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HOLOMORPHIC MORSE INEQUALITIES ON MANIFOLDS WITH BOUNDARY

by Robert BERMAN

1. Introduction.

Let X be a compact n-dimensional complex manifold with boundary. Let ρ be a defining function of the boundary of X, *i.e.* ρ is defined in a neighborhood of the boundary of X, vanishing on the boundary and negative on X. We take a Hermitian metric ω on X such that $d\rho$ is of unit-norm close to the boundary of X. The restriction of the two-form $i\partial\overline{\partial}\rho$ to the maximal complex subbundle $T^{1,0}(\partial X)$ of the tangent bundle of ∂X, is the Levi curvature form of the boundary ∂X. It will be denoted by $\mathcal{L}$. Furthermore, let L be a Hermitian holomorphic line bundle over X with fiber metric ϕ, so that $i\partial\overline{\partial}\phi$ is the curvature two-form of L. It will be denoted by Θ. The line bundle L is assumed to be smooth up to the boundary of X. Strictly speaking, ϕ is a collection of local functions. Namely, let s_i be a local holomorphic trivializing section of L, then locally, $|s_i(z)|^2 = e^{-\phi_i(z)}$. The notation $\eta_p := \eta^p/p!$ will be used in the sequel, so that the volume form on X may be written as ω_n.

When X is a compact manifold without boundary Demailly's (weak) holomorphic Morse inequalities [9] give asymptotic bounds on the dimension of the Dolbeault cohomology groups associated to the k :th tensor power of the line bundle L :

$$\dim_{\mathbb{C}} H^{0,q}(X, L^k) \leqslant k^n(-1)^q\Big(\frac{1}{2\pi}\Big)^n \int_{X(q)} \Theta_n + o(k^n), \tag{1.1}$$

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where $X(q)$ is the subset of X where the curvature-two form Θ has exactly q negative eigenvalues, *i.e.* the set where $\text{index}(\Theta) = q$. Demailly's inspiration came from Witten's analytical proof of the classical Morse inequalities for the Betti numbers of a real manifold [29], where the role of the fiber metric ϕ is played by a Morse function. Subsequently, holomorphic Morse inequalities on manifolds with boundary where studied. The cases of q-convex and q-concave boundary were studied by Bouche [7], and Marinescu [22], respectively, and they obtained the same curvature integral as in the case when X has no boundary. However, it was assumed that, close to the boundary, the curvature of the line bundle L is adapted to the curvature of the boundary. For example, on a pseudoconcave manifold (*i.e.* the Levi form is negative on the boundary) it is assumed that the curvature of L is non-positive close to the boundary. This is related to the well-known fact that in the global L^2-estimates for the $\overline{\partial}$-operator of Morrey-Kohn-Hörmander-Kodaira there is a curvature term from the line bundle as well from the boundary and, in general, it is difficult to control the sign of the total curvature contribution. Morse inequalities over strictly pseudoconvex CR manifolds have been obtained by Getzler [16], who also suggested that one should try to prove similar formulas for the $\overline{\partial}$-Neumann problem on a complex manifold with boundary. This will be done in the present paper.

We will consider an arbitrary holomorphic line bundle L over a complex manifold with boundary and extend Demailly's inequalities to this situation. We will write $h^q(L^k)$ for the dimension of $H^{0,q}(X, L^k)$, the Dolbeault cohomology group of $(0,q)$-forms with values in L^k. The cohomology groups are defined with respect to forms that are smooth up to the boundary. Recall that $X(q)$ is the subset of X where $\text{index}(\Theta) = q$ and we let

$$T(q)_{\rho,x} = \left\{ t > 0 : \text{index}(\Theta + t\mathcal{L}) = q \ \text{ along } \ T^{1,0}(\partial X)_x \right\}.$$

The main theorem we will prove is the following generalization of Demailly's weak holomorphic Morse inequalities.

Theorem 1.1. — *Suppose that X is a compact complex manifold with boundary, such that the Levi form is non-degenerate on the boundary. Then, up to terms of order $o(k^n)$,*

$$h^q(L^k) \leqslant k^n(-1)^q\Big(\frac{1}{2\pi}\Big)^n\Big(\int_{X(q)}\Theta_n + \int_{\partial X}\int_{T(q)_{\rho,x}}(\Theta + t\mathcal{L})_{n-1}\wedge\partial\rho\wedge dt\Big), \tag{1.2}$$

The boundary integral above may also be expressed more directly in terms of symplectic geometry as

$$\int_{X_+(q)} (\Theta + d\alpha)_n \tag{1.3}$$

where $(X_+, d\alpha)$ is the symplectification of the contact manifold ∂X induced by the complex structure of X (Section 7.1).

Examples will be presented that show that the leading constants in the bounds of the theorem are sharp. We will also obtain the corresponding generalization of the strong holomorphic Morse inequalities. The most interesting case is when the manifold is a strongly pseudoconcave manifold X of dimension $n \geqslant 3$ with a positive line bundle L. Then, if the curvature forms of L and ∂X are conformally equivalent along the complex tangential directions of ∂X, we will deduce that

$$h^0(L^k) = k^n\Big(\int_X \Theta_n + \frac{1}{n}\int_{\partial X} (i\partial\overline{\partial}\rho)_{n-1} \wedge i\partial\rho\Big) + o(k^n), \tag{1.4}$$

if the defining function ρ is chosen in an appropriate way. In particular, such a line bundle L is *big* and (1.4) can be expressed as

$$\mathrm{Vol}(L) = \mathrm{Vol}(X) + \frac{1}{n}\,\mathrm{Vol}(\partial X)$$

in terms of the corresponding symplectic volume of X and contact volume of ∂X. Examples are provided that show that Theorem 1.1 is sharp and also compatible with "hole filling".

The proof of Theorem 1.1 will follow from local estimates for the corresponding Bergman function $B_X^{q,k}$ where $B_X^{q,k}$ is the Bergman function of the space $\mathcal{H}^{0,q}(X, L^k)$ of $\overline{\partial}$-harmonic $(0,q)$-forms satisfying $\overline{\partial}$-Neumann boundary conditions (simply referred to as the harmonic forms in the sequel). The point is that the integral of the Bergman function is the dimension of $\mathcal{H}^{0,q}(X, L^k)$. It is shown that, for large k, the Bergman function (or more precisely the corresponding measure) is estimated by the sum of two model Bergman functions, giving rise to the bulk and the boundary integrals in Theorem 1.1. The model at a point x in the interior of X is obtained by replacing the manifold X with flat $\mathbb{C}^n$ and the line bundle L with the constant curvature line bundle over $\mathbb{C}^n$ obtained by freezing the curvature of the line bundle at the point x. Similarly, the model at a boundary point is obtained by replacing X with the unbounded domain X_0 in $\mathbb{C}^n$, whose constant Levi curvature is obtained by freezing the Levi curvature at the boundary point in X. The line bundle L is replaced by the constant curvature line bundle over X_0, obtained by freezing the curvature along the complex tangential directions, while making it flat in the complex normal direction.

The method of proof is an elaboration of the, comparatively elementary, technique introduced in [4] to handle Demailly's case of a manifold without boundary.

Remark 1.2. — The boundary integral in (1.2) is finite precisely when there is no point in the boundary where the Levi form $i\partial\overline{\partial}\rho$ has exactly q negative eigenvalues. Indeed, any sufficiently large t will then be in the complement of the set $T(q)_{\rho,x}$. Since, we have assumed that the Levi form $i\partial\overline{\partial}\rho$ is non-degenerate, this condition coincides with the so-called condition $Z(q)$ [15]. However, for an arbitrary Levi form the latter condition is slightly more general: it holds if the Levi form has at least $q+1$ negative eigenvalues or at least $n-q$ positive eigenvalues everywhere on ∂X. In fact, the proof of Theorem 1.1 only uses that ∂X satisfies condition $Z(q)$ and is hence slightly more general than stated. Furthermore, a function ρ is said to satisfy condition $Z(q)$ at a point x if x is not a critical point of ρ and if $\partial\overline{\partial}\rho$ satisfies the curvature condition at x along the level surface of ρ passing through x.

One final remark about the extension of the Morse inequalities to open manifolds:

Remark 1.3. — The cohomology groups $H^{0,*}(X, L^k)$ associated to the manifold with boundary X occurring in the weak Morse inequalities, Theorem 2.1, are defined with respect to forms that are smooth up to the boundary. Removing the boundary from X we get an open manifold, that we denote by $\dot{X}$. By the Dolbeault theorem [17] the usual Dolbeault cohomology groups $H^{0,*}(\dot{X}, L^k)$ of $\dot{X}$ are isomorphic to the cohomology groups $H^*(\dot{X}, \mathcal{O}(L^k))$ of the sheaf $\mathcal{O}(L^k)$ of germs of holomorphic sections on $\dot{X}$ with values in L^k. Moreover, if we assume that condition $Z(q)$ *and* $Z(q+1)$ hold then $H^{0,q}(X, L^k)$ and $H^{0,q}(\dot{X}, L^k)$ are isomorphic [15]. Furthermore, consider a given *open* manifold Y with a smooth exhaustion function ρ, *i.e.* a function such that the open sublevel sets of ρ are relatively compact in Y for every real number c. Then, if for a fixed regular value c_0, the curvature conditions $Z(q)$ and $Z(q+1)$ hold for ρ when $\rho \geqslant c_0$, the group $H^{0,q}(Y, L^k)$ is isomorphic to $H^{0,q}(X_{c_0}, L^k)$ [20], where X_{c_0} is the corresponding closed sublevel set of ρ. In this way one gets Morse inequalities on certain open manifolds Y.

