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DECAY OF VOLUMES UNDER ITERATION OF MEROMORPHIC MAPPINGS

by Vincent GUEDJ

Introduction.

Let $f : X \to X$ be a meromorphic self-mapping of a compact Kähler manifold X. In this note we address the following question:

How much can f decrease volumes ?

This does not make much sense if f is degenerate, so we assume throughout the paper that f is *dominating*, i.e. that its jacobian determinant does not vanish identically (in any local chart). The mapping f is generically locally open and one-to-one, so that $\mathrm{Vol}(f(\Omega))$ has approximatively the same size as $\mathrm{Vol}(\Omega)$ for most (small) open subsets Ω. This does not necessarily hold however if Ω meets the critical set of f. Further problems may arise when Ω contains points of the indeterminacy locus I_f (the set of points where f is not holomorphic). For applications to complex dynamics, we moreover need estimates that are both uniform in Ω and quantitative with respect to iteration: if Ω does not meet the critical set nor the indeterminacy locus, it may still happen that $f^j(\Omega)$ does, for some $j \in \mathbb{N}$. Our main result takes this into account and gives a rough asymptotic lower bound in terms of the "degrees" $\delta_1(f^j) := \int_{X \setminus I_{f^j}} (f^j)^*\omega \wedge \omega^{k-1}$, where ω is some Kähler form on X and $k = \dim_{\mathbb{C}} X$.

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THEOREM 0.1.— *Assume* $\delta_1(f^j) \preceq \lambda^j$ *where* $\lambda > 1$. *Then* $\forall j \in \mathbb{N}$, $\Omega \subset X$,

$$\mathrm{Vol}\,(f^j\Omega) \geqslant \exp\left(-\frac{C}{\mathrm{Vol}\,(\Omega)}\lambda^j\right),$$

where $C > 0$ *is independent of* j, Ω.

Here $\delta_1(f^j) \preceq \lambda^j$ means that $\delta_1(f^j)\lambda^{-j}$ is bounded from above.

The study of volume estimates has a long history in complex dynamics. It has been used to construct and characterize invariant currents (see [G 03a], [FJ 03], [CG 04] for references). A partial result was obtained in [G 03a] in this direction. The present one has several advantages: it concerns a more general frame ([G 03a] dealt with case $X = \mathbb{P}^k$) in which we do not assume either algebraic stability, or integrability of the logarithm of the jacobian. We also avoid the use of the Green function made in [G 03a]. In fact, the existence of the Green current is not known in this quite general context and one may hope that volume estimates will help to construct it. Indeed after establishing our main result in *section 1*, we give such a construction in *section 2* under a mild cohomological assumption (theorem 2.2). In the last *section 3* we discuss our hypotheses and related open questions.

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1. Volume estimates.

In the whole paper we fix a Kähler form ω on X normalized by $\int_X \omega^k = 1$, where k denotes the (complex) dimension of X. All volumes are computed with respect to the probability volume form ω^k.

1.1. Dynamical degrees.

Given a smooth real form θ of bidegree (l,l), the pull-back of θ by f is defined in the following way: let $\Gamma_f \subset X \times X$ denote the graph of f and consider a desingularization $\tilde{\Gamma}_f$ of Γ_f. We have a commutative diagram

$$\begin{tikzcd} & \tilde{\Gamma}_f \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, "f"] & & X \end{tikzcd}$$

where π_1, π_2 are holomorphic maps. We set $f^*\theta := (\pi_1)_*(\pi_2^*\theta)$ where we push forward the smooth form $\pi_2^*\theta$ by π_1 as a current. Note that $f^*\theta$ is actually a form with L^1_{loc}-coefficients which coincides with the usual smooth pull-back $(f_{|X\setminus I_f})^*\theta$ on $X \setminus I_f$, thus the definition does not depend on the choice of desingularization. In other words, $f^*\theta$ is the trivial extension, as current, of $(f_{|X\setminus I_f})^*\theta$ through I_f. This definition induces a linear action on the cohomology vector space $H^{l,l}(X,\mathbb{R})$ which we denote again by f^*:

$$f^* : \{\theta\} \in H^{l,l}(X,\mathbb{R}) \mapsto \{f^*\theta\} \in H^{l,l}(X,\mathbb{R}).$$

Here $\{\theta\}$ denotes the de Rham cohomology class defined by the smooth closed form θ, while $\{f^*\theta\}$ denotes the de Rham cohomology class defined by the closed current $f^*\theta$.

Similarly one can define the push-forward of a form θ by

$$f_*\theta := (\pi_2)_*(\pi_1^*\theta).$$

This induces a linear action f_* on cohomology which is dual to that of f^*. We set

$$\delta_l(f) := \int_X f^*\omega^l \wedge \omega^{k-l}.$$

This "degree" is comparable to the spectral radius of the linear action induced by f on $H^{l,l}(X,\mathbb{R})$. It does not necessarily behave well under iteration, hence the following definition:

DEFINITION 1.1.— *The l^{th}-dynamical degree of f is*

$$\lambda_l(f) := \limsup_{j\to+\infty} [\delta_l(f^j)]^{1/j}.$$

In this note we shall be mainly interested in the first dynamical degree $\lambda := \lambda_1(f)$. Observe that for $l = k = \dim_{\mathbb{C}} X$, $\lambda_k(f)$ is the *topological degree* of f (the number of preimages of a generic point) which we shall denote by $d_t = d_t(f)$. We refer the reader to [G 03b], [DS 04] for more information on dynamical degrees.

