}_{/n-2}$. But $\mathrm{ext}^* \mathsf{L}_\psi$ is nontrivial on the factor $\mathrm{Hom}(\Omega^{\bullet,n-2}, \Omega_D^{\bullet,n-1}(D))$, and therefore its cohomology is trivial. Thereby we get that $\mathbf{R}\,\mathrm{forget}_{D_1,!}(\mathrm{quot}^*\mathcal{L}_{\mathsf{E}}^d \otimes \mathrm{ext}^* \mathsf{L}_\psi) = 0$ as well.

- *Assume now that* $D_1 \leq D_2$. In this case we can define a map

$$\mathrm{ext}_{D_1} : \Omega_{D_1}\,\mathrm{Ext}_{D_2,\dots,D_n} \longrightarrow \mathbb{A}^1$$

given as the composition

$$\begin{aligned}\Omega_{D_1}\,\mathrm{Ext}_{D_2,\dots,D_n} &\longrightarrow \mathrm{Ext}^1(\Omega^{\bullet,n-2}(D_2), \Omega^{\bullet,n-1}(D_1)) \\ &\quad \times \prod_{i=2}^{n-1} \mathrm{Ext}^1(\Omega^{\bullet,n-i-1}, \Omega^{\bullet,n-i}) \\ &\longrightarrow \prod_{i=1}^{n-1} \mathrm{Ext}^1(\Omega^{\bullet,n-i-1}, \Omega^{\bullet,n-i}) \xrightarrow{\Sigma\mathrm{Res}} \mathbb{A}^1.\end{aligned}$$

We will use this map to write the restriction of $\mathrm{ext}^* \mathsf{L}_\psi$ to $\Omega\text{-}\mathrm{Ext}_{D_1,\dots,D_n}$ as a product of two local systems. To this end note that – because $D_1 \leq D_2$ – for any point in $\Omega_{D_1}\,\mathrm{Ext}_{D_2,\dots,D_n}$ there is a canonical lifting $\tilde{\jmath}\colon \Omega^{\bullet,n-2} \to \mathcal{E}^\bullet/\Omega^{\bullet,n-1}$ (choose any lifting $\Omega^{\bullet,n-2}(D_2) \to \mathcal{E}^\bullet/\Omega^{\bullet,n-1}$ and restrict this to $\Omega^{\bullet,n-2}$ – this is independent of the choice since $D_1 \leq D_2$). Moreover, for any point in $\Omega_{D_1}\,\mathrm{Ext}_{D_2,\dots,D_n}$ its image under ext_{D_1} is the same as ext applied to this canonical lifting.

Furthermore, for any point of this space the torsion sheaf $\mathcal{E}^\bullet/\mathcal{J}_n^\bullet$ is equipped with a filtration induced by the $\mathcal{J}_i^\bullet$'s with subquotients isomorphic to $\Omega_{D_i}^{\bullet,n-i}(D_i)$. Denote by $\mathrm{Ext}(D_n,\dots,D_1)$ the stack of parabolic torsion sheaves together with such a filtration.

Since $D_1 \leq D_2$ we can define a residue map for sheaves in $\mathrm{Ext}(D_2,D_1)$, because we have an exact sequence

$$\begin{aligned}\mathrm{Hom}\big(\Omega^{\bullet,n-2}(D_2), \Omega_{D_1}^{\bullet,n-1}(D_1)\big) &\xrightarrow{0} \mathrm{Hom}\big(\Omega^{\bullet,n-2}, \Omega_{D_1}^{\bullet,n-1}(D_1)\big) \\ &\longrightarrow\!\!\!\!\!\rightarrow \mathrm{Ext}^1\big(\Omega_{D_2}^{\bullet,n-2}(D_2), \Omega_{D_1}^{\bullet,n-1}(D_1)\big),\end{aligned}$$

and therefore the usual residue map $\mathrm{Res} : \mathrm{Hom}(\Omega^{\bullet,n-2}, \Omega_{D_1}^{\bullet,n-1}(D_1)) \to \mathbb{A}^1$ factors through $\mathrm{Ext}(D_2,D_1)$. Let Ψ_{12} be the pull-back of L_ψ via the composition

$$\mathrm{Ext}(D_n,\dots,D_1) \longrightarrow \mathrm{Ext}(D_2,D_1) \overset{\mathrm{Res}}{\to} \mathbb{A}^1.$$

Then we have a diagram

$$\begin{tikzcd}
\Omega\text{-}\operatorname{Ext}_{D_1,\dots,D_n} \arrow[d, "\mathrm{pr}_{\mathrm{Fib}}"] \arrow[dr, "\mathrm{qext}"] \arrow[drr, "\mathrm{quot}"] \arrow[dd, bend right=60, "\mathrm{forget}_{D_1}"'] & & \\
\mathrm{Fib} \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] & \operatorname{Ext}(D_n,\dots,D_1) \arrow[r, "\mathrm{pr}_{\mathrm{big}}"'] \arrow[d, "\mathrm{gr}_{D_1}"] & \mathrm{Coh}^{d}_{0,S} \\
\Omega_{D_1}\operatorname{Ext}_{D_2,\dots,D_n} \arrow[r, "\mathrm{qext}_{n-2}"] & C^{(d_1)} \times \operatorname{Ext}(D_n,\dots,D_2). &
\end{tikzcd}$$

Here Fib is the fibre product making the lower square cartesian, and the maps are the natural projections. The additivity of L_ψ implies that

$$\mathrm{ext}^*\,\mathsf{L}_\psi \cong \mathrm{forget}^*_{D_1}(\mathrm{ext}^*_{D_1}\,\mathsf{L}_\psi) \otimes \mathrm{qext}^*\Psi_{12}.$$

Furthermore the map pr_{Fib} is a $\operatorname{Hom}(\mathcal{E}^\bullet/\Omega^{\bullet,n-1}(D_1), \Omega^{\bullet,n-1}_{D_1}(D_1))$-bundle, because the fibres of $\mathrm{pr}_{\mathrm{Fib}}$ consist of the different choices of the dotted arrow in

$$\begin{tikzcd}
\Omega^{\bullet,n-1} \arrow[d, hook] & & \overline{\mathcal{J}}^\bullet_n \arrow[d, hook] \\
\Omega^{\bullet,n-1}(D_1) \arrow[r, hook] \arrow[d, two heads] & \mathcal{E}^\bullet \arrow[r] \arrow[d, dashed] & \mathcal{E}^\bullet/\Omega^{\bullet,n-1}(D_1) \arrow[d, two heads] \\
\Omega^{\bullet,n-1}_{D_1}(D_1) \arrow[r, hook] & \mathcal{T}^\bullet \arrow[r, two heads] & \mathcal{T}''^\bullet .
\end{tikzcd}$$

Therefore the projection formula and base-change imply that

$$\begin{aligned}
&\mathbf{R}\,\mathrm{forget}_{D_1,!}(\mathrm{ext}^*\,\mathsf{L}_\psi \otimes \mathrm{quot}^*\mathcal{L}^d_{\mathsf{E}}) \\
&\quad\cong \mathbf{R}\,\mathrm{forget}_{D_1,!}(\mathrm{forget}^*_{D_1}\mathrm{ext}^*_{D_1}\,\mathsf{L}_\psi \otimes \mathrm{qext}^*\Psi_{12} \otimes \mathrm{quot}^*\mathcal{L}^d_{\mathsf{E}}) \\
&\qquad\cong \mathrm{ext}^*_{D_1}\,\mathsf{L}_\psi \otimes \mathrm{qext}^*_{n-2}(\mathbf{R}\,\mathrm{gr}_{D_1,!}(\mathrm{pr}^*_{\mathrm{big}}\mathcal{L}^d_{\mathsf{E}} \otimes \Psi_{12}))[-2c_1](-c_1),
\end{aligned}$$

where $c_1 = \dim(\operatorname{Hom}(\mathcal{E}^\bullet/(\Omega^{\bullet,n-1}(D_1)), \Omega^{\bullet,n-1}_{D_1})$.

Now we can inductively apply the same considerations to the maps $\mathrm{forget}_{D_i} : \Omega_{D_1,\dots,D_{i-1}}\operatorname{Ext}_{D_i,\dots,D_n} \to \Omega_{D_1,\dots,D_i}\operatorname{Ext}_{D_i+1,\dots,D_n}$ to prove:

1) $\mathbf{R}\,\mathrm{forget}_{\mathrm{Tor},!}(\mathrm{ext}^*\,\mathsf{L}_\Psi \otimes \mathrm{quot}^*\mathcal{L}^d_{\mathsf{E}}) = 0$ unless $0 \le D_1 \ldots \le D_n$.

2) If we have $0 \leq D_1 \leq \ldots \leq D_n$, then we may define a sheaf Ψ_{Tor} on the stack $\mathrm{Ext}(D_n, D_{n-1}, \ldots, D_1)$ as the tensor product of the sheaves $\Psi_{i,i+1}$ defined as the pull back of L_ψ via the map:

$$\mathrm{Ext}(D_n, \ldots, D_1) \longrightarrow \mathrm{Ext}(D_{i+1}, D_i) \xrightarrow{\mathrm{Res}} \mathbb{A}^1.$$

3) Denote gr the natural map

$$\mathrm{gr} : \mathrm{Ext}(D_n, D_{n-1}, \ldots, D_1) \longrightarrow C^{(d_1)} \times \cdots \times C^{(d_n - d_{n-1})}$$

and define $\mathsf{W}_{\mathsf{E}} := \mathbf{R}\, \mathrm{gr}_!(\mathrm{pr}^*_{\mathrm{big}} \mathcal{L}^d_{\mathsf{E}} \otimes \Psi_{\mathrm{Tor}})$. Then

$$\mathbf{R}\, \mathrm{forget}_{\mathrm{Tor},!}(\mathrm{ext}^*\, \mathsf{L}_\psi \otimes \mathrm{quot}^* \mathcal{L}^d_{\mathsf{E}}) \cong \Psi_{D_1,\ldots,D_n} \otimes \mathrm{div}^*\, \mathsf{W}_{\mathsf{E}}[-2c](-c),$$

where $c = \sum_{i=1}^{n-1} c_i$ and $c_i = \dim(\mathrm{Hom}(\mathcal{E}^\bullet / \mathcal{J}_i^\bullet, \Omega^{\bullet, n-i}_{D_i}))$. □

To compare the trace function of $\mathbf{R}\, \mathrm{forget}_{\mathrm{Tor},!}(\mathrm{ext}^*\, \mathsf{L}_\psi \otimes \mathrm{quot}^* \mathcal{L}^d_{\mathsf{E}})$ and W_{E} we therefore only need to calculate the trace function of W_{E}. Denote the trace of W_{E} at the set of divisors $D_1 \leq \cdots \leq D_n$ by $\mathrm{tr}(\mathrm{Frob}_{D_1,\ldots,D_n}, \mathsf{W}_{\mathsf{E}})$. By construction it is sufficient to calculate this in the case that all D_i's are supported at a single point p, because we can write $D_i = \sum_{p \in \mathrm{supp}(D_i)} D_{i,p}$ with divisors $D_{i,p}$ supported at p and then

$$\mathrm{tr}(\mathrm{Frob}_{D_1,\ldots,D_n}, \mathsf{W}_{\mathsf{E}}) = \prod_{p \in \mathrm{supp}(D_n)} \mathrm{tr}(\mathrm{Frob}_{D_{1,p},\ldots,D_{n,p}}, \mathsf{W}_{\mathsf{E}}).$$

We may also assume that $p \in S$, because for $p \notin S$ we can use the calculations for unramified local systems [11] (note however that a calculation similar to the one we do below (Lemma 5.4) could be applied for $p \notin S$ as well).

5.2. Calculation in the case rank = 2.

We want to compute the trace function of the sheaf W_{E} defined in the preceding paragraph in the case of a 2-step parabolic structure. We use the above reductions, i.e. we take $D_1 = kp \leq D_2 = (d-k)p$ parabolic divisors supported at $p \in S$ with $d \in \mathbb{N}$. Recall from the proof of the last proposition that for such parabolic divisors we have defined a residue map $\mathrm{Res} : \mathrm{Ext}(D_2, D_1) = \mathrm{Ext}(\mathcal{O}^\bullet_{(d-k)p}((d-k)p), \Omega^\bullet_{kp}(kp)) \to \mathbb{A}^1$ and a sheaf $\Psi_{\mathrm{Tor}} = \mathrm{Res}^*\, \mathsf{L}_\psi$. Further, by abuse of notation, we denote the pull-back of $\mathcal{L}^d_{\mathsf{E}}$ to $\mathrm{Ext}(D_2, D_1)$ by the same symbol. Finally we will replace the stack Ext by corresponding set Ext^1 to prove the following formula:

LEMMA 5.4. — *Consider sheaves with 2-step parabolic structure at* $S = \{p\} \in C$. *Denote by* $\lambda_{\mathsf{E}} := \operatorname{tr}(\mathrm{Frob}_p, j_*\mathsf{E})$. *Then for any* $d \in \mathbb{N}$ *and* $k \in \frac{1}{2}\mathbb{N}$ *with* $0 \leq k \leq d-k$ *we have*

$$\sum_{e \in \mathrm{Ext}^1(\mathcal{O}^\bullet_{(d-k)p}((d-k)p), \Omega^\bullet_{kp}(kp))} \operatorname{tr}(\mathrm{Frob}_e, \Psi_{\mathrm{Tor}} \otimes \mathcal{L}^d_{\mathsf{E}}) = \begin{cases} +q^{2k}\lambda^d_{\mathsf{E}} & \textit{for } k \in \mathbb{N}, \\ -q^{2k}\lambda^d_{E} & \textit{for } k \in \frac{1}{2}+\mathbb{N}. \end{cases}$$

Before we prove this lemma we need to use the Hecke property of $\mathcal{L}^d_{\mathsf{E}}$ to give a recursive formula for the trace function $\operatorname{tr}_{\mathcal{L}^d_{\mathsf{E}}}$: as in the unramified situation we know that $\operatorname{tr}(\mathrm{Frob}_{\mathcal{O}^\bullet_{dp}(dp)}, \mathcal{L}^d_{\mathsf{E}}) = \lambda^d_{\mathsf{E}} = \operatorname{tr}(\mathrm{Frob}_{\Omega^\bullet_{dp}(dp)}, \mathcal{L}^d_{\mathsf{E}})$, because the parabolic torsion sheaves $\mathcal{O}^\bullet_{dp}(dp)$ and $\Omega^\bullet_{dp}(dp)$ are both contained in the image of an open embedding $\mathrm{Coh}^d_{0,C} \hookrightarrow \mathrm{Coh}^d_{0,S}$. Therefore, the Hecke property of $\mathcal{L}^d_{\mathsf{E}}$ (Proposition 4.8) implies on the level of functions that

$$\sum_{e \in \mathrm{Ext}^1(\mathcal{O}^\bullet_{(d-k)p}((d-k)p), \Omega^\bullet_{kp}(kp))} \operatorname{tr}(\mathrm{Frob}_e, \mathcal{L}^d_{\mathsf{E}}) = \begin{cases} q^k\lambda^d_{\mathsf{E}} & \text{for } k \in \mathbb{N}, \\ 0 & \text{for } k \in \frac{1}{2}+\mathbb{N}. \end{cases} \tag{5.3}$$

(Note that the set Ext^1 used above differs from the stack $\underline{\mathrm{Ext}}$ by some automorphisms, whereby we obtain the factor q^k in the above formula.) Recall that since $k \leq d-k$ we have an isomorphism

$$\mathrm{Hom}\big(\mathcal{O}^\bullet, \Omega^\bullet_{kp}(kp)\big) \cong \mathrm{Ext}^1\big(\mathcal{O}^\bullet_{(d-k)p}((d-k)p), \Omega^\bullet_{kp}(kp)\big)$$

given by mapping a homomorphism s to the push out of the extension $\mathcal{O}^\bullet \to \mathcal{O}^\bullet((d-k)p) \to \mathcal{O}^\bullet_{(d-k)p}((d-k)p)$ by s. Thus the middle term of the resulting extension of torsion sheaves is

$$\operatorname{coker}\big(\mathcal{O}^\bullet \xrightarrow{(1,s)} \mathcal{O}^\bullet((d-k)p) \oplus \Omega^\bullet_{kp}(kp)\big) =: \mathcal{T}^\bullet_s.$$

Further $\mathrm{Hom}(\mathcal{O}^\bullet, \Omega^\bullet_{kp}(kp)) \cong \mathrm{Hom}(\mathcal{O}, \Omega^{(0,p)}_{kp}(kp))$, therefore we have a filtration of Ext^1 given by

$$\mathrm{Hom}\big(\mathcal{O}^\bullet, \Omega^\bullet_{kp}(kp)\big) \supset \mathrm{Hom}\big(\mathcal{O}^\bullet, \Omega^\bullet_{(k-1)p}((k-1)p)\big) \supset \cdots \supset 0$$

and for any element s of the subset

$$\mathrm{Hom}\big(\mathcal{O}^\bullet, \Omega^\bullet_{(k-i)p}((k-i)p)\big) - \mathrm{Hom}\big(\mathcal{O}^\bullet, \Omega^\bullet_{(k-i-1)p}((k-i-1)p)\big)$$

the corresponding parabolic torsion sheaf $\mathcal{T}_s^\bullet$ is isomorphic to

$$\mathcal{T}_s^\bullet \cong \begin{cases} \Omega^\bullet_{(d-i)p} \oplus \mathcal{O}^\bullet_{ip} & \text{for } 0 \leq i < k \text{ and } k \in \frac{1}{2} + \mathbb{N}, \\ \Omega^\bullet_{(d-i-\frac{1}{2})p} \oplus \mathcal{O}^\bullet_{(i+\frac{1}{2})} & \text{for } 0 \leq i < k \text{ and } k \in \mathbb{N}_{>0}, \\ \mathcal{O}^\bullet_{(d-k)p} \oplus \Omega^\bullet_{kp} & \text{if } s = 0. \end{cases}$$

It might be helpful to write this out in the simplest cases: for $k \in \frac{1}{2} + \mathbb{N}$ we have

$$\Omega^\bullet_{kp}(kp) \cong \big(\Omega_{(k+1/2)p} \twoheadrightarrow \Omega_{(k-1/2)p} \rightarrow\big).$$

Thus, if $i = 0$, i.e. $s : \mathcal{O} \rightarrow \Omega_{kp}(kp)$ induces a surjective map $\mathcal{O}^\bullet \twoheadrightarrow \Omega^\bullet_{kp}(kp)$, the above cokernel is isomorphic to $(\Omega_{dp} \rightarrow \Omega_{dp}(p) \rightarrow)$ and the second map is an isomorphism. In particular for $d = 1, k = \frac{1}{2}$ this extension is of the form $(\Omega_p \xrightarrow{0} \mathcal{O}_p \xrightarrow{\phi_1})$.