Notation 1.4. — The notation $a_k \sim$ (resp. $\lesssim$) b_k will stand for $a_k =$ (resp. $\leqslant$) $C_k b_k$, where C_k tends to one when k tends to infinity. The $\overline{\partial}$-Laplacian [17] will be called just the Laplacian. It is the differential operator defined by $\Delta := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ (where $\overline{\partial}^*$ denotes the formal adjoint of $\overline{\partial}$) acting on smooth forms on X with values in L^k. Similarly, we will call an element in the kernel of Δ harmonic, instead of $\overline{\partial}$-harmonic.

The paper is organized in two parts. In the first part we will state and prove the *weak* holomorphic Morse inequalities. In the second part the *strong* holomorphic Morse inequalities are obtained. Finally, the weak Morse inequalities are shown to be sharp and the relation to hole filling investigated.

Part I. THE WEAK MORSE INEQUALITIES.

2. Setup and a sketch of the proof.

In the first part we will show how to obtain weak holomorphic Morse inequalities for $(0,q)$-forms, with values in a given line bundle L over a manifold with boundary X. In other words we will estimate the dimension of $H^{0,q}(X, L^k)$ in terms of the curvature of L and the Levi curvature of the boundary of X. With notation as in the introduction of the article the theorem we will prove is as follows.

THEOREM 2.1. — *Suppose that X is a compact complex manifold with boundary, such that the Levi form is non-degenerate on the boundary. Then, up to terms of order $o(k^n)$,*

$$h^q(L^k) \leqslant k^n(-1)^q\Big(\frac{1}{2\pi}\Big)^n\Big(\int_{X(q)}\Theta_n + \int_{\partial X}\int_{T(q)_{\rho,x}}(\Theta + t\mathcal{L})_{n-1}\wedge\partial\rho\wedge dt\Big),$$

Note that the last integral is independent of the choice of defining function. Indeed, if $\rho' = f\rho$ is another defining function, where f is a positive function, the change of variables $s = ft$ shows that the integral is unchanged. A more intrinsic formulation of the last integral will be given in Section 7.1. Let us now fix the grade q. Since, the statement of the theorem is vacuous if the Levi form $i\partial\overline{\partial}\rho$ has exactly q negative eigenvalues somewhere on ∂X (compare Remark 1.2) we may assume that this is not the case. Then it is well-known that the Dolbeault cohomology group $H^{0,q}(X, E)$ is finite dimensional for any given vector bundle E over X. The cohomology groups are defined using forms that are smooth up to the boundary. Moreover, the Hodge theorem, in this context, says that $H^{0,q}(X, E)$ is isomorphic to the space $\mathcal{H}^{0,q}(X, E)$, consisting of harmonic

$(0,q)$-forms, that are smooth up to the boundary, where they satisfy $\overline{\partial}$-Neumann boundary conditions [15]. The space $\mathcal{H}^{0,q}(X,E)$ is defined with respect to given metrics on X and E.

The starting point of the proof of Theorem 2.1 is the fact that the dimension of the space $\mathcal{H}^{0,q}(X,L^k)$ may be expressed as an integral over X of the so-called Bergman function $B_X^{q,k}$ defined as

$$B_X^{q,k}(x)=\sum_i \left|\Psi_i(x)\right|^2,$$

where $\{\Psi_i\}$ is any orthonormal base for $\mathcal{H}^{0,q}(X,L^k)$. Indeed, the integral of each term in the sum is equal to one. Note that the Bergman function $B_X^{q,k}$ depends on the metric ω on X. It is convenient to use $k\omega$ as metric for a given k. The point is that the volume of X measured with respect to $k\omega$ is of the order k^n. Hence, the dimension bound in Theorem 2.1 will follow from a point-wise estimate of the corresponding Bergman function $B_X^{q,k}$. Another reason why $k\omega$ is a natural metric on X is that, since $k\phi$ is the induced fiber metric on L^k, the norms of forms on X with values in L^k become more symmetrical with respect to the base and fiber metrics. In fact, we will have to let the metric ω itself depend on k (and on a large parameter R) close to the boundary and we will estimate the Bergman function of the space $\mathcal{H}^{0,q}(X,k\omega_k,L^k)$ in terms of model Bergman functions and compute the model cases explicitly. The sequence of metrics ω_k will be of the following form. First split the manifold X in an *inner region* X_ε, with defining function $\rho+\varepsilon$ and its complement, the *boundary region*, given a small positive number ε. The level sets where $\rho=-Rk^{-1}$ and $\rho=-k^{-1/2}$ divide the boundary region into three regions. The one that is closest to the boundary of X will be called the *first region* and so on. Next, define ω_T, the complex tangential part of ω close to the boundary by

$$\omega_T:=\omega-2i\partial\rho\wedge\overline{\partial}\rho$$

(recall that we assumed that $d\rho$ is of unit-norm with respect to ω close to the boundary of X). The metric ω_k is of the form

$$\omega_k=\omega_T+a_k(\rho)^{-1}2i\partial\rho\wedge\overline{\partial}\rho, \tag{2.1}$$

where the sequence of smooth functions a_k will be chosen so that, basically, the distance to the boundary, when measured with respect to $k\omega_k$, in the three different regions is independent of k. The properties of ω_k that we will use in the two regions will be stated in the proofs below, while the precise definition of ω_k is deferred to Section 5.4.

2.1. A sketch of the proof of the weak Morse inequalities.

To make the sketch of the proof cleaner, we will just show how to estimate the *extremal function*

$$S_X^{q,k}(x) = \sup_{\alpha_k} |\alpha_k(x)|^2 \tag{2.2}$$

closely related to $B_X^{q,k}$, where the supremum is taken over all normalized elements of the space $\mathcal{H}^{0,q}(X, \omega_k, L^k)$. When $q = 0$, *i.e.* the case of holomorphic sections, it is a classical fact that they are actually equal and the general relation is given in Section 3. Let us first see how to get the following bound in the inner region X_ε defined above:

$$S_X^{q,k}(x) \lesssim S^q_{\mathbb{C}^n,x}(0), \tag{2.3}$$

where the right hand side is the extremal function for the model case defined below. Moreover, the left hand side is uniformly bounded by a constant, which is essential when integrating the estimate to get an estimate on the dimension of $\mathcal{H}^{0,q}(X, k\omega_k, L^k)$. The proof of (2.3) proceeds exactly as in the case when X is a compact manifold without boundary [4]. Let us recall the argument, slightly reformulated. Fix a point x in X_ε. We may take local holomorphic coordinates centered at x and a local trivialization of L such that

$$\phi(z) = \sum_{i=1}^n \lambda_i z_i \bar{z}_i + \cdots, \qquad \omega(z) = \frac{i}{2} \sum_{i=1}^n dz_i \wedge \overline{dz}_i + \cdots \tag{2.4}$$

where the dots indicate lower order terms and the leading terms are called model metrics and denoted by ϕ_0 and ω_0, respectively. Hence, the model situation is a line bundle of constant curvature on flat $\mathbb{C}^n$. Note that the unit ball at x with respect to the metric $k\omega_k$ corresponds approximately to the coordinate ball at 0 of radius $k^{-1/2}$. To make this more precise, define a scaling map

$$F_k(z) = k^{-1/2} z$$

and consider a sequence of expanding balls centered at 0 in $\mathbb{C}^n$ of radius r_k, slowly exhausting all of $\mathbb{C}^n$. We will call $F_k^*(k\phi)$ and $F_k^*(k\omega)$ the scaled metrics on the expanding balls. The point is that they converge to the model metrics ϕ_0 and ω_0. This follows immediately from the expressions (2.4) and the fact that the model metrics are invariant when kF_k^* is applied. Next, given a $(0, q)$ form on X with values in L^k, we denote by $\alpha^{(k)}$ the scaled form defined by $\alpha^{(k)} = F_k^* \alpha_k$. Then, by the convergence of the scaled metrics,

$$\|\alpha_k\|^2_{F_k(B_{r_k})} \sim \left\|\alpha^{(k)}\right\|^2_{B_{r_k}}, \tag{2.5}$$

using the norms induced by the model metrics in the right hand side above. Now, if α_k is a normalized sequence of extremals (*i.e.* realizing the extremum in (2.2)) we have

$$S_X^{q,k}(x) = \left|\alpha^{(k)}(0)\right|^2.$$

By (2.5), the norms of the scaled sequence $\alpha^{(k)}$ are less than one, when k tends to infinity. Moreover, $\alpha^{(k)}$ is harmonic with respect to the scaled metrics and since these converge to model metrics, inner elliptic estimates for the Laplacian show that there is a subsequence of $\alpha^{(k)}$ that converges to a model harmonic form β in $\mathbb{C}^n$. In fact, we may assume that the whole sequence $\alpha^{(k)}$ converges. Hence,

$$\limsup_k \left|\alpha^{(k)}(0)\right|^2 = |\beta(0)|^2$$

which in turn is bounded by the model extremal function $S_{X_0,x}(0)$. Moreover, since X_ε may be covered by coordinate balls of radius $k^{-1/2}$, staying inside of X for large k, one actually gets a uniform bound.

