



# ANNALES

DE

# L'INSTITUT FOURIER

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Tome 55, n° 3 (2005), p. 1017-1053.

[http://aif.cedram.org/item?id=AIF\\_2005\\_\\_55\\_3\\_1017\\_0](http://aif.cedram.org/item?id=AIF_2005__55_3_1017_0)

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# HARDER-NARASIMHAN FILTRATIONS AND OPTIMAL DESTABILIZING VECTORS IN COMPLEX GEOMETRY

by Laurent BRUASSE & Andrei TELEMAN

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## 1. Introduction.

A classical result of Harder and Narasimhan states that any non-semistable bundle on a curve admits a canonical filtration of subsheaves with torsion free semistable quotients.

This result was generalized for reflexive sheaves on projective varieties [20], [13], and finally to reflexive sheaves on arbitrary compact Hermitian manifolds [2], [3].

The initial motivation for this paper was to find the analogous statement for other type of complex geometric objects, for instance holomorphic bundles coupled with sections or with endomorphisms (Higgs fields).

The system of semistable quotients associated with the Harder-Narasimhan filtration of a non-semistable bundle can be interpreted as a semistable object with respect to the moduli problem for  $G$ -bundles, where  $G$  is a product of reductive group of the form  $\prod_i GL(r_i)$ .

Therefore, the Harder-Narasimhan result can be understood as an assignment which associates to a non-semistable object a semistable object but for a different moduli problem.

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*Keywords:* Symplectic actions, Hamiltonian actions, stability, Harder-Narasimhan filtration, Shatz stratification, gauge theory.

*Math. classification:* 32M05, 53D20, 14L24, 14L30, 32L05, 32Q15.

We believe that it is a natural and important problem to find a general principle which generalizes this result for arbitrary moduli problems. More precisely, we seek a general rule which associates – in a canonical way – to a non-semistable object  $\mathcal{O}$  with respect to any complex geometric moduli problem  $\mathcal{A}$  a new moduli problem  $\mathcal{B}(\mathcal{A}, \mathcal{O})$  and a semistable object  $\mathcal{O}'(\mathcal{A}, \mathcal{O})$  for  $\mathcal{B}(\mathcal{A}, \mathcal{O})$ .

Our first attempt was to understand this principle in the finite dimensional framework, *i.e.* for moduli problems associated with actions of reductive groups on finite dimensional but in general *non-compact* varieties.

After consulting the available literature dedicated to the algebraic case, we realized that the main tool for understanding the analogue of the Harder-Narasimhan assignment in the finite dimensional framework is *the theory of optimal one-parameter subgroups*, for which we refer to Kirwan [11], Slodowy [21] and Ramanan & Ramanathan [19].

This theory can be sketched as follows: if  $[x] \in \mathbb{P}^n(V)$  is non-semistable point with respect to a linear representation  $\rho : G \rightarrow GL(V)$  of a reductive group  $G$ , then there exists a one parameter subgroup  $\tau : \mathbb{C}^* \rightarrow G$  of “norm” 1 which “destabilizes”  $[x]$  in the strongest possible way, *i.e.*

$$\lambda(x, \tau) \leq \lambda(x, \theta),$$

for any one-parameter subgroup  $\theta : \mathbb{C}^* \rightarrow G$  of norm 1. Here we denoted by  $\lambda$  the maximal weight function which occurs in the Hilbert criterion for stability. A one parameter subgroup (an OPS) with this property is called an *optimal destabilizing OPS* for  $[x]$  and it is essentially unique, in the sense that any other optimal destabilizing OPS  $\tau'$  for  $[x]$  has the same associated parabolic subgroup as  $\tau$ , and is conjugated with  $\tau$  in this parabolic subgroup.

This result has certainly become part of classical GIT. What is (at least for the authors) less standard material is the following crucial remark, which was probably first pointed out by Ramanan & Ramanathan [19] (see also Kirwan [11]):

*If  $\tau : \mathbb{C}^* \rightarrow G$  is an optimal destabilizing OPS for  $[x]$ , then  $\tau(t)$  converges to a point  $[x_0]$  which is semistable with respect to an induced action of the reductive centralizator  $Z(\tau)$  of  $\tau$  on a  $Z(\tau)$ -stable subvariety of  $\mathbb{P}(V)$ .*

We claim that the assignment  $[x] \mapsto [x_0]$  is the GIT model which

should be followed in order to get the correct generalization of the Harder-Narasimhan theorem to the non-compact non-algebraic and gauge theoretical frameworks.

Therefore our final goal is to give a gauge theoretical version of this remark, which applies to all moduli problems obtained by coupling holomorphic bundles (with arbitrary reductive structure groups) with sections in associated bundles (see [15], [12], [17]).

The first step in achieving this goal is to give the *complex analytic* version of the assignment  $[x] \mapsto [x_0]$  explained above and to prove the analogous remark in this framework. Therefore, the main object of this article is a holomorphic action  $\alpha : G \times F \rightarrow F$  of a complex reductive group on a complex manifold  $F$ . Since we are especially interested in the linear case (and later in the infinite dimensional case), we will not assume that  $F$  is compact. We realized that extending the theory of optimal destabilizing OPS to this situation raises substantial technical difficulties.

First of all, in order to have a good stability condition for a holomorphic action  $\alpha : G \times F \rightarrow F$  one has to fix a Kähler metric  $h$  of  $F$  which is invariant under a maximal compact subgroup  $K$  of  $G$  and a moment map for the induced  $K$ -action. Such a data system  $(K, h, \mu)$  provides a generalized maximal weight function  $\lambda : i\mathfrak{k} \times F \rightarrow \mathbb{R} \cup \{\infty\}$ . It is well known (see for instance Mundet i Riera [15]) that the stability condition with respect to  $(K, h, \mu)$  can be expressed in terms of the maximal weight function as in the projective algebraic case. *But there is no way to extend this result for the semistability condition* in the general non-algebraic non-compact case. Moreover, in the algebraic theory of optimal destabilizing OPSs it is very important to have a  $G$ -equivariant maximal weight function, whereas the choice of a triple  $(K, h, \mu)$  only provides a  $K$ -equivariant one.

In order to solve these difficulties one has to impose a certain completeness condition on the triple  $(K, h, \mu)$ , namely *energy completeness* which was introduced in [22] and used in [12]. This condition is always satisfied in both compact and linear case [22] and also for certain moduli problems on curves [5].

Moreover, in order to get a  $G$ -equivariant maximal weight function  $\lambda$ , it is convenient to work with an equivalence class of triples  $(K, h, \mu)$  and to show that  $\lambda$  extends to the union of all subspaces of the form  $i\mathfrak{k}$ . The equivalence is defined by the natural  $G$ -action on the set of such triples. Such an equivalence class will be called a *symplectization* of the action  $\alpha$ , and it plays the same role as a linearization of an action in an ample line

bundle, in classical GIT.

An important tool in our proofs will be the linearization theorems of Heinzner-Huckleberry for Hamiltonian actions [8].

The contents of this article is the following: First we explain the properties of the maximal weight function associated with an energy-complete symplectization. Next we prove one of our main results: the existence and the unicity (up to equivalence) of an optimal destabilizing element  $\xi$  in the Lie algebra of  $G$  for any non-semistable point  $f \in F$ . Following the principle explained in the algebraic case, we next show that the path  $e^{t\xi}f$  converges to a point  $f_0$  which is semistable with respect to a natural action of the reductive centralizer  $Z(\xi)$  on a certain submanifold of  $F$ . Fixing the conjugacy class of  $\xi$  one gets a  $G$ -invariant subset of  $F$ . The subsets of this type give a  $G$ -invariant stratification of  $F$ , which, for a large class of actions, is locally finite with locally Zariski closed strata. This stratification is the analogue of the Shatz stratification in the theory of holomorphic vector bundles (see [11] for the projective case).

At the end, we study the optimal destabilizing vectors of the non-semistable objects of two important gauge theoretical moduli problems: holomorphic bundles and holomorphic pairs (bundles coupled with morphisms with fixed source). Detailed proofs of these results can be found in [4]. In this way, we illustrate our general principle:

*In order to get the analogue of the Harder-Narasimhan theorem for a complex geometric moduli problem one has to give a gauge theoretical formulation of the problem and to study the optimal destabilizing vectors of the non-semistable objects.*

This suggests that this principle also holds in the infinite dimensional gauge theoretical framework. Details on the gauge theoretical examples and generalizations will appear in a future article.

## 2. Background.

### 2.1. Symplectization of a holomorphic action and the maximal weight map $\lambda$ .

Let us recall some definitions introduced in [22].

Let  $G$  be a complex reductive group. One can identify the set  $\text{Hom}(\mathbb{C}^*, G)$  of one parameter subgroups of  $G$  with a subset  $H_{\text{alg}}(G)$  of

the Lie algebra  $\mathfrak{g}$  using the map

$$\lambda \mapsto d_1(\lambda)(1) = \frac{d}{dt}\Big|_{t=0}(\lambda(e^t)),$$

where  $\lambda$  was regarded here as a map between real manifolds and the Lie algebra of  $S^1 \subset \mathbb{C}$  was identified with  $i\mathbb{R}$ . In complex non-algebraic geometry one has to consider a larger subset of  $\mathfrak{g}$ , and to define a "generalized maximal weight function" on this larger set. In the non-algebraic non-compact case the optimal destabilizing vectors do *not* belong in general to  $H_{\text{alg}}(G)$ , and this phenomenon occurs even in the simple case of linear actions (see section 6).

DEFINITION 2.1. — Let  $G$  be a complex reductive group and  $\mathfrak{g}$  its Lie algebra. We denote by  $H(G)$  the subset of  $\mathfrak{g}$  consisting of elements  $s \in \mathfrak{g}$  of Hermitian type, i.e. of elements which satisfy one of the following equivalent properties:

- (1) There exists a compact subgroup  $K \subset G$  such that  $s \in i\mathfrak{k}$ .
- (2) For every embedding  $\rho : G \hookrightarrow GL(r, \mathbb{C})$  the matrix  $\rho_*(s)$  is diagonalizable and has real eigenvalues.
- (3) The closure of the real one parameter subgroup of  $G$  defined by  $is \in \mathfrak{g}$  is compact.

This subset is invariant under the adjoint action of  $G$  on  $\mathfrak{g}$ ; in general it is not closed. Although  $H(G)$  is a subset of  $\mathfrak{g}$ , it *cannot* be defined intrinsically in terms of the Lie algebra  $\mathfrak{g}$ .

One can associate to every  $s \in H(G)$  a parabolic subgroup  $G(s) \subset G$  in the following way:

$$G(s) := \left\{ g \in G \mid \lim_{t \rightarrow +\infty} e^{st} g e^{-st} \text{ exists in } G \right\}.$$

Then  $G(s)$  decomposes as a semi-direct product  $G(s) = Z(s) \ltimes U(s)$ , where  $Z(s)$  is the centralizer of  $s$  in  $G$  and  $U(s)$  is the unipotent subgroup defined by:

$$U(s) := \left\{ g \in G \mid \lim_{t \rightarrow +\infty} e^{st} g e^{-st} = e \right\}.$$

We will denote by  $\mathfrak{g}(s)$ ,  $\mathfrak{z}(s)$  and  $\mathfrak{u}(s)$  the corresponding Lie algebras, and by  $p_{\mathfrak{z}(s)}$ ,  $p_{\mathfrak{u}(s)}$  the corresponding projections.

Recall the following facts from [22]:

PROPOSITION 2.2.

- (1) Let  $\sigma, s \in H(G)$ . The following properties are equivalent:

- (a)  $s$  and  $\sigma$  are conjugated under the adjoint action of  $U(s)$ ;
  - (b)  $s$  and  $\sigma$  are conjugated under the adjoint action of  $G(s)$ ;
  - (c)  $\sigma \in \mathfrak{g}(s)$  and  $p_{\mathfrak{g}(s)}(\sigma) = s$ .
- (2) If one of these conditions is satisfied then  $G(s) = G(\sigma)$ .
- (3) The condition in (1) defines an equivalence relation  $\sim$  on  $H(G)$ .
- (4) Let  $K$  be a maximal compact subgroup of  $G$ . Then  $i\mathfrak{k} \subset H(G)$  is a complete system of representatives for  $\sim$ . Mapping  $s$  to the representative in  $i\mathfrak{k}$  of its equivalence class gives a continuous retraction  $\sigma_K : H(G) \rightarrow i\mathfrak{k}$ .

*Example 2.3.* — Assume that  $G = GL(r, \mathbb{C})$ . The data of an equivalence class of  $H(G)$  is the data of a pair  $(\mathcal{F}, \lambda)$  where  $\mathcal{F}$  is a filtration

$$\mathcal{F} : \{0\} \subset V_1 \subset \dots \subset V_k = \mathbb{C}^r$$

and  $\lambda$  is an increasing sequence  $\lambda_1 < \dots < \lambda_k$  of real numbers. An element  $s \in \mathfrak{gl}(r, \mathbb{C})$  belongs to the equivalence class corresponding to  $(\mathcal{F}, \lambda)$  if it is diagonalisable with spectrum  $(\lambda_1, \dots, \lambda_k)$  and

$$V_i = \bigoplus_{j=1}^i V_{\lambda_j},$$

where  $V_{\lambda_i}$  is the  $\lambda_i$  eigenspace of  $s$ . Here, the parabolic subgroup  $G(s)$  is the subgroup of matrices stabilizing the filtration  $\mathcal{F}$ .