Similarly for $k \in \mathbb{N}_{>0}$,

$$\Omega^\bullet_{kp}(kp) \cong \big(\Omega_{kp}((k-1)p) \rightarrow \Omega_{kp}(kp) \rightarrow\big).$$

And again if $s : \mathcal{O} \twoheadrightarrow \Omega_{kp}$ is surjective we get that the corresponding torsion sheaf is of the form $(\Omega_{dp} \rightarrow \Omega_{(d-1)p} \oplus \mathcal{O}_p \rightarrow)$, because s induces a non-surjective map on the $(1,p)$-component of $s^\bullet : \mathcal{O}^\bullet \rightarrow \Omega^\bullet_{kp}(kp)$.

The general case is proven in the same way, the above considerations already give the isomorphism classes of the $\mathcal{T}_s^{(i,p)}$ and we also know on which summands the homomorphisms $\phi^{(i,p)}$ giving the parabolic structure of $\mathcal{T}_s^\bullet$ are injective or surjective.

Therefore if we rewrite the summation in (5.3) according to the above filtration of Ext^1 we get a recursion relation for the value of the trace at the trivial extension

$$L^d_{\mathsf{E}}(k) := \mathrm{tr}(\mathrm{Frob}_{\mathcal{O}^\bullet_{(d-k)p}((d-k)p) \oplus \Omega^\bullet_{kp}(kp)}, \mathcal{L}^d_{\mathsf{E}}),$$

$$L^d_{\mathsf{E}}(k) = \begin{cases} q^k \lambda^d_{\mathsf{E}} - (q-1) \sum\limits_{i=0}^{k-1} q^i L^d_{\mathsf{E}}(k - \frac{1}{2} - i) & \text{for } k \in \mathbb{N}, \\ -(q-1) \sum\limits_{i=0}^{k-\frac{1}{2}} q^i L_{\mathsf{E}}(k - \frac{1}{2} - i) & \text{for } k \in \frac{1}{2} + \mathbb{N}. \end{cases} \tag{5.4}$$

(To shorten the formula we used that $L^d_{\mathsf{E}}(k) = L^d_{\mathsf{E}}(d-k)$, since the corresponding torsion sheaves differ only by a shift.)

Note further that this recursion relation does not depend on the rank of E.

Proof of Lemma 5.4. — By induction on k (for $k=0$ there is nothing to show). Since $\mathrm{tr}(\mathrm{Frob}_e, \Psi_{\mathrm{Tor}} \otimes \mathcal{L}^d_{\mathsf{E}}) = \psi(\mathrm{Res}(e)) \cdot \mathrm{tr}(\mathrm{Frob}_e, \mathcal{L}^d_{\mathsf{E}})$, all the summands corresponding to elements of

$$\mathrm{Hom}\big(\mathcal{O}^\bullet, \Omega^\bullet_{(k-i+1)p}((k-i+1)kp)\big) - \mathrm{Hom}\big(\mathcal{O}^\bullet, \Omega^\bullet_{(k-i)p}((k-i)p)$$

for $k-i \geq 1$ cancel out, because for these $\sum \psi(\mathrm{Res}(e)) = 0$. Thus

$$\sum_{e \in \mathrm{Ext}^1(\mathcal{O}^\bullet_{(d-k)p}((d-k)p), \Omega^\bullet_{kp}(kp))} \psi\big(\mathrm{Res}(e)\big) L^d_{\mathsf{E}}(e) = L^d_{\mathsf{E}}(k) - L^d_{\mathsf{E}}(k - \tfrac{1}{2}).$$

Apply (5.4) to $L^d_{\mathsf{E}}(k)$, then this equals

$$= \begin{cases} q^k \lambda^d_{\mathsf{E}} - q L^d_{\mathsf{E}}(k-\tfrac{1}{2}) - (q-1) \sum\limits_{i=1}^{k-1} q^i L^d_{\mathsf{E}}(k-\tfrac{1}{2}-i) & \text{for } k \in \mathbb{N}, \\ -q L^d_{\mathsf{E}}(k-\tfrac{1}{2}) - (q-1) \sum\limits_{i=1}^{k-\frac{1}{2}} q^i L^d_{\mathsf{E}}(k-\tfrac{1}{2}-i) & \text{for } k \in \tfrac{1}{2}+\mathbb{N}, \end{cases}$$

$$= \begin{cases} \Big(q^k \lambda^d_{\mathsf{E}} - \sum\limits_{i=0}^{k-1} q^{i+1} \big(L^d_{\mathsf{E}}(k-\tfrac{1}{2}-i) - L^d_{\mathsf{E}}(k-1-i)\big) \\ \qquad - \sum\limits_{i=0}^{k-2} q^{i+1} \big(L^d_{\mathsf{E}}(k-1-i) - L^d_{\mathsf{E}}(k-\tfrac{3}{2}-i)\big) - q^k L^d_{\mathsf{E}}(0)\big)\Big) & \text{for } k \in \mathbb{N}, \\ -\sum\limits_{i=0}^{k-\frac{1}{2}} q^{i+1} L^d_{\mathsf{E}}(k-\tfrac{1}{2}-i) + \sum\limits_{i=0}^{k-\frac{3}{2}} q^{i+1} L^d_{\mathsf{E}}(k-\tfrac{3}{2}-i) & \text{for } k \in \tfrac{1}{2}+\mathbb{N}. \end{cases}$$

By induction $L^d_{\mathsf{E}}(k-\tfrac{1}{2}-i) - L^d_{\mathsf{E}}(k-1-i) = -q^{2(k-i-\frac{1}{2})} \lambda^d_{\mathsf{E}}$ for $k \in \mathbb{N}$:

$$= \begin{cases} -\sum\limits_{i=0}^{k-1} q^{i+1}(-q^{2(k-i)-1}\lambda^d_{\mathsf{E}}) - \sum\limits_{i=0}^{k-2} q^{i+1}(q^{2(k-1)-2}\lambda^d_{\mathsf{E}}) & \text{for } k \in \mathbb{N}, \\ -\sum\limits_{i=0}^{k-\frac{1}{2}} q^{i+1}(q^{2(k-i)-1}\lambda^d_{\mathsf{E}}) - \sum\limits_{i=0}^{k-\frac{3}{2}} q^{i+1}(-q^{2(k-i-1)}\lambda^d_{\mathsf{E}}) & \text{for } k \in \tfrac{1}{2}+\mathbb{N}, \end{cases}$$

$$= \begin{cases} +q^{2k}\lambda^d_{\mathsf{E}} & \text{for } k \in \mathbb{N}, \\ -q^{2k}\lambda^d_{\mathsf{E}} & \text{for } k \in \tfrac{1}{2}+\mathbb{N}. \end{cases} \qquad \square$$

COROLLARY 5.5. — *Let E be a local system on $C - S$ with indecomposable unipotent ramification at S, denote by $\lambda_{\mathsf{E}} := \prod_{p \in S} \mathrm{tr}(\mathrm{Frob}_p, j_* \mathsf{E})$ and let W_{E} be the Whittaker function defined in Section* 1.1.

1) *If E is of rank 2, then for any point $x \in \Omega$-Plücker we have*

$$\mathrm{tr}\big(\mathrm{Frob}_x, \mathbf{R}\,\mathrm{forget}_{\mathrm{Tor},!}(\mathrm{ext}^* \mathsf{L}_\psi \otimes \mathrm{quot}^* \mathcal{L}^d_{\mathsf{E}})\big) = \lambda_{\mathsf{E}} \cdot q^{|S|} \cdot W_{\mathsf{E}}(x).$$

In particular for any point $\bar{x} \in \mathrm{Hom}_2^{\mathrm{inj}}$ *we have*

$$\mathrm{tr}(\mathrm{Frob}_{\bar{x}}, \mathsf{F}^2_{\mathsf{E},!}) = \lambda_{\mathsf{E}} \cdot q^{|S|} \cdot \Phi(W_{\mathsf{E}})(\bar{x}).$$

2) *If* E *is of rank* 3, *then for any point* $x \in \Omega$-Plücker *with* $D_1 = 0$ *we have*

$$\mathrm{tr}\big(\mathrm{Frob}_x, \mathbf{R}\,\mathrm{forget}_{\mathrm{Tor},!}(\mathrm{ext}^* \mathsf{L}_\psi \otimes \mathrm{quot}^* \mathcal{L}^d_{\mathsf{E}})\big) = \lambda^3_{\mathsf{E}} q^{3|S|} W_{\mathsf{E}}(x).$$

In particular, for any point $\bar{x} \in \mathrm{Hom}_3^{\mathrm{inj}}$ *corresponding to a maximal embedding* $\Omega^{\bullet,2} \hookrightarrow \mathcal{E}^\bullet$ *we have*

$$\mathrm{tr}(\mathrm{Frob}_{\bar{x}}, \mathsf{F}^3_{\mathsf{E},!}) = \lambda_{\mathsf{E}} \cdot q^{3|S|} \cdot \Phi(W_{\mathsf{E}})(\bar{x}).$$

Proof. — Comparing the above lemma with the calculation of W_{E} we get the first assertion. Note that since the power of λ_{E} appearing on either side of the equation depends only on the degree, we just have to compare these for the trivial bundle. Similarly the power of q only depends on the difference $D_1 - D_2$.

For the second assertion note that for a maximal embedding, the quotient sheaf $\mathcal{E}^\bullet/\Omega^{\bullet,n-1}$ may be viewed as a bundle with $(n-1)$-step parabolic structure since the n-th morphism in the parabolic structure is an isomorphism. Thus for rank 3 bundles we may apply the calculation given in part 1). □

6. Constructing A_{E} under the assumption $\mathsf{F}^n_{\mathsf{E}} = \mathsf{F}^n_{\mathsf{E},!}$.

In this section we give a proof of the main Theorem 2.5 under the additional assumption that $\mathsf{F}^n_{\mathsf{E}} = \mathsf{F}^n_{\mathsf{E},!}$. Here the proofs are almost identical to the ones in the case of unramified local systems (see [19], [11]): first we show that the Hecke property for $\mathcal{L}^d_{\mathsf{E}}$ implies that $\mathsf{F}_{\mathsf{E},!}$ is a Hecke eigensheaf as well. The second step is to deduce from Lafforgue's theorem and the calculation of the previous section, that the function $t_{\mathsf{F}^n_{\mathsf{E},!}}$ (on $\mathrm{Hom}_n^{\mathrm{inj}}$) descends to a function on $\mathrm{Bun}^d_{n,S}$. Therefore we can argue as in [11] that the sheaf F_{E} also descends to the space of parabolic vector bundles. The resulting sheaf A_{E} inherits the Hecke property from $\mathsf{F}_{\mathsf{E},!}$, and we show that this property implies the one stated in the theorem.

6.1. The Hecke operators on the "fundamental diagram".

We want to check that Laumon's arguments in [19] carry over to our situation. We define operators analogous to the operators H^i_k on the spaces occurring in the fundamental diagram (2.2).

We start with $\mathrm{Hom}_k^{\mathrm{inj}}$ (recall that this is the stack classifying *good* coherent sheaves $\mathcal{F}^\bullet$ of generic rank k together with an injection $\Omega^{\bullet,k-1} \hookrightarrow \mathcal{F}^\bullet$). We define a diagram

$$\begin{tikzcd} \left\langle \begin{smallmatrix} \mathcal{F}'^\bullet \subset \mathcal{F}^\bullet \\ \Omega^{\bullet,k-1} \hookrightarrow \mathcal{F}^\bullet \end{smallmatrix} \right\rangle \arrow[d, "\pi_{\mathrm{big}}"] & \langle \Omega^{\bullet,k-1} \hookrightarrow \mathcal{F}'^\bullet \subset \mathcal{F}^\bullet \rangle \arrow[l, hook', "\iota"'] \arrow[d, "\pi_{\mathrm{small}} \times \mathrm{quot}"] \\ \mathrm{Hom}_k^{\mathrm{inj}} & \mathrm{Hom}_k^{\mathrm{inj}} \times \mathrm{Coh}_{0,S}^{\underline{i}} \end{tikzcd}$$

and the corresponding Hecke operator

$$\begin{aligned} H^{\underline{i}}_{k,\mathrm{Hom}^{\mathrm{inj}}} : D^b(\mathrm{Hom}_k^{\mathrm{inj}}) &\longrightarrow D^b(\mathrm{Hom}_k^{\mathrm{inj}} \times \mathrm{Coh}_{0,S}^{\underline{i}}), \\ \mathsf{K} &\longmapsto \mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_! \iota^* \pi_{\mathrm{big}}^* \mathsf{K}. \end{aligned}$$

Analogously, replacing $\mathrm{Hom}_k^{\mathrm{inj}}$ by Hom_k, we define an operator

$$H^{\underline{i}}_{k,\mathrm{Hom}} : D^b(\mathrm{Hom}_k) \longrightarrow D^b(\mathrm{Hom}_k \times \mathrm{Coh}_{0,S}^{\underline{i}}).$$

We used a shorthand notation to describe the algebraic stacks occurring in the above diagram, e.g. $\left\langle \begin{smallmatrix} \mathcal{F}'^\bullet \subset \mathcal{F}^\bullet \\ \Omega^{\bullet,k-1} \hookrightarrow \mathcal{F}^\bullet \end{smallmatrix} \right\rangle$ denotes the algebraic stack classifying objects $(\Omega^{\bullet,k-1} \hookrightarrow \mathcal{F}^\bullet) \in \mathrm{Hom}_k^{\mathrm{inj}}$, together with a coherent parabolic subsheaf $\mathcal{F}'^\bullet \subset \mathcal{F}^\bullet$ such that the quotient $\mathcal{F}^\bullet/\mathcal{F}'^\bullet \in \mathrm{Coh}_{0,S}^{\underline{i}}$ is a parabolic torsion sheaf of degree $\underline{i}$. And the maps are the natural ones, e.g. π_{small} is the map forgetting everything but the smaller bundle $\mathcal{F}'^{,\bullet}$ and quot forgets everything but the quotient $\mathcal{F}^\bullet/\mathcal{F}'^\bullet$. To make this easier to read we use the following conventions:

1) $\mathcal{F}^\bullet$ will always be a coherent parabolic sheaf; oftentimes a subscript will be used to specify its generic rank.

2) $\mathcal{E}^\bullet$ is a parabolic vector bundle, i.e. it is torsion free; again $\mathcal{E}_k^\bullet$ is a parabolic vector bundle of rank k.

3) By $\mathcal{T}^\bullet$ we will always denote a parabolic torsion sheaf.

4) Three term sequences will always be short exact sequences.