Let us now move on to the boundary region $X - X_\varepsilon$ that we split into three regions as above. Fix a point σ in the boundary of X. We may take local coordinates centered at σ and orthonormal at σ, so that

$$\rho(z,w) = v - \sum_{i=1}^{n-1} \mu_i \left|z_i\right|^2 + \cdots$$

where v is the imaginary part of w [8]. The leading term of ρ will be denoted by ρ_0, and will be referred to as the defining function of the *model domain* X_0 in $\mathbb{C}^n$. Observe that the model domain X_0 is invariant under the holomorphic anisotropic scaling map

$$F_k(z,w) = (z/k^{1/2}, w/k).$$

Moreover, the scaled fiber metric on L^k now tends to the new model fiber metric $\phi_0(z,0)$, since the terms in ϕ involving the coordinate w are suppressed by the anisotropic scaling map F_k. Now, the bound (2.3) is replaced by

$$S_X^{q,k}(0, iv/k) \lesssim S(0, iv), \tag{2.6}$$

in terms of the new model case. To see this one replaces the balls of decreasing radii used before with $F_k(D_k)$ intersected with X, where D_k is a sequence of slowly expanding polydiscs. Moreover, we have to let the initial metric ω on X depend on k in the normal direction in order that the scaled metric converge to a non-degenerate model metric. In the first region we will essentially let

$$\omega_k = \omega_T + k2i\partial\rho \wedge \overline{\partial}\rho.$$

As a *model metric* in X_0 we will essentially use

$$\omega_0 = \frac{i}{2}\partial\overline{\partial}\left|z\right|^2 + 2i\partial\rho_0 \wedge \overline{\partial}\rho_0.$$

Then clearly

$$F_k^*(k\omega_k) = \omega_0 \tag{2.7}$$

in the model case and it also holds asymptotically in k, in the general case. Replacing the inner elliptic estimates used in the inner part X_ε with subelliptic estimates for the $\overline{\partial}$-Laplacian close to the boundary one gets the bound (2.6) more or less as before. Finally, using similar scaling arguments, one shows that the contribution from the second and third region to the total integral of $B_X^{q,k}$ is negligible when k tends to infinity.

This gives the bound

$$\int_X B_X^{q,k}(k\omega_k)_n \lesssim k^n\Big(\int_X B_{\mathbb{C}^n,x}^q\omega_n + \int_{\partial X}\int_{-\infty}^0 B_{X_0,\sigma}^q(iv)dvd\sigma\Big)$$

integrating over an infinite ray in the model region X_0 in the second integral (after letting R tend to infinity). Computing the model Bergman functions explicitly then finishes the proof of the theorem.

3. Bergman kernel forms.

Let us now turn to the detailed proof of Theorem 2.1. First we introduce Bergman kernel forms to relate the Bergman function $B_X^{q,k}$ to extremal functions taking account of the components of a form (see [5] for proofs). Let (ψ_i) be an orthonormal base for a finite dimensional Hilbert space $\mathcal{H}^{0,q}$ of $(0,q)$-forms with values in L. Denote by π_1 and π_2 the projections on the factors of $X \times X$. The *Bergman kernel form* of the Hilbert space $\mathcal{H}^{0,q}$ is defined by

$$\mathbb{K}^q(x,y) = \sum_i \bar{\psi}_i(x) \wedge \psi_i(y).$$

Hence, $\mathbb{K}^q(x,y)$ is a form on $X \times X$ with values in the pulled back line bundle $\bar{\pi}_1^*(L) \otimes \pi_2^*(L)$. For a fixed point x we identify $\mathbb{K}_x^q(y) := \mathbb{K}^q(x,y)$ with a $(0,q)$-form with values in $L \otimes \Lambda^{0,q}(X,\overline{L})_x$. The definition of $\mathbb{K}^q$ is made so that $\mathbb{K}^q$ satisfies the following reproducing property:

$$\alpha(x) = c_{n,q}\int_X \alpha \wedge \overline{\mathbb{K}}_x^q \wedge e^{-\phi}\omega_{n-q}, \tag{3.1}$$

for any element α in $\mathcal{H}^{0,q}$, using a suggestive notation and where $c_{n,q}$ is a complex number of unit norm that ensures that (3.1) may be interpreted as a scalar product. Properly speaking, $\alpha(x)$ is equal to the push forward $\pi_{2*}(c_{n,q}\alpha \wedge \mathbb{K}^q \wedge \omega_{n-q}e^{-\phi})(x)$. The restriction of $\mathbb{K}^q$ to the diagonal can be identified with a (q,q)-form on X with values in $\overline{L} \otimes L$. The *Bergman form* is defined as $\mathbb{K}^q(x,x)e^{-\phi(x)}$, *i.e.*

$$\mathbb{B}^q(x) = \sum_i \bar{\psi}_i(x) \wedge \psi_i(x)e^{-\phi(x)} \tag{3.2}$$

and it is a globally well-defined (q,q)-form on X. The following notation will turn out to be useful. For a given form α in $\Omega^{0,q}(X,L)$ and a decomposable form in $\Omega^{0,q}(X)_x$ of unit norm, let $\alpha_\theta(x)$ denote the element of $\Omega^{0,0}(X,L)_x$ defined as

$$\alpha_\theta(x) = \langle \alpha, \theta \rangle_x$$

where the product takes values in L_x. We call $\alpha_\theta(x)$ the *value of α at the point x, in the direction θ*. Similarly, let $B_\theta^q(x)$ denote the function obtained by replacing (3.2) by the sum of the squared pointwise norms of $\psi_{i,\theta}(x)$. Then $B_\theta^q(x)$ has the following useful extremal property:

$$B_\theta^q(x) = \sup_\alpha |\alpha_\theta(x)|^2 , \tag{3.3}$$

where the supremum is taken over all elements α in $\mathcal{H}^{0,q}$ of unit norm. The supremum will be denoted by $S_\theta^q(x)$ and an element α realizing the supremum will be referred to as an *extremal form for the space $\mathcal{H}^{0,q}$ at the point x, in the direction θ*. The reproducing formula (3.1) may now be written as

$$\alpha_\theta(x) = (\alpha, \mathbb{K}_{x,\theta}^q).$$

Finally, note that the Bergman function B is the trace of $\mathbb{B}^q$, *i.e.*

$$B^q\omega_n = c_{n,q}\mathbb{B}^q \wedge \omega_{n-q}.$$

Using the extremal characterization (3.3) we have the following useful expression for B :

$$B(x) = \sum_\theta S_\theta(x), \tag{3.4}$$

where the sum is taken over any orthonormal base of direction forms θ in $\Lambda^{0,q}(X)_x$.

4. The model boundary case.

In the sketch of the proof of the weak holomorphic Morse inequalities (Section 2.1) it was explained how to bound the Bergman function on X by model Bergman functions. In this section we will compute the Bergman kernel explicitly in the model boundary case. Consider $\mathbb{C}^n$ with coordinates (z, w), where z is in $\mathbb{C}^{n-1}$ and $w = u + iv$. Let X_0 be the domain with defining function

$$\rho_0(\mathrm{z}, w) = v + \psi_0(z) := v + \sum_{i=1}^{n-1} \mu_i \left|z_i\right|^2,$$

and with the metric

$$\omega_0 = \frac{i}{2}\partial\overline{\partial}\left|z\right|^2 + a(\rho)^{-1} 2i\partial\rho_0 \wedge \overline{\partial}\rho_0.$$

Note that the corresponding volume element $(\omega_0)_n$ is given by $a(\rho)^{-1}$ times the usual Euclidean volume element on $\mathbb{C}^n$. We will take $a(\rho_0)$ to be comparable to $(1-\rho_0)^2$ (compare Section 5.4) but we will only use that the corresponding metric ω_0 is "relatively complete" (compare Section 4.1). We fix the q and assume that condition $Z(q)$ holds on ∂X_0, *i.e.* that at least $q+1$ of the eigenvalues μ_i are negative or that at least $n-q$ of them are positive.

Let $\mathcal{H}^{0,q}(X_0, \phi_0)$ be the space of all $(0, q)$- forms on X that have finite L^2-norm with respect to the norms defined by the metric ω_0 and the weight $e^{-\phi_0(z)}$, where ϕ_0 is quadratic, and that are harmonic with respect to the corresponding Laplacian. Moreover, we assume the the forms are smooth up to the boundary of X_0 where they satisfy $\overline{\partial}$-Neumann boundary conditions (in fact, the regularity properties are automatic, since we have assumed that condition $Z(q)$ holds [15]). The Bergman kernel form of the Hilbert space $\mathcal{H}^{0,q}(X_0, \phi_0)$ will be denoted by $\mathbb{K}^q_{X_0}$. We will show how to expand $\mathbb{K}^q_{X_0}$ in terms of Bergman kernels on $\mathbb{C}^{n-1}$, and then compute it explicitly. Note that the metric ω_0 is chosen so that the pullback of any form on $\mathbb{C}^{n-1}$ satisfies $\overline{\partial}$-Neumann boundary conditions. Conversely, we will show that any form in $\mathcal{H}^{0,q}(X_0, \phi_0)$ can be written as a superposition of such pulled-back forms.

By the very definition of the metric ω_0, the forms dz_i and $a^{-1/2}\partial\rho_0$ together define an orthogonal frame of $(1, 0)$-forms. Any $(0, q)$-form α on X may now be uniquely decomposed in a *tangential* and a *normal* part:

$$\alpha = \alpha_T + \alpha_N,$$

where $\alpha = \alpha_T$ modulo the algebra generated by $\overline{\partial}\rho_0$. A form α without normal part will be called *tangential.* The proof of the following proposition is postponed till the end of the section.