Following [22], we introduce the notion of *symplectization* of an holomorphic action. A symplectization of a holomorphic action  $\alpha$  plays the same role as a linearization of an algebraic action in an ample line bundle in the classical GIT. This notion will allow us to define a  $G$ -equivariant maximal weight function on the set  $H(G)$ .

**DEFINITION 2.4** ([12]). — A *symplectization* of the action  $\alpha$  is an equivalence class of triples  $(K, h, \mu)$ , where  $K$  is a maximal compact subgroup of  $G$ ,  $h$  is a  $K$ -invariant Kähler metric on  $F$  and  $\mu : F \rightarrow \mathfrak{k}$  is a moment map for the  $K$ -action with respect to the symplectic structure  $\omega_h$  defined by  $h$ .

Two 3-tuples  $(K, h, \mu)$  and  $(K', h', \mu')$  will be considered equivalent if there exists  $\gamma \in G$  such that:

$$K' = \text{Ad}_\gamma(K), h' = (\gamma^{-1})^*h, \mu' = \text{ad}_{\gamma^{-1}}^t \circ \mu \circ \gamma^{-1}.$$

A symplectization of a holomorphic action  $\alpha$  should be regarded as a *complex geometric datum*, which allows one to define a stability condition

independently of the choice of a maximal compact subgroup of  $G$ . A triple  $(K, h, \mu) \in \sigma$  should be regarded as a *symplectic geometric* parameter compatible with the complex geometric data  $\sigma$ .

Let  $f \in F$  and  $u \in \mathfrak{g}$ . We denote by  $c_f^u$  the path in  $F$  defined by

$$c_f^u : [0, \infty) \rightarrow F, c_f^u(t) := e^{tu} f.$$

In order to define the “maximal weight” map  $\lambda$  associated with a symplectization, let us introduce the following definition:

**DEFINITION 2.5** ([12]). — A symplectization  $\sigma$  of the action  $\alpha$  will be called *energy-complete* if, for a representative  $(K, h, \mu) \in \sigma$  (and hence for any representative) the following implication holds: if  $s \in i\mathfrak{k}$ ,  $f \in F$  and the energy  $E_h(c_f^s)$  with respect to the Riemannian metric  $h$  is finite, then  $c_f^s$  has a limit as  $t \rightarrow +\infty$ .

Let  $\alpha : G \times V \rightarrow V$  be a linear action. A symplectization of  $\alpha$  given by a triple  $(K, h, \mu)$ , where  $h$  is a Hermitian structure on the vector space  $V$ , will be called a *linear symplectization* of  $\alpha$ .

*Remark 2.6.* — Any linear symplectization and any symplectization of an action on a compact complex manifold is energy-complete [22]. The natural symplectization of the action of the complex gauge group on the configuration space associated to certain moduli problems on curves is also energy complete [5].

If we choose a representative  $(K, h, \mu) \in \sigma$ , we can associate to every pair  $(s, t) \in i\mathfrak{k} \times \mathbb{R}$  the map

$$\begin{aligned} \lambda_t^s : F &\rightarrow \mathbb{R} \\ f &\mapsto \mu^{-is}(e^{ts} f) \end{aligned}$$

where we use the notation  $\mu^s := \langle \mu, s \rangle : F \rightarrow \mathbb{R}$  for any  $s \in \mathfrak{k}$ .

It is easy to see that the map  $t \mapsto \lambda_t^s(f)$  is increasing so that one can put

$$\lambda^s(f) := \lim_{t \rightarrow +\infty} \lambda_t^s(f) \in \mathbb{R} \cup \{\infty\}.$$

The energy-completeness condition allows one to prove the following technical result.

**PROPOSITION 2.7** ([22]). — Assume that  $\sigma$  is energy-complete and let  $s \in H(G)$ . The map  $\lambda^s : F \rightarrow \mathbb{R} \cup \{\infty\}$  does not depend on the choice



of a representative  $(K, h, \mu) \in \sigma$  with  $s \in \mathfrak{k}$  and gives rise to a well defined map

$$\begin{aligned} \lambda : H(G) \times F &\rightarrow \mathbb{R} \cup \{\infty\} \\ (s, f) &\mapsto \lambda(s, f) = \lambda^s(f) \end{aligned}$$

The following properties of the map  $\lambda$  will be useful in our study:

PROPOSITION 2.8 ([22]). — Assume that  $\sigma$  is energy-complete. The map  $\lambda$  introduced above has the following properties:

- (1) homogeneity:  $\lambda(ts, f) = t\lambda(s, f)$  for any  $t \in \mathbb{R}^+$ ;
- (2)  $\lambda$  is  $G$ -equivariant:  $\lambda(s, f) = \lambda(\text{ad}_{\gamma^{-1}}(s), \gamma^{-1}.f)$ ;
- (3)  $\lambda$  is  $\sim$  invariant:  $\lambda^s(f) = \lambda^\sigma(f)$  if  $s \sim \sigma$ ;
- (4) semi-continuity :  
if  $(f_n, s_n)_n \rightarrow (f, s)$ , then  $\lambda^s(f) \leq \liminf_{n \rightarrow \infty} \lambda^{s_n}(f_n)$ .

Remark 2.9. — One can work with a similar equivalence relation  $\simeq$  on  $H(G)$ , using the parabolic subgroups

$$G^-(s) := \left\{ g \in G \mid \lim_{t \rightarrow -\infty} e^{ts} g e^{-ts} \text{ exists in } G \right\}.$$

It holds

$$\sigma \sim s \Leftrightarrow -\sigma \simeq -s.$$

In general, our map  $\lambda$  will *not* be  $\simeq$  invariant (see Property 2.8 above). On the other hand, the “opposite” maximal weight map given by  $\lambda_-^s = \lambda^{-s}$  will be invariant with respect to this relation. Using  $\simeq$  and  $\lambda_-$ , one will get a completely parallel theory.

### 2.2. Analytic and symplectic stability.

Let also  $\alpha$  be an action of a reductive group  $G$  on a complex Kähler manifold  $F$ , let us choose an energy-complete symplectization  $\sigma$ , and let  $\lambda : H(G) \times F \rightarrow \mathbb{R} \cup \{\infty\}$  be the associated maximal weight map.

We will denote by  $s^\#$  the vector field on  $F$  defined by  $s$ . We will denote by  $\mathfrak{g}_f$  the Lie algebra of the stabilizer of a point  $f \in F$ , hence the Lie subalgebra of  $\mathfrak{g}$  consisting of those elements  $s$  such that  $s_f^\# = 0$ .

DEFINITION 2.10. — A point  $f \in F$  will be called

- (1) analytically  $\sigma$ -semistable if  $\lambda^s(f) \geq 0$  for all  $s \in H(G)$ .
- (2) analytically  $\sigma$ -stable if it is semistable and  $\lambda^s(f) > 0$  for any  $s \in H(G) \setminus \{0\}$ .
- (3) analytically  $\sigma$ -polystable if it is semistable,  $\mathfrak{g}_f$  is a reductive subalgebra and  $\lambda^s(f) > 0$  for every  $s$  which is not equivalent to an element of  $\mathfrak{g}_f$ .

In this definition we used the following convention: A subalgebra  $\mathfrak{g}' \subset \mathfrak{g}$  is called a reductive subalgebra if it has the form  $\mathfrak{g}' = \mathfrak{k}'^{\mathbb{C}}$ , where  $\mathfrak{k}'$  is the Lie algebra of a compact subgroup of  $G$ . This is more restrictive than the condition that  $\mathfrak{g}'$  is isomorphic to the Lie algebra of a reductive group.

*Remark 2.11.* — The property of stability (semistability and polystability) for  $f \in F$  depends only on the complex orbit  $Gf$  of  $f$ .

Note that the proof of this fact for semistable points requires energy-completeness.

Let us remind the classical definition of (semi)stability for symplectic actions (see [10], [11], [9], [8]). The polystability condition was first introduced in [16] in the algebraic framework, as a natural generalization of the polystability condition for bundles.

**DEFINITION 2.12.** — Let  $\sigma$  be a symplectization of the action  $\alpha : G \times F \rightarrow F$ . A point  $f \in F$  is called

- (1) symplectically  $\sigma$ -semistable if, choosing any representative  $(K, h, \mu) \in \sigma$ , one has  $\overline{G \cdot f} \cap \mu^{-1}(0) \neq \emptyset$ .
- (2) symplectically  $\sigma$ -stable if  $G \cdot f \cap \mu^{-1}(0) \neq \emptyset$  and  $\mathfrak{g}_f = \{0\}$ .
- (3) symplectically  $\sigma$ -polystable if  $G \cdot f \cap \mu^{-1}(0) \neq \emptyset$ .

These conditions do not depend on the chosen representative  $(K, h, \mu) \in \sigma$  and they are obviously  $G$ -invariant conditions with respect to  $f$ . Note also that the polystability condition is not open in general.

The following result of Heinzner and Loose (see [9], [8]) show that one can always construct a good quotient of the semistable locus. No condition on the symplectization is needed.

**THEOREM 2.13.** — *The set  $F^{ss}(\sigma)$  of symplectically  $\sigma$ -semistable*

points is open. Moreover, there is a categorical quotient

$$F^{ss}(\sigma) \rightarrow Q_\sigma$$

where  $Q_\sigma$  is a Hausdorff space with the property that two  $G$ -orbits have the same image in  $Q_\sigma$  if and only if their closure contains a common symplectically  $\sigma$ -polystable orbit.

Choose a representative  $(K, h, \mu) \in \sigma$ , then every  $\sigma$ -polystable orbit intersects  $\mu^{-1}(0)$  along a  $K$ -orbit and the induced map

$$\mu^{-1}(0)/K \rightarrow Q_\sigma$$

is a homeomorphism.

The following fundamental result links these two notions of stability:

**THEOREM 2.14** ([15], [22]). — *Assume that  $\sigma$  is energy-complete. A point  $f$  is symplectically  $\sigma$ -stable (polystable) if and only if it is analytically  $\sigma$ -stable (polystable).*

Our goal here is merely to study the behavior of non semistable points. So from our point of view, the most important fact is that the concepts of analytic semistability and symplectic semistability coincide. This is a rather difficult technical result ([22] for details). The main tool is the so-called integral of the moment map, whose existence is assured by the following

**LEMMA 2.15.** — *Let  $(K, h, \mu)$  be a representative of the symplectization  $\sigma$ , then there exists a unique smooth function  $\Psi : F \times G \rightarrow \mathbb{R}$  with the following properties:*

- $\frac{d}{dt}\Psi(f, e^{ts}) = \lambda_t^s(f)$ ;
- $\Psi(f, k) = 0$  for all  $k \in K$ ;
- $\Psi(f, gh) = \Psi(f, h) + \Psi(hf, g)$ , for all  $h, g \in G, f \in F$ .

*Proof.* — This is a well-known result (see for instance [15]). □

**Remark 2.16.**

(1) The map  $t \mapsto \Psi(f, e^{ts})$  is convex for all  $s \in i\mathfrak{k}, f \in F$ .

(2) The two following properties are equivalent:

- $g \in G$  is a critical point of the map  $g \mapsto \Psi(f, g)$ ;
- $\mu(gf) = 0$ .

**THEOREM 2.17** (see [22]). — *Let  $(F, h)$  be a Kähler manifold,  $\alpha : G \times F \rightarrow F$  a complex reductive Lie group action and let  $\sigma$  be an energy complete symplectization for this action. Then, for any point  $f \in F$  the following properties are equivalent :*

- (1) *the point  $f$  is analytically  $\sigma$ -semistable;*
- (2) *the map  $g \rightarrow \Psi(f, g)$  associated to any representative  $(K, h, \mu)$  of  $\sigma$  is bounded from below over  $G$ ;*
- (3) *the point  $f$  is symplectically  $\sigma$ -semistable.*

Note that energy completeness plays an essential role in the proof. In the sequel we will speak of  $\sigma$ -semistability (stability, polystability) without precisising if the analytical or symplectical condition is meant.

### 3. The reductive quotient associated to a class of Hermitian type elements and its canonical action.

We have seen that any equivalence class of elements of Hermitian type defines a parabolic subgroup  $G(\mathcal{S})$  of  $G$ . In this section, our purpose is to associate to any non trivial equivalence class  $\mathcal{S}$  of Hermitian type elements a new factorization problem with symmetry group  $G(\mathcal{S})/U(\mathcal{S})$ , where  $U(\mathcal{S})$  is the unipotent subgroup associated with  $\mathcal{S}$ . This quotient is a reductive group. The new manifold, that we introduce is isomorphic to a submanifold of  $F$ , but the identification is not canonical. Then we will show that for any choice of a symplectization  $\sigma$  for the factorization problem  $(F, G, \alpha)$ , we may define a natural symplectization for our new problem associated to the class  $\mathcal{S}$ .