We have a similar diagram for Ext_k^1:

$$\begin{tikzcd} \left\langle \begin{smallmatrix} \mathcal{F}_k'^\bullet \subset \mathcal{F}_k^\bullet \\ \Omega^{\bullet,k-1} \to \mathcal{F}_{k+1}^\bullet \to \mathcal{F}_k^\bullet \end{smallmatrix} \right\rangle \arrow[r, two heads, "p"] \arrow[d, "\pi_{\mathrm{big}}"] & \left\langle \begin{smallmatrix} \mathcal{F}_k'^\bullet \subset \mathcal{F}_k^\bullet \\ \Omega^{\bullet,k-1} \to \mathcal{F}_{k+1}'^\bullet \to \mathcal{F}_k'^\bullet \end{smallmatrix} \right\rangle \arrow[d, "\pi_{\mathrm{small}} \times \mathrm{quot}"] \\ \mathrm{Ext}_k^1 & \mathrm{Ext}_k^1 \times \mathrm{Coh}_{0,S}^{\underline{i}} \end{tikzcd}$$

$$H^{\underline{i}}_{k,\mathrm{Ext}^1} : D^b(\mathrm{Ext}^1_k) \longrightarrow D^b(\mathrm{Ext}^1_k \times \mathrm{Coh}^{\underline{i}}_{0,S}),$$
$$\mathsf{K} \longmapsto \mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_! \mathbf{R}p_! \pi^*_{\mathrm{big}} \mathsf{K}.$$

And finally on $\mathrm{Ext}^{1,\mathrm{good}}_k$ we have:

$$\begin{tikzcd}
\left\langle \begin{matrix} \mathcal{F}'^\bullet_k \subset \mathcal{F}^\bullet_k \\ \Omega^{\bullet,k-1} \to \mathcal{F}^\bullet_{k+1} \to \mathcal{F}^\bullet_k \\ \mathcal{F}'^\bullet_k \times_{\mathcal{F}^\bullet_k} \mathcal{F}^\bullet_{k+1} \ \text{ good} \end{matrix} \right\rangle \arrow[r, two heads, "p"] \arrow[d, "\pi_{\mathrm{big}}"] & \left\langle \begin{matrix} \mathcal{F}'^\bullet_k \subset \mathcal{F}^\bullet_k \\ \Omega^{\bullet,k-1} \to \mathcal{F}'^\bullet_{k+1} \to \mathcal{F}'^\bullet_k \\ \mathcal{F}'^\bullet_{k+1} \ \text{ good} \end{matrix} \right\rangle \arrow[d, "\pi_{\mathrm{small}} \times \mathrm{quot}"] \\
\mathrm{Ext}^{1,\mathrm{good}}_k & \mathrm{Ext}^{1,\mathrm{good}}_k \times \mathrm{Coh}^{\underline{i}}_{0,S}
\end{tikzcd}$$

$$H^{\underline{i}}_{k,\mathrm{Ext}^{1,\mathrm{good}}} : D^b(\mathrm{Ext}^{1,\mathrm{good}}_k) \longrightarrow D^b(\mathrm{Ext}^{1,\mathrm{good}}_k \times \mathrm{Coh}^{\underline{i}}_{0,S}),$$
$$\mathsf{K} \mapsto \mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_! \mathbf{R}p_! \pi^*_{\mathrm{big}} \mathsf{K}.$$

6.2. The Hecke property of $\mathsf{F}^k_{\mathsf{E},!}$.

We want to show that these Hecke operators commute with the functors used to construct $\mathsf{F}^k_{\mathsf{E},!}$. We will use the notations of Definition 2.3, in particular the maps $j_{\mathrm{Hom}}, j_{\mathrm{Ext}}, I$. Let further $i^{(k,S)} := \deg(\mathcal{T}^{(k,S)})$ for any $\mathcal{T}^\bullet \in \mathrm{Coh}^{\underline{i}}_{0,S}$. And denote by $\mathrm{pr}_{\mathrm{Coh}^{\underline{d}}_{0,S}} : \mathrm{Ext}^1_0 \to \mathrm{Coh}^{\underline{d}}_{0,S}$ the projection.

Proposition 6.1. — *Let $\underline{i}$ be any parabolic degree and let $\underline{d} = (d, \dots, d)$ be a constant parabolic degree, then we have:*

1) *For any $\mathsf{K} \in D^b(\mathrm{Coh}^{\underline{d}}_{0,S})$,*

$$H^{\underline{i}}_{0,\,\mathrm{Ext}^1} \mathrm{pr}^*_{\mathrm{Coh}^{\underline{d}}_{0,S}} \mathsf{K} = (\mathrm{pr}^*_{\mathrm{Coh}^{\underline{d}-\underline{i}}_{0,S}} \times \mathrm{id}_{\mathrm{Coh}^{\underline{i}}_{0,S}})^* H^{\underline{i},\underline{d}-\underline{i}}_0 \mathsf{K}[-2i^{(0,S)}](-i^{(0,S)}).$$

2) $H^{\underline{i}}_{k,\,\mathrm{Ext}^{1,\mathrm{good}}} j^*_{\mathrm{Ext}} \mathsf{K} = j^*_{\mathrm{Ext}} H^{\underline{i}}_{k,\,\mathrm{Ext}^1} \mathsf{K}$ *for any $\mathsf{K} \in D^b(\mathrm{Ext}^1_k)$.*

3) $H^{\underline{i}}_{k,\,\mathrm{Hom}^{\mathrm{inj}}} I^* \mathsf{K} = I^* H^{\underline{i}}_{k-1,\,\mathrm{Ext}^{1,\mathrm{good}}} \mathsf{K}$ *for any $\mathsf{K} \in D^b(\mathrm{Ext}^{1,\mathrm{good}}_k)$.*

4) $H^{\underline{i}}_{k,\,\mathrm{Hom}} j_{\mathrm{Hom},!} \mathsf{K} = j_{\mathrm{Hom},!} H^{\underline{i}}_{k,\,\mathrm{Hom}^{\mathrm{inj}}} \mathsf{K}$ *for any $\mathsf{K} \in D^b(\mathrm{Hom}^{\mathrm{inj}}_k)$.*

5) *For any $\mathsf{K} \in D^b(\mathrm{Hom}_k)$,*

$$H^{i}_{k,\,\mathrm{Ext}^1} \circ \mathcal{F}our\, \mathsf{K} = \mathcal{F}our \circ H^{i}_{k,\,\mathrm{Hom}} \mathsf{K}[-i^{(k(n-1),S)}](-i^{(k(n-1),S)}).$$

Proof. — 1) Write down the definition of the correspondences:

$$\begin{tikzcd}
\left\langle \begin{smallmatrix} \mathcal{T}'^\bullet \subset \mathcal{T}^\bullet \\ \mathcal{O}^\bullet \to \mathcal{F}_1^\bullet \to \mathcal{T}^\bullet \end{smallmatrix} \right\rangle \arrow[rr, two heads, "p"] \arrow[d, "\pi_{\mathrm{big}}"'] \arrow[drr, "\mathrm{pr}_{\mathrm{left}}"] & & \left\langle \begin{smallmatrix} \mathcal{T}'^\bullet \subset \mathcal{T}^\bullet \\ \mathcal{O}^\bullet \to \mathcal{F}_1'^\bullet \to \mathcal{T}'^\bullet \end{smallmatrix} \right\rangle \arrow[d, "\mathrm{pr}_{\mathrm{right}}"'] \arrow[dr, "\pi_{\mathrm{small}} \times \mathrm{quot}"] & \\
\mathrm{Ext}_1^1 \arrow[d, "\mathrm{pr}_{\mathrm{Coh}^{d}_{0,S}}"'] & & \langle \mathcal{T}^{\bullet\prime} \subset \mathcal{T}^\bullet \rangle \arrow[dll, "\mathrm{forget}"] \arrow[dr, "\mathrm{gr}"] & \mathrm{Ext}_1^1 \times \mathrm{Coh}^{i}_{0,S} \arrow[d, "\mathrm{pr}_{\mathrm{Coh}^{d-i}_{0,S}} \times \mathrm{id}"] \\
\mathrm{Coh}^{d}_{0,S} & & & \mathrm{Coh}^{d-i}_{0,S} \times \mathrm{Coh}^{i}_{0,S}
\end{tikzcd}$$

The left- and right-hand "squares" are cartesian and p is an affine space bundle (an $\mathrm{Ext}^1(\mathcal{T}^\bullet/\mathcal{T}'^\bullet, \mathcal{O}^\bullet)$-torsor), therefore we get our claim:

$$\begin{aligned}
H^{i}_{0,\mathrm{Ext}^1}\, \mathrm{pr}^*_{\mathrm{Coh}^{d}_{0,S}}\, \mathsf{K} &= \mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_! \mathbf{R}p_! (\pi_{\mathrm{big}} \circ \mathrm{pr}_{\mathrm{Coh}_{0,S}})^* \mathsf{K} \\
&= \mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_! \mathbf{R}p_! (\mathrm{forget} \circ \mathrm{pr}_{\mathrm{right}} \circ p)^* \mathsf{K} \\
&= \mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_! (\mathrm{forget} \circ \mathrm{pr}_{\mathrm{right}})^* \mathsf{K}[-2i^{(0,S)}](-i^{(0,S)}) \\
&\qquad\qquad\qquad\qquad \text{as } p \text{ is a bundle} \\
&= (\mathrm{pr}_{\mathrm{Coh}^{d-i}_{0,S}} \times \mathrm{id})^* (\mathbf{R}\,\mathrm{gr}_!\, \mathrm{forget}^*\, \mathsf{K})[-2i^{(0,S)}](-i^{(0,S)}) \\
&\qquad\qquad\qquad\qquad \text{by base-change.}
\end{aligned}$$

2) This holds, because extensions of good sheaves by torsion sheaves are good.

3) This is true, because there is an isomorphism of the diagrams defining the two Hecke functors given by

$$\begin{pmatrix} \mathcal{F}'^\bullet_{k-1} \subset \mathcal{F}^\bullet_{k-1} \\ \Omega^{\bullet,k-1} \to \mathcal{F}^\bullet_k \to \mathcal{F}^\bullet_{k-1} \\ \mathcal{F}'^\bullet_k := \mathcal{F}^\bullet_k \times_{\mathcal{F}^\bullet_{k-1}} \mathcal{F}'^\bullet_{k-1} \mathrm{good} \end{pmatrix} \longmapsto \left(\Omega^{\bullet,k-1} \hookrightarrow \mathcal{F}'^\bullet_k \subset \mathcal{F}^\bullet_k\right).$$

4) By definition.

5) Again Laumon's proof can be copied word by word, the only thing used is the compatibility of the Fourier transform with bundle maps: $\mathcal{F}our(\iota^* \mathsf{K}) = \mathbf{R}p_!\, \mathcal{F}our\, \mathsf{K}[i^{(k(n-1),S)}](i^{(k(n-1),S)})$ (see [21] Thm 1.2.2.1 and 1.2.2.4). ☐

COROLLARY 6.2. — *The sheaf* $\mathsf{F}^k_{\mathsf{E}!}$ *is a Hecke eigensheaf on* $\mathrm{Hom}_k^{\mathrm{inj}}$, *i.e.*

$$H^{\underline{i}}_{k,\,\mathrm{Hom}^{\mathrm{inj}}}\mathsf{F}^k_{\mathsf{E}!} = \begin{cases} \mathsf{F}^k_{\mathsf{E}!} \boxtimes \mathcal{L}^{\underline{i}}_{\mathsf{E}}[-ki](-ki) & \text{if } \underline{i} \text{ is constant,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — By the above Proposition 6.1 this follows from the Hecke property of $\mathcal{L}^{\underline{i}}_{\mathsf{E}}$ (Proposition 4.8). □

6.3. Comparison of the Hecke operators and the generalized Hecke operators.

In the same way as in [11], Proposition 8.4 we want to show that for some sheaves on $\mathrm{Coh}_{n,S}$ the eigensheaf property with respect to $H_n^{\underline{d}-\underline{i},\underline{i}}$ implies that the restriction of the sheaf to $\mathrm{Bun}_{n,S}$ has the eigensheaf property for $H^{\underline{\epsilon}}$ and H^1_C. To do this we need to note some general properties of the maps π_{small} and π_{big} used in the definition of the operators $H_n^{\underline{d}-\underline{i},\underline{i}}$.

Fix a degree $\underline{d} = (d^{(j,p)})$ of parabolic sheaves, and let $\underline{i}$ some positive degree. We have defined a diagram

$$\mathrm{Coh}^{\underline{d}}_{n,S} \xleftarrow{\pi_{\mathrm{big}}} \mathrm{Hecke}_n^{\underline{d}-\underline{i},\underline{i}} \xrightarrow{\pi_{\mathrm{small}}\times\mathrm{quot}} \mathrm{Coh}^{\underline{d}-\underline{i}}_{n,S} \times \mathrm{Coh}^{\underline{i}}_{0,S}\,.$$

Denote further

$$\mathrm{Coh}^{\underline{d},\leq\underline{i}}_{n,S} := \big\langle \mathcal{F}^\bullet \in \mathrm{Coh}^{\underline{d}}_{n,S} \mid \mathrm{length}(\mathrm{torsion}(\mathcal{F}^\bullet)) \leq \underline{i} \,\big\rangle.$$

Then we have:

Remark 6.3. — 1) The map $\pi_{\mathrm{small}} \times \mathrm{quot}$ is a generalized vector bundle, in particular it is smooth.

2) The map π_{small} is smooth.

3) The map π_{big} is representable and projective.

4) The restriction of π_{big} to the pre-image is smooth:

$$(\pi_{\mathrm{small}} \times \mathrm{quot})^{-1}(\mathrm{Bun}^{\underline{d}-\underline{i}}_{n,S} \times \mathrm{Coh}^{\underline{i}}_{0,S}).$$

5) Assertions 2), 3) and the second part of 1) are true for the analogous maps defined by replacing $\mathrm{Coh}^{\underline{d}}_{n,S}$ and $\mathrm{Coh}^{\underline{d}-\underline{i}}_{n,S}$ by $\mathrm{Bun}^{\underline{d}}_{n,S}$ and $\mathrm{Bun}^{\underline{d}-\underline{i}}_{n,S}$, respectively.

Proof. — 1) The map $\pi_{\mathrm{small}} \times \mathrm{quot}$ is the projection from the generalized vector bundle

$$\mathbb{V}(\mathbf{R}\,\mathrm{pr}_{12,*}\,\mathrm{Hom}(\mathrm{pr}_{23}^*\mathcal{T}^\bullet_{\mathrm{univ}}, \mathrm{pr}_{13}^*\mathcal{F}^\bullet_{\mathrm{univ}})) \longrightarrow \mathrm{Coh}^{\underline{d}-\underline{i}}_{n,S} \times \mathrm{Coh}^{\underline{i}}_{0,S},$$

where $\mathrm{pr}_{j\ell}$ are the projections from $\mathrm{Coh}^{\underline{d}-\underline{i}}_{n,S} \times \mathrm{Coh}^{\underline{i}}_{0,S} \times C$ on the j and ℓ-th factors, and $\mathcal{T}^\bullet_{\mathrm{univ}}$ and $\mathcal{F}^\bullet_{\mathrm{univ}}$ are the universal bundles on $\mathrm{Coh}^{\underline{i}}_{0,S} \times C$ and $\mathrm{Coh}^{\underline{d}-\underline{i}}_{n,S} \times C$ respectively.

2) By 1) we only need to note that $\mathrm{Coh}^{\underline{i}}_{0,S}$ is a smooth stack (Lemma 3.7).

3) The fibres of π_{big} are closed subschemes in the scheme $\prod \mathrm{Quot}_{\mathrm{rank}\,n, \deg d^{(j,p)}}(\mathcal{F}^{(j,p)})$ which is projective (see [15]).

4) This is as in [11]: the given pre-image is smooth, since it is a vector bundle over a smooth stack. Its image under π_{big} is $\mathrm{Coh}^{\underline{d},\leq \underline{i}}_{n,S}$ which is smooth as well. Furthermore, π_{big} is representable, and therefore it is sufficient to prove that it induces a surjective map on all tangent spaces. Thus we need to show that at every point in a fibre of π_{big} the kernel of the induced map is of the correct, constant dimension.

We claim that for any point $(\mathcal{E}^\bullet \hookrightarrow \mathcal{F}^\bullet \twoheadrightarrow \mathcal{T}^\bullet := \mathcal{F}^\bullet/\mathcal{E}^\bullet)$ this kernel is isomorphic to $\mathrm{Hom}_{\mathrm{para}}(\mathcal{E}^\bullet, \mathcal{T}^\bullet)$. In Lemma 3.5 we have shown that this space is of constant dimension, and in case that $\mathcal{F}^\bullet$ is torsion free the map is certainly smooth at this point, thus it is smooth on the whole subset.(7)

To prove the claim, take a point in the tangent space to the fibre of π_{big}, i.e. a deformation to $k[\epsilon]/(\epsilon^2)$, such that the deformation of the middle term is trivial:

$$\begin{array}{ccccc} \mathcal{E}^\bullet & \xrightarrow{\psi} & \mathcal{F}^\bullet_0 \otimes_k k[\epsilon]/\epsilon^2 & \longrightarrow & \mathcal{T}^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}^\bullet_0 & \longrightarrow & \mathcal{F}^\bullet_0 & \longrightarrow & \mathcal{T}^\bullet_0. \end{array}$$

But then $\mathcal{E}^\bullet \cong \mathcal{E}^\bullet_0 \times_{\mathcal{F}^\bullet_0} \mathcal{F}^\bullet_0 \otimes k[\epsilon]/\epsilon^2 \cong \mathcal{E}^\bullet_0 \otimes k[\epsilon]/\epsilon^2$. And therefore the choices of ψ are given by $\mathrm{Hom}(\mathcal{E}^\bullet_0, \mathcal{T}^\bullet_0)$, as claimed.

5) Since $\mathrm{Bun}^{\underline{d}}_{n,S} \subset \mathrm{Coh}^{\underline{d}}_{n,S}$ is open the maps are still smooth. The restriction of π_{big} is still projective because subsheaves of vector bundles on curves are automatically vector bundles. □

(7) Alternatively one could use Lemma 3.5 to calculate the dimensions of the spaces involved, but one has to be careful in case $\underline{i}$ is not constant.