Proposition 4.1. — *Suppose that α is in $\mathcal{H}^{0,q}(X_0,\phi_0)$. Then α is tangential, closed and coclosed (with respect to $\overline{\partial}$).*

By the previous proposition any form α in $\mathcal{H}^{0,q}(X_0,\phi_0)$ may be written as

$$\alpha(z,w) = \sum_I f_I d\bar{z}_I.$$

Moreover, since α is in $L^2(X_0,\phi_0)$ and $\overline{\partial}$- closed, the components f_I are in $L^2(X_0,\phi_0)$ and holomorphic in the w-variable. We will have use for the following basic lemma: (1)

Lemma 4.2. — *Let $m(v)$ be a positive function on $[0,\infty[$ with polynomial growth at infinity. If $f(w)$ is a a holomorphic function in $\{v < c\}$ with finite L^2-norm with respect to the measure $m(v)dudv$, then there exists a function $\widehat{f}(t)$ on $]0,\infty[$ such that*

$$f(w) = \int_0^\infty \widehat{f}(t)e^{-\frac{i}{2}wt}dt.$$

Moreover,

$$\int_{v<c}\int_{u=-\infty}^{\infty} |f(w)|^2\, m(v)dudv = 4\pi\int_{v<c}\int_{t=0}^{\infty} \left|\widehat{f}(t)\right|^2 e^{vt}m(v)dtdv. \tag{4.1}$$

We will call $\widehat{f}(t)$ the Fourier transform of $f(w)$. Now, fix z in $\mathbb{C}^{n-1}$ and take $c = -\psi_0(z)$ and $m(v) = a(\rho_0)^{-1} = a(v+\psi_0(z))^{-1}$. Then f_I, as a function of w, must satisfy the requirements in the lemma above for almost all z. Fixing such a z we write $\widehat{f}_{I,t}(z)$ for the function of t obtained by taking the Fourier transformation with respect to w. Hence, we can write

$$\alpha(z,w) = \int_0^\infty \widehat{\alpha}_t(z)e^{-\frac{i}{2}wt}dt \tag{4.2}$$

for almost all z, if we extend the Fourier transform and the integral to act on forms coefficient wise. Note that the equality (4.2) holds in $L^2(X_0,\phi_0)$. The following proposition describes the space $\mathcal{H}^{0,q}(X_0,\phi_0)$ in terms of the spaces $\mathcal{H}^{0,q}(\mathbb{C}^{n-1},t\psi_0+\phi_0)$, consisting of all harmonic $(0,q)$-forms in $L^2(\mathbb{C}^{n-1},t\psi_0+\phi_0)$ (with respect to the Euclidean metric in $\mathbb{C}^n$). The corresponding scalar products over $\mathbb{C}^{n-1}$ are denoted by $(\cdot,\cdot)_t$.

(1) This lemma can be reduced to the Payley-Wiener Theorem 19.2 in [25].

Proposition 4.3. — *Suppose that α is a tangential $(0,q)$-form on X_0 with coefficients holomorphic with respect to w. Then $\widehat{\alpha}_t$ is in $L^2(\mathbb{C}^{n-1}, t\psi_0+\phi_0)$ for almost all t and*

$$(\alpha,\alpha)_{X_0} = 4\pi \int (\widehat{\alpha}_t, \widehat{\alpha}_t)_t b(t)dt, \tag{4.3}$$

where $b(t) = \int_{s<0} e^{st} a(\rho_0)^{-1} ds$. Moreover, if α is in $\mathcal{H}^{0,q}(X_0,\phi_0)$, then $\widehat{\alpha}_t$ is in $\mathcal{H}^{0,q}(\mathbb{C}^{n-1}, t\psi_0+\phi_0)$ for almost all t.

Proof. — It is clearly enough to prove (4.3) for the components f_I of α, *i.e.* for a function f in X_0 that is holomorphic with respect to w. When evaluating the norm $(f,f)_{X_0}$ over X_0 we may first perform the integration over u, using (4.1), giving

$$(f,f)_{X_0} = 4\pi \int_{\rho_0<0} \left|\widehat{f}_t(z)\right|^2 e^{vt} e^{-\phi(z)} a(\rho_0)^{-1} dz dt dv,$$

where dz stands for the Euclidean volume form $\left(\frac{i}{2}\partial\overline{\partial}\,|z|^2\right)_{n-1}$ on $\mathbb{C}^{n-1}$. If we now fix z and make the change of variables $s := v + \psi_0(z)$ and integrate with respect to s we get

$$4\pi \int \left|\widehat{f}_t(z)\right|^2 b(t) e^{-(t\psi(z)+\phi(z))} dt dz = 4\pi \int (\widehat{f}_t, \widehat{f}_t)_t b(t) dt.$$

Since this integral is finite, it follows that $(\widehat{f}_t, \widehat{f}_t)$ is finite for almost all t.

Next, assume that α is in $\mathcal{H}^{0,q}(X_0,\phi_0)$. By Proposition 4.1 α is $\overline{\partial}$-closed, so that (4.2) gives that $\widehat{\alpha}_t$ is $\overline{\partial}$-closed for almost all t. Let us now show that $\widehat{\alpha}_t$ is $\overline{\partial}$-coclosed with respect to $L^2(\mathbb{C}^{n-1}, t\psi_0+\phi_0)$ for almost all t. Fix an interval I in the positive half-line and let β be a form in X_0 that can be written as

$$\beta(z,w) = \int_{t\ \in I} \eta_t(z) e^{-\frac{i}{2}wt} dt$$

where η_t is a smooth $(0,q-1)$-form with compact support on $\mathbb{C}^{n-1}$ for a fixed t (and measurable with respect to t for z fixed). In particular β is a smooth form in $L^2(X_0,\phi_0)$ that is tangential and holomorphic with respect to w. According to (4.2) $\widehat{\beta}_t$ is equal to η_t for $t \in I$ and vanishes otherwise. By Proposition 4.1 α is $\overline{\partial}$-coclosed (with respect to $L^2(X_0,\phi_0)$). Using (4.3) we get that

$$0 = (\overline{\partial}^* \alpha, \beta) = (\alpha, \overline{\partial}\beta) = 4\pi \int_{t\in I} (\widehat{\alpha}_t, \overline{\partial}\eta_t)_t b(t) dt,$$

where we have used Lemma 4.6 proved in the next section to get the second equality. Since this holds for any choice of form β and interval I as above we conclude that $\overline{\partial}^* \widehat{\alpha}_t = 0$ for almost all t. Hence $\widehat{\alpha}_t$ is in $\mathcal{H}^{0,q}(\mathbb{C}^{n-1}, t\Psi_0 + \phi_0)$ for almost all t. □

Denote by $\mathbb{K}_t^q$ the Bergman kernel of the Hilbert space $\mathcal{H}^{0,q}(\mathbb{C}^{n-1}, t\Psi_0 + \phi_0)$.

Lemma 4.4. — *The Bergman kernel $\mathbb{K}_{X_0}^q$ may be expressed as*

$$\mathbb{K}_{X_0}^q(z, w, z', w') = \frac{1}{4\pi} \int_0^\infty \mathbb{K}_t^q(z, z') e^{\frac{i}{2}(\bar{w} - w')t} b(t)^{-1} dt$$

In particular, the Bergman form $\mathbb{B}_{X_0}^q$ is given by

$$\mathbb{B}_{X_0}^q(z, w) = \frac{1}{4\pi} \int_0^\infty \mathbb{B}_t^q(z, z) e^{\rho_0 t} b(t)^{-1} dt.$$

Proof. — Take a form α is in $\mathcal{H}^{0,q}(X_0, \phi_0)$ and expand it in terms of its Fourier transform as in (4.2). According to the previous lemma $\widehat{\alpha}_t$ is in $\mathcal{H}^{0,q}(\mathbb{C}^{n-1}, t\Psi_0 + \phi_0)$ for almost all t. Hence, we can express it in terms of the corresponding Bergman kernel $\mathbb{K}_t^q$, giving

$$f_I(z, w) = \int \widehat{f}_{I,t}(z) e^{-\frac{i}{2}wt} dt = \int (\widehat{\alpha}_t(z), \mathbb{K}_{t,z,I}^q) e^{\frac{i}{2}wt} dt,$$

where $\mathbb{K}_{t,z,I}^q$ denotes the Bergman kernel form $\mathbb{K}_t^q$ at the point z in the direction $d\bar{z}_I$ (see Section 3) and where have used the reproducing property (3.1) of the Bergman kernel. Now, using the relation between the different scalar products in the previous lemma we get

$$f_I(z, w) = \frac{1}{4\pi} \Big(\alpha(z', w'), \int_t \mathbb{K}_{t,z,I}^q(z') e^{\frac{i}{2}\bar{w}t} e^{-\frac{i}{2}w't} b(t)^{-1} dt \Big)_{X_0}$$

where (z', w') are the integration variables in the scalar product. But this means exactly that $\mathbb{K}_{X_0}^q$ as defined in the statement of the lemma is the Bergman kernel form of the space $\mathcal{H}^{0,q}(X_0, \phi_0)$ since α was chosen arbitrarily. Finally, by definition we have that

$$\mathbb{B}_{X_0}^q(z, w) = \mathbb{K}_{X_0}^q(z, w, z, w) e^{-\phi_0(z)}, \ \mathbb{B}_t^q(z, z) = \mathbb{K}_t^q(z, z') e^{-(t\Psi_0 + \phi_0)(z)}.$$

Hence, the expression for $\mathbb{B}_{X_0}^q$ is obtained. □

Now we can give an explicit expression for the Bergman kernel form and the Bergman function. In the formulation of the following theorem we consider X_0 as a fiber bundle of infinity rays $]-\infty, 0]$ over (or rather under) the boundary ∂X_0. Then we can consider the fiber integral over 0, *i.e.* the push forward at 0, of forms on X_0. Moreover, given a real-valued function

η on $\mathbb{C}^{n-1}$ such that $\frac{i}{2}\partial\overline{\partial}\eta$ has exactly q negative eigenvalues, we define an associated (q,q)- form $\chi^{q,q}$ by

$$\chi^{q,q} := (i/2)^q e^1 \wedge \cdots \wedge e^q \wedge \bar{e}^1 \wedge \cdots \wedge \bar{e}^q$$

where e^i is a orthonormal $(1,0)$-frame that is dual to a base e_i of the direct sum of eigen spaces corresponding to negative eigenvalues of $i\partial\overline{\partial}\eta$ (compare [5]). The (q,q)-form associated to $\frac{i}{2}\partial\overline{\partial}\phi_0 + t\frac{i}{2}\partial\overline{\partial}\rho_0$ is denoted by $\chi_t^{q,q}$ in the statement of the following theorem.