#### 3.1. Natural action of the canonical reductive quotient.

First of all, let us remind that for any  $s, s' \in H(G)$  such that  $s \sim s'$ , we have  $G(s) = G(s')$  and  $U(s) = U(s')$  (because  $U(s)$  is a normal subgroup in  $G(s)$ ), so that we may associate to any equivalence class  $\mathcal{S} \in H(G)/\sim$  of Hermitian type elements a unique parabolic subgroup  $G(\mathcal{S})$  of  $G$  and a unique unipotent subgroup  $U(\mathcal{S}) \subset G(\mathcal{S})$ .

For every  $s \in H(G)$ , let us denote by

$$V_s := \{f \in F \mid (s^\#)_f = 0\}$$

the zero locus of the vector field  $s^\sharp$ . Locally this set consists of fixed points under the action of the compact torus  $T = \overline{\{e^{its} \mid t \in \mathbb{R}\}}$  and therefore, using the slice theorem [18], we see that  $V_s$  is a smooth submanifold of  $F$ , in general *not* of pure dimension. Being the vanishing locus of the holomorphic tangent field associated with  $s^\sharp$ , it inherits a structure of complex manifold (of possibly non-pure dimension).

Let us now remark that for any  $s$  and  $s'$  in  $H(G)$ , if  $s \sim s'$  then there exists  $u \in U(s) = U(s')$  such that  $s' = \text{ad}_u(s)$  and we have an associated isomorphism  $\alpha(u) : V_s \xrightarrow{\simeq} V_{s'} = V_{\text{ad}_u(s)}$ .

One can easily prove that the element  $u \in U(s) = U(s')$  such that  $s' = \text{ad}_u(s)$  is unique, so that one gets a canonical identification  $V_s \simeq V_{s'}$ . Indeed, if there exists two elements  $u, v \in U(s)$  such that  $\text{ad}_u(s) = \text{ad}_v(s) = s'$ , we get  $w = v^{-1}u \in U(s)$  and  $\text{ad}_w(s) = s$ . Then we have,  $w \in Z(s) \cap U(s) = \{e\}$  so that the induced isomorphism is the identity.

Therefore we can associate to any non trivial equivalence class  $\mathcal{S}$  of  $H(G)$  a canonically defined complex manifold

$$\mathcal{V}(\mathcal{S}) := \left\{ \prod_{s \in \mathcal{S}} V_s \right\} / \sim$$

where  $\sim$  is induced by the previous identifications. One has, for every  $s \in \mathcal{S}$ , a natural identification  $\mathcal{V}(\mathcal{S}) \simeq V_s$ .

*Remark 3.1.* — One can wonder why is it important to consider the copy  $\mathcal{V}(\mathcal{S})$  of  $V_s$ . The reason is the following: we will associate to any non-semistable point  $f \in F$  a well defined class  $\mathcal{S}_f \in H(G)/\sim$  of so-called optimal destabilizing vectors, but there is *no way* to associate a well defined such destabilizing vector. Therefore, we get a well defined assignment  $f \mapsto \mathcal{V}(\mathcal{S}_f)$ . In the next step we will see that  $f$  also defines canonically a point  $f_0 \in \mathcal{V}(\mathcal{S}_f)$  which is semistable with respect to a certain (again canonically associated) symplectization.

The action  $\alpha$  induces an action of the parabolic group  $G(\mathcal{S})$  over the complex manifold  $\mathcal{V}(\mathcal{S})$  defined by:

$$\begin{aligned} G(\mathcal{S}) \times \mathcal{V}(\mathcal{S}) &\rightarrow \mathcal{V}(\mathcal{S}) \\ (g, [x]) &\mapsto [g(x)] \end{aligned}$$

where  $x \in V_s$  and  $g(x) \in V_{\text{ad}_g(s)}$  for any  $s \in \mathcal{S}$ .

Of course  $G(\mathcal{S})$  is not reductive but it is easy to see, using the definition of  $\mathcal{V}(\mathcal{S})$ , that the unipotent subgroup  $U(\mathcal{S})$  acts trivially on  $\mathcal{V}(\mathcal{S})$ .

So that we get a well-defined action

$$\alpha_{\mathcal{S}} : G(\mathcal{S})/U(\mathcal{S}) \times \mathcal{V}(\mathcal{S}) \rightarrow \mathcal{V}(\mathcal{S})$$

of the canonical reductive quotient  $G(\mathcal{S})/U(\mathcal{S})$ .

Let us remark that if we choose any representative  $s \in \mathcal{S}$ , the action of  $G(\mathcal{S})/U(\mathcal{S})$  over the representative  $V_s$  of  $\mathcal{V}(\mathcal{S})$  is just the induced action of the reductive Lie group  $Z(s)$  over the complex submanifold  $V_s \subset F$ .

### 3.2. A natural symplectization for the action $\alpha_{\mathcal{S}}$ .

An *ad-invariant inner product of Euclidian type* on the Lie algebra  $\mathfrak{g}$  is an  $\text{ad}_G$ -invariant non-degenerate complex symmetric bilinear form  $h$  on  $\mathfrak{g}$  which restricts to an inner product on a subspace of the form  $i\mathfrak{k}$  (and hence on any subspace of this form as any two such subspaces are conjugated).

The data of such an inner product is equivalent to the data of:

- a multiple of the Killing form  $k_{\mathfrak{s}}$  of each simple summand  $\mathfrak{s}$  of the semisimple part  $\mathfrak{g}^s$  of  $\mathfrak{g}$  and

- an inner product on  $i\mathfrak{t}_0$ , where  $\mathfrak{t}_0$  is the Lie algebra of the maximal compact subgroup of the complex torus  $\mathfrak{z}(\mathfrak{g})$ .

We fix such an inner product  $\langle \cdot, \cdot \rangle$  on our Lie algebra  $\mathfrak{g}$ .

For any choice of a symplectization  $\sigma$  of the factorization problem  $(F, G, \alpha)$ , we may define a canonical symplectization for the action  $\alpha_{\mathcal{S}}$  in the following way: let  $\rho = (K, h, \mu) \in \sigma$ , and let us take the unique representative  $s \in i\mathfrak{k} \cap \mathcal{S}$  and the corresponding copy  $V_s$  of  $\mathcal{V}(\mathcal{S})$ , then we can define an associated symplectization of  $V_s \simeq \mathcal{V}(\mathcal{S})$  using the triple

$$\rho_{\mathcal{S}} := (K \cap Z(s), h|_{V_s}, i^*(\mu|_{V_s}) + \tau)$$

where  $i : \mathfrak{k} \cap \mathfrak{z}(s) \hookrightarrow \mathfrak{k}$  is the inclusion and  $\tau$  is the locally constant  $\mathfrak{z}(s)^{\vee}$ -valued function over  $V_s$  defined by

$$\tau(x) = -(\mu^{-is}(x))\langle is, \cdot \rangle.$$

To see that the map  $\tau$  above is indeed locally constant, note that the map  $x \rightarrow \mu^{-is}(x)$  is locally constant over  $V_s$  since

$$d\mu^{-is}(\cdot) = \omega_h((-is)^{\sharp}, \cdot) = h(s^{\sharp}, \cdot) = 0.$$

The reason for this particular choice of the moment map will appear later in section 5.

This definition is coherent with the identifications defined above : let  $\rho' = (K', h', \mu') \in \sigma$  be another representative of  $\sigma$  and let  $s' \in i\mathfrak{k}' \cap \mathcal{S}$ . Then, there exists  $u \in U(\mathcal{S})$  such that  $\text{ad}_u(s') = s$ . The application  $\alpha(u)$  defines an isomorphism from  $V_{s'}$  onto  $V_s = V_{\text{ad}_u(s)}$  and conjugates  $\rho'$  to another representative  $\rho'' = u_*(\rho') \in \sigma$  defined by

$$\rho'' = (K'', h'', \mu'') = (\text{Ad}_u(K'), (u^{-1})^*(h'), \text{ad}_{u^{-1}}^t \circ \mu' \circ u^{-1}).$$

It is sufficient to show that  $\rho_{\mathcal{S}}$  and  $\rho''_{\mathcal{S}}$  define the same symplectization for the action  $\alpha_{\mathcal{S}}$ . Let us remark that, by the definition of  $\tau$ ,  $\rho''_{\mathcal{S}} = u_*(\rho'_{\mathcal{S}})$ . Moreover, we have  $s \in i\mathfrak{k} \cap i\mathfrak{k}''$  and there exists  $\gamma \in G$  which conjugates  $\rho$  and  $\rho''$ , i.e.

$$\text{Ad}_{\gamma}(K'') = K, h = (\gamma^{-1})^*(h''), \mu = \text{ad}_{\gamma^{-1}}^t \circ \mu'' \circ \gamma^{-1}.$$

We now use the following lemma:

LEMMA 3.2. — *Let  $K$  be a maximal compact subgroup of  $G$  and let  $g \in G$ ,  $s \in \mathfrak{k}$  such that  $\text{ad}_g(s) \in \mathfrak{k}$ . Then the decomposition  $g = kl$ , where  $l \in \exp(i\mathfrak{k})$ ,  $k \in K$  satisfies  $\text{ad}_l(s) = s$ , i.e.  $l \in Z(s)$ .*

*Proof.* — Let us decompose  $g$  as  $g = kl$  with  $l \in \exp(i\mathfrak{k})$  and  $k \in K$ . Then

$$\gamma := \text{ad}_l(s) = \text{ad}_{k^{-1}}(\text{ad}_g(s)) \in \mathfrak{k}.$$

If we choose an embedding  $G \hookrightarrow GL(r, \mathbb{C})$  mapping  $K$  to  $U(r)$ , then the image of  $l$  is Hermitian with positive eigenvalues, whereas the images of  $s$  and  $\gamma$  are anti-Hermitian. We have:

$$\text{ad}_l(s)^* = -\text{ad}_{l^{-1}}(s) = \gamma^* = -\gamma = -\text{ad}_l(s),$$

hence  $\text{ad}_{l^2}(s) = s$ . This implies that the eigenspaces of  $l^2$  and hence of  $l$  are invariant under  $s$ , so that one also has  $\text{ad}_l(s) = s$ . □

Therefore, since  $s \in i\mathfrak{k}$  and  $\text{ad}_{\gamma}(s) \in i\mathfrak{k}$ , we have the decomposition  $\gamma = kl$ ,  $k \in K$  and  $l \in Z(s)$  so that

$$\begin{aligned} K'' &= \text{Ad}_{l^{-1}k^{-1}}(K) = \text{Ad}_{l^{-1}}(K), \\ h'' &= \gamma^*(h) = l^*(h) \end{aligned}$$

because  $h$  is by definition  $K$ -invariant and

$$\mu'' = \text{ad}_l^t \circ \text{ad}_k^t \circ \mu \circ k \circ l = \text{ad}_l^t \circ \mu \circ l$$

because a moment map is always  $K$ -equivariant. We conclude that  $\rho$  and  $\rho''$  are conjugated by an element of  $l \in Z(s)$ . One has  $s'' = s$  because they

are both representatives in  $i\mathfrak{k}''$  of  $\mathcal{S}$ , therefore  $\tau$  and  $\tau''$  are conjugated, so that the two induced triple  $\rho_{\mathcal{S}}$  and  $\rho''_{\mathcal{S}}$  are equivalent for the action  $\alpha_{\mathcal{S}}$ .

In the sequel, we will denote by  $\sigma_{\mathcal{S}}$  this natural symplectization for the factorization problem  $(\mathcal{V}(\mathcal{S}), G(\mathcal{S})/U(\mathcal{S}), \alpha_{\mathcal{S}})$ .

#### 4. Optimal destabilizing vector for a non semistable point.

In this section we will associate to every non  $\sigma$ -semistable point  $f \in F$ , an optimal destabilizing element  $s \in H(G)$  which minimize the weight function  $\lambda(\cdot, f)$ . We will also see that this element is unique up to equivalence.

So, let us consider a holomorphic action  $\alpha : G \times F \rightarrow F$  of a reductive group  $G$  on the Kähler manifold  $F$ . We choose a symplectization  $\sigma$  for this action and we assume in the sequel that  $\sigma$  is energy-complete (see def. 2.5) so that the map  $\lambda : H(G) \times F \rightarrow \mathbb{R} \cup \{\infty\}$  is well defined.

Fix again an *ad-invariant inner product of Euclidian type*  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ . Such a structure gives a well defined real application  $\|\cdot\| : H(G) \rightarrow \mathbb{R}$  defined by  $\|s\| = \sqrt{\langle s, s \rangle}$  (in fact all the elements of  $H(G)$  lie in a Lie algebra of the form  $i\mathfrak{k}$  for a certain maximal compact subgroup  $K$ , on which  $\langle \cdot, \cdot \rangle$  is a scalar product). Let us remark that  $\langle \cdot, \cdot \rangle$  is constant on the equivalence classes of  $H(G)$ , so that we may speak of a “normalized class”  $\mathcal{S}$ .

We consider in this section a given  $\sigma$ -non semistable point  $f \in F$  and we set

$$\lambda_{\inf} := \inf_{\substack{s \in H(G) \\ \|s\|=1}} \lambda(s, f).$$

Let us remark that this lower bound is not  $-\infty$  as

$$\lambda(s, f) \geq \lambda_0(s, f) + E_h(c_f^s) \geq \lambda_0(s, f) = \langle \mu(f), -is \rangle.$$

Let us define the set of normalized destabilizing elements of  $f$  :

$$\Lambda_f := \{\xi \in H(G) \mid \|\xi\| = 1 \text{ and } \lambda(\xi, f) = \lambda_{\inf}\}.$$

**THEOREM 4.1.** — *Let  $f \in F$  be a non  $\sigma$ -semistable point. Then  $\Lambda_f$  is non empty and consists of exactly a normalized equivalence class  $\mathcal{S}_f \subset H(G)$ .*



*Proof of the theorem.*

LEMMA 4.2 (Existence).