1.2. The estimates.

Our main result is a consequence of the following proposition.

PROPOSITION 1.2. — *There exists $C > 0$ such that*

$$\mathrm{Vol}\,(f^j\Omega) \geqslant \frac{\mathrm{Vol}\,(\Omega)}{d_t^j} \exp\left(-\frac{C}{\mathrm{Vol}\,(\Omega)} \sum_{l=0}^{j-1} \sum_{p=0}^{l} \delta_1(f^p)\right)$$

for all $j \in \mathbb{N}$ and for all $\Omega \subset X$.

Proof. — Let $\mathrm{Jac}(f)$ denote the complex jacobian of f with respect to ω^k, defined by $f^*\omega^k = |\mathrm{Jac}(f)|^2\omega^k$. It satisfies the chain rule

$$\mathrm{Jac}\,(f^j) = \Pi_{l=0}^{j-1} \mathrm{Jac}(f) \circ f^l.$$

A straightforward computation shows that $\log|\mathrm{Jac}(f)|$ can be written as a difference of quasiplurisubharmonic (qpsh) functions. Let us recall that a qpsh function is an upper semi-continuous (u.s.c.) function $\varphi \in L^1(X, \mathbb{R} \cup \{-\infty\})$ which is locally given as the sum of a psh and a smooth function. Thus $dd^c\varphi$ is a well defined real current of bidegree $(1,1)$ on X which is bounded from below by a smooth form, in particular $dd^c\varphi \geqslant -A\omega$ for some large $A > 0$. Here $d = \partial + \overline{\partial}$ and $d^c = \frac{i}{2\pi}[\overline{\partial} - \partial]$.

Write $\log|\mathrm{Jac}(f)| = u - v$, where u, v both are qpsh. Since u, v are u.s.c. on X which is compact, we can assume without loss of generality that $u, v \leqslant 0$. By the change of variables formula, we get

$$\mathrm{Vol}(f^j\Omega) \geqslant \frac{1}{d_t^j} \int_\Omega |\mathrm{Jac}(f^j)|^2\omega^k \geqslant \frac{\mathrm{Vol}(\Omega)}{d_t^j} \exp\left(\frac{2}{\mathrm{Vol}(\Omega)} \int_\Omega \log|\mathrm{Jac}(f^j)|\omega^k\right),$$

where the last inequality follows from the concavity of the logarithm. Using the chain rule and $\log|\mathrm{Jac}(f)| \geqslant u$, we obtain

$$\int_\Omega \log|\mathrm{Jac}(f^j)|\omega^k \geqslant \sum_{l=0}^{j-1} \int_X (u \circ f^l)\omega^k.$$

Our lower bound is thus a consequence of the next proposition which is of independent interest. □

PROPOSITION 1.3. — *There exists $C > 0$ such that for all qpsh function φ, $dd^c\varphi \geqslant -\omega$, and for all $n \in \mathbb{N}$,*

$$0 \leqslant ||\varphi \circ f^n||_{L^1(X)} := \int_X (|\varphi| \circ f^n)\omega^k \leqslant C \sum_{j=0}^{n} \delta_1(f^j).$$

Proof. — Let φ be a qpsh function such that $dd^c\varphi \geqslant -\omega$. We can assume without loss of generality that $\varphi \leqslant 0$.

Fix Θ a smooth probability measure whose support is concentrated near a point a which is neither critical, nor a point of indeterminacy. Thus f is a local biholomorphism from a neighborhood of a (containing the support of Θ) onto a neighborhood of $f(a)$ (containing the support of $f_*\Theta$). Therefore $f_*\Theta$ is yet another smooth probability measure on X. Since X is Kähler, we can find a smooth form R of bidegree $(k-1, k-1)$ on X such that $f_*\Theta = \Theta + dd^cR$. Adding a large multiple of ω^{k-1}, we can further assume $0 \leqslant R \leqslant C_1\omega^{k-1}$. Iterating the previous functional equation yields

$$(1) \qquad (f^n)_*\Theta = \Theta + dd^cR_n, \text{ where } R_n := \sum_{j=0}^{n-1} (f^j)_*R$$

is an increasing sequence of positive currents of bidimension $(1,1)$ on X whose mass $||R_n||$ is controlled by

$$(2) \qquad ||R_n|| := \int_X R_n \wedge \omega \leqslant C_1 \sum_{j=0}^{n-1} \int_X (f^j)_*\omega^{k-1} \wedge \omega = C_1 \sum_{j=0}^{n-1} \delta_1(f^j).$$

It follows from Stokes theorem and (1) that

$$(3) \quad \int_X (-\varphi \circ f^n)\,\Theta = \int_X (-\varphi)\Theta + \int_X -dd^c\varphi \wedge R_n \leqslant \int_X (-\varphi)\Theta + ||R_n||,$$

since $-dd^c\varphi \leqslant \omega$ and $R_n \geqslant 0$. Now Θ and ω^k are both smooth probability measures, so we can fix S, a smooth form of bidegree $(k-1, k-1)$ on X, such that $\omega^k = \Theta + dd^cS$ and $0 \leqslant S \leqslant C_2\omega^{k-1}$. We infer

$$(4) \qquad \int_X (-\varphi \circ f^n)\,\omega^k = \int_X (-\varphi \circ f^n)\,\Theta + \int_X (-\varphi \circ f^n)\,dd^cS.$$