THEOREM 4.5. — *The Bergman form $\mathbb{B}^q_{X_0}$ can be written as an integral over a parameter t :*

$$\mathbb{B}^q_{X_0}(0,u+iv) = \frac{1}{4\pi}\frac{1}{\pi^{n-1}}\int_{T(q)} \chi_t^{q,q} \det(\frac{i}{2}\partial\overline{\partial}\phi_0 + t\frac{i}{2}\partial\overline{\partial}\rho_0)e^{vt}b(t)^{-1}dt,$$

where $b(t) = \int_{\rho<0} e^{\rho t}a(\rho)^{-1}d\rho$. In particular, the fiber integral over 0 of the Bergman function B_{X_0} times the volume form is given by

$$(4.4)\quad \int_{v=-\infty}^{0} B^q_{X_0}(0,iv)(\omega_0)_n = \Big(\frac{i}{2\pi}\Big)^n(-1)^q\int_{T(q)}(\partial\overline{\partial}\phi_0+t\partial\overline{\partial}\rho_0)_{n-1}\wedge\partial\rho\wedge dt.$$

Proof. — Let us first show how to get the expression for $\mathbb{B}^q_{X_0}$. Using the previous proposition we just have to observe that in $\mathbb{C}^m$, with η a quadratic weight function, the Bergman form is given by

$$(4.5)\qquad \mathbb{B}^q_\eta = \frac{1}{\pi^m}1_{X(q)}\chi^{q,q}\det\Big(\frac{i}{2}\partial\overline{\partial}\eta\Big),$$

where the constant function $1_{X(q)}$ is equal to one if $\frac{i}{2}\partial\overline{\partial}\eta$ has precisely q negative eigenvalues and is zero otherwise (see [4], [5], [6]). Next, from Section 3 we have that $B^q_{X_0}(\omega_0)_n$ is given by $\mathbb{B}^q_{X_0}(\omega_0)_{n-q}$. Note that

$$\chi^{q,q}\wedge(\omega_0)_{n-q} = \Big(\frac{i}{2}\partial\overline{\partial}\,|z|^2\Big)_{n-1}\wedge a(\rho_0)^{-1}(2i)\partial\rho_0\wedge d\rho_0,$$

Thus, the fiber integral of $B^q_{X_0}(\omega_0)_n$ over 0 reduces to (4.4) since the factor $b(t)$ is cancelled by the integral of $e^{vt}a(v)^{-1}$. □

Let us finally prove Proposition 4.1.

4.1. The proof of Proposition 4.1: all harmonic forms are tangential, closed and coclosed.

We may write

$$\omega_0 = \frac{i}{2}\partial\overline{\partial}\,|z|^2 + 2i\partial\rho'\wedge\overline{\partial}\rho'$$

for a certain function ρ' of ρ_0. The forms $2^{-1/2}dz_i$ and $2^{1/2}\partial\rho'$ together define a orthonormal frame of $(1,0)$-forms. However, we will use the orthogonal frame consisting of all dz_i and $\partial\rho'$ in order not to clutter the formulas. A dual frame of $(1,0)$-vector fields is obtained as

$$Z_i := \frac{\partial}{\partial z_i} - 2i\mu_i\bar{z}_i\frac{\partial}{\partial w}, \quad i = 1, 2, \ldots, n, \quad N := ia^{1/2}\frac{\partial}{\partial w}, \tag{4.6}$$

where Z_i is tangential to the level surfaces of ρ, while N is a complex normal vector field. We decompose any form α as

$$\alpha = \alpha_T + \alpha_N = \sum f_I d\bar{z}_I + \overline{\partial}\rho' \wedge g^{0,q-1}$$

Similarly, we decompose the $\overline{\partial}$-operator acting on the algebra of forms $\Omega^{0,*}(X_0)$ as

$$\overline{\partial} = \overline{\partial}_T + \overline{\partial}_N = \sum_{i=1}^{n-1} \overline{Z}_i d\bar{z}_i \wedge + \overline{N\partial\rho'}\wedge, \tag{4.7}$$

where the vector fields $\overline{Z}_i$ *etc.* act on forms over X_0 by acting on the coefficients where $d\bar{z}_i\wedge$ etc denotes the operator acting on forms on X, obtained by wedging with $d\bar{z}_i$. The adjoint operator will be denoted by $d\bar{z}_i^*$. Note that the expression for $\overline{\partial}$ is independent of the ordering of the operators, since the elements in the corresponding frame of $(0,1)$-forms are $\overline{\partial}$-closed. We denote by Δ_T and Δ_N the corresponding Laplace operators, *i.e.*

$$\Delta_T = \overline{\partial}_T\overline{\partial}_T^* + \overline{\partial}_T^*\overline{\partial}_T, \;\; \Delta_N = \overline{\partial}_N\overline{\partial}_N^* + \overline{\partial}_N^*\bar{\partial}_N,$$

Recall that α is said to satisfy $\overline{\partial}$-Neumann boundary conditions if $\overline{\partial}\rho'^*$ applied to α and $\overline{\partial}\alpha$ vanishes on the boundary of X_0, or equivalently if α_N and $(\overline{\partial}\alpha)_N$ vanishes there.

Lemma 4.6. — *Denote by $\overline{Z}_i^*$ and $\overline{N}^*$ the formal adjoint operators of the operators $\overline{Z}_i$ and $\overline{N}$ acting on $\Omega^{0,*}(X_0)$. Then*

$$\begin{aligned} \overline{Z}_i^* &= -e^{\phi_0}Z_ie^{-\phi_0} \\ \overline{N}^* &= -a^{1/2}Na^{-1/2} \end{aligned} \tag{4.8}$$

Moreover, if the form α has relatively compact support in X_0 then for any form smooth form β in X_0 we have that $(\overline{\partial}_T^*\alpha, \beta) = (\alpha, \overline{\partial}_T\beta)$ and if furthermore α satisfies $\overline{\partial}$-Neumann boundary conditions, then $(\overline{\partial}_N^*\alpha, \beta) = (\alpha, \overline{\partial}_N\beta)$ (in terms of the formal adjoint operators).

Proof. — It is clearly enough to prove (4.8) for the action of the operators on smooth *functions* with compact support (*i.e.* we write $\alpha = f$

and $\beta = g$, where f and g are smooth functions with compact support). To prove the first statement in (4.8) it is, using Leibniz rule, enough to show that

$$\int_X \overline{Z}_i(f\bar{g}e^{-\phi_0})(\omega_0)_n = 0.$$

But this follows from Stokes theorem since the integrand can be written as a constant times the form

$$d\Big(f\bar{g}e^{-\phi_0}a^{-1}\Big(\bigwedge_{j\neq i} dz_j \wedge d\bar{z}_j\Big) \wedge dz_i \wedge dw \wedge d\bar{w}\Big),$$

using that $\overline{Z}_i(a^{-1}) = 0$, since $\overline{Z}_i$ is tangential. Similarly, to prove the second statement in (4.8) it is, using Stokes theorem, enough to observe that

$$\int_X d(f\bar{g}e^{-\phi_0}a^{-1/2}(\partial\overline{\partial}\,|z|^2)^{n-1} \wedge dw) = 0.$$

Indeed, we have that $\overline{N} := -ia^{1/2}\frac{\partial}{\partial\bar{w}}$, so the statement now follows from Leibniz' rule. Finally, the last two statements follow from the arguments above, since the boundary integrals obtained from Stokes theorem vanish. □

Lemma 4.7. — *The $\overline{\partial}$-Laplacian Δ acting on $\Omega^{0,*}(X_0)$ decomposes as*

$$\Delta = \Delta_T + \Delta_N.$$

Proof. — Expanding with respect to the decomposition (4.7) we just have to show that the sum of the mixed terms

$$\big(\overline{\partial}_N\overline{\partial}_T^* + \overline{\partial}_T^*\overline{\partial}_N\big) + \big(\overline{\partial}_N^*\overline{\partial}_T + \overline{\partial}_T\overline{\partial}_N^*\big)$$

vanishes. Let us first show that the first term vanishes. Observe that the following anti-commutation relations hold:

$$d\bar{z}_i \wedge \overline{\partial}\rho'^* + \overline{\partial}\rho'^* d\bar{z}_i \wedge = 0.$$

Indeed, this is equivalent to the corresponding forms being orthogonal. Using this and the expansion (4.7) we get that

$$\big(\overline{\partial}_N\overline{\partial}_T^* + \overline{\partial}_T^*\overline{\partial}_N\big) = \sum_i [\overline{N}, \overline{Z}_i^*]\overline{\partial}\rho'^* d\bar{z}_i \wedge .$$

But this equals zero since the commutators $[\overline{N}, \overline{Z}_i^*]$ vanish, using the expressions in Lemma 4.6. To see that the second terms vanishes one can go through the same argument again, now using that the commutators $[\overline{N}^*, \overline{Z}_i]$ vanish. □

We will call a sequence χ_i of non-negative functions on X_0 a *relative exhaustion sequence* if there is sequence of balls B_{R_i} centered at the origin and exhausting $\mathbb{C}^n$, such that χ_i is identically equal to 1 on $B_{R_i/2}$ and with support in B_{R_i}. Moreover, if the metric ω is such that the sequence χ_i can be chosen to make $|d\chi_i|$ uniformly bounded then (X_0,ω) is called *relatively complete*. The point is that when (X_0,ω) is relatively complete, one can integrate partially without getting boundary terms "at infinity". For a complete manifold this was shown in [18] and the extension to the relative case is straightforward.