- (1) If  $s \in \Lambda_f$  and  $s' \in H(G)$  with  $s' \sim s$  then  $s' \in \Lambda_f$ .
- (2)  $\Lambda_f \neq \emptyset$ .

*Proof.* — The first point follows directly from the equivariance properties of  $\lambda$  (see prop. 2.8) and the ad-invariance of  $\langle \cdot, \cdot \rangle$ .

For the second point, let us fix a maximal compact subgroup  $K$  of  $G$ . Then we know that  $i\mathfrak{k} \subset \mathfrak{g}$  is a complete system of representatives for  $\sim$ . By invariance, the application  $\lambda$  restricts to a map  $\tilde{\lambda} : i\mathfrak{k} \rightarrow \mathbb{R} \cup \{\infty\}$ . Take now a sequence  $(s_n)_n \in H(G)$  such that  $\lambda(s_n, f)$  converges to  $\lambda_{\text{inf}}$  and  $\|s_n\| = 1$  for all  $n$ . We take  $\tilde{s}_n$  to be the representative in  $i\mathfrak{k}$  which is in the same equivalence class as  $s_n$ . We still have  $\|\tilde{s}_n\| = 1$  and  $\tilde{\lambda}(\tilde{s}_n) \rightarrow \lambda_{\text{inf}}$ . Now  $i\mathfrak{k}$  is a closed finite dimensional vector space in  $\mathfrak{g}$  so that its unit sphere is compact. Thus, we can extract a converging subsequence  $\tilde{s}_m \rightarrow \tilde{s}$ . Now, the semi-continuity property of  $\lambda$  (prop. 2.8) implies

$$\tilde{\lambda}(\tilde{s}, f) \leq \liminf_{n \rightarrow \infty} \lambda(s_n, f) = \lambda_{\text{inf}},$$

i.e. all the elements of the class  $\tilde{s}$  are elements of  $\Lambda_f$ . □

LEMMA 4.3 (Unicity). — *The optimal destabilizing element is unique up to equivalence:*

$$\exists \xi \in H(G) \text{ s.t. } \Lambda_f = \{s \in H(G) \mid \xi \sim s\} = \mathcal{S}(\xi).$$

*Proof.* — Let us choose a representative  $(K, h, \mu) \in \sigma$  and let  $\Psi : F \times G \rightarrow \mathbb{R}$  the associated integral of the moment map (see prop. 2.15). We must prove that there exists only one maximal element in  $i\mathfrak{k}$ .

Our first step is to prove the result when  $K = T$  is a real torus.

LEMMA 4.4. — *If  $K = T$  is a real torus then there exists a unique  $\xi_T(f) \in i\mathfrak{t}$  such that  $\Lambda_f \cap i\mathfrak{t} = \{\xi_T(f)\}$ .*

*Proof.* — The proof is based on the following lemma:

LEMMA 4.5. — *The map  $\Phi_f : i\mathfrak{t} \rightarrow \mathbb{R}$  defined by  $\Phi_f(s) = \Psi(f, e^s)$  is convex on  $i\mathfrak{t}$ .*

*Proof.* — This is a well-known property of  $\Psi$  that the maps  $t \mapsto \Psi(f, e^{ts})$  are convex for all  $s \in it$  (prop. 2.15). Let  $\xi, s \in it$ , then using the fact that  $\xi$  and  $s$  commute, we get:

$$\Psi(f, e^{\xi+ts}) = \Psi(f, e^{\xi}e^{ts}) = \Psi(f, e^{\xi}) + \Psi(e^{\xi}f, e^{ts})$$

so that  $t \mapsto \Phi(\xi + ts)$  is convex for every  $s \in it$ . To conclude we use the following easy lemma:

LEMMA 4.6. — *Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  a smooth function such that for all  $x_0, x \in \mathbb{R}^n$  the map  $t \mapsto f(x_0 + tx)$  is convex. Then  $f$  is convex on  $\mathbb{R}^n$ .*

□

By definition we have

$$\lambda(\xi, f) = \lim_{t \rightarrow +\infty} \frac{d}{dt} \phi_f(t\xi).$$

Assume that there exist two distinct optimal destabilizing elements  $\xi_1, \xi_2 \in it \cap \Lambda_f$  and let  $\xi = \frac{\xi_1 + \xi_2}{2} \in it$ . Of course we have  $\|\xi\| < 1$ . The convexity of the function  $\phi_f$  implies that  $\phi_f(t\xi) \leq \frac{1}{2}(\phi_f(t\xi_1) + \phi_f(t\xi_2))$  for all  $t \in \mathbb{R}$ . We get

$$\begin{aligned} \frac{\phi_f(\theta\xi) - \phi_f(t\xi)}{\theta - t} &\leq \frac{\frac{1}{2}(\phi_f(\theta\xi_1) + \phi_f(\theta\xi_2)) - \frac{1}{2}(\phi_f(t\xi_1) + \phi_f(t\xi_2))}{\theta - t} \\ &\quad + \frac{\frac{1}{2}(\phi_f(t\xi_1) + \phi_f(t\xi_2)) - \phi_f(t\xi)}{\theta - t} \end{aligned}$$

and so

$$\begin{aligned} \limsup_{\theta \rightarrow +\infty} \left[ \frac{\phi_f(\theta\xi) - \phi_f(t\xi)}{\theta - t} \right] \\ \leq \limsup_{\theta \rightarrow \infty} \left[ \frac{\frac{1}{2}(\phi_f(\theta\xi_1) - \phi_f(t\xi_1)) + \frac{1}{2}(\phi_f(\theta\xi_2) - \phi_f(t\xi_2))}{\theta - t} \right]. \end{aligned}$$

The regularity and the convexity of  $\Psi$  implies that for all  $\theta, t \in \mathbb{R}$

$$\frac{d}{ds} \Big|_{s=\theta} \Psi(f, e^{s\xi}) \geq \frac{\phi_f(\theta\xi) - \phi_f(t\xi)}{\theta - t} \geq \frac{d}{ds} \Big|_{s=t} \Psi(f, e^{s\xi})$$

thus we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{d}{dt} \Psi(f, e^{t\xi}) \\ \leq \limsup_{t \rightarrow +\infty} \left[ \limsup_{\theta \rightarrow +\infty} \left( \frac{\frac{1}{2}(\phi_f(\theta\xi_1) - \phi_f(t\xi_1))}{\theta - t} + \frac{\frac{1}{2}(\phi_f(\theta\xi_2) - \phi_f(t\xi_2))}{\theta - t} \right) \right] \\ = \frac{\lambda_{\inf} + \lambda_{\sup}}{2}. \end{aligned}$$

We deduce from this that

$$\lambda\left(\frac{\xi}{\|\xi\|}, f\right) = \frac{\lambda(\xi, f)}{\|\xi\|} \leq \frac{\lambda_{\text{inf}}}{\|\xi\|} < \lambda_{\text{inf}},$$

because  $\|\xi\| < 1$ . This leads to a contradiction. □

*Remark 4.7.* — Note that in this argument one essentially needs the fact that  $f$  is non semistable (i.e.  $\lambda_{\text{inf}} < 0$ ).

Let us now come back to our main proof for an arbitrary compact Lie group  $K$ .

LEMMA 4.8. — *Let  $f$  a non  $\sigma$ -semistable point and  $\xi \in \Lambda_f$ . Let  $T$  be a maximal torus in  $G(\xi)$ . Then  $f$  is non semistable with respect to the induced symplectization of the  $T^{\mathbb{C}}$ -action, and  $\xi$  is conjugated to  $\xi_T(f)$  by an element of  $G(\xi)$ .*

*Proof.* — Let  $S$  be a maximal torus of  $G(\xi)$  whose Lie algebra is containing  $\xi$ . All maximal tori of  $G(\xi)$  are conjugated to each other, so there exists  $p \in G(\xi)$  such that  $\text{Ad}_p(S) = T$  then we have  $\text{ad}_p(\xi) \in \mathfrak{it} \cap \Lambda_f$  (see proposition 4.2). We deduce from this that  $f$  is  $T^{\mathbb{C}}$  non semistable and  $\text{ad}_p(\xi)$  is an optimal destabilizing element with respect to the  $T^{\mathbb{C}}$ -action. Therefore, from the previous unicity lemma 4.4,  $\xi_T(f) = \text{ad}_p(\xi)$ . □

Now we make use of the following well-known lemma (see [21]) :

LEMMA 4.9. — *Let  $P$  and  $P'$  be parabolic subgroups of  $G$ . Then there is a maximal torus  $T$  of  $G$  contained in the intersection  $P \cap P'$ .*

We use now the same method as in the algebraic case [21].

LEMMA 4.10. — *Let  $f$  be a non semistable point and let  $\xi_1$  and  $\xi_2$  be two elements of  $\Lambda_f$ . Then we have  $G(\xi_1) = G(\xi_2)$  and there exists  $p \in G(\xi_1)$  such that  $\text{ad}_p(\xi_1) = \xi_2$ , i.e.  $\xi_1 \sim \xi_2$ .*

*Proof.* — Let  $T$  be a maximal torus contained in  $G(\xi_1) \cap G(\xi_2)$ . By the previous lemma, there exists  $g \in G(\xi_1)$  and  $k \in G(\xi_2)$  such that  $\text{ad}_g(\xi_1) = \xi_T(v) = \text{ad}_k(\xi_2)$ . Then we get  $G(\xi_T(v)) = \text{Ad}_g(G(\xi_1)) = G(\xi_1)$  and the same thing for  $G(\xi_2)$ . We get  $G(\xi_1) = G(\xi_2)$  and  $\xi_2 = \text{ad}_{(k^{-1}g)} \xi_1$ , so that  $\xi_1 \sim \xi_2$ . □

This concludes the proof of the theorem. □

## 5. Associating a semistable point to a non-semistable one. Shatz stratification associated with a Hamiltonian action.

Using the results of the two previous sections, we will show here that it is possible to associate naturally to any non  $\sigma$ -semistable point a semistable point for the new factorization problem defined in section 3.1 and thus a point in the associated Hamiltonian quotient. This leads to the stratification of  $F$  by  $G$  invariant subsets described in the main Theorem. 5.4. The main stratum is the semi-stable locus and it is open. The other strata are obtained by fixing the conjugacy class (with respect to the  $\text{ad}_G$ -action on  $H(\mathfrak{g})$ ) of the optimal destabilizing element.

Let  $(F, G, \alpha)$  be a factorization problem with an associated symplectization  $\sigma$ . Let us choose a  $\text{ad}_G$ -invariant inner product of Euclidian type over  $\mathfrak{g}$ . Using Theorem. 4.1, we define for any normalized equivalence class  $\mathcal{S} \in H(G)/\sim$  the following subset of  $F$ :

$$\mathcal{Z}_{\mathcal{S}} := \{f \in F \mid f \text{ is non } \sigma\text{-semistable and } \Lambda_f = \mathcal{S}\}.$$

Therefore  $\mathcal{Z}_{\mathcal{S}}$  is the locus of points with optimal destabilizing class  $\mathcal{S}$ .

Now let us choose  $f \in \mathcal{Z}_{\mathcal{S}}$ , and let us fix any representative  $\rho = (K, h, \mu) \in \sigma$  and take the unique representative  $\xi_f^\rho \in \Lambda_f \cap i\mathfrak{k}$  of  $\mathcal{S}$ . The point  $f$  being non  $\sigma$ -semistable, we get from the formula

$$\lambda(\xi_f^\rho, f) = \lambda_0(\xi_f^\rho, f) + E_h(c_f^{\xi_f^\rho})$$

that  $E_h(c_f^{\xi_f^\rho}) < +\infty$ . The symplectization  $\sigma$  is supposed to be energy complete, so that there exists a limit element  $f_0^\rho = \lim_{t \rightarrow +\infty} e^{t\xi_f^\rho} f \in F \cap \overline{Gf}$ .