The first term is controlled by (2) and (3). The second can be estimated by using Stokes theorem again,

$$(5)\qquad \int_X (-\varphi \circ f^n)\, dd^c S = \int_X (f^n)^*(-dd^c\varphi) \wedge S \leqslant C_2\delta_1(f^n).$$

Using (2), (3), (4), (5) we obtain

$$0 \leqslant \int_X (-\varphi \circ f^n)\, \omega^k \leqslant C_3 \sum_{j=0}^{n} \delta_1(f^j)$$

for some large constant $C_3 > 0$. This concludes the proof. □

We now specialize the previous estimates when the behaviour of sequence $(\delta_1(f^j))$ is under control. Our assumptions will be discussed in section 3.

THEOREM 1.4. — *Assume $\delta_1(f^j) \preceq \lambda^j$ with $\lambda > 1$. Then $\exists C > 0$ such that*

$$\mathrm{Vol}\,(f^j\Omega) \geqslant \exp\left(-\frac{C}{\mathrm{Vol}(\Omega)}\lambda^j\right)$$

for all $j \in \mathbb{N}$ and $\Omega \subset X$.

Proof. — This is a straightforward consequence of the previous proposition. Indeed $\sum_{l=0}^{j-1}\sum_{p=0}^{l}\delta_1(f^p)$ grows at most like $c_1\lambda^j$ while $d_t^{-j} \geqslant \exp(-c_2\lambda^j)$ if $c_2 > 0$ is chosen large enough. Thus proposition 1.2 yields

$$\mathrm{Vol}(f^j\Omega) \geqslant \mathrm{Vol}(\Omega) \exp\left(\frac{-c_3}{\mathrm{Vol}(\Omega)}\lambda^j\right),$$

for some appropriate constant $c_3 \geqslant 1$. The conclusion follows by observing that $\alpha \exp(-x/\alpha) \geqslant \exp(-2x/\alpha)$ for all $\alpha > 0$ and all $x \geqslant 1/e$. □

1.3. Sharpness of the estimates.

Our volume estimates (theorem 1.4) are sharp in the sense that $\mathrm{Vol} f^j(\Omega) \simeq (\mathrm{Vol}\Omega)^{\lambda^j}$ for many mappings. Here is an elementary example: consider $f : (z,w) \in \mathbb{C}^2 \mapsto f(z,w) = (z^\lambda, w^\lambda) \in \mathbb{C}^2$, where $\lambda \in \mathbb{N}$, $\lambda \geqslant 2$. Then f gives rise to an holomorphic endomorphism of $X = \mathbb{P}^2$ with $\lambda = \lambda_1(f)$. Simple computations show that

$$\mathrm{Vol}(f^j\Delta^2(r)) \simeq r^{4\lambda^j} \simeq \mathrm{Vol}(\Delta^2(r))^{\lambda^j},$$

where $\Delta^2(r)$ denotes the bidisk of radius (r,r) centered at the origin in $\mathbb{C}^2$. The interested reader may consult [FG 01], [FJ 03] for further examples.

However one expects $\mathrm{Vol} f^j(\Omega)$ to decrease much slower for "generic" open sets Ω (or for "generic" mappings f). This is the point of view of [FG 01], [DF 01] where finer volume estimates are established (through a much harder analysis) in case f is a 2-dimensional birational mapping.

2. Green current.

Our goal is now to show how volume estimates can be used to construct the Green current of an algebraically stable mapping f.

2.1. Minimal singularities.

We start by recalling a few facts about the linear action induced by f^* on $H^{1,1}(X,\mathbb{R})$.

DEFINITION 2.1.— *The mapping $f : X \to X$ is* **algebraically stable** *if the linear actions $(f^n)^* : H^{1,1}(X,\mathbb{R}) \to H^{1,1}(X,\mathbb{R})$ satisfy*

$$(f^{n+1})^* = f^* \circ (f^n)^*, \text{for all } n \in \mathbb{N}.$$

When f is algebraically stable, its first dynamical degree $\lambda_1(f)$ equals the spectral radius of $f^* : H^{1,1}(X,\mathbb{R}) \to H^{1,1}(X,\mathbb{R})$.

Let us recall that one can define the pull-back of any positive closed current S of bidegree $(1,1)$ on X. The mapping $S \mapsto f^*S$ is continuous for the weak topology of (positive) currents (see [S 99]). The induced property on cohomology classes reads

$$f^* H^{1,1}_{psef}(X,\mathbb{R}) \subset H^{1,1}_{psef}(X,\mathbb{R}),$$

where $H^{1,1}_{psef}(X,\mathbb{R})$ denotes the cone generated by *pseudoeffective* cohomology classes, i.e. classes that can be represented by a positive closed current. This cone is closed and strict (i.e. $H^{1,1}_{psef}(X,\mathbb{R}) \cap -H^{1,1}_{psef}(X,\mathbb{R}) = \{0\}$). It follows from Perron-Frobenius theory that the spectral radius $\lambda_1(f)$ of

$f^* : H^{1,1}(X,\mathbb{R}) \to H^{1,1}(X,\mathbb{R})$ is an eigenvalue of f^* which dominates all the other eigenvalues, and that there exists an eigenvector $\alpha \in H^{1,1}_{psef}(X,\mathbb{R})$ with $f^*\alpha = \lambda_1(f)\alpha$.