LEMMA 4.8. — *Suppose that (X_0,ω) is relatively complete. Then there is a relative exhaustion sequence χ_i of X_0 such that, if α is a smooth form in $L^2(X_0)$, then*

$$\lim_i(\chi_i\Delta_T\alpha,\alpha)=(\overline{\partial}_T^*\alpha,\overline{\partial}_T^*\alpha)+(\overline{\partial}_T\alpha,\overline{\partial}_T\alpha).$$

Moreover, if α satisfies $\overline{\partial}$-Neumann boundary conditions on ∂X_0, then

$$\lim_i(\chi_i\Delta_N\alpha,\alpha)=(\overline{\partial}_N^*\alpha,\overline{\partial}_N^*\alpha)+(\overline{\partial}_N\alpha,\overline{\partial}_N\alpha). \tag{4.9}$$

Proof. — Since (X_0,ω) is relatively complete, following Section 1.1 B in [18] it is enough to prove the statements for a form α with relatively compact support, with χ_i identically equal to 1 (this is called the Gaffney cutoff trick in [18]). Assuming this, the first statement then follows immediately from Lemma 4.6. To prove the second statement we assume that α satisfies $\overline{\partial}$-Neumann boundary conditions, *i.e.* $\alpha_N=(\overline{\partial}\alpha)_N=0$ on ∂X_0. According to Lemma 4.6 the first term in the right hand side of (4.9) may be written as $(\overline{\partial}_N\overline{\partial}_N^*\alpha,\alpha)$, since $\alpha_N=0$ on ∂X_0 by assumption. To show that the second term may be written as $(\overline{\partial}_N^*\overline{\partial}_N\alpha,\alpha)$ we just have to show that $(\overline{\partial}_N\alpha)_N=0$ on ∂X_0. To this end, first observe that $\overline{\partial}_N\alpha=\overline{\partial}_N\alpha_T$ and $(\overline{\partial}\alpha)_N=\overline{\partial}_T\alpha_N+\overline{\partial}_N\alpha_T$. Now, by assumption $(\overline{\partial}\alpha)_N=0$ on ∂X_0. Combing this with the previous two identities we deduce that $\overline{\partial}_N\alpha=-\overline{\partial}_T\alpha_N$ on ∂X_0. But $\alpha_N=0$ on ∂X_0 and $\overline{\partial}_T$ is a tangential operator, so it follows that $\overline{\partial}_T\alpha_N=0$ on ∂X_0. This proves that $(\overline{\partial}_N\alpha)_N=0$ on ∂X_0. □

Finally, to finish the proof of Proposition 4.1, first observe that the model metric ω_0 corresponding to $a(\rho_0)=(1-\rho_0)^2$ is relatively complete. Now take an arbitrary form α in $\mathcal{H}^{0,q}(X_0,\phi_0)$. Then $(\chi_i\Delta\alpha,\alpha)=0$ for each i. Hence, using Lemma 4.7 together with Lemma 4.8 we deduce, after letting i tend to infinity, that

$$0=\left\|\overline{\partial}_T\alpha\right\|^2+\left\|\overline{\partial}_T^*\alpha\right\|+\left\|\overline{\partial}_N\alpha\right\|^2+\left\|\overline{\partial}_N^*\alpha\right\|^2. \tag{4.10}$$

In particular, $\overline{\partial}_N^* \alpha$ vanishes in X_0. If we write $\alpha_N = \overline{\partial}\rho' \wedge \sum_I g_I d\bar{z}_I$ this means that

$$\overline{N}^* g_I = 0$$

in X_0 for all I. Moreover, since α satisfies $\overline{\partial}$-Neumann boundary conditions, each function g_I vanishes on the boundary of X_0. It follows that $g_I = 0$ in all of X_0. Indeed, let $g_I' := a(\rho)^{-1/2}\bar{g}_I$ and consider the restriction of g_I' to the half planes in $\mathbb{C}$ obtained by freezing the z_i-variables. Then g_I' is holomorphic in the half plane, vanishing on the boundary. It is a classical fact that g_I' then actually vanishes identically. Moreover, (4.10) also gives that α is $\overline{\partial}$-closed and coclosed. This finishes the proof of Proposition 4.1.

5. Contributions from the three boundary regions.

In this section we will estimate the integral of the Bergman function over the three different boundary regions. The contribution from the inner part of X was essentially computed in [4].

5.1. The first region.

Recall that the first region is the set where $\rho \geqslant -R/k)$. Fix a point σ in ∂X and take local holomorphic coordinates (z, w), where z is in $\mathbb{C}^{n-1}$ and $w = u + iv$. By an appropriate choice [8], we may assume that the coordinates are orthonormal at 0 and that

$$\rho(z,w) = \sum_{i=1}^{n-1} v + \mu_i |z_i|^2 + O(|(z,w)|^3) =: \rho_0(z,w) + O(|(z,w)|^3). \tag{5.1}$$

In a suitable local holomorphic trivialization of L close to the boundary point σ, the fiber metric may be written as

$$\phi(z) = \sum_{i,j=1}^{n-1} \lambda_{ij} z_i \bar{z}_j + O(|w|)O(|z|) + O(|w|^2) + O(|(z,w)|^3).$$

Denote by F_k the holomorphic scaling map

$$F_k(z,w) = (z/k^{1/2}, w/k),$$

so that

$$X_k = F_k(D_{\ln k}) \bigcap X$$

is a sequence of decreasing neighborhoods of the boundary point σ, where $D_{\ln k}$ denotes the polydisc of radius $\ln k$ in $\mathbb{C}^n$. Note that

$$F_k^{-1}(X_k) \to X_0,$$

in a certain sense, where X_0 is the model domain with defining function ρ_0. On $F_k^{-1}(X_k)$ we have the *scaled metrics* $F_k^* k\omega_k$ and $F_k^* k\phi$ that tend to the model metrics ω_0 and ϕ_0 on X_0, when k tends to infinity, where

$$\omega_0 = \frac{i}{2}\partial\overline{\partial}\,|z|^2 + a(\rho_0)^{-1}2i\partial\rho_0 \wedge \overline{\partial}\rho_0 \quad \text{and} \quad \phi_0(z) = \sum_{i,j=1}^{n-1} \lambda_{ij} z_i \bar{z}_j, \tag{5.2}$$

for a smooth function $a(\rho_0)$ that is positive on $]-\infty,0]$. The factor $a(\rho_0)$, and hence the model metric ω_0, really depends on the number R (used in the definition of the boundary regions). However, the dependence on R will play no role in the proofs, since R will be fixed when k tends to infinity. Recall that the space of model harmonic $(0,q)$-forms in $L^2(X_0,\omega_0,\phi_0)$ satisfying $\overline{\partial}$-Neumann boundary conditions is denoted by $\mathcal{H}^{0,q}(X_0,\phi_0)$.

Lemma 5.1. — *For the component-wise uniform norms on $F_k^{-1}(X_k)$ we have that*

$$\|F_k^* k\rho - \rho_0\|_\infty \to 0$$
$$\|F_k^* k\omega_k - \omega_0\|_\infty \to 0$$
$$\|F_k^* k\phi - \phi_0\|_\infty \to 0$$

and similarly for all derivatives.

Proof. — The convergence for ρ and ϕ is straightforward (compare [4]) and the convergence for ω_k will be showed in Section 5.4 once ω_k has been constructed. □

The Laplacian on $F_k^{-1}(X)$ taken with respect to the scaled metrics will be denoted by $\Delta^{(k)}$ and the corresponding formal adjoint of $\overline{\partial}$ will be denoted by $\overline{\partial}^{*(k)}$. The Laplacian on X_0 taken with respect to the model metrics will be denoted by Δ_0. Because of the convergence property of the metrics above it is not hard to check that

$$\Delta^{(k)} = \Delta_0 + \varepsilon_k \mathcal{D}_k, \tag{5.3}$$

where $\mathcal{D}_k$ is a second order partial differential operator with bounded variable coefficients on $F_k^{-1}(X_k)$ and ε_k is a sequence tending to zero with k. Next, given a $(0,q)$-form α_k on X_k with values in L^k, define the *scaled form* $\alpha^{(k)}$ on $F_k^{-1}(X_k)$ by

$$\alpha^{(k)} := F_k^* \alpha_k.$$

Then

$$F_k^* |\alpha_k|^2 = \left|\alpha^{(k)}\right|^2, \tag{5.4}$$

where the norm of α_k is the one induced by the metrics $k\omega_k$ and $k\phi$ and the norm of the scaled form $\alpha^{(k)}$ is taken with respect to the scaled metrics $F_k^* k\omega_k$ and $F_k^* k\phi$. This is a direct consequence of the definitions. Moreover, the next lemma gives the transformation of the Laplacian.