As on $\mathrm{Hom}_n^{\mathrm{inj}}$, we say that a perverse sheaf A_{E} on $\mathrm{Coh}_{n,S}$ is a (generalized) *Hecke eigensheaf* for E if

$$H^{\underline{d}-\underline{i},\underline{i}}\mathsf{A}_{\mathsf{E}} = \begin{cases} \mathsf{A}_{\mathsf{E}} \boxtimes \mathcal{L}_{\mathsf{E}}^{i}[-(n-1)i](-(n-1)i) & \text{if } \underline{i} \text{ is constant,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if the sheaf $\mathsf{F}_{\mathsf{E}}^{n}$ descends to $\mathrm{Coh}_{n,S}$, then this is the Hecke property of the descended sheaf (twisted by $\mathbb{Q}_\ell(d)$ on the component of degree $d = d^{(0,p)}$, the additional shift coming from the fact that the dimensions of the connected components of Hom_n are different). Using the definition of the operators $H^{\underline{\epsilon}}$ and the sheaf $\bar{\mathcal{L}}_{\mathsf{E}}^{\underline{1}}$ on $\overline{\mathrm{Coh}}_{0,S}^{\underline{1}}$ given in Section 2.6 we claim:

Proposition 6.4. — *Assume that A_{E} is a Hecke eigensheaf for E on $\mathrm{Coh}_{n,S}$, such that $\mathbb{D}A_{\mathsf{E}}$ is a Hecke eigensheaf for $\mathbb{D}\mathsf{E} =: \mathsf{E}^\vee$. Then $A_{\mathsf{E}|\mathrm{Bun}_{n,S}}$ is an eigensheaf for $H^{\underline{1}}$, i.e.*

$$\begin{cases} H^{\underline{1}}A_{\mathsf{E}|\mathrm{Bun}_{n,S}} = A_{\mathsf{E}|\mathrm{Bun}_{n,S}} \boxtimes \bar{\mathcal{L}}_{\mathsf{E}}^{\underline{1}}[-n+1](-n+1), \\ H^{\underline{\epsilon}}A_{\mathsf{E}|\mathrm{Bun}_{n,S}} = 0 \quad \textit{for } 0 < \underline{\epsilon} < (1,\dots,1). \end{cases}$$

Proof. — Look at the generalized Hecke correspondence restricted to $\mathrm{Bun}_{n,S}^{\underline{d}-\underline{1}}$:

$$\mathrm{Coh}_{n,S}^{\underline{d},\leq\underline{1}} \xleftarrow{\pi_{\mathrm{big}}} \langle \mathcal{E}'^{\bullet} \subset \mathcal{F}^{\bullet} \twoheadrightarrow \mathcal{T}^{\bullet} \rangle \xrightarrow{\pi_{\mathrm{small}}\times \mathrm{quot}} \mathrm{Bun}_{n,S}^{\underline{d}-\underline{1}} \times \mathrm{Coh}_{0,S}^{\underline{1}}$$

We know by Remark 6.3, 2) that in this diagram the map π_{big} is smooth of relative dimension n. Therefore on $\mathrm{Coh}_{n,S}^{\underline{d},\leq\underline{1}}$:

$$\begin{aligned} A_{\mathsf{E}} \boxtimes \mathcal{L}_{\mathsf{E}}^{\underline{1}} &= (\mathbb{D}H^{\underline{1}}\mathbb{D}A_{\mathsf{E}})[-n+1](-n+1) \qquad (\mathbb{D}A_{\mathsf{E}} \text{ eigensheaf}) \\ &= (\mathbb{D}\mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_! \pi_{\mathrm{big}}^{*}\mathbb{D}A_{\mathsf{E}})[-n+1](-n+1) \\ &= \mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_* \pi_{\mathrm{big}}^{!}A_{\mathsf{E}}[-n+1](-n+1) \\ &= \mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_* \pi_{\mathrm{big}}^{*}A_{\mathsf{E}}[-n+1+2n](1) \qquad (\pi_{\mathrm{big}} \text{ smooth}) \\ &= \mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_* \pi_{\mathrm{big}}^{*}A_{\mathsf{E}}[n+1](1) \end{aligned}$$

In other words, for A_{E} we can replace $\mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_!$ by $\mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{quot})_*$ in the definition of the Hecke operators. Note that the same consideration applies to the operators $H^{\underline{\epsilon}}$ for any $\underline{\epsilon}$ with $(0,\dots,0) < \underline{\epsilon} < (1,\dots,1)$.

In this case we even know that $H^{\underline{\epsilon}}A_{\mathsf{E}} = 0$, and this helps to prove:

LEMMA 6.5. — *Under the assumptions of* 6.4 *the restriction of the sheaf* A_{E} *to the stack* $\mathrm{Coh}_{n,S}^{d,\leq\underline{\epsilon}} - \mathrm{Bun}_{n,S}^{\underline{d}}$ *is zero for all* $(0,\dots,0) < \underline{\epsilon} < (1,\dots,1)$.

Proof. — The map

$$\pi_{\text{small}} \times \text{quot} : \langle \mathcal{E}'^{\bullet} \subset \mathcal{F}^{\bullet} \twoheadrightarrow \mathcal{T}^{\bullet} \rangle \longrightarrow \mathrm{Bun}_{k,S}^{\underline{d}-\underline{\epsilon}} \times \mathrm{Coh}_{0.S}^{\underline{\epsilon}}$$

is a vector bundle projection, let c be its relative dimension. Furthermore $\pi_{\text{big}}^{*} A_{\mathsf{E}}$ is $\mathbb{G}_m$-equivariant, and thus we can apply Lemma 0.3 to get that

$$s_0^{*}\pi_{\text{big}}^{*} A_{\mathsf{E}} = \mathbf{R}(\pi_{\text{small}} \times \text{quot})_{*}\pi_{\text{big}}^{*} A_{\mathsf{E}} = H^{\underline{\epsilon}} A_{\mathsf{E}}[-2c](-c) = 0. \qquad \square$$

Now we can apply Lemma 8.5 of [11] – which says that in the situation of Lemma 0.3, i.e. we have a vector bundle projection p and some $\mathbb{G}_m$-equivariant perverse sheaf K if both $\mathbf{R}p_{*}\mathsf{K}[-1]$ and $\mathbf{R}p_{!}\mathsf{K}[1]$ are perverse, then $\mathbf{R}p_{!}\mathsf{K} \cong \mathbf{R}\overline{p}_{!}\overline{\mathsf{K}}$ – to get that

$$\mathbf{R}(\overline{\pi_{\text{small}} \times \text{quot}})_{!}\overline{\pi_{\text{big}}^{*} A_{\mathsf{E}}} = A_{\mathsf{E}} \boxtimes \overline{\mathcal{L}}_{\mathsf{E}}^{\underline{1}}[-n+1](-n+1).$$

The fibres of the projectivized bundle $\overline{\pi_{\text{small}} \times \text{quot}}$ are $\mathbb{P}(\mathrm{Ext}^1(\mathcal{T}^{\bullet}, \mathcal{E}^{\bullet}))$ and by the above lemma we even know that the stalk of A_{E} is zero at sheaves $\mathcal{F}^{\bullet}$ with $\underline{0} < \deg(\text{torsion}(\mathcal{F}^{\bullet})) < \underline{1}$, therefore in the above equation we may restrict $\overline{\pi_{\text{small}} \times \text{quot}}$ to the space of torsion free extensions. But on this substack the base change to $\mathrm{Bun}_{n,S}^{\underline{d}-\underline{1}} \times \overline{\mathrm{Coh}}_{0,S}^{\underline{1}}$ gives the map used to define $H^{\underline{1}}$. $\square$

COROLLARY 6.6. — *Assume that* A_{E} *is a Hecke eigensheaf for* E *on* $\mathrm{Coh}_{n,S}^{\underline{d}}$, *such that* $\mathbb{D}A_{\mathsf{E}}$ *is a Hecke eigensheaf for* $\mathbb{D}\mathsf{E} := \mathsf{E}^{\vee}$. *Then the corresponding function* $t_{A_{\mathsf{E}}}$ *on* $\mathrm{Bun}_{n,S}^{\underline{d}}$ *is an eigenfunction for the Iwahori-Hecke algebra.*

Proof. — We just have proven the Hecke-property of the restriction of A_{E} to $\mathrm{Bun}_{n,S}$. Therefore we only need to compare the result with the computation of $\overline{\mathcal{L}}_{\mathsf{E}}^{\underline{1}}$ on $\overline{\mathrm{Coh}}_{0,S}^{\underline{1}}$ (Lemma 4.5) and note that the Iwahori-Hecke-algebra at S is generated by elements corresponding to the points of $\mathrm{Coh}_{0,S}^{\underline{1}}$. And for the Hecke operators supported in $C - S$ the situation is the same as in the unramified situation (see [11]). $\square$

6.4. Descent of the sheaf $\mathsf{F}^n_{\mathsf{E}}$.

PROPOSITION 6.7. — *Assume that we know that $\mathsf{F}^n_{\mathsf{E}} = \mathsf{F}^n_{\mathsf{E},!}$, then Lafforgue's theorem implies that $\mathsf{F}^n_{\mathsf{E}}$ descends to a Hecke eigensheaf on $\mathrm{Bun}^{\mathrm{good}}_{n,S}$, and this sheaf can be extended to a non zero Hecke eigensheaf A_{E} on $\mathrm{Bun}_{n,S}$.*

Proof. — By definition $\mathsf{F}^n_{\mathsf{E}}$ is an irreducible perverse sheaf and by our assumption $\mathsf{F}^n_{\mathsf{E}} = \mathsf{F}^n_{\mathsf{E},!}$ is a Hecke eigensheaf (by Corollary 6.6).

We first want to explain why the function $\Phi(W_{\mathsf{E}})$ does not depend on the section $\Omega^{\bullet,n-1} \hookrightarrow \mathcal{E}^\bullet$, but only on the bundle $\mathcal{E}^\bullet$: On the one hand by Lafforgue's theorem [18] there is a (cuspidal) Hecke eigenfunction on $\mathrm{Bun}_{n,S}(\mathbb{F}_q)$ with eigenvalues given by $\mathrm{tr}_{\mathcal{L}_{\mathsf{E}}}$ (1.2). On the other hand by Shalika's result (see [25], Theorem 5.9), every Hecke eigenfunction on $\mathrm{Hom}^{\mathrm{inj}}_n(\mathbb{F}_q)$ is in the image of Φ and there is a unique such function in the Whittaker space. Therefore the function $f_{\mathsf{E}} = \Phi(W_{\mathsf{E}})$ is the pull back of a function on $\mathrm{Bun}_{n,S}(\mathbb{F}_q)$.

Assume for the moment that $n \leq 3$. In this case we know that restricted to the maximal embeddings $\mathrm{Hom}^{\max}_n \subset \mathrm{Hom}^{\mathrm{inj}}_n$ the function $\mathrm{tr}_{\mathsf{F}^n_{\mathsf{E},!}} = \mathrm{const} \cdot \Phi(W_{\mathsf{E}})$ for some non-zero constant. In particular, this function is not identically zero on $\mathrm{Hom}^{\max}_n$ and descends to $\mathrm{Bun}_{n,S}(\mathbb{F}_q)$.

Thus to show that this implies the descent of $\mathsf{F}^n_{\mathsf{E}}$, we can apply a variant of the argument given in [11]: since $\mathsf{F}^n_{\mathsf{E}}$ is a $\mathbb{G}_m$-equivariant irreducible perverse sheaf it descends to the projective bundle $\mathbb{P}\,\mathrm{Hom}^{\mathrm{inj}}$ and there is a constructible subset $V \xhookrightarrow{j_V} \mathbb{P}\,\mathrm{Hom}^{\max}(\Omega^{\bullet,n-1},\mathcal{E}^\bullet)$ such that $\mathsf{F}^n_{\mathsf{E}|V}$ is an irreducible local system and $\mathsf{F}^n_{\mathsf{E}} = j_{V,!*}(\mathsf{F}^n_{\mathsf{E}|V})$.

Further the restriction of $\mathsf{F}^n_{\mathsf{E}}$ to V is constant on the fibres over $\mathrm{Bun}_{n,S}$, because the trace of $\mathsf{F}^n_{\mathsf{E}}$ is constant on the fibres (for any extension $\mathbb{F}_{q^n}$ of the base field). And the two pull backs of F_{E} to

$$\mathbb{P}\,\mathrm{Hom}^{\max} \times_{\mathrm{Bun}_{n,d}} \mathbb{P}\,\mathrm{Hom}^{\max}$$

are irreducible ($\mathbb{P}\,\mathrm{Hom}^{\max}$ is an open subset of a projectivized bundle) and isomorphic, because the corresponding trace functions are the same. Since the two systems are irreducible, there is only one isomorphism of these sheaves which induces the identity on the points of the diagonal $V \subset V \times_{\mathrm{Bun}_n} V$. Hence $\mathsf{F}^n_{\mathsf{E}|V}$ descends to a perverse sheaf $A_{\mathsf{E},V}$ on $\mathrm{pr}_{\mathrm{Bun}_{n,S}}(V)$. Further, since $\mathsf{F}^n_{\mathsf{E}} = j_{V,!*}(\mathsf{F}^n_{\mathsf{E}|V})$, we also know that $\mathsf{F}^n_{\mathsf{E}} = \mathrm{pr}^*_{\mathrm{Bun}_{n,S}} j_{\mathrm{pr}(V),!*} A_{\mathsf{E},V}$, i.e. $\mathsf{F}^n_{\mathsf{E}}$ descends to a sheaf $\mathsf{A}^{\mathrm{good}}_{\mathsf{E}}$ on $\mathrm{Bun}^{\mathrm{good}}_{n,S}$.

Note that in particular we have shown that $\mathrm{tr}_{\mathsf{F}_{\mathsf{E},!}} = \mathrm{const} \cdot \Phi(W_{\mathsf{E}})$ on the whole of $\mathrm{Hom}_n^{\mathrm{inj}}$. Therefore we may apply Φ^{-1} to see that the trace function of the sheaf $\mathbf{R}\,\mathrm{forget}_{\mathrm{Tor},!}(\mathrm{quot}^*(\mathcal{L}_{\mathsf{E}}^{d_0}) \otimes \mathrm{ext}^*\, \mathsf{L}_\psi)$ on Ω-Plücker must be equal to W_{E}. This allows us to drop the temporary assumption that $n \le 3$, because we can apply the argument of Lemma 5.5 to show that the trace of $\mathsf{F}_{\mathsf{E}}^n$ is equal to $\Phi(W_{\mathsf{E}})$ on the space of maximal embeddings for $n \le 4$, and this gives an inductive argument for all n.

To finish the proof of the theorem we only need to extend the resulting sheaf $\mathsf{A}_{\mathsf{E}}^{\mathrm{good}}$ to the whole of $\mathrm{Bun}_{n,S}$. Again this works as in [11], Section 7.8: for $q \in C - S$ (we might allow $q \in S$) the maps $\otimes\mathcal{O}(-rq)\colon \mathrm{Bun}_{n,S}^{d+r,\mathrm{good}} \to \mathrm{Bun}_{n,S}^{d}$ are a covering of $\mathrm{Bun}_{n,S}$. We define

$$\mathsf{A}_{\mathsf{E}} := \varinjlim_{r} \big(\otimes\mathcal{O}(-rq)\big)_* \mathsf{A}_{\mathsf{E}}^{\mathrm{good}} \otimes \big(\det(\mathsf{E})_{|q}\big)^{-\otimes r}.$$

The Hecke property of $\mathsf{A}_{\mathsf{E}}^{\mathrm{good}}$ (together with the S_2-equivariance of the isomorphism $H^1 \circ H^1 \mathsf{A}_{\mathsf{E}}^{\mathrm{good}} \cong \mathsf{A}_{\mathsf{E}}^{\mathrm{good}} \boxtimes \mathsf{E} \boxtimes \mathsf{E}$) gives that this is a well-defined Hecke eigensheaf on Bun_n^d. $\square$

7. The analogue of the vanishing theorem for $n \le 3$.

The aim of the last two sections of this article is to prove that our assumption $\mathsf{F}_{\mathsf{E}}^k = \mathsf{F}_{\mathsf{E},!}^k$ holds for $k \le \min(3, n)$ (Proposition 8.2). To do so we need an analogue of the vanishing theorem in [11] which is given below (Proposition 7.1):

For any $i \in \mathbb{Z}_{>0}$ consider the *total Hecke-* or *averaging functor* $H_{\mathsf{E},\mathrm{tot}}^{-i}$ defined as follows ($\underline{i} := (i, \dots, i)$):

$$\begin{tikzcd}
& \langle \mathcal{E}'^\bullet \subset \mathcal{E}^\bullet \mid \mathcal{E}'^\bullet \in \mathrm{Bun}_{k,S}^{\underline{d}-\underline{i}},\ \mathcal{E}^\bullet \in \mathrm{Bun}_{k,S}^{\underline{d}} \rangle \arrow[dl, "\pi_{\mathrm{small}}"'] \arrow[d, "\mathrm{quot}"] \arrow[dr, "\pi_{\mathrm{big}}"] & \\
\mathrm{Bun}_{k,S}^{\underline{d}-\underline{i}} & \mathrm{Coh}_{0,S}^{\underline{i}} & \mathrm{Bun}_{k,S}^{\underline{d}}
\end{tikzcd}$$

We set

$$H_{\mathsf{E},\mathrm{tot}}^{-i}\colon D^b(\mathrm{Bun}_{k,S}^{\underline{d}-\underline{i}}) \longrightarrow D^b(\mathrm{Bun}_{k,S}^{\underline{d}}),$$
$$\mathsf{K} \longmapsto H_{\mathsf{E},\mathrm{tot}}^{-i}\mathsf{K} := \mathbf{R}\pi_{\mathrm{big},!}(\pi_{\mathrm{small}}^*\mathsf{K} \otimes \mathrm{quot}^*\mathcal{L}_{\mathsf{E}}^{\underline{i}}).$$

Remark. — This definition is used for any $\underline{d} = (d_{i,p})_{0 \le i < n, p \in S}$, therefore it includes the case of bundles with not necessarily full parabolic structure. In particular for $\underline{d} = (d)_{0 \le i < n, p \in S}$ the stack $\mathrm{Bun}_{k,S}^{\underline{d}} \cong \mathrm{Bun}_k^d$ is the stack of vector bundles without extra structure.