The point  $f_0^\rho$  lies in the vanishing subset  $V_{\xi_f^\rho} = V((\xi_f^\rho)^\sharp)$ . For another choice  $\rho' \in \sigma$  we get  $\xi_{f'}^{\rho'} = \text{ad}_u(\xi_f^\rho)$ , where  $u \in U(\mathcal{S})$ , and  $f_0^{\rho'} = \alpha(u)(f_0^\rho) \in V_{\xi_{f'}^{\rho'}}$ . Thus, we obtain a well defined point  $f_0 \in \mathcal{V}(\mathcal{S})$  canonically associated to  $f$ . Our claim is the following :

**THEOREM 5.1.** — *Let  $f \in F$  be a non  $\sigma$ -semistable point, and  $\mathcal{S} = \Lambda_f$  the class of its optimal destabilizing element. Then the canonically associated point  $f_0 \in \mathcal{V}(\mathcal{S})$  is  $\sigma_{\mathcal{S}}$ -semistable for the action of the canonical reductive quotient  $G(\mathcal{S})/U(\mathcal{S})$  over  $\mathcal{V}(\mathcal{S})$ .*

*Proof.* — For our purpose, we may fix a representative  $\rho = (K, h, \mu) \in \sigma$  and the element  $\xi_f^\rho \in \Lambda_f \cap i\mathfrak{k}$ . Let us remark first that, by definition,

$$\lambda_{\text{inf}} = \lim_{t \rightarrow +\infty} \mu^{-i\xi_f^\rho}(e^{t\xi_f^\rho} f) = \mu^{-i\xi_f^\rho}(f_0^\rho).$$

Let

$$\rho_S = (K \cap Z(\xi_f^\rho), h|_{V_{\xi_f^\rho}}, \mu' = i^*(\mu|_{V_{\xi_f^\rho}}) + \tau)$$

be the associated triple representing  $\sigma_S$  (see 3.2), then  $\tau$  is given on the connected component containing  $f_0^\rho$  by

$$\tau = -\lambda_{\inf} \langle i\xi_f^\rho, \cdot \rangle.$$

Let  $\lambda'$  be the map associated to the symplectization  $\sigma_S$ . We must show that  $f_0^\rho$  is  $\sigma_S$  semistable.

An element  $s \in i\mathfrak{k} \cap \mathfrak{z}(\xi_f^\rho)$  has an orthogonal decomposition  $s = \beta\xi_f^\rho + s^\perp$ .

From now on, we assume that  $\lambda'(s, f_0^\rho) < 0$  and we will get a contradiction.

Let  $\xi_\varepsilon = \xi_f^\rho + \varepsilon s^\perp$  for  $\varepsilon > 0$ . Then we get

$$\begin{aligned} \lambda\left(\frac{\xi_\varepsilon}{\|\xi_\varepsilon\|}, f\right) &= \frac{\lambda(\xi_\varepsilon, f)}{\|\xi_\varepsilon\|} \\ &= \frac{\lim_{t \rightarrow \infty} \mu^{-i\xi_\varepsilon}(e^{t\xi_\varepsilon} f)}{\|\xi_f^\rho + \varepsilon s^\perp\|} \\ &= \frac{\lim_{t \rightarrow \infty} \mu^{-i\xi_f^\rho}(e^{t\xi_\varepsilon} f) + \varepsilon \lim_{t \rightarrow \infty} \mu^{-is^\perp}(e^{t\xi_\varepsilon} f)}{\|\xi_f^\rho + \varepsilon s^\perp\|}. \end{aligned}$$

So we are reduced to study the orbit of  $f$  under the flow of  $\xi_\varepsilon^\sharp$ . We begin with the remark that the hypothesis  $\lambda'(s, f_0^\rho) < 0$  implies that  $E_h(c_{f_0^\rho}^{s^\perp}) < \infty$ , so that, the action being energy complete, we know that the curve  $c_{f_0^\rho}^{s^\perp}$  converges to some point  $f_1 = \lim_{t \rightarrow \infty} e^{ts^\perp} f_0^\rho \in F \cap V_{\xi_f^\rho}$ .

The main point of the proof is the following

CLAIM. — *For any sufficiently small  $\varepsilon > 0$ , the orbit of  $f$  under the one-parameter subgroup generated by  $\xi_\varepsilon$  converges to  $f_1$ , i.e.  $\lim_{t \rightarrow \infty} e^{t\xi_\varepsilon} f = f_1$ .*

*Proof of the claim.* — We consider first the compact torus

$$T := \overline{\left\{ e^{it\xi_f^\rho} e^{i\theta s^\perp} \mid t, \theta \in \mathbb{R} \right\}} \subset K$$

and the induced action  $T^{\mathbb{C}} \times F \rightarrow F$  of its complexification  $T^{\mathbb{C}} \subset G$ .

Now we use a fundamental result of Heinzner and Huckleberry, which allows us to “linearize” this action around  $f_1$ . Indeed, up to a modification

of the moment map  $\mu_T$  by a constant in  $\mathfrak{t} = \mathfrak{z}(\mathfrak{t})$ , we may always assume that  $\mu_T(f_1) = 0$ . Now following [8] (p. 346), we may find an open  $T^{\mathbb{C}}$ -stable Stein neighborhood of  $f_1$ . Using the fact that  $T^{\mathbb{C}}$  is reductive, we can apply Theorem 3.3.14 in [8] and get the existence of an open  $T^{\mathbb{C}}$ -invariant Stein neighborhood  $U$  of  $f_1$ , a linear representation  $\rho : T^{\mathbb{C}} \times V \rightarrow V$  and a closed  $T^{\mathbb{C}}$ -equivariant embedding  $a : U \rightarrow V$ . Since  $U$  is open and  $T^{\mathbb{C}}$ -invariant, it follows easily that it contains the points  $f_0^\rho$  and  $f$ . Put  $v_1 := a(f_1)$ ,  $v_0 := a(f_0^\rho)$ ,  $v := a(f)$ .

We decompose  $V$  as

$$V = \bigoplus_{\chi \in R} V_\chi,$$

where  $R \subset \text{Hom}(T, S^1)$  and  $\rho(t)|_{V_\chi} = \chi(t)\text{id}_{V_\chi}$  for all  $t \in T$ .

Since  $\lim_{t \rightarrow \infty} e^{t\xi_f^\rho} f = f_0^\rho$ , we deduce that

$$v = v_0 + v_-$$

where

$$v_0 \in \bigoplus_{d_e(\chi)(\xi_f^\rho)=0} V_\chi, v_- \in \bigoplus_{d_e(\chi)(\xi_f^\rho)<0} V_\chi.$$

For sufficiently small  $\varepsilon > 0$  we get that  $d_e(\chi)(\xi_f^\rho + \varepsilon s^\perp) < 0$  for all  $\chi \in R$  for which  $d_e(\chi)(\xi_f^\rho) < 0$ .

For such  $\varepsilon$  we get that

$$\lim_{t \rightarrow \infty} e^{t(\xi_f^\rho + \varepsilon s^\perp)} v = \lim_{t \rightarrow \infty} e^{t(\xi_f^\rho + \varepsilon s^\perp)} v_0 = \lim_{t \rightarrow \infty} e^{t\varepsilon s^\perp} v_0 = v_1.$$

□

According to the above claim, if  $\varepsilon$  is sufficiently small, our computation gives

$$\lambda\left(\frac{\xi_\varepsilon}{\|\xi_\varepsilon\|}, f\right) = \frac{\mu^{-i\xi_f^\rho}(f_1) + \varepsilon\mu^{-is^\perp}(f_1)}{\|\xi_f^\rho + \varepsilon s^\perp\|}.$$

Using the same methods as before we have

$$\mu^{-i\xi_f^\rho}(f_1) = \mu^{-i\xi_f^\rho}(f_0^\rho) = \lambda(\xi_f^\rho, f) = \lambda_{\text{inf}}$$

and moreover, by Lemma 5.2 below, we get

$$\mu^{-is^\perp}(f_1) = \mu'^{-is^\perp}(f_1) = \lambda'(s^\perp, f_0^\rho) < 0.$$

We obtain

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(\lambda\left(\frac{\xi_\varepsilon}{\|\xi_\varepsilon\|}, f\right)) = \mu^{-is^\perp}(f_1) < 0.$$

Thus, by taking  $\varepsilon$  small enough, we get a normalized element  $\frac{\xi_\varepsilon}{\|\xi_\varepsilon\|} \in H(G)$  with  $\lambda(\frac{\xi_\varepsilon}{\|\xi_\varepsilon\|}, f) < \lambda_{\text{inf}}$  which is a contradiction.  $\square$

LEMMA 5.2. — *If  $s = \beta \xi_f^\rho + s^\perp$  then  $\lambda'(s, f_0^\rho) = \lambda'(s^\perp, f_0^\rho)$ .*

*Proof.* — We have

$$\begin{aligned} \lambda'(s, f_0^\rho) &= \lim_{t \rightarrow \infty} \mu'^{-is}(e^{ts} f_0^\rho) \\ &= \lim_{t \rightarrow \infty} \mu^{-is}(e^{ts} f_0^\rho) - \lambda_{\text{inf}} \langle \xi_f^\rho, s \rangle \\ &= \lim_{t \rightarrow \infty} (\mu^{-is^\perp}(e^{ts} f_0^\rho) + \mu^{-i\beta \xi_f^\rho}(e^{ts} f_0^\rho)) - \lambda_{\text{inf}} \langle \xi_f^\rho, s \rangle. \end{aligned}$$

Now keep in mind that  $\xi_f^\rho$  and  $s$  commute so that

$$e^{st} f_0^\rho = e^{s^\perp t} (e^{\beta \xi_f^\rho t} f_0^\rho) = e^{s^\perp t} f_0^\rho.$$

we get

$$\begin{aligned} \lambda'(s, f_0^\rho) &= \lim_{t \rightarrow \infty} \mu^{-is^\perp}(e^{ts^\perp} f_0^\rho) + \lim_{t \rightarrow \infty} \mu^{-i\beta \xi_f^\rho}(e^{ts} f_0^\rho) - \lambda_{\text{inf}} \langle \xi_f^\rho, s \rangle \\ &= \lim_{t \rightarrow \infty} \mu'^{-is^\perp}(e^{ts^\perp} f_0^\rho) + \lim_{t \rightarrow \infty} \mu^{-i\beta \xi_f^\rho}(e^{ts} f_0^\rho) - \lambda_{\text{inf}} \beta \\ &= \lambda'(s^\perp, f_0^\rho) + \beta \lim_{t \rightarrow \infty} \mu^{-i\xi_f^\rho}(e^{ts} f_0^\rho) - \lambda_{\text{inf}} \beta. \end{aligned}$$

Note that

$$\mu^{-i\xi_f^\rho}(e^{ts} f_0^\rho) = \mu^{-i\xi_f^\rho}(f_0^\rho) + \int_0^t \frac{d}{d\tau} \mu^{-i\xi_f^\rho}(e^{\tau s} f_0^\rho) d\tau.$$

Using the definition of the moment map, we get

$$\frac{d}{d\tau} \mu^{-i\xi_f^\rho}(e^{\tau s} f_0^\rho) = d(\mu^{-i\xi_f^\rho})(\mathbf{v}_\tau) = \omega_h(-i\xi_f^{\rho\sharp}, \mathbf{v}_\tau) = h(\xi_f^{\rho\sharp}, \mathbf{v}_\tau)$$

where  $\mathbf{v}_\tau$  is the speed vector along the curve  $c_{f_0^\rho}^s$ . But the vector field  $\xi_f^{\rho\sharp}$  vanishes identically along the curve  $c_{f_0^\rho}^s$ , because

$$e^{t\xi_f^\rho}(e^{\tau s} f_0^\rho) = e^{\tau s}(e^{t\xi_f^\rho} f_0^\rho) = e^{\tau s} f_0^\rho$$

so that each point  $c_{f_0^\rho}^s(\tau)$  of the curve is a fixed point of the flow of the vector field  $\xi_f^{\rho\sharp}$ . We get  $h(\xi_f^{\rho\sharp}, \mathbf{v}_\tau) = 0$  and  $\mu^{-i\xi_f^\rho}(e^{ts} f_0^\rho) = \mu^{-i\xi_f^\rho}(f_0^\rho) = \lambda_{\text{inf}}$ . The above formula shows that

$$\lambda'(s, f_0^\rho) = \lambda'(s^\perp, f_0^\rho).$$

$\square$

COROLLARY 5.3. — *The subsets  $\mathcal{Z}_S$  are  $G(\mathcal{S})$ -invariant and there is a natural quotient map  $\mathcal{Z}_S \rightarrow Q_{\sigma_S}$  where  $Q_{\sigma_S}$  denotes the Hamiltonian quotient associated to the factorization problem  $(\mathcal{V}(\mathcal{S}), G(\mathcal{S})/U(\mathcal{S}))$  and to the symplectization  $\alpha_S$ .*

*Proof.* — The invariance is a direct consequence of the ad-invariance properties of  $\lambda$  (see prop. 2.8).  $\square$

To get a  $G$  invariant stratification we have to glue these subsets together in the following way:  $H(G)$  is  $\text{ad}_G$  invariant and we denote by  $\Sigma_{\text{ad}}$  the set of all orbits for this action. Then for any non trivial orbit  $\delta \in \Sigma_{\text{ad}}$ , we define

$$\mathcal{X}_\delta := \{f \in F \mid f \text{ non-semistable, } \mathcal{S}_f \subset \delta\} = \coprod_{S \subset \delta} \mathcal{Z}_S.$$

For  $\delta = \{0\}$ , we put  $\mathcal{X}_{\{0\}} = F^{ss}$ . Clearly the  $\mathcal{X}_\delta$  are disjoint  $G$ -invariant subsets such that

$$F = \coprod_{\delta} \mathcal{X}_\delta.$$

For any  $\mathcal{S}, \mathcal{S}'$  in the same class  $\delta \in \Sigma_{\text{ad}}$ , we may define an isomorphism between the manifolds  $\mathcal{V}(\mathcal{S})$  and  $\mathcal{V}(\mathcal{S}')$  by choosing suitable representatives  $V_s$  and  $V_{s'}$ . This gives an isomorphism between the Hamiltonian quotients  $Q_{\sigma_S}$  and  $Q_{\sigma_{S'}}$ .