LEMMA 5.2. — *The following relation between the Laplacians holds:*

$$\Delta^{(k)}\alpha^{(k)} = (\Delta_k \alpha_k)^{(k)}. \tag{5.5}$$

Proof. — Since the Laplacian is naturally defined with respect to any given metric it is invariant under pull-back, proving the lemma. □

In the following, all norms over $F_k^{-1}(X_k)$ will be taken with respect to the model metrics ω_0 and ϕ_0. The point is that these norms anyway coincide, asymptotically in k, with the norms defined with respect to the scaled metrics used above, by the following lemma.

LEMMA 5.3. — *We have that uniformly on $F_k^{-1}(X_k)$*

$$F_k^* |\alpha_k| \sim \left|\alpha^{(k)}\right|$$
$$\|\alpha_k\|_{X_k} \sim \left\|\alpha^{(k)}\right\|_{F_k^{-1}(X_k)}.$$

Moreover, for any sequence θ_k^I of ω_k-orthonormal bases of direction forms in $\Lambda^{0,q}(X)_x$ at $F_k(x)$, there is a bases of ω_0-orthonormal direction forms at x, such that the following asymptotic equality holds, when k tends to infinity:

$$F_k^* \left|\alpha_{k,\theta_k^I}\right| \sim \left|\alpha_{\theta^I}^{(k)}\right|$$

for each index I.

Proof. — The lemma follows immediately from (5.4) and the convergence of the metrics in the previous lemma. □

Now we can prove the following lemma that makes precise the statement that, in the large k limit, harmonic forms α_k are harmonic with respect to the model metrics and the model domain on a small scale close to the boundary of X.

Lemma 5.4. — *Suppose that the boundary of X satisfies condition $Z(q)$ (see Remark 1.2). For each k, suppose that $\alpha^{(k)}$ is a $\overline{\partial}$-closed smooth $(0,q)$-form on $F_k^{-1}(X_k)$ such that $\overline{\partial}^{*(k)}\alpha^{(k)} = 0$ and that $\alpha^{(k)}$ satisfies $\overline{\partial}$-Neumann boundary conditions on $F_k^{-1}(\partial X)$. Identify $\alpha^{(k)}$ with a form in $L^2(\mathbb{C}^n)$ by extending with zero. Then there is a constant C_R independent of k such that*

$$\sup_{D_R\cap F_k^{-1}(X_k).} \left|\alpha^{(k)}\right|^2 \leqslant C_R\left\|\alpha^{(k)}\right\|^2_{D_{2R}\cap F_k^{-1}(X_k)}.$$

Moreover, if the sequence of norms $\left\|\alpha^{(k)}\right\|^2_{F_k^{-1}(X_k)}$ is bounded, then there is a subsequence of $\left\{\alpha^{(k)}\right\}$ which converges uniformly with all derivatives on any compactly included set in X_0 to a smooth form β, where β is in $\mathcal{H}^{0,q}(X_0)$. The convergence is uniform on $D_R \cap F_k^{-1}(X_k)$.

Proof. — Fix a k and consider the intersection of the polydisc D_R of radius R with $F_k^{-1}(X_k)$. It is well-known that the Laplace operator $\Delta^{(k)}$ acting on $(0,q)$-forms is sub-elliptic close to a point x in the boundary satisfying the condition $Z(q)$ (see [15]). In particular, sub-elliptic estimates give for any smooth form $\beta^{(k)}$ satisfying $\overline{\partial}$-Neumann boundary conditions on $F_k^{-1}(\partial X)$ that

$$\left\|\beta^{(k)}\right\|^2_{D_R,m-1} \leqslant C_{k,R}(\left\|\beta^{(k)}\right\|^2_{D_{2R}} + \left\|\Delta^{(k)}\beta^{(k)}\right\|^2_{D_{2R},m}), \tag{5.6}$$

where the subscript m indicates a Sobolev norm with m derivatives in L^2 and where the norms are taken over $F_k^{-1}(X)$ with respect to the scaled metrics. The k-dependence of the constants $C_{k,R}$ comes from the boundary $F_k^{-1}(\partial X)$ and the scaled metrics $F_k^* k\omega_k$ and $F_k^* k\phi$. However, thanks to the convergence of the metrics in Lemma 5.1 one can check that the dependence is uniform in k. Hence, applying the subelliptic estimates (5.6) to $\alpha^{(k)}$ we get

$$\left\|\alpha^{(k)}\right\|^2_{D_R\cap F_k^{-1}(X),m} \leqslant C_R\left\|\alpha^{(k)}\right\|^2_{D_{2R}\cap F_k^{-1}(X)} \tag{5.7}$$

and the continuous injection $L^{2,l} \hookrightarrow C^0$, $l > n$, provided by the Sobolev embedding theorem, proves the first statement in the lemma. To prove the second statement assume that $\left\|\alpha^{(k)}\right\|^2_{F_k^{-1}(X)}$ is uniformly bounded in k. Take a sequence of sets K_n, compactly included in X_0, exhausting X_0 when n tends to infinity. Then the estimate (5.6) (applied to polydiscs of increasing radii) shows that

$$\left\|\alpha^{(k)}\right\|^2_{K_n,m} \leqslant C'_n. \tag{5.8}$$

Since this holds for any $m \geqslant 1$, Rellich's compactness theorem yields, for each n, a subsequence of $\{\alpha^{(k)}\}$, which converges in all Sobolev spaces $L^{2,l}(K_n)$ for $l \geqslant 0$ for a fixed n. The compact embedding $L^{2,l} \hookrightarrow C^p$, $k > n + \frac{1}{2}p$, shows that the sequence converges in all $C^p(K_n)$. Choosing a diagonal sequence with respect to k and n, yields convergence on any compactly included set K. Finally, we will prove that the limit form β is in $\mathcal{H}(X_0)$. First observe that by weak compactness we may assume that the sequence $1_{X_0}\alpha^{(k)}$ tends to β weakly in $L^2(\mathbb{C}^n)$, where 1_{X_0} is the characteristic function of X_0 and β is extended by zero to all of $\mathbb{C}^n$. In particular, the form β is weakly $\overline{\partial}$-closed in X_0. To prove that β is in $\mathcal{H}(X_0)$ it will now be enough to show that

$$(\beta, \overline{\partial}\eta)_{X_0} = 0 \tag{5.9}$$

for any form η in X_0 that is smooth up to the boundary and with a relatively compact support in X_0. Indeed, it is well-known that β then is in the kernel of the Hilbert adjoint of the densely defined operator $\overline{\partial}$. Moreover, the regularity theory then shows that β is smooth up to the boundary, where it satisfies $\overline{\partial}$-Neumann boundary conditions (actually this is shown using sub-elliptic estimates as in (5.6)) [20], [15]. To see that (5.9) holds, we write the left hand side, using the weak convergence of $1_{X_0}\alpha^{(k)}$, as

$$\lim_k (\alpha^{(k)}, \overline{\partial}\eta)_{X_0} = \lim_k (\alpha^{(k)}, \overline{\partial}\eta)_{X_0 \cap F_k^{-1}(X)}. \tag{5.10}$$

Extending η to a smooth form on some neighborhood of X_0 in $\mathbb{C}^n$ we may now write this as a scalar product, with respect to the scaled metrics, over $F_k^{-1}(X)$, thanks to the convergence in Lemma 5.1 of the scaled metrics and the scaled defining function. Since, $\alpha^{(k)}$ is assumed to satisfy $\overline{\partial}$-Neumann-boundary conditions on $F_k^{-1}(\partial X)$ and be in the kernel of the formal adjoint of $\overline{\partial}$, taken with respect to the scaled metrics, this means that the right hand side of (5.10) vanishes. This proves (5.9) and finishes the proof of the lemma. □

The following proposition will give the boundary contribution to the holomorphic Morse inequalities in Theorem 2.1.

Proposition 5.5. — *Let*

$$I_R := \limsup_k \int_{-\rho < Rk^{-1}} B_X^{q,k} \omega_n.$$

Then

$$\limsup_R I_R \leqslant (-1)^q \left(\frac{1}{2\pi}\right)^n \int_{\partial X} \int_{T(q)_{\rho,x}} (\Theta + t\mathcal{L})_{n-1} \wedge \partial\rho \wedge dt$$

where $T(q)_{\rho,x} = \left\{t > 0 : \text{index}(\Theta + t\mathcal{L}) = q \text{ along } T^{1,0}(\partial X)_x\right\}.$

Proof. — We may assume that the boundary of X satisfies condition $Z(q)$ (compare Remark 1.2). Using the expression (2.1) for the metric ω_k, the volume form $(\omega_k)_n$ may be written as $a_k(\rho)^{-1}(\omega_T)_{n-1} \wedge 2i\partial\rho \wedge d\rho$. Hence, I_R can be expressed as

$$\limsup_k \int_{\partial X} (\omega_T)_{n-1} \wedge 2i\partial\rho \int_{-R/k}^{0} a_k(\rho)^{-1} B_X^k d\rho.$$

Now, fix a point in the boundary of X and take local coordinates as in the beginning of the section. To make the argument cleaner we will first assume that the restriction of ρ to the ray close to the boundary where z and the real part of w vanish, coincides with v. Then, after a change of variables, the inner integral along the ray in the first region becomes

$$1/k \int_0^R a_k(v/k)^{-1} B_X^{q,k}(v/k) dv. \tag{5.11}$$

Moreover, by the scaling properties of the metrics $k\omega_k$ in Lemma 5.1we have the uniform convergence

$$ka_k(v/k) \to a(v).$$

on the segment $[0, R]$ (see also Section 5.4). Thus,

$$I_R = \limsup_k \int_{\partial X} (\omega_T)_{n-1} \wedge 2i\partial\rho \int_0^R a(v)^{-1} B_X^{q,k}(v/k) dv.$$