Our aim is now to construct a canonical positive closed current T_α such that $\{T_\alpha\} = \alpha$ and $f^*T_\alpha = \lambda_1(f)T_\alpha$. Fix θ a smooth closed real $(1,1)$-form with $\{\theta\} = \alpha$ and consider

$$v(x) := \sup\{\varphi(x) \,/\, \varphi \in L^1(X,[-\infty,0]),\ \varphi \text{ is u.s.c. and } dd^c\varphi \geqslant -\theta\}.$$

This is an extremal function with respect to the family of θ-psh functions normalized by $\sup_X \varphi \leqslant 0$ (see [GZ 04]). The function v is not necessarily upper semi-continuous (u.s.c.), so we consider its upper semi-continuous regularization,

$$v^\theta_{\min}(x) := \limsup_{y\to x} v(y).$$

The current $\theta_{\min} := \theta + dd^c v^\theta_{\min}$ is a positive closed current cohomologous to θ with "minimal singularities" (see theorem 1.5 in [DPS 01]): if S is any positive closed current cohomologous to θ, then S writes $S = \theta + dd^c w$, where $w - v^\theta_{\min}$ is bounded from above, so that w is more singular than $v^\theta_{\min}$.

Similarly if T a positive closed current cohomologous to θ which is invariant, $f^*T = \lambda_1(f)T$, we say that T has minimal singularities among such invariant currents if whenever S is another invariant positive closed current cohomologous to θ, the potential of S is dominated from above by that of T (up to an additive constant).

We make the following assumption on the cohomology class α: $\forall t > 0$,

$$(\mathbf{H}_\alpha) \qquad \mathrm{Vol}\left(v^\theta_{\min} < -t\right) \leqslant \exp\left(-h(t)t\right), \quad \text{where } \int_1^{+\infty} \frac{\ln t\, dt}{h(e^t)} < +\infty.$$

This is clearly an assumption on α rather than on θ: if θ' is another smooth closed real $(1,1)$-form such that $\{\theta'\} = \alpha$, then $\theta' = \theta + dd^c u$ with u smooth hence bounded, so that $v^\theta_{\min} - v^{\theta'}_{\min}$ is bounded on X (see [GZ 04]). The assumption $(\mathbf{H}_\alpha)$ will be discussed in section 3. Observe that the integrability condition is satisfied if h grows fast enough to infinity, as $t \to +\infty$, e.g. if $h(t) \succeq [\log(1+t)]^{1+\delta}$, $\delta > 0$.

2.2. Construction of invariant currents.

THEOREM 2.2.— *Assume $f : X \to X$ is algebraically stable with $\delta_1(f^j) \preceq \lambda_1(f)^j$, $\lambda_1(f) > 1$. If $\alpha \in H^{1,1}_{psef}(X, \mathbb{R})$ satisfies $f^*\alpha = \lambda_1(f)\alpha$ and (H_α), then for any smooth representative θ of α,*

$$\frac{1}{\lambda^n}(f^n)^*\theta \longrightarrow T_\alpha \text{ in the weak sense of currents,}$$

where T_α is a positive closed current such that $f^*T_\alpha = \lambda_1(f)T_\alpha$, $\{T_\alpha\} = \alpha$.

The current T_α has minimal singularities among invariant closed currents whose cohomology class is α. It is extremal within the cone of positive closed invariant $(1,1)$-currents whose cohomology class belongs to the ray $\mathbb{R}\alpha$.

Proof.— Set for simplicity $\lambda = \lambda_1(f)$. Let θ be a smooth closed real $(1,1)$-form such that $\{\theta\} = \alpha$. Let $\theta_{\min} = \theta + dd^c v^\theta_{\min} \geqslant 0$ be the corresponding positive current with minimal singularities. Since X is Kähler, the invariance relation $f^*\alpha = \lambda\alpha$ reads

$$\frac{1}{\lambda} f^*\theta_{\min} = \theta_{\min} + dd^c\gamma, \tag{6}$$

where $\gamma \in L^1(X)$ is locally given as the sum of a psh function and $-v^\theta_{\min}$. We normalize γ by requiring $\int_X \gamma\omega^k = 0$. Observe that $\gamma \leqslant -v^\theta_{\min} + C$ for some constant $C \in \mathbb{R}$. Since f is algebraically stable, we can pull-back (6) by f^n and obtain this way

$$\frac{1}{\lambda^n}(f^n)^*\theta_{\min} = \theta_{\min} + dd^c\gamma_n, \quad \gamma_n = \sum_{j=0}^{n-1} \frac{1}{\lambda^j}\gamma \circ f^j.$$