PROPOSITION 7.1. — *Let E be a (pure) irreducible rank n local system with indecomposable unipotent ramification at S. Then for any $k < \min(3,n)$ and any (mixed) complex $\mathsf{K} \in D^b(\mathrm{Bun}_{k,S})$ we have*

$$H^{-i}_{\mathsf{E},\,\mathrm{tot}}\mathsf{K} = 0 \quad \textit{for all } i > (2g-2)nk + |S| \cdot k.$$

Note that by Lafforgue's theorem we may assume that E is pure, since every irreducible sheaf is pure up to a twist.

Proof (almost the same as in [11], *using the conventions given in Section 6.1 for the definition of stacks occurring in the proof).* — We use that, by induction we already know the proposition for all $k' < k$.

Reductions: without loss of generality, we may assume that K is a pure complex, because any mixed complex has a filtration with pure filtration quotients.

For a pure complex K the complex $H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K}$ is pure as well, because $H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K} = \mathbf{R}\pi_{\mathrm{big},!}(\pi^*_{\mathrm{small}}\mathsf{K} \otimes \mathrm{quot}^*\mathcal{L}^i_{\mathsf{E}})$ and π_{small} is smooth (Lemma 6.3), therefore π^*_{small} preserves purity (i.e. smoothness implies $\pi^*_{\mathrm{small}} = \pi^!_{\mathrm{small}}[2d](d)$). The same is true for quot^* and finally π_{big} is proper (Lemma 6.3), therefore Deligne's theorem (see [7], 6.2.6) implies that $\mathbf{R}\pi_{\mathrm{big},*} = \mathbf{R}\pi_{\mathrm{big},!}$ also preserves purity.

Furthermore, a pure complex $H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K}$ is zero if and only if the associated function $\mathrm{tr}_{H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K}}$ on $\mathbb{F}_{q^\ell}$-points is zero for all ℓ. Hence it is enough to prove that $h := \mathrm{tr}_{H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K}}$ is the zero-function.

Finally, to show that a function h on $\mathrm{Bun}_{k,S}(\mathbb{F}_q)$ is zero it is sufficient to show that 1) h is cuspidal and 2) for all cuspidal functions f on $\mathrm{Bun}_{k,S}(\mathbb{F}_q)$ the scalar product $< h, f >= 0$ – the product being defined since cuspidal functions have finite support on every connected component of $\mathrm{Bun}_{k,S}$. In the proof of these statements we will reduce back to a statement for sheaves.

- *First step*: $H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K}$ *is a cuspidal complex, therefore* $\mathrm{tr}_{H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K}}$ *is a cuspidal function*, i.e. for all $k_1 + k_2 = k$ and all $\underline{d}_1 + \underline{d}_2 = \underline{d}$ let $C^{\underline{d}_1,\underline{d}_2}_{k_1,k_2}$ be the functor defined as follows

$$\mathrm{Bun}^{\underline{d}}_{k,S} \xleftarrow{\text{forget}} \langle \mathcal{E}^\bullet_{k_1} \hookrightarrow \mathcal{E}^\bullet_{k} \twoheadrightarrow \mathcal{E}^\bullet_{k_2} \rangle \xrightarrow[(\mathcal{E}^\bullet_{k_i},\mathcal{E}^\bullet_{k})\mapsto(\mathcal{E}^\bullet_{k_1},\mathcal{E}^\bullet_{k_2})]{\mathrm{gr}} \mathrm{Bun}^{\underline{d}_1}_{k_1,S} \times \mathrm{Bun}^{\underline{d}_2}_{k_2,S}$$

$$C^{\underline{d}_1,\underline{d}_2}_{k_1,k_2} : D^b(\mathrm{Bun}^{\underline{d}}_{k,S}) \longrightarrow D^b(\mathrm{Bun}^{\underline{d}_1}_{k_1,S} \times \mathrm{Bun}^{\underline{d}_2}_{k_2,S}),$$

$$\mathsf{K} \longmapsto C^{\underline{d}_1,\underline{d}_2}_{k_1,k_2}\mathsf{K} := \mathbf{R}\,\mathrm{gr}_!\,\mathrm{forget}^*\,\mathsf{K}.$$

DEFINITION 7.1. — A complex $\mathsf{K} \in D^b(\mathrm{Bun}_{k,S})$ is called *cuspidal* if for all $\underline{d}_1, \underline{d}_2$ and any non trivial partition $k_1 + k_2 = k$ we have $C^{\underline{d}_1,\underline{d}_2}_{k_1,k_2}\mathsf{K} = 0$.

PROPOSITION 7.2. — *Let E be a irreducible local system of arbitrary rank $n \geq k$ on $C - S$ with indecomposable unipotent ramification at S. Then for all $\underline{d}_1, \underline{d}_2$ and any non trivial partition $k_1 + k_2 = k$ the complex $C^{\underline{d}_1,\underline{d}_2}_{k_1,k_2} \circ H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K}$ has a filtration with subquotients isomorphic to $(H^{-i_1}_{\mathsf{E},\mathrm{tot}} \times H^{-i_2}_{\mathsf{E},\mathrm{tot}}) \circ C^{\underline{d}_1,\underline{d}_2}_{k_1,k_2}\mathsf{K}$ for some $i_1 + i_2 = i$.*

Note that by induction on k we can assume that the vanishing Theorem 7.1 holds for all $k_i < k$. Therefore we know that the filtration subquotients occurring in the above proposition are all zero, because $k_1, k_2 < k$ and either i_1 or i_2 is sufficiently big. Therefore the proposition proves that $H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K}$ is cuspidal if $i > (2g-2)nk + |S|k$.

Proof of Proposition 7.2. — We define a diagram using the conventions given in Section 6.1, all three term sequences occurring in the diagram are short exact sequences

$$\begin{tikzcd}
\mathrm{Bun}^{\underline{d}}_{k,S} & \langle \mathcal{E}^\bullet_{k_1} \to \mathcal{E}^\bullet_k \to \mathcal{E}^\bullet_{k_2}\rangle \arrow[l, "\mathrm{forget}"'] \arrow[r, "\mathrm{gr}"] & \mathrm{Bun}^{\underline{d}_1}_{k_1,S} \times \mathrm{Bun}^{\underline{d}_2}_{k_2,S} \\
\langle \mathcal{E}'^\bullet_k \subset \mathcal{E}^\bullet_k \rangle \arrow[u, "\pi_{\mathrm{big}}"] \arrow[d, "\pi_{\mathrm{small}}"] & \underbrace{\langle (\mathcal{E}'^\bullet_k \subset \mathcal{E}^\bullet_k, \mathcal{E}^\bullet_{k_1} \to \mathcal{E}^\bullet_k \to \mathcal{E}^\bullet_{k_2})\rangle}_{=:\mathrm{Middle}} \arrow[l, "\mathrm{forget}'"'] \arrow[u, "\pi'_{\mathrm{big}}"] & \\
\mathrm{Bun}^{\underline{d}-\underline{i}}_{k,S} & &
\end{tikzcd}$$

to compute

$$\begin{aligned}
C^{\underline{d}_1,\underline{d}_2}_{k_1,k_2} \circ H^{-\underline{d}}_{\mathsf{E},\mathrm{tot}}\mathsf{K} &\overset{\text{def.}}{=\!=} \mathbf{R}\mathrm{gr}_!\,\mathrm{forget}^*\,\mathbf{R}\pi_{\mathrm{big},!}(\pi^*_{\mathrm{small}}\mathsf{K} \otimes \mathrm{quot}^*\,\mathcal{L}^{\underline{i}}_{\mathsf{E}}) \\
&\underset{\text{change}}{\overset{\text{base-}}{=\!=}} \mathbf{R}(\mathrm{gr}\circ\pi'_{\mathrm{big}})_!\,\underbrace{(\mathrm{forget}'^*\,\pi^*_{\mathrm{small}}\mathsf{K} \otimes \mathrm{forget}'^*\,\mathrm{quot}^*\,\mathcal{L}^{\underline{i}}_{\mathsf{E}})}_{=:\mathsf{K}_1}.
\end{aligned}$$

The stack Middle is stratified by substacks indexed by $0 \leq \underline{i}_1 \leq \underline{i}$, given by the condition $\deg(\mathcal{E}'^{(j,p)}_k \cap \mathcal{E}^{(j,p)}_{k_1}) = d_1^{(j,p)} - i_1^{(j,p)}$:

$$\mathrm{Middle}_{\underline{i}_1} := \left\langle \begin{matrix} \mathcal{E}'^\bullet_{k_1} & \to & \mathcal{E}'^\bullet_k & \to & \mathcal{E}'^\bullet_{k_2} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}^\bullet_{k_1} & \to & \mathcal{E}^\bullet_k & \to & \mathcal{E}^\bullet_{k_2} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{T}^\bullet_1 & \to & \mathcal{T}^\bullet & \to & \mathcal{T}^\bullet_2 \end{matrix} \;\middle|\; \begin{matrix} \mathcal{E}'^\bullet_{k_1} = \mathcal{E}^\bullet_{k_1} \cap \mathcal{E}'^\bullet_k \text{ and} \\ \deg(\mathcal{E}'^\bullet_{k_1}) = \underline{d}_1 - \underline{i}_1 \end{matrix} \right\rangle$$

This stratification will induce the filtration we are looking for.

Now $\mathrm{gr} \circ \pi'_{\mathrm{big}}$ restricted to $\mathrm{Middle}_{\underline{i}_1}$ is the map forgetting everything but $\mathcal{E}^\bullet_{k_1}$ and $\mathcal{E}^\bullet_{k_2}$. We factor this as follows: first consider the map $\mathrm{forget}_{\mathcal{E}^\bullet_n}$ forgetting $\mathcal{E}^\bullet_k$. This is an affine fibration, the fibres being homogeneous spaces for $\mathrm{Ext}^1_{\mathrm{para}}(\mathcal{T}^\bullet_2, \mathcal{E}'^\bullet_{k_1})$ (because of the exact square of Ext^1 groups we get from the extensions of the $\mathcal{T}^\bullet_i$ by the $\mathcal{E}'^\bullet_{k_i}$).

Furthermore, both the map $\pi_{\mathrm{small}} \circ \mathrm{forget}'$ and $\mathrm{quot} \circ \mathrm{forget}'$ factor through $\mathrm{forget}_{\mathcal{E}^\bullet_n}$, i.e. $\mathsf{K}_{1|\mathrm{Middle}_{\underline{i}_1}} = \mathrm{forget}^*_{\mathcal{E}^\bullet_k} \mathsf{K}_2$ for some complex K_2 and thus $\mathbf{R}\,\mathrm{forget}_{\mathcal{E}^\bullet_k,!} \mathsf{K}_1 = \mathsf{K}_2[2c](c)$ for some c.

We can compose the map $\mathrm{forget}_{\mathcal{E}^\bullet_k}$ with the forgetful map $\mathrm{forget}_{\mathcal{T}^\bullet}$. This map is just the pull back of the corresponding map in the Hecke correspondence of torsion sheaves, and still $\pi_{\mathrm{small}} \circ \mathrm{forget}'$ factors through this map. Therefore, by the Hecke property of $\mathcal{L}_{\mathsf{E}}$ we get that $\mathbf{R}\,\mathrm{forget}_{\mathcal{T}^\bullet,!} \mathsf{K}_2$ is zero if $\underline{i}_1$ is not constant.

But if $\underline{i}_1 = (i_1)$ is constant, the Hecke property implies that

$$\mathbf{R}(gr \circ \pi'_{\mathrm{big}})_!(\mathsf{K}_{|\mathrm{Middle}_{\underline{i}_1}}) = H^{-i_1}_{\mathsf{E},\mathrm{tot}} \times H^{-(i-i_1)}_{\mathsf{E},\mathrm{tot}} \circ C^{d_1,d_2}_{k_1,k_2}(\mathsf{K}).$$

Thus the stratification of the stack Middle induces a filtration as claimed. □

- *Second step*: *for every cuspidal function f we have $\langle t_{H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K}}, f\rangle = 0$.*

Using the same diagram as in the definition of $H^{-i}_{\mathsf{E},\mathrm{tot}}$ at the beginning of this section, we define $H^{i}_{\mathsf{E},\mathrm{tot}}\mathsf{K} := \mathbf{R}\pi_{\mathrm{small},!}(\pi^*_{\mathrm{big}}\mathsf{K} \otimes \mathrm{quot}^* \mathcal{L}^i_{\mathsf{E}})$, and denote the analogous operator for functions on $\mathrm{Bun}^d_{k,S}$ (i.e. the sheaf $\mathcal{L}^i_{\mathsf{E}}$ is replaced by its trace function, pull-backs are considered as pull-backs of functions, the tensor product is replaced by the product of functions and $\mathbf{R}\pi_{\mathrm{small},!}$ is replaced by summation over the fibres of π_{small}) by the same symbol. Then for any cuspidal function f

$$\langle \mathrm{tr}_{H^{-i}_{\mathsf{E},\mathrm{tot}}\mathsf{K}}, f\rangle = \langle \mathrm{tr}_{\mathsf{K}}, H^i_{\mathsf{E},\mathrm{tot}} f\rangle,$$

the brackets $\langle .,. \rangle$ again denote scalar products.

We want to show that $H^i_{\mathsf{E},\mathrm{tot}} f = 0$ for all cuspidal functions f. Using the Langlands correspondence for $k < n$, we know that the space of cuspidal functions on $\mathrm{Bun}_{k,S}$ is spanned by cuspidal Hecke eigenfunctions $f_{\mathsf{E}'}$ corresponding to local systems E' of dimension k with at most unipotent ramification at S and their images under the action of the Iwahori-Hecke algebra (note that for unramified local systems E' on C

these functions do not have an eigenfunction property for the Iwahori-Hecke algebra). Furthermore, since $k < n$, we know that these $f_{\mathsf{E}'}$ are the traces of irreducible perverse sheaves $A_{\mathsf{E}'}$ on $\mathrm{Bun}_{k,S'}$ for some $S' \subset S$. For this argument we need that $n \leq 3$, because for $k \geq 3$ we have not given a construction for representations with reducible unipotent monodromy.

To prove the second step it is therefore sufficient to show:

1) For all irreducible local systems E' on $C - S'$ with indecomposable unipotent ramification at $S' \subset S$ we have

$$H^i_{\mathsf{E},\mathrm{tot}}\, \mathrm{pr}^*_{\mathrm{Bun}_{k,S'}} A_{\mathsf{E}'} = 0 \quad \text{for } i > (2g-2)nk + |S|k,$$

where $\mathrm{pr}_{\mathrm{Bun}_{k,S'}} : \mathrm{Bun}_{k,S} \to \mathrm{Bun}_{k,S'}$ is the map forgetting the parabolic structure at $S - S'$ and A_{E} is the automorphic Hecke-eigensheaf already constructed for $k < n$.

2) Any element of the Iwahori-Hecke algebra commutes with the operator $H^i_{\mathsf{E},\mathrm{tot}}$ on the level of functions.

We need another Hecke-operator $H^i_{\mathsf{E},C}$. As before set $\underline{i} := (i, \dots, i)$.

$$\begin{array}{ccccc} \mathrm{Bun}^{\underline{d}}_{k,S} & \xleftarrow{\pi_{\mathrm{big}}} & \langle \mathcal{E}'^\bullet \subset \mathcal{E}^\bullet \,|\, \mathcal{E}'^\bullet \in \mathrm{Bun}^{\underline{d}-\underline{i}}_{k,S}, \mathcal{E}^\bullet \in \mathrm{Bun}^{\underline{d}}_{k,S} \rangle & \xrightarrow{\pi_{\mathrm{small}} \times \mathrm{supp}} & \mathrm{Bun}^{\underline{d}-\underline{i}}_{k,S} \times C^{(i)} \\ & & \Big\downarrow {\scriptstyle \mathrm{quot}} & & \\ & & \mathrm{Coh}^{\underline{i}}_{0,S} & & \end{array}$$

Here $\mathrm{supp}(\mathcal{E}'^\bullet \subset \mathcal{E}^\bullet) := \mathrm{supp}(\mathcal{E}^\bullet / \mathcal{E}'^\bullet)$. We set

$$\begin{aligned} H^i_{\mathsf{E},C} : D^b(\mathrm{Bun}^{\underline{d}}_{k,S}) &\longrightarrow D^b(\mathrm{Bun}^{\underline{d}-\underline{i}}_{k,S} \times C^{(i)}), \\ \mathsf{K} &\longmapsto H^i_{\mathsf{E},C}\mathsf{K} := \mathbf{R}(\pi_{\mathrm{small}} \times \mathrm{supp})_!(\pi^*_{\mathrm{big}}\mathsf{K} \otimes \mathrm{quot}^* \mathcal{L}^{\underline{i}}_{\mathsf{E}}). \end{aligned}$$

Note that in the above we may assume that we are concerned with k-step parabolic structures since the image of quot is contained in the image of (k-step parabolic sheaves) $\subset$ (n-step parabolic sheaves). Thus to prove the first claim we have to show:

Proposition 7.3. — *Let E' a local system of rank $k < n$, possibly with unipotent ramification at $S' \subset S$, and let $A_{\mathsf{E}'}$ be a Hecke eigensheaf for E' on $\mathrm{Bun}_{k,S'}$. Then*

$$H^i_{\mathsf{E},\,\mathrm{tot}}\, \mathrm{pr}^*_{\mathrm{Bun}_{k,S'}} A_{\mathsf{E}'} = 0 \quad \textit{for } i > (2g-2)nk + |S|k.$$

More precisely,

$$H^i_{\mathsf{E},C}\, \mathrm{pr}^*_{\mathrm{Bun}_{k,S'}} A_{\mathsf{E}'} = \big(j_*(\mathsf{E} \otimes \mathsf{E}')\big)^{(i)} \boxtimes \mathrm{pr}^*_{\mathrm{Bun}_{k,S'}} A_{\mathsf{E}'} \quad \textit{for all } i.$$

Proof. — The first statement follows from the second, as in the proof of Deligne's Lemma in [8]: one has $H^0(C, j_*(\mathsf{E} \otimes \mathsf{E}')) = 0$, because E is irreducible and not isomorphic to any subquotient of E'. By Poincaré duality therefore $H^2(C, j_*(\mathsf{E} \otimes \mathsf{E}')) = 0$ and thus

$$\dim\big(H^1(C, j_*(\mathsf{E} \otimes \mathsf{E}'))\big) = -\chi\big(j_*(\mathsf{E} \otimes \mathsf{E}')\big) = kn(2g-2) + |S|k$$

by the formula for the Euler characteristic of Grothendieck-Ogg-Shafarevich (see [17], Exp. X, 7.1).