We have proved:

THEOREM 5.4. — *Let  $(F, G, \alpha)$  be a general factorization problem with an energy-complete symplectization  $\sigma$ . Then we may define a stratification*

$$F = \coprod_{\delta \in \Sigma_{\text{ad}}} \mathcal{X}_\delta$$

by  $G$ -invariant subsets defined by:

- $\mathcal{X}_{\{0\}}$  consists of the subset  $F^{ss}$  of  $\sigma$ -semistable elements;
- for a non trivial class  $\delta$ , the stratum  $\mathcal{X}_\delta$  is a disjoint union

$$\mathcal{X}_\delta = \coprod_{\substack{S \in (H(G)/\sim) \\ S \subset \delta}} \mathcal{Z}_S$$

where

$$\mathcal{Z}_S = \{f \in F \mid f \text{ is non } \sigma\text{-semistable and } \Lambda_f = S\}.$$



We have natural quotient maps  $Z_S \rightarrow \mathcal{Q}_{\sigma_S}$ , where  $\mathcal{Q}_{\sigma_S}$  is the Hamiltonian quotient associated to the factorization problem  $(\mathcal{V}_S, G(\mathcal{S})/U(\mathcal{S}), \alpha_S)$  and to the symplectization  $\alpha_S$ .

For any  $Z_S, Z_{S'}$  in  $\mathcal{X}_\delta$ , the Hamiltonian quotient  $\mathcal{Q}_{\sigma_S}$  and  $\mathcal{Q}_{\sigma_{S'}}$  are isomorphic.

As we will see in the last section, for the examples we have computed, it remains that there are only a finite number of classes in  $\Sigma_{\text{ad}}$  which may correspond to the class of an optimal destabilizing element, so that the number of stratum is finite. We believe that, at least for a large class of actions, this is the general behavior.

### 6. Linear actions.

We focus here our attention on linear actions. This is a special case of the previous chapter. In this case, it is possible to be more accurate concerning the definition of the associated factorization problem. Indeed, it can be built as a quotient vector subspace. Moreover the induced action is much more understandable.

So, let  $\rho : G \rightarrow GL(V)$  be a linear action of a reductive group  $G$  on a finite dimensional vector space  $V$ .

Fix a maximal compact subgroup  $K$  of  $G$  and an  $\text{ad}_G$ -invariant inner product of real type on  $\mathfrak{g}$ . If  $h$  is a  $K$ -invariant Hermitian inner product on  $V$ , one has a standard moment map for the  $K$  action which is given by

$$\mu_0(v) = \rho^*\left(-\frac{i}{2}v \otimes v^*\right)$$

and any other moment map has the form

$$\mu_\tau = \mu_0 - i\tau$$

with  $\tau \in iz(\mathfrak{k})$ . So we get a symplectization  $\sigma = (K, h, \mu_\tau)$  for the  $\rho$  action. Let us remark that in the case of a linear action, the symplectization is always energy-complete and thus produces a well defined weight map  $\lambda^\tau : H(G) \rightarrow \mathbb{R} \cup \{\infty\}$ .

Now, for each  $\xi \in i\mathfrak{k}$ , we can decompose  $V$  into eigenspaces  $V = \bigoplus_{i=1}^k V_i$  where  $\rho_*(\xi)|_{V_i} = \xi_i id_{V_i}$  and  $\xi_i$  are the distinct eigenvalues of  $\xi$ . Now we have a very simple expression for  $\lambda^\tau(\xi, v)$ : put

$$V_\xi^\pm := \bigoplus_{\pm \xi_i > 0} V_i, V_\pm^\xi := \bigoplus_{\pm \xi_i \geq 0} V_i.$$

Any  $v \in V$  decomposes as  $v = \sum_{i=1}^k v_i$  with  $v_i \in V_i$ . Then, we can compute the map  $\lambda$  in the following way :

$$\lambda^\tau(v, \xi) := \lim_{t \rightarrow +\infty} \langle \mu_\tau(\rho(e^{t\xi})v), -i\xi \rangle = \begin{cases} +\infty & \text{if } \exists i \text{ s.t. } \xi_i > 0 \text{ and } v_i \neq 0; \\ \langle \tau, \xi \rangle & \text{otherwise.} \end{cases}$$

Let  $\mathcal{S}$  be a non trivial equivalence class of normalized Hermitian type elements and let  $\xi \in \mathcal{S} \cap i\mathfrak{k}$  with  $\langle \tau, \xi \rangle < 0$ . Then

$$\mathcal{Z}_\mathcal{S} = \left\{ v \in V \mid v \in V_-^\xi \text{ and } \langle \tau, \xi \rangle = \min_{\substack{\zeta \in i\mathfrak{k}, \|\zeta\|=1 \\ v \in V_-^\zeta}} \langle \tau, \zeta \rangle \right\}.$$

The complex manifold associated to  $\mathcal{S}$  is the complex space

$$\mathcal{V}(\mathcal{S}) = V_-^\xi / V_\xi^-.$$

Let us remark that this vector space comes with a natural action of  $G(\mathcal{S}) = G(\xi)$  since this parabolic subgroup leaves the flag  $V_\xi^- \subset V_-^\xi$  invariant and that  $U(\xi)$  acts trivially on the quotient. So we get a well defined action  $\alpha_\mathcal{S}$  of  $G(\mathcal{S})/U(\mathcal{S})$  over  $V_-^\xi/V_\xi^-$ .

We may take as a representative for the symplectization  $\sigma_\mathcal{S}$  introduced above (see. 3.2) the triple

$$(K \cap Z(\xi), h_{|(V_\xi^-)^{\perp h}}, \mu' = i^* \mu_{|(V_\xi^-)^{\perp h}} - \langle \tau, \xi \rangle \langle i\xi, \cdot \rangle)$$

where  $(V_\xi^-)^{\perp h}$  denotes the orthogonal of  $V_\xi^-$  in  $V_-^\xi$ .

Let  $v \in \mathcal{Z}_\mathcal{S}$ , and let  $v_0$  be the projection onto  $V_-^\xi/V_\xi^-$ .

In this framework, our general result 5.1 becomes

PROPOSITION 6.1. — *The vector  $v_0 \in V_-^\xi/V_\xi^-$  is  $\sigma_\mathcal{S}$ -semistable.*

We give below a simple self-contained proof of this result.

*Proof.* — Denote by  $\lambda'$  the map associated to the symplectization  $\sigma_\mathcal{S}$ . Let  $s \in i\mathfrak{k} \cap \mathfrak{z}(\xi)$ , then,  $s$  admits an orthogonal decomposition as  $s = \beta\xi + s^\perp$ .

Assume that  $\lambda'(s, v_0) < 0$  so that  $v_0 \in \mathcal{V}(\mathcal{S})_-^s$ . Using the fact that  $s$  and  $\xi$  commute and so are simultaneously diagonalizable we get  $v_0 \in \mathcal{V}(\mathcal{S})_-^{s^\perp}$  and:

$$\begin{aligned} \lambda'(s, v_0) &= \langle \tau, s \rangle - \langle \xi, s \rangle \lambda_{inf} \\ &= \langle \tau, s \rangle - \langle \xi, s \rangle \langle \tau, \xi \rangle \\ &= \langle \tau, s^\perp \rangle \\ &= \lambda'(s^\perp, v_0). \end{aligned}$$

Let  $\xi_\varepsilon = \xi + \varepsilon s^\perp$  for  $\varepsilon > 0$ . Using again the fact that  $\xi$  and  $s^\perp$  are simultaneously diagonalizable, it is easy to see that for  $\varepsilon$  small enough  $v \in V_-^{\xi_\varepsilon}$ . Then we get

$$\begin{aligned} \lambda\left(\frac{\xi_\varepsilon}{\|\xi_\varepsilon\|}, v\right) &= \frac{\lambda(\xi_\varepsilon, v)}{\|\xi_\varepsilon\|} \\ &= \frac{\langle \tau, \xi \rangle + \varepsilon \langle \tau, s^\perp \rangle}{\|\xi + \varepsilon s^\perp\|}. \end{aligned}$$

Now we get

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(\lambda(\frac{\xi_\varepsilon}{\|\xi_\varepsilon\|}, f)) = \langle \tau, s^\perp \rangle - \langle \tau, \xi \rangle \langle s^\perp, \xi \rangle = \lambda'(s, v_0) < 0.$$

Thus, by taking  $\varepsilon$  small enough, we get a normalized element  $\frac{\xi_\varepsilon}{\|\xi_\varepsilon\|} \in H(G)$  with  $\lambda(\frac{\xi_\varepsilon}{\|\xi_\varepsilon\|}, v) < \lambda_{\inf}$  which is a contradiction.  $\square$

We retrieve here the natural quotient maps  $\mathcal{V}(\mathcal{S}) \rightarrow \mathcal{Q}_{\sigma_s}$  defined in section 5. We give in the last section examples of such linear actions, associated stratifications and quotients maps.

Let us consider the example of a linear action  $\rho$  of a complex torus  $T$  over a complex vector space  $V$ .

*We aim to show here that even in this very simple case, the optimal destabilizing vector  $\xi$  may not be algebraic (that is may not lie in the subset  $OPS(G) \subset \mathfrak{g}$ ).*

We have a decomposition

$$V = \bigoplus_{\chi \in R} V_\chi,$$

where  $R \subset \text{Hom}(T, S^1)$  and  $\rho(t)|_{V_\chi} = \chi(t) \text{id}_{V_\chi}$  for all  $t \in T$ .

The following picture explains geometrically how we find the optimal destabilizing vector associated to a symplectization  $\sigma = (K, h, \mu_\tau)$  and to a nonsemistable vector  $v$ :

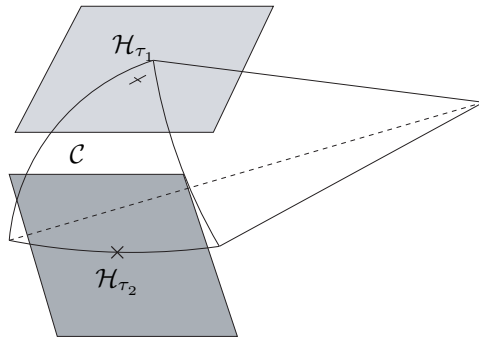
Using the expression of the weight map  $\lambda^\tau$ , we see that this optimal vector as to be search in the subset

$$\mathcal{C} = \{s \in \mathfrak{g} \mid d_e \chi(s) \leq 0, \forall \chi \in R \text{ s.t. } v_\chi \neq 0\} \cap \{s \in \mathfrak{g} \mid \|s\| = 1\}.$$

The two planes  $\mathcal{H}_{\tau_1}$  and  $\mathcal{H}_{\tau_2}$  in the picture represents the hyperplanes defined by the equation

$$\langle \tau, s \rangle = \lambda_{\inf}^\tau$$

for two distinct values  $\tau_1, \tau_2 \in \mathfrak{t}$ .



As  $\tau$  can be chosen freely in  $\mathfrak{z}(\mathfrak{t}) = \mathfrak{t}$ , the optimal vector may be reached at any point of  $\mathcal{C}$ .

### 7. Linear examples.

#### 7.1. Non-semistable points in the factorization problems which yield the Grassmannians.

Let  $V, V_0$  be two Hermitian vector spaces of dimensions  $r = \dim(V)$ ,  $r_0 := \dim(V_0)$ . Consider the natural action  $\alpha_{\text{can}}$  of  $GL(V)$  on the space of linear morphisms  $F := \text{Hom}(V, V_0)$ , given by  $(u, f) \mapsto f \circ u^{-1}$ . A moment map for the restricted  $U(V)$ -action has the form

$$\mu_t(f) = \frac{i}{2} f^* \circ f - it \text{id}_V, t \in \mathbb{R},$$

and the corresponding Hamiltonian quotients of  $F$  are

$$Q_{\mu_t}^F = \begin{cases} \mathbb{G}r_r(V_0) & \text{if } t > 0 \\ \{*\} & \text{if } t = 0 \\ \emptyset & \text{if } t < 0. \end{cases}$$

Fix  $t > 0$ . With respect to the moment map  $\mu_t$  a point  $f \in F$  is not semistable if and only if  $\ker f \neq \{0\}$ . In this case, and an element  $s \in iu(V)$  destabilizes  $f$  if and only if the following two conditions are satisfied

- $V_s^- \subset \ker(f)$ , where  $V_s^- := \bigoplus_{\substack{\lambda \in \text{Spec}(s) \\ \lambda < 0}} V_\lambda$ ,
- $\lambda^s(f) = t \text{Tr}(s) < 0$ .

This shows that the unique normalized optimal destabilizing element of  $iu(V)$  is  $s_f := -\frac{1}{\sqrt{\dim(\ker(f))}} \text{Pr}_{\ker(f)}$ .