We want to show that (γ_n) converges in $L^1(X)$. Observe that

$$\gamma_n = \varphi_n - v_n + C\sum_{j=0}^{n-1}\lambda^{-j}, \text{ where } \varphi_n = \sum_{j=0}^{n-1}\frac{1}{\lambda^j}(\gamma + v^\theta_{\min} - C) \circ f^j$$

and $v_n = \sum_{j=0}^{n-1} \lambda^{-j} v^\theta_{\min} \circ f^j$ are both decreasing sequences of L^1-functions. We show in lemma 2.3 below that (v_n) is a convergent sequence in $L^1(X)$. It is therefore sufficient to get a lower bound on $\int_X \gamma_n\omega^k$. We establish it in the same vein as what was done in the proof of proposition 1.3. Observe first that it is sufficient to evaluate γ_n against a properly chosen smooth probability measure Θ. Indeed we can write $\omega^k = \Theta + dd^c S$ with $S \geqslant 0$ smooth, so that

$$\int_X \gamma_n \omega^k = \int_X \gamma_n \Theta + \int_X S \wedge dd^c \gamma_n \geqslant \int_X \gamma_n \Theta - \int_X S \wedge \theta_{\min}.$$

We choose Θ as in the proof of proposition 1.3 and use (1). Recall that $R_j = \sum_{l=0}^{j-1} (f^l)_* R$ is an increasing sequence of positive currents of bidimension $(1,1)$ on X such that $(f^j)_*\Theta = \Theta + dd^c R_j$. Therefore

$$\begin{aligned}\int_X \frac{1}{\lambda^j} \gamma \circ f^j \Theta &= \frac{1}{\lambda^j} \int_X \gamma \Theta + \int_X dd^c \gamma \wedge \frac{R_j}{\lambda^j} \\ &= \frac{1}{\lambda^j} \int_X \gamma \Theta + \int_X \left[\frac{1}{\lambda} f^* \theta_{\min} - \theta_{\min}\right] \wedge \frac{R_j}{\lambda^j} \\ &= \frac{1}{\lambda^j} \left(\int_X \gamma \Theta - \int_X \theta_{\min} \wedge \frac{R}{\lambda}\right) + \int_X \theta_{\min} \wedge \left[\frac{R_{j+1}}{\lambda^{j+1}} - \frac{R_j}{\lambda^j}\right].\end{aligned}$$

Adding these inequalities and observing that $\theta_{\min} \wedge R_n \geqslant 0$, we infer

$$\begin{aligned}\int_X \gamma_n \Theta &= \sum_{j=0}^{n-1} \frac{1}{\lambda^j} \left(\int_X \gamma \Theta - \int_X \theta_{\min} \wedge \frac{R}{\lambda}\right) + \int_X \theta_{\min} \wedge \left(\frac{R_n}{\lambda^n} - R\right) \\ &\geqslant \frac{-\lambda}{\lambda - 1} \left(\int_X (-\gamma)\Theta + \int_X \theta_{\min} \wedge R \right) > -\infty.\end{aligned}$$

It follows that $\gamma_n \to g_\theta := \sum_{j \geqslant 0} \lambda^{-j} \gamma \circ f^j$ in $L^1(X)$, hence

$$\theta_n := \frac{1}{\lambda^n} (f^n)^* \theta_{\min} = \theta_{\min} + dd^c \gamma_n \to T_\alpha := \theta_{\min} + dd^c g_\theta.$$

The current T_α is positive (as a limit of positive currents) and invariant since $f^* \theta_n = \lambda \theta_{n+1}$. It is obviously closed and cohomologous to $\theta_{\min}$ hence $\{T_\alpha\} = \alpha$. Observe also that $\lambda^{-n}(f^n)^*\theta \to T_\alpha$ since $\lambda^{-n} v_{\min}^\theta \circ f^n \to 0$ (by lemma 2.3).

We now claim that T_α is the invariant current with minimal singularities within the compact set of positive invariant closed $(1,1)$-currents S with $\{S\} = \alpha$. Indeed let $S = \theta_{\min} + dd^c w \geqslant 0$ be such a current. Then $w \leqslant -v_{\min}^\theta + C_1$ by definition of $v_{\min}^\theta$. Shifting w, we can assume $C_1 = 0$.

The invariance $f^*S = \lambda S$ reads on potentials

$$w - \frac{1}{\lambda} w \circ f = \gamma + C_2,$$

for some constant $C_2 \in \mathbb{R}$. Therefore

$$w - \frac{1}{\lambda^n} w \circ f^n = \gamma_n + C_2 \sum_{j=0}^{n-1} \frac{1}{\lambda^j}.$$

Now $\lambda^{-n} w \circ f^n \leqslant -\lambda^{-n} v_{\min}^{\theta} \circ f^n + C_1 \lambda^{-n} \to 0$ hence

$$w \leqslant g_\theta + C_2 \frac{\lambda}{\lambda - 1},$$

as claimed.