Furthermore we can apply the symmetric Künneth formula (see [2], Exp. XVII, 5.5.21) and – because $h^0 = h^2 = 0$ – we get that

$$H^*\big(C, (j_*(\mathsf{E}\otimes\mathsf{E}'))^{(i)}\big) = \overset{i}{\wedge} H^1\big(C, j_*(\mathsf{E}\otimes\mathsf{E}')\big) = 0 \quad \text{for } i > kn(2g-2)+|S|k.$$

We are left with proving the second statement.

Reduction to the case $i = 1$. — Consider the resolution

$$\begin{tikzcd}
 & \left\langle \begin{matrix} \mathcal{E}'^\bullet \subset \mathcal{E}^\bullet \\ \mathcal{T}_1^\bullet \subset \cdots \subset \mathcal{T}_i^\bullet = \mathcal{E}^\bullet/\mathcal{E}'^\bullet \end{matrix} \right\rangle \arrow[d, "\mathrm{flag}'"] \arrow[dr, "\pi_{\mathrm{small}} \times \widetilde{\mathrm{quot}}"'] \arrow[drr, "\pi_{\mathrm{small}} \times \widetilde{\mathrm{supp}}"] & & \\
 & \langle \mathcal{E}'^\bullet \subset \mathcal{E}^\bullet \rangle \arrow[dl, "\pi_{\mathrm{big}}"'] \arrow[dr, "\pi_{\mathrm{small}} \times \mathrm{quot}"'] & \mathrm{Bun}^{d-i}_{k,S} \times \widetilde{\mathrm{Coh}}^{\,i}_{0,S} \arrow[r] \arrow[d, "\mathrm{flag}"] & \mathrm{Bun}^{d-i}_{k,S} \times C^i \arrow[d, "\mathrm{sym}"] \\
\mathrm{Bun}^d_{k,S} & & \mathrm{Bun}^{d-i}_{k,S} \times \mathrm{Coh}^i_{0,S} \arrow[r] & \mathrm{Bun}^{d-i}_{k,S} \times C^{(i)}.
\end{tikzcd}$$

Note that $(H^1_{\mathsf{E},C})^{\circ i}\mathsf{K} = \mathbf{R}\pi \times \widetilde{\mathrm{supp}}_!((\mathrm{flag}' \circ \pi_{\mathrm{big}})^*\mathsf{K} \otimes \widetilde{\mathrm{quot}}^* \mathrm{gr}^*(\mathcal{L}^1_{\mathsf{E}})^{\boxtimes i})$ for any complex K on $\mathrm{Bun}^d_{k,S}$. Further, by Lemma 4.10 the sheaf $\mathbf{R}\mathrm{flag}'_!(\widetilde{\mathrm{quot}}^* \mathrm{gr}^*(\mathcal{L}^1_{\mathsf{E}})^{\boxtimes i})$ carries an S_i-action and

$$\big(\mathbf{R}\mathrm{flag}'_!(\widetilde{\mathrm{quot}}^* \mathrm{gr}^*(\mathcal{L}^1_{\mathsf{E}})^{\boxtimes i})\big)^{S_i} = \mathrm{quot}^* \mathcal{L}^i_{\mathsf{E}}.$$

Therefore the projection formula implies that the complex

$$\mathbf{R}\mathrm{flag}'_!\big((\pi_{\mathrm{big}} \circ \mathrm{flag}')^*\mathsf{K} \otimes \widetilde{\mathrm{quot}}^* \mathrm{gr}^*(\mathcal{L}^1_{\mathsf{E}})^{\boxtimes i}\big) = (H^1_{\mathsf{E},C})^{\circ i}\mathsf{K}$$

carries an S_i action as well and that $((H^1_{\mathsf{E},C})^{\circ i}\mathsf{K})^{S_i} = H^i_{\mathsf{E},C}\mathsf{K}$.

Thus putting $\mathsf{K} := \mathrm{pr}^*_{\mathrm{Bun}_{k,S'}} \mathsf{A}_{\mathsf{E}}$ we are reduced to prove:

LEMMA 7.4. — *With the notation of Proposition* 7.3 *we have*

$$H^1_{\mathsf{E},C}(\mathrm{pr}^*_{\mathrm{Bun}^d_{k,S'}} A_{\mathsf{E}'}) = \mathrm{pr}^*_{\mathrm{Bun}^{d-1}_{k,S'}} A_{\mathsf{E}'} \boxtimes j_*(\mathsf{E}\otimes\mathsf{E}').$$

Proof. — In the proof we will denote sheaves with parabolic structure at S by $\mathcal{E}^{\bullet_S}$, and sheaves with parabolic structure at S' will be denoted $\mathcal{E}^{\bullet_{S'}}$ to distinguish the two sets of data. We have a morphism of the Hecke correspondences for S- and S'-parabolic sheaves:

$$\begin{CD}
\mathrm{Bun}^d_{k,S} @<{\pi_{\mathrm{big}}}<< \langle \mathcal{E}'^{\bullet_S} \subset \mathcal{E}^{\bullet_S}\rangle @>{\pi_{\mathrm{small}}\times \mathrm{pr}_C}>> \mathrm{Bun}^{d-1}_{k,S}\times C\\
@V{\mathrm{pr}_{\mathrm{Bun}^d_{k,S'}}}VV @VV{\mathrm{forget}_{\mathrm{Hecke}}}V @VVV\\
\mathrm{Bun}^d_{k,S'} @<{\pi'_{\mathrm{big}}}<< \langle \mathcal{E}'^{\bullet_{S'}} \subset \mathcal{E}^{\bullet_{S'}}\rangle @>>> \mathrm{Bun}^{d-1}_{k,S'}\times C.
\end{CD}$$

The right hand square induces a map to the fibre product:

$$\mathrm{pr}_{\mathrm{Fib}} : \langle \mathcal{E}'^{\bullet_S} \subset \mathcal{E}^{\bullet_S}\rangle \longrightarrow \langle \mathcal{E}'^{\bullet_{S'}} \subset \mathcal{E}^{\bullet_{S'}}\rangle \times_{\mathrm{Bun}^{d-1}_{k,S'}\times C} (\mathrm{Bun}^{d-1}_{k,S}\times C) =: \mathrm{Fib}.$$

Denote by $\mathrm{pr}_1, \mathrm{pr}_2$ the projections from this fibre product to its factors, and let $\mathrm{quot}, \mathrm{quot}'$ be the quotient maps from the Hecke correspondence to $\mathrm{Coh}^{1}_{0,S}$ and $\mathrm{Coh}^{1}_{0,S'}$ respectively. We can apply the projection formula to rewrite

$$H^1_{\mathsf{E},C}(\mathrm{pr}^*_{\mathrm{Bun}^d_{k,S'}} A_{\mathsf{E}'}) = \mathbf{R}\,\mathrm{pr}_{2,!}\big((\mathbf{R}\,\mathrm{pr}_{\mathrm{Fib},!}\,\mathrm{quot}^*\,\mathcal{L}_{\mathsf{E}}\big)\otimes(\pi'_{\mathrm{big}}\circ\mathrm{pr}_1)^* A_{\mathsf{E}'}\big).$$

The calculation of $\mathbf{R}\,\mathrm{pr}_{\mathrm{Fib},!}\,\mathrm{quot}^*\,\mathcal{L}^1_{\mathsf{E}}$ can be reduced to a calculation for torsion sheaves as follows. We have a map:

$$q\colon \mathrm{Fib}\longrightarrow \mathrm{Coh}^{\underline{e}_k}_{0,S},\quad (\mathcal{E}'^{\bullet_S},\mathcal{E}^{\bullet_{S'}})\longmapsto \mathcal{T}^{(i,p)}$$

where

$$\mathcal{T}^{(i,p)} := \begin{cases} \mathcal{E}^{(i,p)}/\mathcal{E}'^{(i,p)} & \text{if } p\in S' \text{ or } i=0,\\ \mathcal{E}^{(0,p)}(p)/\mathcal{E}'^{(i,p)} & \text{if } p\in S-S' \text{ and } i\neq 0,\end{cases}$$

$$(\underline{e}_k^{(*,p)})_{*=0,\dots,k-1} = \begin{cases}(1,\dots,1) & \text{if } p\in S',\\ (1,k,\dots,2) & \text{if } p\in S-S'.\end{cases}$$

This gives rise to the cartesian diagram

$$\begin{CD}
\langle \mathcal{E}'^{\bullet_S} \subset \mathcal{E}^{\bullet_S}\rangle @>{\tilde q}>> \left\langle (\mathcal{T}'^{\bullet}\subset\mathcal{T}^{\bullet}) \,\middle|\, \begin{smallmatrix}\mathcal{T}'^{\bullet}\in\mathrm{Coh}^{1}_{0,S}\\ \mathcal{T}^{\bullet}\in\mathrm{Coh}^{\underline{e}_k}_{0,S}\end{smallmatrix}\right\rangle @>{\mathrm{pr}_{\mathcal{T}'^{\bullet}}}>> \mathrm{Coh}^{1}_{0,S}\\
@V{\mathrm{pr}_{\mathrm{Fib}}}VV \square @VV{\mathrm{forget}_{\mathcal{T}'^{\bullet}}}V\\
\mathrm{Fib} @>{q}>> \mathrm{Coh}^{\underline{e}_k}_{0,S} @>{\mathrm{forget}_{S-S'}}>> \mathrm{Coh}^{1}_{0,S'},
\end{CD}$$

where $\tilde q(\mathcal{E}'^{\bullet_S}\subset\mathcal{E}^{\bullet_S}) = \big(\mathcal{E}^{\bullet_S}/\mathcal{E}'^{\bullet_S}\subset q(\mathrm{pr}_{Fib}(\mathcal{E}'^{\bullet_S},\mathcal{E}^{\bullet_S}))\big)$. By the base change formula it will be sufficient to calculate:

LEMMA 7.5. — *One has*

$$(\mathbf{R}\,\mathrm{forget}_{\mathcal{T}'^\bullet,!}\,\mathrm{pr}^*_{\mathcal{T}'^\bullet}\,\mathcal{L}^1_{\mathsf{E}})_{|\mathrm{Im(q)}} \cong (\mathrm{forget}^*_{S-S'}\,\mathcal{L}^1_{\mathsf{E}})_{|\mathrm{Im(q)}},$$

where by abuse of notation we denoted by $\mathcal{L}^1_{\mathsf{E}}$ the middle extensions of E *on $C-S$ to* $\mathrm{Coh}^{\,1}_{0,S}$ *and* $\mathrm{Coh}^{\,1}_{0,S'}$.

Proof. — First, we want to show that the image of q is the open substack of $\mathrm{Coh}^{\,\underline{e}_k}_{0,S}$ defined by the condition that the maps $\phi^{i,p}$ are surjective for $1 < i \leq k$ and $p \in S - S'$. By definition q maps into this substack and we can easily describe the torsion sheaves in the image of q. Given a point $(\mathcal{E}'^{\bullet_S}, \mathcal{E}^{\bullet_{S'}}) \in \mathrm{Fib}$, let $\mathcal{T}^\bullet := q(\mathcal{E}'^{\bullet_S}, \mathcal{E}^{\bullet_{S'}})$. Locally at $p \in S - S'$ write

$$\mathcal{E}'^\bullet \cong \bigoplus_{i=0}^{k-1} \mathcal{O}^\bullet(\frac{i}{k}p) \quad \text{and} \quad \mathcal{E}^{(0,p)} = \mathcal{O}^{\oplus k-1} \oplus \mathcal{O}(p_1)$$

such that the cokernel $\mathcal{E}^{(0,p)}/\mathcal{E}'^{(0,p)} \cong k_{p_1}$. If $p \neq p_1$ we see that

$$\mathcal{T}^\bullet \cong \bigoplus_{i=1}^{k-1} \mathcal{O}^\bullet_{ip/k}(\tfrac{k-1}{k}p) \oplus \mathcal{T}'^\bullet$$

where $\mathrm{supp}(\mathcal{T}'^\bullet) = p_1$. And if $p_1 = p$ there exists $0 \leq i_0 < k$ such that

$$\mathcal{T}^\bullet \cong \textstyle\bigoplus_{i=0, i\neq i_0}^{k-1} \mathcal{O}^\bullet_{ip/k}(\tfrac{k-1}{k}p) \oplus \mathcal{O}^\bullet_{(k+i_0)p/k}(\tfrac{k-1}{k}p).$$

By the structure of torsion sheaves of degree $\underline{e}_k$ (Lemma 3.2) this shows that the image of the map q exhausts the claimed substack. Denote by

$$j_{S-S'} : \mathrm{Coh}^{\,1}_{0,C-(S-S'),S'} \longrightarrow \mathrm{Im}(q), \quad \mathcal{T}^\bullet_1 \longmapsto \mathcal{T}^\bullet_1 \oplus \bigoplus_{i=1}^{k-1} \mathcal{O}^\bullet_{ip/k}(\tfrac{k-1}{k}p)$$

and note that by the above this is almost an open embedding (i.e. the image is an open substack isomorphic to the quotient of $\mathrm{Coh}^{\,1}_{0,C-(S-S'),S'}$ by a trivial group action).

Further, note that the map $\mathrm{pr}_{\mathcal{T}'^\bullet}$ is smooth, since it can be factored into a generalized vector bundle over $\mathrm{Coh}^{\,\underline{e}_k-1}_{0,S} \times \mathrm{Coh}^{\,1}_{0,S}$ and the projection onto the second factor. Therefore $\mathrm{pr}^*_{\mathcal{T}'^\bullet}\,\mathcal{L}^1_{\mathsf{E}}$ is the middle extension of its restriction to the subset where $\mathrm{supp}(\mathcal{T}^{(0,p)}) \not\in S$. The map $\mathrm{forget}_{\mathcal{T}'}$ is projective because the fibres are closed in a product of projective spaces and therefore $\mathbf{R}\,\mathrm{forget}_{\mathcal{T}'^\bullet,*} = \mathbf{R}\,\mathrm{forget}_{\mathcal{T}'^\bullet,!}$.

Combining the two remarks above we get a canonical morphism

$$F : j_{S-S',*}\mathcal{L}^1_{\mathsf{E}} \longrightarrow \mathbf{R}\,\mathrm{forget}_{\mathcal{T}'^\bullet,*}\,\mathrm{pr}^*_{\mathcal{T}'^\bullet}\,\mathcal{L}^1_{\mathsf{E}} = \mathbf{R}\,\mathrm{forget}_{\mathcal{T}'^\bullet,!}\,\mathrm{pr}^*_{\mathcal{T}'^\bullet}\,\mathcal{L}^1_{\mathsf{E}},$$

and $j_{S-S',*}\mathcal{L}^1_{\mathsf{E}} \cong \mathrm{forget}^*_{S-S'}\,\mathcal{L}^1_{\mathsf{E}}$ (note that $j_{S-S'.*}$ makes sense, because $\mathcal{L}^1_{\mathsf{E}}$ is a sheaf (and not a complex) at points with support outside S).

We have to prove that F is an isomorphism over the image of q. First note that $\mathrm{forget}_{\mathcal{T}'^\bullet}$ is an isomorphism over the open substack where $\mathrm{supp}(\mathcal{T}^{(0,p)}) \notin (S - S')$, so the above sheaves are isomorphic on this substack.

We are left to check that F is an isomorphism on the fibres over points $\mathcal{T}^\bullet$ with $\mathcal{T}^{(0,p)} = k_p$ and $p \in S - S'$. Since this problem is local on $\mathrm{Coh}_{0,S}^{\underline{e}k}$ we may assume that $(C, S, S') = (\mathbb{A}^1, \{0\}, \emptyset)$ and $\mathsf{E} = \mathsf{E}_n$ (see Section 4.1).