Let  $\mathcal{S} \in H(\text{gl}(V))/\sim$  be the equivalence class of  $s_f$ . The vector space  $V_{\mathcal{S}}^- = V_{s_f}^-$  depends only of  $\mathcal{S}$  and the set  $\mathcal{Z}_{\mathcal{S}}$  is given by

$$\mathcal{Z}_{\mathcal{S}} = \{u \in \text{Hom}(V, V_0) \mid \ker(u) = V_{\mathcal{S}}^-\}$$

The canonically associated manifold  $\mathcal{V}(\mathcal{S})$  provided by Theorem 5.4 is

$$\mathcal{V}(\mathcal{S}) := F/\{u \in \text{Hom}(V, V_0) \mid u|_{V_{\mathcal{S}}^-} = 0\} = \text{Hom}(V/V_{\mathcal{S}}^-, V_0).$$

whereas the reductive quotient  $G(\mathcal{S})/U(\mathcal{S})$  is the product

$$G_{\mathcal{S}} := GL(V_{\mathcal{S}}^-) \times GL(V/V_{\mathcal{S}}^-).$$

The reductive group  $G_{\mathcal{S}}$  acts on  $\mathcal{V}(\mathcal{S})$  in the obvious way such that the first factor of  $G_{\mathcal{S}}$  operates trivially.

The moment map  $\mu'$  associated with this new action (see 3.2), is

$$\mu'_t : \mathcal{V}(\mathcal{S}) \rightarrow u(V_{\mathcal{S}}^-) \oplus u(V/V_{\mathcal{S}}^-)$$

given by

$$\mu'_t(\varphi) = (0, \frac{i}{2}\varphi^* \circ \varphi - it \text{id}_{V/V_{\mathcal{S}}^-})$$

and the quotient  $\mathcal{Q}_{\sigma_{\mathcal{S}}}$  is just the Grassmannian  $\mathbb{G}r_{r-\dim(V_{\mathcal{S}}^-)}(V_0)$ .

Therefore, applying our general result to the factorization problem

$$(\text{Hom}(V, V_0), GL(V), \alpha_{\text{can}})$$

with the symplectization defined by  $\mu_t, t > 0$ , one gets the stratification

$$\text{Hom}(V, V_0) = \coprod_{\rho \leq r} \text{Hom}(V, V_0)_{\rho}$$

with

$$\begin{aligned} \text{Hom}(V, V_0)_{\rho} &:= \{f \in \text{Hom}(V, V_0) \mid \text{rk}(f) = \rho\} \\ &= \coprod_{\dim W = r-\rho} \{f \in \text{Hom}(V, V_0) \mid \ker(f) = W\} \end{aligned}$$

of  $\text{Hom}(V, V_0)$  and the natural quotient maps

$$\text{Hom}(V, V_0)_{\rho} \longrightarrow \mathbb{G}r_{\rho}(V_0)$$

on the strata. This is the Shatz stratification of the factorization problem  $(\text{Hom}(V, V_0), GL(V), \alpha_{\text{can}})$ .

**7.2. Non-semistable points in the factorization problems which yield the flag manifolds.**

Let  $V_1, \dots, V_m, V = V_{m+1}$  be Hermitian vector spaces. Put

$$d_i := \dim(V_i), d := \dim(V), F := \bigoplus_{i=1}^m \text{Hom}(V_i, V_{i+1}), K := \prod_{i=1}^m \text{U}(V_i),$$

and consider the  $K$ -action  $\alpha_{\text{can}}$  on  $F$  given by

$$\alpha_{\text{can}}(g_1, \dots, g_m)(f_1, \dots, f_m) = (g_2 \circ f_1 \circ g_1^{-1}, \dots, g_m \circ f_{m-1} \circ g_{m-1}^{-1}, f_m \circ g_m^{-1}).$$

The general form of a moment map for the restricted  $K$ -action on  $F$  is

$$\mu_t(f_1, \dots, f_m) = \frac{i}{2} \begin{pmatrix} f_1^* \circ f_1 \\ f_2^* \circ f_2 - f_1 \circ f_1^* \\ \dots \\ f_m^* \circ f_m - f_{m-1} \circ f_{m-1}^* \end{pmatrix} - i \begin{pmatrix} t_1 \text{id}_{V_1} \\ t_2 \text{id}_{V_2} \\ \dots \\ t_m \text{id}_{V_m} \end{pmatrix}$$

where  $t \in \mathbb{R}^m$ . To every  $f = (f_1, \dots, f_m) \in F$  we associate the subspaces

$$W_i(f) := (f_m \circ \dots \circ f_i)(V_i) \subset V, 1 \leq i \leq m.$$

One obviously has  $W_i \subset W_{i+1}$  and the map

$$f \mapsto (W_i(f))_{1 \leq i \leq m}$$

is constant on orbits. We refer to [17] for the following simple result

PROPOSITION 7.1. — *Suppose that  $t_i > 0$ , for all  $1 \leq i \leq m$ .*

(1) *Let  $f \in F$ . Then the following conditions are equivalent:*

- (a)  *$f$  is  $\mu_t$ -semistable*
- (b)  *$f$  is  $\mu_t$ -stable*
- (c) *all maps  $f_i$  are injective.*

(2) *The map*

$$w : f \mapsto (W_i(f))_{1 \leq i \leq m}$$

*identifies the Hamiltonian quotient  $Q_{\mu_t}^F$  with the flag manifold*

$$\mathbb{F}_{d_1, \dots, d_m}(V) := \{(W_1, \dots, W_m) \mid W_1 \subset \dots \subset W_m \subset V, \dim(W_i) = d_i\}.$$

Fix  $t = (t_1, \dots, t_m) \in \mathbb{R}_{>0}^m$ . We assume  $d_1 \leq d_2 \leq \dots \leq d_m$ , which insures that  $\mathbb{F}_{d_1, \dots, d_m}(V)$  is non-empty. We do not require strict inequalities; when some of the  $d_i$ -s coincide, the corresponding flag manifold  $\mathbb{F}_{d_1, \dots, d_m}(V)$  can be identified with a flag manifold associated with a smaller

$m$ . More precisely  $\mathbb{F}_{d_1, \dots, d_m}(V) \simeq \mathbb{F}_{d_{i_1}, \dots, d_{i_k}}(V)$  if  $i_1 < i_k < \dots < i_k$  and  $\{d_1, \dots, d_m\} = \{d_{i_1}, \dots, d_{i_k}\}$ .

Suppose that  $t_i > 0$ , for all  $1 \leq i \leq m$ , let  $f = (f_1, \dots, f_m)$  be a non-semistable point with respect to  $\mu_t$  and denote by  $\mathcal{S}$  the class of its optimal destabilizing element. The associated manifold  $\mathcal{V}(\mathcal{S})$  is

$$\mathcal{V}(\mathcal{S}) := \bigoplus_{i=1}^m \text{Hom}(V_i^{\mathcal{S}}, V_{i+1}^{\mathcal{S}}),$$

where  $V_{m+1}^{\mathcal{S}} = V$  and  $V_i^{\mathcal{S}} := V_i / E_i^{\mathcal{S}}$  with  $E_i^{\mathcal{S}} := \ker(f_m \circ \dots \circ f_i)$  (this does not depend of the choice of  $f \in \mathcal{Z}_{\mathcal{S}}$ ).

The reductive group  $G_{\mathcal{S}}$  associated with  $\mathcal{S}$  is the product

$$G_{\mathcal{S}} := \prod_{i=1}^m GL(E_i^{\mathcal{S}}) \times \prod_{i=1}^m GL(V_i^{\mathcal{S}})$$

and the first factor operates trivially. We put  $\bar{G}_{\mathcal{S}} := \prod_{i=1}^m GL(E_i^{\mathcal{S}})$ .

The point  $f_0$  of  $\mathcal{V}(\mathcal{S})$  associated with the non-semistable point  $f$  is just  $f_0 = (\bar{f}_1, \dots, \bar{f}_m)$ , where  $\bar{f}_i \in \text{Hom}(V_i^{\mathcal{S}}, V_{i+1}^{\mathcal{S}})$  is induced by  $f_i$ . It is easy to see that  $\bar{f}_i$  is injective, so the system  $f_0$  defines indeed a  $(t_1, \dots, t_m)$ -stable point with respect to the  $\bar{G}_{\mathcal{S}}$ -action on  $\mathcal{V}(\mathcal{S})$ . The corresponding point in the  $\bar{G}_{\mathcal{S}}$ -quotient of  $\mathcal{V}(\mathcal{S})$  is just  $(W_1(f), \dots, W_m(f)) \in \mathbb{F}_{\bar{d}_1, \dots, \bar{d}_k}(V)$ , where  $\bar{d}_i := \text{rk}(f_m \circ \dots \circ f_i)$ .

Therefore, our general result applied to the factorization problem

$$\left( \bigoplus_{i=1}^m \text{Hom}(V_i, V_{i+1}), \prod_{i=1}^m GL(V_i), \alpha_{\text{can}} \right)$$

with the symplectization defined by  $\mu_t$  yields the natural rank-stratification

$$\bigoplus_{i=1}^m \text{Hom}(V_i, V_{i+1}) = \prod_{\substack{(\rho_1, \dots, \rho_m) \\ \rho_1 \leq \dots \leq \rho_m \\ 0 \leq \rho_i \leq d_i}} F_{\rho_1, \dots, \rho_m},$$

of  $\bigoplus_{i=1}^m \text{Hom}(V_i, V_{i+1})$ . The Shatz strata are

$$\begin{aligned} F_{\rho_1, \dots, \rho_m} &:= \left\{ (f_1, \dots, f_m) \in \bigoplus_{i=1}^m \text{Hom}(V_i, V_{i+1}) \mid \text{rk}(f_m \circ \dots \circ f_i) = \rho_i \right\} \\ &= \prod_{\substack{(E_1, \dots, E_m) \\ \dim(E_i) = d_i - \rho_i}} \left\{ (f_1, \dots, f_m) \in \text{Hom}(V_i, V_{i+1}) \mid \ker(f_m \circ \dots \circ f_i) = E_i \right\}. \end{aligned}$$

The natural quotient maps provided by our general construction are just the obvious maps  $F_{\rho_1, \dots, \rho_m} \rightarrow \mathbb{F}_{\rho_1, \dots, \rho_m}(V)$ .

## 8. Optimal destabilizing vectors in Gauge theory.

In order to avoid the complications related to singular sheaves, we will treat here the case when the base manifold is a complex curve  $Y$ . Another reason for choosing this framework is the following: the natural Hamiltonian action of the complex gauge group on the configuration space associated with a *linear* moduli problem on a complex curve is *formally* energy complete, so it is natural to expect that all our results above can be easily generalized to this infinite dimensional framework.

### 8.1. Holomorphic fibre bundles.

Let  $E$  be a complex vector bundle of rang  $r$  over the Hermitian curve  $(Y, g)$ . We denote by  $\mathcal{G}$  the complex gauge group  $\mathcal{G} := \text{Aut}(E)$ . Its formal Lie algebra is  $A^0(\text{End}(E))$ .

The groups which play the role of the maximal compact subgroups in our gauge theoretical framework are the subgroups of the form

$$\mathcal{K}_h := U(E, h) \subset \mathcal{G},$$

where  $U(E, h)$  stands for the group of unitary automorphisms of  $E$  with respect to a Hermitian structure  $h$  on  $E$ .

Following our general terminology developed in the finite dimensional case, we will say that an element  $s \in A^0(\text{End}(E))$  is of Hermitian type if there exists a Hermitian metric  $h$  on  $E$  such that  $s \in A^0(\text{Herm}(E, h))$ .

We are interested in the stability theory for the  $\mathcal{G}$ -action on the space  $\mathcal{H}(E)$  of holomorphic structures (semiconnections) on  $E$  (see [12]). Fixing a Hermitian metric  $h$ , our moment map for the  $\mathcal{K}_h$ -action on  $\mathcal{H}(E)$  has the form

$$\mu(\mathcal{E}) = \Lambda_g(F_{\mathcal{E}, h}) + \frac{2\pi i}{\text{Vol}_g(Y)} \frac{\text{deg}(E)}{r} \text{id}_E.$$

One has an explicit formula for the maximal weight map  $\lambda$  in this case (see [15]).

We will need the following notation: If  $a$  is an endomorphism of a vector space  $V$ , and  $\lambda \in \mathbb{R}$ , we will put

$$V_a(\lambda) := \bigoplus_{\lambda' \leq \lambda} \text{Eig}(a, \lambda').$$



The notation extends for endomorphisms with constant eigenvalues on vector bundles in an obvious way.

If  $\mathcal{E} \in \mathcal{H}$  and  $s \in A^0(\text{Herm}(E, h))$ , then

$$\lambda^s(\mathcal{E}) = \begin{cases} \lambda_k \deg(\mathcal{E}) + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg(\mathcal{E}_i) - \frac{\deg(\mathcal{E})}{r} \text{Tr}(s) \\ \text{if the eigenvalues } \lambda_1 < \dots < \lambda_k \text{ of } s \text{ are constant and} \\ \mathcal{E}_i := \mathcal{E}_s(\lambda_i) \text{ are holomorphic} \\ \infty \text{ if not.} \end{cases}$$

Suppose that  $\mathcal{E}$  is not semistable. Let

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_k = \mathcal{E}$$

be the Harder-Narasimhan filtration of  $\mathcal{E}$  (see [6], [3] for the non-algebraic case). We recall that this filtration is characterized by the two conditions:

- The quotients  $\mathcal{E}_{i+1}/\mathcal{E}_i$  are torsion free and semistable.
- The slope sequence  $(\mu(\mathcal{E}_{i+1}/\mathcal{E}_i))_i$  is strictly decreasing.