It follows easily that T_α is extremal among invariant positive closed $(1,1)$-currents whose cohomology class belongs to $\mathbb{R}\alpha$. Indeed let S be a positive closed current such that $0 \leqslant S \leqslant T_\alpha$, $f^*S = \lambda S$, and $\{S\} = a\alpha$ where $0 \leqslant a \leqslant 1$. Then S writes

$$S = a\theta_{\min} + dd^c w, \quad \text{where } \sup_X w = 0.$$

Similarly one can fix $u \in L^1(X)$ u.s.c. such that $S' := T_\alpha - S = (1-a)\theta_{\min} + dd^c u \geqslant 0$ with $u + v = g_\theta$. Note that u is bounded from above on X, $u \leqslant C_3$. We infer

$$-\frac{C_3}{\lambda^n} + \frac{1}{\lambda^n} g_\theta \circ f^n \leqslant \frac{1}{\lambda^n} w \circ f^n \leqslant 0.$$

Now $\lambda^{-n} g_\theta \circ f^n \to 0$ by construction, hence so does $\lambda^{-n} w \circ f^n$, which yields

$$S = \frac{1}{\lambda^n}(f^n)^* S = \frac{a}{\lambda^n}(f^n)^* \theta_{\min} + dd^c \left(\frac{1}{\lambda^n} w \circ f^n \right) \to aT_\alpha,$$

so that S actually equals aT_α, as claimed. □

LEMMA 2.3. — *Assume α satisfies $(\mathbf{H}_\alpha)$ and $\delta_1(f^j) \preceq \lambda^j$, $\lambda > 1$. Then*

$$\sum_{j \geqslant 0} \lambda^{-j} v_{\min}^{\theta} \circ f^j \text{ converges in } L^1(X).$$

Proof. — For fixed $j \in \mathbb{N}$, $\varepsilon > 0$, we set

$$v_j := \frac{1}{\lambda^j} v_{\min}^{\theta} \circ f^j \text{ and } \Omega_{j,\varepsilon} := \{x \in X \,/\, v_j(x) < -\varepsilon\}.$$

Observe that $f^j(\Omega_{j,\varepsilon}) \subset \{v_{\min}^{\theta} < -\varepsilon\lambda^j\}$, so that by theorem 1.4 and $(\mathbf{H}_\alpha)$,

$$\exp\left(-\frac{C}{\mathrm{Vol}(\Omega_{j,\varepsilon})}\lambda^j\right) \leqslant \mathrm{Vol}(f^j(\Omega_{j,\varepsilon})) \leqslant \exp\left(-h(\varepsilon\lambda^j)\varepsilon\lambda^j\right),$$

hence

$$\mathrm{Vol}(\Omega_{j,\varepsilon}) \leqslant \frac{C}{\varepsilon h(\varepsilon\lambda^j)}.$$

It follows that

$$\int_X |v_j|\omega^k = \int_0^{+\infty} \mathrm{Vol}(v_j < -t)\mathrm{d}t \leqslant \lambda^{-j/2} + C\int_{\lambda^{-j/2}}^{+\infty} \frac{\mathrm{d}t}{th(t\lambda^j)}.$$

Using an elementary change of variables, we infer

$$\sum_{j\geqslant 0}\int_X |v_j|\omega^k \leqslant C' + C\sum_{j\geqslant 0}\int_{\lambda^{j/2}}^{+\infty} \frac{\mathrm{d}x}{h(e^x)}.$$

The double sum is comparable to

$$\int_{y=0}^{+\infty}\int_{x=\exp\left(\frac{\ln\lambda}{2}y\right)}^{+\infty} \frac{\mathrm{d}x\,\mathrm{d}y}{h(e^x)} = \int_{x=1}^{+\infty}\int_{y=0}^{2\frac{\ln x}{\ln\lambda}} \frac{\mathrm{d}x\,\mathrm{d}y}{h(e^x)} = \frac{2}{\ln\lambda}\int_1^{+\infty} \frac{\ln x}{h(e^x)}\mathrm{d}x$$

which is finite by $(\mathbf{H}_\alpha)$. □

Remark 2.4. — When $X = \mathbb{CP}^k$, the complex projective space of dimension k, then all our assumptions are trivially satisfied. Indeed if we take $\omega = \omega_{FS}$, the Fubini-Study Kähler form, then $\delta_1(f^j) = \lambda^j$, $\alpha = \{\omega_{FS}\}$ is Kähler hence $v_{\min}^{\theta} \equiv 0$ for $\theta = \omega_{FS}$ (so $(\mathbf{H}_\alpha)$ becomes trivial). In this case the Green current has been constructed by N.Sibony in [S 99] and our theorem 2.2 provides an alternative proof of his result. See also [FG 01], [Ca 01], [G 02] for related constructions.

3. Concluding remarks.

3.1. Growth of $\delta_1(f^j)$.

Throughout this article we assumed that $\lambda_1(f) > 1$. When $\lambda_1(f) = 1$, it follows from concavity properties of the dynamical degrees that f is bimeromorphic with topological entropy zero (see §1 in [G 03b], [DS 04]). The assumption $\lambda_1(f) > 1$ is therefore dynamically significant.

We moreover assumed $\delta_1(f^j) \preceq \lambda_1(f)^j$. In general there may be a further polynomial growth, $\delta_1(f^j) \simeq cj^s\lambda_1(f)^j$ as the following example shows:

Example 3.1. — Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and f be the compactification of the following polynomial endomorphism of $\mathbb{C}^2$,

$$(z,w) \in \mathbb{C}^2 \mapsto (z^\lambda, zw^\lambda) \in \mathbb{C}^2.$$

The linear action induced by f^* on $H^{1,1}(X,\mathbb{R}) \simeq \mathbb{R}^2$ is given by the 2-by-2 matrix $A_f = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ and we obtain in this case $\delta_1(f^j) \simeq j\lambda^{j-1}$.