We know that $k_p \cong \mathcal{T}'^{(i,p)} \subset \mathcal{T}^{(i,p)}$, and we may factorize $\mathrm{forget}_{\mathcal{T}'^\bullet}$, into the maps forgetting the choice of the subspaces $\mathcal{T}'^{,(i,p)} \subset \mathcal{T}^{(i,p)}$ for $i > k$. Consider for example the map forgetting the choice of $\mathcal{T}'^{(k-1,p)}$. Its fibre is either a single point, if $\phi^{k-1,p}(\mathcal{T}'^{(k-2,p)}) \neq 0$, or it is isomorphic to the projective space $\mathbb{P}(H^0(C, \mathcal{T}^{(k-1,p)}))$, where the kernel of $\phi^{(k,p)}$ defines a linear subspace of codimension 1. Thus we can apply the calculation of $\mathcal{L}^1_{\mathsf{E}_n}$ (Lemma 4.4) to conclude that $\mathrm{pr}^*_{\mathcal{T}'^\bullet} \mathcal{L}^1_{\mathsf{E}}$ restricted to this projective space is the direct image $(\mathbf{R}j_*)$ of its restriction to the the complement of the kernel of $\phi^{(k,p)}$. Thus the cohomology of this fibre is isomorphic to the fibre of $\mathcal{L}^1_{\mathsf{E}}$ at $\mathcal{T}'^\bullet$ for any choice of $\mathcal{T}'^\bullet$ not contained in the linear subspace. By induction we therefore get the claimed isomorphism. □

Continuing the proof of Lemma 7.4 we can factor pr_2 as

$$\mathrm{Fib} \xrightarrow{\widetilde{\mathrm{pr}_2}} \mathrm{Bun}^{d-1}_{2,S} \times \mathrm{Coh}^1_{0,S'} \xrightarrow{\mathrm{id} \times \mathrm{pr}_C} \mathrm{Bun}^{d-1}_{2,S} \times C$$

and apply the projection formula again:

$$\begin{aligned}
H^1_{\mathsf{E},C}(\mathrm{pr}^*_{\mathrm{Bun}^d_{k,S'}} \mathsf{A}_{\mathsf{E}'}) &= \mathbf{R}\,\mathrm{pr}_{2,!}(\mathrm{pr}^*_{\mathrm{Coh}^1_{0,S'}} \mathcal{L}^1_{\mathsf{E}} \otimes \mathrm{pr}_1^* \pi'^*_{\mathrm{big}} \mathsf{A}_{\mathsf{E}'}) \\
&= \mathbf{R}(\mathrm{id} \times \mathrm{pr}_C)_!(\mathrm{pr}^*_{\mathrm{Coh}^1_{0,S'}} \mathcal{L}^1_{\mathsf{E}} \otimes \mathbf{R}\widetilde{pr}_{2,!}\, \mathrm{pr}_2^* \pi'^*_{\mathrm{big}} \mathsf{A}_{\mathsf{E}'}) \\
&= \mathbf{R}(\mathrm{id} \times \mathrm{pr}_C)_!\big(\mathrm{pr}^*_{\mathrm{Coh}^1_{0,S'}} \mathcal{L}^1_{\mathsf{E}} \otimes (\mathrm{pr}^*_{\mathrm{Bun}^{d-1}_{k,S'}} \mathsf{A}_{\mathsf{E}'} \boxtimes \mathcal{L}^1_{\mathsf{E}'})\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(base-change)} \\
&= \mathrm{pr}^*_{\mathrm{Bun}^{d-1}_{k,S'}} \mathsf{A}_{\mathsf{E}'} \boxtimes (\mathbf{R}\,\mathrm{pr}_{C,!}(\mathcal{L}^1_{\mathsf{E}} \otimes \mathcal{L}^1_{\mathsf{E}'})) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(projection formula)} \\
&= \mathrm{pr}^*_{\mathrm{Bun}^{d-1}_{k,S'}} \mathsf{A}_{\mathsf{E}'} \boxtimes (j_*(\mathsf{E} \otimes \mathsf{E}')) \qquad \text{(Corollary 4.7).} \ \square
\end{aligned}$$

To finish the proof of the vanishing Theorem 7.1 we have to show that the operator $H^i_{\mathsf{E},\mathrm{tot}}$ commutes with all other Hecke operators (at least on the level of functions).

Fix a parabolic torsion sheaf $\mathcal{T}^\bullet$ and define the Hecke operator $H_{\overline{\mathcal{T}^\bullet}}$ as the sum over all Hecke operators corresponding to torsion sheaves contained in the closure of $\mathcal{T}^\bullet$, i.e. let $\overline{\langle \mathcal{T}^\bullet \rangle} \subset \mathrm{Coh}_{0,S}^{\deg(\mathcal{T}^\bullet)}$ be the closure of the substack classifying parabolic torsion sheaves which are locally isomorphic to $\mathcal{T}^\bullet$. And define the stack

$$\mathrm{Hecke}_{\overline{\mathcal{T}^\bullet}} := \langle \mathcal{E}'^\bullet \subset \mathcal{E}^\bullet \mid \mathcal{E}^\bullet/\mathcal{E}'^\bullet \in \overline{\langle \mathcal{T}^\bullet \rangle} \subset \mathrm{Coh}_{0,S}^{\deg(\mathcal{T}^\bullet)} \rangle.$$

As before this provides a Hecke operator

$$H_{\overline{\mathcal{T}^\bullet}} : D^b(\mathrm{Bun}_{n,S}^{\underline{d}}) \longrightarrow D^b(\mathrm{Bun}_{n,S}^{\underline{d}-deg(\mathcal{T}^\bullet)}).$$

By induction on the codimension of $\langle \mathcal{T}^\bullet \rangle \subset \mathrm{Coh}_{0,S}^{\deg(\mathcal{T}^\bullet)}$ it is sufficient to prove that $H^i_{\mathsf{E},\mathrm{tot}}$ commutes with $H_{\overline{\mathcal{T}^\bullet}}$ for all $\mathcal{T}^\bullet$.

We may apply the reduction of Proposition 7.3 to reduce ourselves to prove this for the operator $H^1_{\mathsf{E},C}$.

Lemma 7.6. — *For any $\mathsf{K} \in D^b(\mathrm{Bun}_{n,S}^{\underline{d}})$ we have*

$$H^1_{\mathsf{E},C} \circ H_{\overline{\mathcal{T}^\bullet}} \mathsf{K} \cong H_{\overline{\mathcal{T}^\bullet}} \circ H^1_{\mathsf{E},C} \mathsf{K}$$

in $D^b(\mathrm{Bun}_{n,S}^{\underline{d}-\underline{1}-\deg(\mathcal{T}^\bullet)} \times C)$.

Proof. — We may assume that $\mathrm{supp}(\mathcal{T}^\bullet) = p$ for a single point $p \in S$, since every torsion sheaf is the direct sum of sheaves supported at a single point and for $p \notin S$ the lemma is easy to prove (and we do not use it in this case).

As in Lemma 7.4 the claim is easily reduced to the following lemma formulated on the stack of parabolic torsion sheaves (apply the projection formula once more): denote by

$$\mathrm{Flag}_{1,\mathcal{T}^\bullet} := \left\langle (0 \to \mathcal{T}'^\bullet \to \mathcal{Q}^\bullet \to \mathcal{T}''^\bullet \to 0) \;\middle|\; \begin{array}{l} \mathcal{T}'^\bullet \in \mathrm{Coh}_{0,S}^{\underline{1}},\ \mathcal{T}''^\bullet \in \overline{\langle \mathcal{T}^\bullet \rangle} \\ \mathcal{Q}^\bullet \in \mathrm{Coh}_{0,S}^{\underline{1}+\deg(\mathcal{T}^\bullet)} \end{array} \right\rangle.$$

Further, denote by $\mathrm{pr}_{\mathcal{T}'^\bullet}$, $\mathrm{pr}_{\mathcal{Q}^\bullet}$ the projections and by pr_C the projection to the curve C defined by the support of $\mathcal{T}'^\bullet$.

Let $\mathrm{Flag}_{\mathcal{T}^\bullet,1}$ be the stack defined as above with the roles of $\mathcal{T}'^\bullet$ and $\mathcal{T}''^\bullet$ interchanged, i.e. $\mathcal{T}''^\bullet \in \mathrm{Coh}_{0,S}^{\underline{1}}$ and $\mathcal{T}'^\bullet \in \overline{\langle \mathcal{T}^\bullet \rangle}$, and denote its projections by $\mathrm{rp}_{\mathcal{Q}^\bullet}$, etc.

LEMMA 7.7. — *We have a canonical isomorphism of complexes*

$$\mathbf{R}(\mathrm{pr}_{\mathcal{Q}^\bullet} \times \mathrm{pr}_C)_! \, \mathrm{pr}^*_{\mathcal{T}'^\bullet} \, \mathcal{L}^1_{\mathsf{E}} \cong \mathbf{R}(\mathrm{rp}_{\mathcal{Q}^\bullet} \times \mathrm{rp}_C)_! \, \mathrm{rp}^*_{\mathcal{T}''^\bullet} \, \mathcal{L}^1_{\mathsf{E}}$$

in $D^b(\mathrm{Coh}_{0,S}^{1+\deg(\mathcal{T})} \times C)$.

Proof. — This is similar to the proof of Lemma 7.5: over the open substack of $\mathrm{Coh}_{0,S}^{1+\deg(\mathcal{T}^\bullet)}$ where the support of the torsion sheaf is not equal to $\mathrm{supp}(\mathcal{T}^\bullet) = p$ the stacks $\mathrm{Flag}_{1,\mathcal{T}^\bullet}$ and $\mathrm{Flag}_{\mathcal{T}^\bullet,1}$ are isomorphic, because there are no extensions between sheaves supported at different points. Therefore the claimed isomorphism exists over this subset. To extend it, we again reduce to the case $(C, S) = (\mathbb{A}^1, \{0\})$ and note that the maps $\mathrm{pr}_{\mathcal{Q}^\bullet}, \mathrm{rp}_{\mathcal{Q}^\bullet}$ are projective and the map $\mathrm{pr}_{\mathcal{T}'^\bullet}$ (resp. $\mathrm{rp}_{\mathcal{T}''^\bullet}$) can be factored as

$$\mathrm{Flag}_{1,\mathcal{T}^\bullet} \longrightarrow \mathrm{Coh}^1_{0,S} \times \langle \overline{\mathcal{T}^\bullet} \rangle \to \mathrm{Coh}^1_{0,S} .$$

The first map is a generalized vector bundle, and the second one is the projection of a product, therefore both maps are locally acyclic. Hence we can use the exact triangle

$$\to \mathcal{L}^1_{\mathsf{E}} \longrightarrow \mathbf{R}j_* \mathsf{E}_\infty \longrightarrow j_! \mathsf{E}_\infty(-n) \xrightarrow{[1]}$$

of Proposition 4.1 once more. If we replace $\mathcal{L}^1_{\mathsf{E}}$ by $j_! \mathsf{E}_\infty(-n)$, then the statement of the lemma is obvious. Further, if we replace $\mathcal{L}^1_{\mathsf{E}}$ by $\mathbf{R}j_* \mathsf{E}_\infty$, then the lemma follows from the Leray spectral sequence, because we just saw that $\mathbf{R}j_*$ commutes with $\mathrm{pr}^*_{\mathcal{T}'^\bullet}$ (and $\mathrm{rp}^*_{\mathcal{T}''^\bullet}$) and we may replace $(\mathbf{R}\,\mathrm{pr}_{\mathcal{Q}^\bullet} \times \mathrm{pr}_C)_!$ by $(\mathbf{R}\,\mathrm{pr}_{\mathcal{Q}^\bullet} \times \mathrm{pr}_C)_*$ because this map is projective. Therefore the lemma follows for $\mathcal{L}^1_{\mathsf{E}}$ as well. □

8. The vanishing theorem implies that $j_{\mathrm{Hom},!} \mathsf{F}^k_{\mathsf{E}} = j_{\mathrm{Hom},!*} \mathsf{F}^k_{\mathsf{E}} = \mathbf{R}j_{\mathrm{Hom},*} \mathsf{F}^k_{\mathsf{E}}$.

With the notations of the fundamental diagram (2.2) of Section 2 we have:

PROPOSITION 8.1. — *Assume that the vanishing theorem 7.1 holds for $k < n$. Then for $k < n$ and $d >> 0$ we have $j_{\mathrm{Hom},!} \mathsf{F}^k_{\mathsf{E}} = j_{\mathrm{Hom},!*} \mathsf{F}^k_{\mathsf{E}}$ and thus for $k \leq n$ we have $\mathsf{F}^k_{\mathsf{E},!} = \mathsf{F}^k_{\mathsf{E}}$.*

Since we have shown the vanishing theorem for local systems of rank ≤ 3, we get in particular:

COROLLARY 8.2. — *For $k \leq n \leq 3$ the sheaves $\mathsf{F}^k_{\mathsf{E}} \cong \mathsf{F}^k_{\mathsf{E},!}$ are isomorphic.* □

Proof of Proposition 8.1. — The Hecke-property of $\mathcal{L}^d_{\mathsf{E}}$ allows us to copy the proof in [11] with some minor changes. We use induction, and assume that the proposition is true for all $k' < k$ thus, in particular $\mathsf{F}^k_{\mathsf{E}} \cong \mathsf{F}^k_{\mathsf{E},!}$.

- *Step 1.* — The claim is true over the substack of parabolic vector bundles. Here every nontrivial homomorphism from $\Omega^\bullet$ into a vector bundle is injective, that is

$$\mathrm{Hom}^{\mathrm{inj}}_k = \mathrm{Hom}_k - (\text{zero-section}) \text{ over } \mathrm{Bun}^{\mathrm{good}}_{k,S}\,.$$

Furthermore $\mathsf{F}^k_{\mathsf{E}}$ is $\mathbb{G}_m$-invariant, since the Fourier transform preserves this property by [21], Proposition 1.2.3.4. Therefore we can apply Lemma 0.3 and get

$$j_{\mathrm{Hom},!}\mathsf{F}^k_{\mathsf{E}} = \mathbf{R}j_{\mathrm{Hom},*}\mathsf{F}^k_{\mathsf{E}} = j_{\mathrm{Hom},!*}\mathsf{F}^k_{\mathsf{E}} \iff \mathbf{R}\pi_!\mathsf{F}^k_{\mathsf{E}} = 0,$$

where $\pi\colon \mathrm{Hom}^{\mathrm{inj}}_k \to \mathrm{Bun}^{\mathrm{good}}_{k,S}$ is the projection.

Recall from Formula (5.1), that we can calculate $\mathsf{F}^k_{\mathsf{E}} = \mathsf{F}^k_{\mathsf{E},!}$ with the following diagram:

$$\begin{tikzcd}
 & \left\langle\begin{smallmatrix} \mathcal{J}^\bullet_1 \subset \cdots \subset \mathcal{J}^\bullet_k \subset \mathcal{E}^\bullet \\ \mathcal{J}^\bullet_i/\mathcal{J}^\bullet_{i-1} \xrightarrow{\sim} \Omega^{\bullet,k-i} \end{smallmatrix}\right\rangle \arrow[dl, bend right, "\text{forget}"'] \arrow[d] \arrow[dr, "\mathrm{pr}_{\mathcal{J}^\bullet_i} \times \mathrm{quot}"] \arrow[rrd, bend left, "\mathrm{ext}"] \arrow[ddl, bend right, "\widetilde{\pi}"] & & \\
\mathrm{Hom}^{\mathrm{inj}}_k \arrow[d, "\pi"'] & \langle \mathcal{J}^\bullet_k \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet \rangle \arrow[dl] \arrow[dr] & \left\langle\begin{smallmatrix} \mathcal{J}^\bullet_1 \subset \cdots \subset \mathcal{J}^\bullet_k \\ \mathcal{J}^\bullet_i/\mathcal{J}^\bullet_{i-1} \xrightarrow{\sim} \Omega^{\bullet,k-i} \end{smallmatrix}\right\rangle \times \mathrm{Coh}^d_{0,S} \arrow[r, "\mathrm{ext}'"] \arrow[d, "\pi'"] & \mathbb{A}^1 \\
\mathrm{Bun}^{d_k,\mathrm{good}}_{k,S} & & \mathrm{Bun}^{d_k-d}_{k,S} \times \mathrm{Coh}^d_{0,S}\,. &
\end{tikzcd}$$

Where $d_k - d = \deg(\mathcal{J}^{(0,p)}_k)$ and we know that $\mathsf{F}^k_{\mathsf{E}} = \mathbf{R}\,\mathrm{forget}_!(\mathrm{ext}^*\mathsf{L}_\psi \otimes \mathrm{quot}^*\mathcal{L}^d_{\mathsf{E}})$. Therefore

$$(\mathbf{R}\pi_!\mathsf{F}^k_{\mathsf{E}})_{|\mathrm{Bun}^{\mathrm{good}}_{k,S}} = \mathbf{R}\widetilde{\pi}_!(\mathrm{ext}^*\mathsf{L}_\psi \otimes \mathrm{quot}^*\mathcal{L}_{\mathsf{E}}) = H^{-d}_{\mathsf{E},\mathrm{tot}}(\mathbf{R}\pi'_!\,\mathrm{ext}'^*\,\mathsf{L}_\psi),$$

and the vanishing theorem 7.1 implies that $H^{-d}_{\mathsf{E},\mathrm{tot}}(\mathbf{R}\pi'_!\,\mathrm{ext}^*\,\mathsf{L}_\psi) = 0$ for $d \gg 0$.