Put  $r_i := \text{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$ . For any Hermitian metric  $h$  on  $E$  the optimal destabilizing element  $s \in A^0(\text{Herm}(E, h))$  is given by the formula

$$s = \frac{1}{\sqrt{\sum_{i=1}^k r_i \left[ \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{r_i} - \frac{\deg(\mathcal{E})}{r} \right]^2}} \sum_{i=1}^k \left[ \frac{\deg(\mathcal{E})}{r} - \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{r_i} \right] \text{id}_{F_i},$$

where  $F_i$  is the  $h$ -orthogonal complement of  $\mathcal{E}_{i-1}$  in  $\mathcal{E}_i$ .

It is not difficult to show that the holomorphic structure  $e^{st}(\mathcal{E})$  converges to the direct sum holomorphic structure  $\bigoplus_{i=1}^k \mathcal{E}_i/\mathcal{E}_{i-1}$  as  $t \rightarrow \infty$ . This holomorphic structure is indeed semistable with respect to the smaller gauge group  $\prod_{i=1}^k \text{Aut}(E_i/E_{i-1})$  and a suitable moment map.

The Shatz stratum of  $\mathcal{E}$  is the space of all holomorphic structures  $\mathcal{F} \in \mathcal{H}(E)$  whose Harder-Narasimhan filtration has the same topological type as the Harder-Narasimhan filtration of  $\mathcal{E}$ .

### 8.2. Holomorphic pairs.

Let  $\mathcal{F}_0$  be a fixed holomorphic bundle of rank  $r_0$  with a fixed Hermitian metric  $h_0$  and  $E$  a complex bundle of rank  $r$  on the Hermitian curve  $(Y, g)$ .

We are interested in the following classification problem: classify pairs  $(\mathcal{E}, \varphi)$ , where  $\mathcal{E}$  is a holomorphic structure on  $E$  and  $\varphi$  is a holomorphic morphism  $\varphi : \mathcal{F}_0 \rightarrow \mathcal{E}$ . Such a pair will be called a holomorphic pair of type  $(E, \mathcal{F}_0)$ , and we will denote by  $\mathcal{H}(E, \mathcal{F}_0)$  the space of such holomorphic pairs.

Our complex gauge group is  $\mathcal{G} := \text{Aut}(E)$  and the role of the maximal compact subgroups of  $\mathcal{G}$  are played by the groups  $\mathcal{K}_h := U(E, h)$  associated with Hermitian metrics on  $E$ .

For any Hermitian metric  $h$  on  $E$  the moment map for the  $\mathcal{K}_h$ -action on  $\mathcal{H}(E, \mathcal{F}_0)$  has the form:

$$\mu(\mathcal{E}, \varphi) = \Lambda_g F_{\mathcal{E}, h} - \frac{i}{2} \varphi \circ \varphi^* + \frac{i}{2} t \text{id}_E.$$

Suppose that the following topological condition holds:

$$\mu(\mathcal{E}) \leq \tau$$

(this is the obvious topological condition implied by the equation  $\mu(\mathcal{E}, \varphi) = 0$  when one integrates its trace over  $Y$ ).

It is well-known ([1]) that in this case a holomorphic pair  $(\mathcal{E}, \varphi)$  with  $\varphi \neq 0$  is semistable with respect to this moment map if and only if it is  $\tau := \frac{1}{4\pi} t \text{Vol}_g(Y)$ -semistable in the following sense:

- (1)  $\frac{\text{deg}(\mathcal{F})}{\text{rk}(\mathcal{F})} \leq \tau$  for all reflexive subsheaves  $\mathcal{F} \subset \mathcal{E}$  with  $0 < \text{rk}(\mathcal{F}) < r$ .
- (2)  $\frac{\text{deg}(\mathcal{E}/\mathcal{F})}{\text{rk}(\mathcal{E}/\mathcal{F})} \geq \tau$  for all reflexive subsheaves  $\mathcal{F} \subset \mathcal{E}$  with  $0 < \text{rk}(\mathcal{F}) < r$  and  $\varphi \in H^0(\text{Hom}(\mathcal{F}_0, \mathcal{F}))$ .

Note that in the case  $\mu(\mathcal{E}) = \tau$ , the  $\tau$ -semistability of the pair  $(\mathcal{E}, \varphi)$  is equivalent to the semistability of the bundle  $\mathcal{E}$ . Such a pair can be  $\tau$ -polystable only if  $\varphi = 0$ .

Using the same method as in the case of bundles one obtains the following analogue of the Harder-Narasimhan theorem (see [5]).

**THEOREM 8.1.** — *Let  $(\mathcal{E}, \varphi)$  be a non  $\tau$ -semistable holomorphic pair of type  $(E, \mathcal{F}_0)$  with  $\mu(\mathcal{E}) \leq \tau$ . Then there exists a unique holomorphic filtration with torsion free quotients*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m \subset \mathcal{E}_{m+1} \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

of  $\mathcal{E}$  such that:

(1) *The slopes sequence satisfies:*

$$\begin{aligned} \frac{\deg(\mathcal{E}_1/\mathcal{E}_0)}{\text{rk}(\mathcal{E}_1/\mathcal{E}_0)} &> \dots > \frac{\deg(\mathcal{E}_m/\mathcal{E}_{m-1})}{\text{rk}(\mathcal{E}_m/\mathcal{E}_{m-1})} > \tau \\ &\geq \frac{\deg(\mathcal{E}_{m+1}/\mathcal{E}_m)}{\text{rk}(\mathcal{E}_{m+1}/\mathcal{E}_m)} > \dots > \frac{\deg(\mathcal{E}_k/\mathcal{E}_{k-1})}{\text{rk}(\mathcal{E}_k/\mathcal{E}_{k-1})}. \end{aligned}$$

(2) *The quotients  $\mathcal{E}_{i+1}/\mathcal{E}_i$  are semistable for  $i \neq m$ .*

(3) *One of the following properties holds:*

- (a)  $\text{im}(\varphi) \not\subset \mathcal{E}_m, \tau > \frac{\deg(\mathcal{E}_{m+1}/\mathcal{E}_m)}{\text{rk}(\mathcal{E}_{m+1}/\mathcal{E}_m)}$  and the pair  $(\mathcal{E}_{m+1}/\mathcal{E}_m, \bar{\varphi})$  is  $\tau$ -semistable, where  $\bar{\varphi}$  is the  $\mathcal{E}_{m+1}/\mathcal{E}_m$ -valued morphism induced by  $\varphi$ .
- (b)  $\text{im}(\varphi) \not\subset \mathcal{E}_m, \tau = \frac{\deg(\mathcal{E}_{m+1}/\mathcal{E}_m)}{\text{rk}(\mathcal{E}_{m+1}/\mathcal{E}_m)}$  and  $\mathcal{E}_{m+1}/\mathcal{E}_m$  is semistable of slope  $\tau$ . This implies that the pair  $(\mathcal{E}_{m+1}/\mathcal{E}_m, \bar{\varphi})$  is  $\tau$ -semistable.
- (c)  $\text{im}(\varphi) \subset \mathcal{E}_m$  and  $\mathcal{E}_{m+1}/\mathcal{E}_m$  is semistable.

Moreover, in the cases (b) and (c) the obtained filtration coincides with the Harder-Narasimhan filtration of  $\mathcal{E}$ .

One can again give an explicit formula for the maximal weight function which corresponds to our gauge theoretical problem. The result is

$$\lambda^s(\mathcal{E}) = \begin{cases} \lambda_k \deg(\mathcal{E}) + \sum_{i=1}^{k-1} (\lambda_i - \lambda_{i+1}) \deg(\mathcal{E}_i) - \tau \text{Tr}(s) \\ \text{if the eigenvalues } \lambda_1 < \dots < \lambda_k \text{ of } s \text{ are constant, } \mathcal{E}_i := \mathcal{E}_s(\lambda_i) \\ \text{are holomorphic, and } \varphi \in H^0(\text{Hom}(\mathcal{F}_0, \mathcal{E}_s(0))). \\ \infty \text{ if not.} \end{cases}$$

Put again  $r_i := \text{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$ .

One can prove the following result:

**THEOREM 8.2.** — *For any Hermitian metric  $h$  on  $E$  the optimal destabilizing element  $s \in A^0(\text{Herm}(E, h))$  of the holomorphic pair  $(\mathcal{E}, \varphi)$  is given by*

(1) *If  $\text{im}(\varphi) \subset \mathcal{E}_m$  then*

$$s = \frac{1}{\sqrt{\sum_{i=1}^k r_i \left[ \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{r_i} - \tau \right]^2}} \sum_{i=1}^k \left[ \tau - \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{r_i} \right] \text{id}_{F_i}.$$

(2) If  $\text{im}(\varphi) \not\subset \mathcal{E}_m$

$$s = \frac{1}{\sqrt{\sum_{\substack{i=1 \\ i \neq m+1}}^k r_i \left[ \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{r_i} - \tau \right]^2}} \sum_{\substack{i=1 \\ i \neq m+1}}^k \left[ \tau - \frac{\deg(\mathcal{E}_i/\mathcal{E}_{i-1})}{r_i} \right] \text{id}_{F_i},$$

where  $F_i$  is the  $h$ -orthogonal complement of  $\mathcal{E}_{i-1}$  in  $\mathcal{E}_i$ .

Note that in the second case the  $(m+1)$ -th eigenvalue of  $s$  vanishes.

The two pictures below explain geometrically why the optimal destabilizing vector is given by different formulae in the two cases  $\text{im}(\varphi) \subset \mathcal{E}_m$ ,  $\text{im}(\varphi) \not\subset \mathcal{E}_m$ .

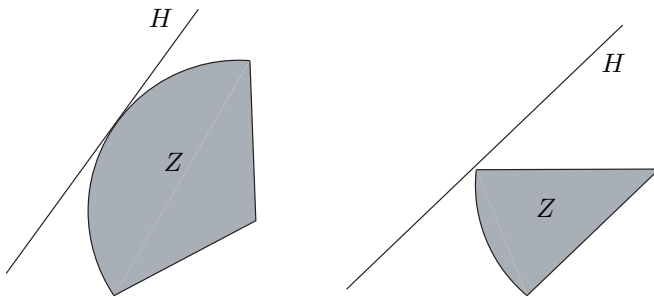
The gray region represents the set  $Z$  of those  $\zeta \in A^0(\text{Herm}(E, h))$  of norm less or equal than 1 with constant eigenvalues  $\zeta_1 < \zeta_2 < \dots < \zeta_k$  such that:

- the associated filtration  $(\mathcal{E}(\zeta_i))_i$  coincides with the filtration given by Theorem 8.1, and
- $\lambda^\zeta(\mathcal{E}, \varphi) < \infty$ .

This region is a convex subset  $Z$  of the space  $\mathbb{R}^k$ .

The second condition means that  $\zeta_j \leq 0$  for those  $j$  for which the projection of  $\varphi$  on  $F_j$  does not vanish.

The line in the two pictures represents the hyperplane  $H \subset \mathbb{R}^k$  given by the equation  $\zeta_k \deg(\mathcal{E}) + \sum_{i=1}^{k-1} (\zeta_i - \zeta_{i+1}) \deg(\mathcal{E}_i) - \tau \text{Tr}(s) = \lambda^{\min}(\mathcal{E}, \varphi)$ .



In the first case  $H$  touches  $Z$  in a smooth point of its boundary which belongs to the interior of the spherical region of this boundary, whereas in the second case  $H$  touches  $Z$  in a singular point of its boundary.

One can show that  $e^{st}(\mathcal{E}, \varphi)$  converges either to the object

$$(\mathcal{E}_1/\mathcal{E}_0, \dots, \mathcal{E}_m/\mathcal{E}_{m-1}, (\mathcal{E}_{m+1}/\mathcal{E}_m, \bar{\varphi}), \mathcal{E}_{m+2}/\mathcal{E}_{m+1}, \dots, \mathcal{E}_k/\mathcal{E}_{k-1}),$$

if  $\text{im}(\varphi) \not\subset \mathcal{E}_m$ , or to the object

$$(\mathcal{E}_1/\mathcal{E}_0, \dots, \mathcal{E}_m/\mathcal{E}_{m-1}, \mathcal{E}_{m+1}/\mathcal{E}_m, \mathcal{E}_{m+2}/\mathcal{E}_{m+1}, \dots, \mathcal{E}_k/\mathcal{E}_{k-1}),$$

if  $\text{im}(\varphi) \subset \mathcal{E}_m$ .

In both cases the limit object is semistable with respect to the gauge group  $\prod_{i=1}^k \text{Aut}(E_i/E_{i-1})$  and a suitable moment map.

Therefore our principle holds again: the optimal destabilizing vector of a non-semistable pair gives the generalized Harder-Narasimhan filtration and the limit object in the direction of this vector is a semistable object for a new moduli problem.

Theorem 8.1 allows one to define a Shatz stratification on the space of holomorphic pairs of type  $(E, \mathcal{F}_0)$ .

Details will appear in a forthcoming article [4].

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Manuscrit reçu le 20 novembre 2003,  
révisé le 11 octobre 2004,  
accepté le 23 novembre 2004.

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