Observe that f has topological degree $d_t(f) = \lambda^2 > \lambda = \lambda_1(f)$.

However in all known examples such that $s \geqslant 1$, the mapping f preserves a fibration. This motivates the following

QUESTION 3.2. — *Assume f does not preserve any fibration. Is it true then that $\delta_1(f^j) \preceq \lambda_1(f)^j$?*

The question makes sense even when $\lambda_1(f) = 1$. When $\dim_{\mathbb{C}} X = 2$, J.Diller and C.Favre [DF 01] gave a positive answer to this question when f is bimeromorphic. They also observed that a bimeromorphic mapping can not preserve a fibration when $\lambda_1(f) > 1$. More generally one can ask the following

QUESTION 3.3. — *Assume $\lambda_1(f)$, the first dynamical degree, strictly dominates all the other dynamical degrees. Is it true then that $\delta_1(f^j) \preceq \lambda_1(f)^j$?*

When f preserves a fibration, one can show that the first dynamical degree is not the largest dynamical degree. The construction of the invariant current T_α is of crucial importance precisely when the first dynamical degree is the largest.

3.2. Properties of invariant cohomology classes.

We have considered in section 2 a psef class α such that $f^*\alpha = \lambda_1(f)\alpha$. There may be several (linearly independent) eigenvectors $\alpha_1, \alpha_2, \ldots$:

Example 3.4. — Let $f = (g_1, g_2) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the direct product of two rational mappings $g_i : \mathbb{P}^1 \to \mathbb{P}^1$ of the same degree $\lambda \geqslant 2$. Then $\lambda_1(f) = \lambda$ and there are two eigenvectors α_1, α_2 associated to $\lambda_1(f)$ which are given by the fibers of the two natural fibrations $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ (projection onto the i^{th} factor). The corresponding invariant current T_{α_i} is then the pull-back of the Lyubich measure of $g_i : \mathbb{P}^1 \to \mathbb{P}^1$ under the projection π_i.

Observe that here again the topological degree $d_t(f) = \lambda^2$ strictly dominates the first dynamical degree $\lambda_1(f) = \lambda$.

However when $\dim_{\mathbb{C}} X = 2$ and the first dynamical degree is the largest dynamical degree (i.e. $\lambda_1(f) > d_t$), J.Diller and C.Favre have proved that the eigenspace associated to $\lambda_1(f)$ is one-dimensional (see remark 5.2 in [DF 01]). One expects similar results to hold true in higher dimension.

We now discuss further positivity properties of the invariant class α. Let $H^{1,1}_{nef}(X,\mathbb{R})$ and $H^{1,1}_{big}(X,\mathbb{R})$ denote the cones generated respectively by nef and big cohomology classes. Recall that a class α is *numerically eventually free* (nef for short) if $\alpha + \varepsilon\{\omega\}$ is a Kähler class for all $\varepsilon > 0$. The class α is *big* if it contains a Kähler current, i.e. if there exists a positive closed $(1,1)$-current T on X such that $\{T\} = \alpha$ and $T \geqslant \varepsilon_0\omega$ for some $\varepsilon_0 > 0$. These notions coincide with the corresponding classical notions in algebraic geometry when X is projective and $\alpha \in H^2(X,\mathbb{Z})$. We refer the reader to [D 90] for more information on these positivity conditions.

PROPOSITION 3.5. — *The cone $H^{1,1}_{big}(X,\mathbb{R})$ is preserved by f^*.*

If $\dim_{\mathbb{C}} X = 2$, then $H^{1,1}_{nef}(X,\mathbb{R})$ is also preserved by f^*.

The proof is an easy application of proposition 4.12 in [B 02] and proposition 1.11 in [DF 01].

Since the cone $H^{1,1}_{nef}(X,\mathbb{R})$ is closed and strict, it follows from the Perron-Frobenius theory that the invariant class α is nef when $\dim_{\mathbb{C}} X = 2$.

The same argument does not apply to $H^{1,1}_{big}(X,\mathbb{R})$ because the latter is not closed. It seems however reasonable to expect α being big when X is e.g. rational. For 2-dimensional bimeromorphic mappings, α is not big precisely when f is conjugate to an automorphism (see theorem 0.4 in [DF 01]): this rarely happens on a rational surface (see [Ca 01]). It would be interesting to establish similar facts for non invertible mappings and/or in any dimension.