- *Step 2: induction on the length of the torsion of* $\mathcal{F}^\bullet$. — Recall that in Section 6 we introduced for any $\underline{r} = (r_{i,p})$

$$\mathrm{Coh}_{k,S}^{\underline{d},\leq\underline{r}} := \langle \mathcal{F}^\bullet \in \mathrm{Coh}_{k,S}^{\underline{d}} \mid \mathrm{length}(\mathrm{torsion}(\mathcal{F}^\bullet)) \leq \underline{r}\rangle$$

the stack of parabolic sheaves such that the length of the torsion of the coherent sheaves $\mathcal{F}^{(i,p)}$ is bounded by $r_{i,p}$. And by induction we need to compare $j_{\mathrm{Hom},!}\mathsf{F}_{\mathsf{E}}^k$ and $\mathbf{R}j_{\mathrm{Hom},*}\mathsf{F}_{\mathsf{E}}^k$ above the points of this stack, where the parabolic sheaf is good and the length of the torsion is exactly $\underline{r}$. Furthermore, note that the torsion free part of a good sheaf is good as well.

It is sufficient to prove the proposition after a smooth base change. To get a map to torsion free parabolic sheaves (we want to apply the vanishing theorem again) we use the stack

$$\widetilde{\mathrm{Coh}}_{k,S}^{\underline{d},\leq\underline{r}} := \big\langle \mathcal{E}^\bullet \subset \mathcal{F}^\bullet \to \mathcal{T}^\bullet \\ \mid \mathcal{E}^\bullet \in \mathrm{Bun}_{k,S}^{\underline{d}-\underline{r},\mathrm{good}}, \mathcal{F}^\bullet \in \mathrm{Coh}_{k,S}^{\underline{d},\mathrm{good}}, \mathcal{T}^\bullet \in \mathrm{Coh}_{0,S}^{\underline{r}}\big\rangle.$$

From Remark 6.3 we know that the forgetful map $\widetilde{\mathrm{Coh}}_{k,S}^{\underline{d},\leq\underline{r}} \to \mathrm{Coh}_{k,S}^{\underline{d},\leq\underline{r}}$ is smooth. And the map

$$\mathrm{gr} : \widetilde{\mathrm{Coh}}_{k,S}^{\underline{d},\leq\underline{r}} \longrightarrow \mathrm{Bun}_{k,S}^{\underline{d}-\underline{r},\mathrm{good}} \times \mathrm{Coh}_{0,S}^{\underline{r}}, \quad (\mathcal{E}^\bullet \subset \mathcal{F}^\bullet) \longmapsto (\mathcal{E}^\bullet, \mathcal{F}^\bullet/\mathcal{E}^\bullet)$$

is a vector bundle, since $\dim(\mathrm{Ext}^1_{\mathrm{para}}(\mathcal{T}^\bullet, \mathcal{E}^\bullet))$ depends only on the degree of $\mathcal{T}^\bullet$ and on the rank and the degree of $\mathcal{E}^\bullet$ by Lemma 3.5. Furthermore, over any point of $\widetilde{\mathrm{Coh}}_{k,S}^{\underline{d}\leq\underline{r}}$ we have $\mathrm{Ext}^1(\Omega^{\bullet,k-1}, \mathcal{E}^\bullet) = 0$ (by assumption $\mathcal{E}^\bullet \in \mathrm{Bun}_{k,S}^{\underline{d}-\underline{r},\mathrm{good}}$), therefore the dimension of $\mathrm{Hom}(\Omega^{\bullet,k-1}, \mathcal{E}^\bullet)$ is constant, so $\mathrm{Hom}(\Omega^{\bullet,k-1}, \mathcal{E}^\bullet)$ is a vector bundle over this stack.

Consider the base change $\widetilde{\mathrm{Hom}}_k$ of Hom_k to $\widetilde{\mathrm{Coh}}_{k,S}^{\underline{d},\leq\underline{r}}$, and analogously define

$$\widetilde{\mathrm{Hom}}_k^{\mathrm{inj}} := \mathrm{Hom}_k^{\mathrm{inj}} \times_{\mathrm{Coh}_{k,S}^{\underline{d}}} \widetilde{\mathrm{Coh}}_{k,S}^{\underline{d},\leq\underline{r}}.$$

By the above, the map

$$\mathrm{gr}_{\widetilde{\mathrm{Hom}}} : \widetilde{\mathrm{Hom}}_k \longrightarrow \mathrm{Bun}_{0,S}^{\underline{d}-\underline{r},\mathrm{good}} \times \mathrm{Hom}(\Omega^{\bullet,k-1}, \mathcal{T}^\bullet),$$
$$(\Omega^{\bullet,k-1} \xrightarrow{s} \mathcal{F}^\bullet, \mathcal{E}^\bullet \subset \mathcal{F}^\bullet \xrightarrow{p} \mathcal{T}^\bullet) \longmapsto (\mathcal{E}^\bullet, \Omega^{\bullet,k-1} \xrightarrow{pos} \mathcal{T}^\bullet)$$

is also vector bundle, because it is the composition of the map induced by composing $p \circ s$, which has fibres $\mathrm{Hom}(\Omega^{\bullet,k-1},\mathcal{E}^\bullet)$, and quot. The zero section

$$(\mathcal{E}^\bullet,\Omega^{\bullet,k-1} \xrightarrow{s} \mathcal{T}^\bullet) \longmapsto (\mathcal{T} \xrightarrow{(0,s)} \mathcal{E}^\bullet \oplus \mathcal{T}^\bullet, \mathcal{E}^\bullet \subset \mathcal{E}^\bullet \oplus \mathcal{T}^\bullet \to \mathcal{T}^\bullet)$$

of this bundle is the substack [8]

$$\left\langle \begin{pmatrix} \Omega^{\bullet,k-1} \xrightarrow{s} \mathcal{F}^\bullet \\ \mathcal{E}^\bullet \subset \mathcal{F}^\bullet \xrightarrow{p} \mathcal{T}^\bullet \end{pmatrix} \middle| \ \mathrm{length}(\mathrm{torsion}(\mathcal{F}^\bullet)) = \underline{r} \text{ and } \Omega^{\bullet,k-1} \to \mathcal{T}^\bullet \to \mathcal{F}^\bullet \right\rangle$$

and this is by induction hypothesis the substack to which we have to extend $\mathsf{F}^k_{\mathsf{E}}$. Thus, denote $\mathrm{gr}_{\widetilde{\mathrm{Hom}}^{\mathrm{inj}}} := \mathrm{gr}_{\widetilde{\mathrm{Hom}}}|_{\mathrm{Hom}^{\mathrm{inj}}}$ and again we have to show that

$$\mathbf{R}\,\mathrm{gr}_{\widetilde{\mathrm{Hom}}^{\mathrm{inj}},!}\,\mathrm{pr}^*_{\mathrm{Hom}^{\mathrm{inj}}}\,\mathsf{F}^k_{\mathsf{E}} = 0.$$

Since this can be checked fibre wise, we fix a point

$$(\mathcal{E}^\bullet,\Omega^{\bullet,k-1} \to \mathcal{T}^\bullet) \in \mathrm{Bun}^{\underline{d}-\underline{r},\mathrm{good}}_{k,S} \times \mathrm{Hom}(\Omega^{\bullet,k-1},\mathcal{T}^\bullet)$$

and denote by $\mathrm{Fibre}_{\mathcal{E}^\bullet,\Omega^{\bullet,k-1}\to\mathcal{T}^\bullet}$ the fibre of $\widetilde{\mathrm{Hom}}^{\mathrm{inj}}_k$ over this point.

▷ *Step 2.1: reduction to the case $\Omega^{\bullet,k-1} \twoheadrightarrow \mathcal{T}^\bullet$ is surjective.*

Factor $s\colon \Omega^{\bullet,k-1} \to \mathrm{Im}(s) =: \mathcal{T}'^\bullet \subset \mathcal{T}^\bullet$, denote $\mathcal{T}^\bullet/\mathcal{T}'^\bullet =: \mathcal{T}''^\bullet$ and set $\underline{r}'' := \deg(\mathcal{T}''^\bullet)$. Then for any $(\Omega^{\bullet,k-1} \to \mathcal{F}^\bullet \twoheadrightarrow \mathcal{T}^\bullet)$ in $\mathrm{Fibre}_{\mathcal{E}^\bullet,\Omega^{\bullet,k-1}\xrightarrow{s}\mathcal{T}^\bullet}$ we get an extension $\mathcal{F}'^\bullet \subset \mathcal{F}^\bullet \twoheadrightarrow \mathcal{T}''^\bullet$. Consider the Hecke operator for

$$\mathrm{Hom}^{\mathrm{inj}}_k \longleftarrow \mathrm{Hecke}_{\mathrm{Hom}^{\mathrm{inj}}} := \left\langle \begin{matrix} \Omega^{\bullet,k-1} \hookrightarrow \mathcal{F}'^\bullet \\ \mathcal{F}'^\bullet \hookrightarrow \mathcal{F}^\bullet \twoheadrightarrow \mathcal{T}''^\bullet \end{matrix} \right\rangle \longrightarrow \mathrm{Hom}^{\mathrm{inj}}_k \times \mathrm{Coh}^{\underline{r}''}_{0,S}.$$

We know by Proposition 6.2 that $\mathsf{F}^k_{\mathsf{E}!}$ is a Hecke eigensheaf and that

$$\mathrm{Fibre}_{\mathcal{E}^\bullet,\Omega^{\bullet,k-1}\xrightarrow{s}\mathcal{T}^\bullet} = \mathrm{Fibre}_{\mathcal{E}^\bullet,\Omega^{\bullet,k-1}\to\mathcal{T}'^\bullet} \times_{\mathrm{Hom}^{\mathrm{inj}}_k \times \mathrm{Coh}^{\underline{r}''}_{0,S}} \mathrm{Hecke}_{\mathrm{Hom}^{\mathrm{inj}}}.$$

Thus, in case that $\underline{r}''$ is not constant, we can establish our claim that

$$H^*_c(\mathrm{Fibre}_{\mathcal{E}^\bullet,\Omega^{\bullet,k-1}\xrightarrow{s}\mathcal{T}}, \mathsf{F}^k_{\mathsf{E}}) = 0,$$

[8] Note that, if there is a splitting of $\mathcal{F}^\bullet \twoheadrightarrow \mathcal{T}^\bullet$, then there is a unique one, since $\mathrm{torsion}(\mathcal{F}^\bullet)$ is a subsheaf of $\mathcal{F}^\bullet$.

since the above Hecke operator is zero by 6.1.

If on the other hand $\underline{r}''$ is constant, we know that it is sufficient to prove the claim for $\mathrm{Fibre}_{\mathcal{E}^\bullet,\Omega^{\bullet,k-1}\to\mathcal{T}'^\bullet}$. This has already been done in the case that $\mathcal{T}'^\bullet \neq \mathcal{T}^\bullet$. Therefore we may assume that $\mathrm{Im}(\Omega^{\bullet,k-1}) = \mathcal{T}'^\bullet = \mathcal{T}^\bullet$.

$\triangleright$ *Step 2.2. Assume that $\Omega^{\bullet,k-1} \twoheadrightarrow \mathcal{T}^\bullet$ is surjective, i.e. $\mathcal{T}^\bullet \cong \Omega^{\bullet,k-1}/\Omega^{\bullet,k-1}(-D)$ for some effective parabolic divisor D.*

In this case, giving an element $(\Omega^{\bullet,k-1} \hookrightarrow \mathcal{F}^\bullet) \in \mathrm{Fibre}_{\mathcal{E}^\bullet,\Omega^{\bullet,k-1}\to\mathcal{T}^\bullet}$ is the same as to give a map $\Omega^{k-1\bullet}(-D) \hookrightarrow \mathcal{E}^\bullet$, because we can define a map

$$(\Omega^{\bullet,k-1}(-D) \hookrightarrow \mathcal{E}^\bullet, \mathcal{T}^\bullet) \longmapsto (\mathcal{E}^\bullet \subset (\mathcal{E}^\bullet \oplus \Omega^{\bullet,k-1})/\Omega^{\bullet,k-1}(-D))$$

And indeed for any square

$$\begin{tikzcd}
\Omega^{\bullet,k-1}(-D) \arrow[r, hook] \arrow[d, hook] & \Omega^{\bullet,k-1} \arrow[r, two heads] \arrow[d, hook] & \mathcal{T}_\bullet \arrow[d, hook, "\cong"] \\
\mathcal{E}^\bullet \arrow[r, hook] & \mathcal{F}^\bullet \arrow[r, two heads] & \mathcal{T}^\bullet
\end{tikzcd}$$

we automatically get that $\mathcal{F}^\bullet = \mathcal{E}^\bullet \bigoplus_{\Omega^{\bullet,k-1}(-D)} \Omega^{\bullet,k-1}$.

Thus we get an isomorphism

$$\mathrm{Fibre}_{\mathcal{E}^\bullet,\Omega^{\bullet,k-1}\to\mathcal{T}^\bullet} \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{inj}}(\Omega^{\bullet,k-1}(-D), \mathcal{E}^\bullet).$$

Furthermore under this isomorphism $\mathsf{F}^k_{\mathsf{E}|\mathrm{Fibre}}$ becomes the sheaf on $\mathrm{Bun}^{\underline{d}-\underline{r},\mathrm{good}}_{k,S}$ constructed in the same way as $\mathsf{F}^k_{\mathsf{E}}$, by replacing $\Omega^{\bullet,k-1}$ by $\Omega^{\bullet,k-1}(-D)$. More precisely, since

$$\mathcal{E}^\bullet/\Omega^{\bullet,k-1}(-D) \cong \mathcal{F}^\bullet/\Omega^{\bullet,k-1},$$

we have again

$$\begin{tikzcd}
 & \left\langle \begin{matrix} \mathcal{J}_1^\bullet \subset \cdots \subset \mathcal{J}_k^\bullet \subset \mathcal{E}^\bullet \\ \mathcal{J}_1^\bullet \xrightarrow{\sim} \Omega^{\bullet,k-1}(-D) \\ \mathcal{J}_{i+1}^\bullet/\mathcal{J}_i^\bullet \xrightarrow{\sim} \Omega^{\bullet,k-i+1} \end{matrix} \right\rangle \arrow[rr, "\mathrm{ext}"] \arrow[dl, "\mathrm{forget}"'] \arrow[d] \arrow[dr] \arrow[ddl, bend right=60, "\widetilde{\pi}"'] & & \mathbb{A}^1 \\
\mathrm{Hom}^{\mathrm{inj}}(\Omega^\bullet(-D), \mathcal{E}^\bullet) \arrow[d, "\pi"] & \langle \mathcal{J}_k^\bullet \hookrightarrow \mathcal{E}^\bullet \twoheadrightarrow \mathcal{T}^\bullet \rangle \arrow[dl] \arrow[dr] & \left\langle \begin{matrix} \mathcal{J}_1^\bullet \subset \cdots \subset \mathcal{J}_k^\bullet \\ \mathcal{J}_1^\bullet \xrightarrow{\sim} \Omega^{\bullet,k-1}(-D) \\ \mathcal{J}_{i+1}^\bullet/\mathcal{J}_i^\bullet \xrightarrow{\sim} \Omega^{\bullet,k-i+1} \end{matrix} \right\rangle \times \mathrm{Coh}^{\underline{d}}_{0,S} \arrow[ur, "\mathrm{ext}'"'] \arrow[d, "\pi'"] & \\
\mathrm{Bun}^{\underline{d}-\underline{r},\mathrm{good}}_{k,S} & & \mathrm{Bun}^{-\underline{r}}_{k,S} \times \mathrm{Coh}^{\underline{d}}_{0,S} &
\end{tikzcd}$$

Here ext' is the composition

$$\begin{tikzcd}
\langle \mathcal{J}_i^\bullet, \mathrm{gr}(\mathcal{J}_k^\bullet) \cong \Omega^{\bullet,k-1}(-D) \oplus \bigoplus_{j=0}^{k-2} \Omega^{\bullet,j} \rangle \arrow[r] \arrow[ddr, "\mathrm{ext}'"'] & H^1(C,\Omega^1(-D)) \oplus \bigoplus_{j=0}^{k-2} H^1(C,\Omega^1) \arrow[d] \\
 & H^1(C,\Omega) \oplus \bigoplus_{j=0}^{k-2} H^1(C,\Omega) \arrow[d, "\Sigma\,\mathrm{Res}"'] \\
 & H^1(C,\Omega) \cong \mathbb{A}^1
\end{tikzcd}$$

and therefore

$$\mathsf{F}^k_{\mathsf{E}|\mathrm{Fibre}_{\mathcal{E}^\bullet,\Omega^{\bullet,k-1}\twoheadrightarrow\mathcal{T}}} \cong (\mathbf{R}\,\mathrm{forget}_!(\mathrm{quot}^*\,\mathcal{L}_{\mathsf{E}} \otimes \mathrm{ext}^*\,\mathsf{L}_\psi))_{|\mathrm{Fibre\ of\ }\pi\mathrm{\ over\ }\mathcal{E}^\bullet}.$$

But here we can apply the vanishing theorem again, because

$$\mathbf{R}(\pi \circ \mathrm{forget})_!(\mathrm{quot}^*\,\mathcal{L}_{\mathsf{E}} \otimes \mathrm{ext}^*\,\mathsf{L}_\psi) = H^{-d}_{\mathsf{E},\mathrm{tot}}(\mathbf{R}\pi'_!\,\mathrm{ext}'^*\,\mathsf{L}_\psi) = 0. \qquad \square$$

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Jochen HEINLOTH,
Mathematisches Institut der
Universität Göttingen
Bunsenstr. 3–5
37073 Göttingen (Germany).
heinloth@uni–math.gwdg.de