3.3. Volumes of sublevel sets.

Let φ be a quasiplurisubharmonic function (qpsh for short) on X, i.e. a function that is locally given by the sum of a psh and a smooth function. We let $\nu(\varphi, x)$ denote its Lelong number at point $x \in X$. It follows from Skoda's integrability theorem [Sk 72] that $\exp(-\varphi) \in L^1(X)$ when $\sup_{x\in X}\nu(\varphi,x) < 2$. Thus, by homogeneity of $\varphi \mapsto \nu(\varphi,x)$, $\exp(-2\varphi/[\varepsilon + \sup_{x\in X}\nu(\varphi,x)]) \in L^1(X)$ for all $\varepsilon > 0$. It follows therefore from the Chebyshev inequality that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $t > 0$,

$$\mathrm{Vol}\,(x \in X \,/\, \varphi(x) < -t) \leqslant C_\varepsilon \exp\left(-\frac{-2}{\varepsilon + \sup_{x\in X}\nu(\varphi,x)}t\right).$$

This observation was first done by C.Kiselman [K 00]. When φ has zero Lelong number at every point of X this can be reformulated as

$$\mathrm{Vol}\,(\varphi < -t) \leqslant \exp(-h(t)t),\ \forall t > 0, \tag{7}$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $\lim_{t\to+\infty} h(t) = +\infty$. Observe that this is always satisfied when φ is a potential with minimal singularities of a big and nef cohomology class.

LEMMA 3.6.— *Let θ be a smooth closed real $(1,1)$-form such that $\alpha = \{\theta\} \in H^{1,1}_{nef}(X,\mathbb{R}) \cap H^{1,1}_{big}(X,\mathbb{R})$. Then $v^\theta_{\min}$ has zero Lelong number at every point.*

This is well-known to complex geometers, at least when $\alpha = c_1(L)$ is the first Chern class of a big and nef holomorphic line bundle L on X (see proposition 1.6 in [DPS 01]). We nevertheless include a proof for the reader's convenience.

Proof.— Since α is big, we can fix a positive closed current S of bidegree $(1,1)$ on X such that $\{S\} = \alpha$ and $S \geqslant \varepsilon_0\omega$ for some $\varepsilon_0 > 0$. Thus

$S - \varepsilon_0\omega$ is still a positive current. Fix $\psi \in L^1(X)$ an u.s.c. function such that $S - \varepsilon_0\omega = \theta - \varepsilon_0\omega + dd^c\psi \geqslant 0$; we normalize ψ by $\sup_X \psi = 0$.

Since α is nef, $N\alpha + \varepsilon_0\{\omega\}$ is Kähler for all $N \in \mathbb{N}$. Fix ψ_N smooth functions such that $N\theta+\varepsilon_0\omega+dd^c\psi_N > 0$ and normalized by $\sup_X \psi_N = 0$. We set $\varphi_N := \frac{1}{N+1}[\psi + \psi_N] \leqslant 0$. These are u.s.c L^1-functions such that

$$\theta + dd^c\varphi_N = \frac{1}{N+1}\left[(\theta - \varepsilon_0\omega + dd^c\psi) + (N\theta + \varepsilon_0\omega + dd^c\psi_N)\right] \geqslant 0,$$

whence $\varphi_N \leqslant v_{\min}^{\theta}$. Now ψ_N is smooth so that

$$0 \leqslant \nu(v_{\min}^{\theta}, x) \leqslant \nu(\varphi_N, x) = \frac{1}{N+1}\nu(\psi, x).$$

We infer $\nu(v_{\min}^{\theta}, x) = 0$ for all $x \in X$. □

Our hypothesis $(\mathbf{H}_\alpha)$ asks for more precise information than $\lim h(t) = +\infty$. It would be satisfied if e.g. $h(t) \succeq (\log[1 + t])^{1+\delta}$, $\delta > 0$. This is trivially true in all cases considered so far:

- When α is semi-positive (i.e. when it can be represented by a *smooth* closed non-negative form), then $(\mathbf{H}_\alpha)$ is trivially satisfied: indeed $\{x \in X \,/\, v_{\min}^{\theta}(x) < -t\}$ is empty for $t > 0$ large enough, hence $h(t) \equiv +\infty$ for $t >> 1$. This is the case considered in [S 99], [FG 01], [G 02]: when X is a complex homogeneous manifold (i.e. when the group of biholomorphisms $Aut(X)$ acts transitively on X), every psef class is actually semi-positive ($X = \mathbb{P}^k$ in [S 99] and $X = \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_s}$ in [FG 01]). There are psef classes that are not semi-positive on a Hirzebruch surface (the situation considered in [G 02]), however every nef class is semi-positive on these minimal rational surfaces.

- When f is holomorphic (i.e. when $I_f = \emptyset$), α admits a positive closed representative with *continuous* potential, so $(\mathbf{H}_\alpha)$ is trivially satified again (this is the situation considered in [Ca 01]). Indeed one can write in this case
$$\frac{1}{\lambda}f^*\theta = \theta + dd^c\psi,$$
where ψ is smooth. Therefore
$$T_\alpha = \lim \frac{1}{\lambda^n}(f^n)^*\theta = \theta + dd^c\psi_\infty,$$

where $\psi_\infty = \sum_{j \geqslant 0} \lambda^{-j} \psi \circ f^j$ is uniformly convergent hence continuous. It follows that $v_{\min}^{\theta} \geqslant \psi_\infty - C$ is bounded from below on X.

QUESTION 3.7. — *Is* $(\mathbf{H}_\alpha)$ *always satisfied ?*

This may be a question of interest, even for complex geometers. Indeed the notion of metric with minimal singularities is of crucial importance in complex geometry (see [DPS 01]), but it is yet poorly understood. One may hope that complex dynamics will provide interesting examples where the corresponding potentials $v_{\min}^{\theta}$ can be accurately described.

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