Let us now show that

$$\text{(5.12)} \qquad \begin{aligned} &\text{(i)} \quad B_X^{q,k}(0, iv/k) \lesssim B_{X_0}^q(0, iv) \\ &\text{(ii)} \quad B_X^{q,k}(0, iv/k) \leqslant C_R. \end{aligned}$$

We first prove (i). According to the extremal property (3.4) it is enough to show that

$$S_{X,\theta_k}^{q,k}(0, iv/k) \lesssim S_{X_0,\theta}^q(0, iv) \tag{5.13}$$

for any sequence of direction forms θ_k at $(0, iv/k)$ as in Lemma 5.3. Given this, the bound (5.12) is obtained after summing over the base elements θ_k. To prove (5.13) we have to estimate

$$\left|\alpha_{k,\theta_k}(0, iv/k\right|^2,$$

where α_k is a normalized harmonic form with values in L^k that is extremal at $(0, iv/k)$ in the direction θ_k. Moreover, it is clearly enough to estimate some subsequence of α_k. By Lemma 5.3 it is equivalent to estimate

$$\left|\alpha_\theta^{(k)}(0, iv)\right|^2,$$

where the scaled form $\alpha^{(k)}$ is defined on $F_k^{-1}(X_k)$ and extended by zero to all of X_0. Note that, according to Lemma 5.3 the norms of the sequence of scaled forms $\alpha^{(k)}$ are asymptotically less than one:

$$\left\|\alpha^{(k)}\right\|^2_{F_k^{-1}(X)} \sim \left\|\alpha_k\right\|^2_{X_k} \leqslant 1, \tag{5.14}$$

since the global norm of α_k is equal to one. Hence, by Lemma 5.4 there is a subsequence $\alpha^{(k_j)}$ that converges uniformly to β with all derivatives on the segment $0 \leqslant v \leqslant R$ in X_0 and where the limit form β is in $\mathcal{H}^{0,q}(X_0, \phi_0)$ and its norm is less than one (by (5.14)). This means that

$$\left|\alpha_{k,\theta}(0, iv/k)\right|^2 \sim \left|\alpha_\theta^{(k_j)}(0, iv)\right|^2 \sim \left|\beta_\theta(0, iv)\right|^2.$$

Since the limit form β is a contender for the model extremal function $S^q_{X_0,\theta}(0)$, this proves (5.13), and hence we obtain (i). To show (ii), just observe that Lemma 5.4 says that there is a constant C_R such that there is a uniform estimate

$$\left|\alpha^{(k)}(0, iv)\right|^2 \leqslant C_R.$$

By the extremal characterization (3.4) of $B_X^{q,k}$ this proves (ii). Now using (5.21) and Fatou's lemma to interchange the limits, I_R may be estimated by

$$\int_{\partial X}\int_0^\infty B^q_{X_0}(0, iv)(\omega_0)_n, \tag{5.15}$$

in terms of the model metric ω_0 on X_0. By Theorem 4.5 this equals

$$(\frac{i}{2\pi})^n(-1)^q \int_{\partial X}\int_{T(q)} (\partial\overline{\partial}\phi_0 + t\partial\overline{\partial}\rho_0)_{n-1} \wedge \partial\rho \wedge dt.$$

This finishes the proof of the proposition under the simplifying assumption that the restriction of ρ to the ray introduced above, coincides with the restriction of v. In general this is only true up to terms of order $O\left|(z, w)\right|^3$, given the expression (5.1). To handle the general case one writes the integral (5.11) as

$$1/k \int_{I_k} a_k B_X^{q,k} dv, \tag{5.16}$$

where I_k is the inverse image under F_k of the ray. Clearly, I_k tends to the segment $[0, R]$ in X_0 obtained by keeping all variables except v equal to zero. Moreover, since the sequence $\alpha^{(k_j)}$ above converges uniformly with all derivatives on $D_R \cap F_k^{-1}(X_k)$ it forms an equicontinuous family, so the same argument as above gives that (5.16) may be estimated by (5.15). This finishes the proof of the proposition. □

5.2. The second and third region.

Let us first consider the second region, *i.e.* where $-1/k^{1/2} \leqslant \rho \leqslant -R/k$. Given a k, consider a fixed point $(0, iv) = (0, ik^{-s})$, where $1/2 \leqslant s < 1$. Any point in the second region may be written in this way. Let (z', w') be coordinates on the unit polydisc D. Define the following holomorphic map from the unit polydisc D to a neighborhood of the fixed point:

$$F_{k,s}(z', w') = \left(k^{-1/2}z', k^{-s} + \frac{1}{2}k^{-s}w'\right)$$

so that

$$X_{k,s} := F_{k,s}(D),$$

is a neighborhood of the fixed point, staying away from the boundary of X. On D we will use the scaled metrics $F_{k,s}^{*}k\omega_k$ and $kF_{k,s}^{*}k\phi$ that have bounded derivatives and are comparable to flat metrics in the following sense:

$$\begin{aligned} C^{-1}\omega_E &\leqslant F_{k,s}^{*}k\omega_k \leqslant C\omega_E \\ \left|F_{k,s}^{*}k\phi\right| &\leqslant C \end{aligned} \tag{5.17}$$

where ω_E is the Euclidean metric. Note that the scaling property of ω_k is equivalent to

$$C^{-1}k\rho^2 \leqslant a_k(\rho) \leqslant Ck\rho^2. \tag{5.18}$$

These properties will be verified in Section 5.4 once ω_k is defined. The Laplacian on D taken with respect to the scaled metrics will be denoted by $\Delta^{(k,s)}$ and the Laplacian on X_0 taken with respect to the model metrics will be denoted by Δ_0. Next, given a $(0,q)$-form α_k on X_k with values in L^k, define the scaled form $\alpha^{(k)}$ on D by

$$\alpha^{(k,s)} := F_{k,s}^{*}\alpha_k.$$

Using (5.17) one can see that the following equivalence of norms holds:

$$\begin{aligned} C^{-1}\left|\alpha^{(k,s)}\right|^2 &\leqslant F_{k,s}^{*}\left|\alpha_k\right|^2 \leqslant C\left|\alpha^{(k,s)}\right|^2 \\ C^{-1}\left\|\alpha^{(k,s)}\right\|_D^2 &\leqslant \left\|\alpha_k\right\|_{X_k}^2 \leqslant C\left\|\alpha^{(k,s)}\right\|_D^2. \end{aligned} \tag{5.19}$$

In the following, all norms over $F_{k,s}^{-1}(X_{k,s})$ will be taken with respect to the Euclidean metric ω_E and the trivial fiber metric.

Proposition 5.6. — *Let*

$$II_R := \limsup_k \int_{Rk^{-1}<-\rho<k^{-1/2}} B_X^{q,k}(\omega_k)_n.$$

Then

$$\lim_R II_R = 0.$$

Proof. — Fix a k and a point $(0, iv) = (0, ik^{-s})$, where s is in $[1/2, 1[$. From the scaling properties (5.18) of ω_k it follows that at the point $(0, ik^{-s})$

$$\omega_k^n \leqslant Ck^{(2s-1)}\omega^n.$$

Next, observe that

$$B_X^{q,k}(0, ik^{-s}) \leqslant C. \tag{5.20}$$

Accepting this for the moment, it follows that

$$B^{q,k}(0, iv)\omega_k^n \leqslant Ck^{-1}v^{-2}\omega^n,$$

since we have assumed that $v = k^{-s}$. Hence, the integral in II_R may be estimated by

$$\int_{\partial X} Ck^{-1} \int_{-k^{-1/2}}^{-Rk^{-1}} v^{-2}\mathrm{d}v = Ck^{-1}(R^{-1}k - k^{1/2})$$

which tends to CR^{-1} when k tends to infinity. This proves that II_R tends to zero when R tends to infinity, which proves the proposition, given (5.20).

Finally, let us prove the claim (5.20). For a given k consider the point $(0, ik^{-s})$ as above. As in the proof of the previous proposition we have to prove the estimate

$$\left|\alpha_k(0, ik^{-s})\right|^2 \leqslant C, \tag{5.21}$$

where α_k is a normalized harmonic section with values in L^k that is extremal at $(0, ik^{-s})$. By the equivalence of norms (5.19), it is equivalent to prove

$$\left|\alpha^{(k,s)}(0)\right|^2 \leqslant C,$$

where the scaled form $\alpha^{(k,s)}$ is defined on D. Note that, according to (5.19)

$$\left\|\alpha^{(k,s)}\right\|_D^2 \sim \left\|\alpha_k\right\|_{X_k}^2 \leqslant C, \tag{5.22}$$

since the global norm of α_k is equal to 1. Moreover, a simple modification of Lemma 5.2 gives

$$\Delta^{(k,s)}\alpha^{(k,s)} = 0$$

on D. Since $\Delta^{(k,s)}$ is an elliptic operator on the polydisc D, inner elliptic estimates (*i.e.* Gøarding's inequality) and the Sobolev embedding theorem can be used as in [4] to get

$$\left|\alpha^{(k,s)}(0)\right|^2 \leqslant C\left\|\alpha^{(k,s)}\right\|_D^2,$$

where the constant C is independent of k and s thanks to the equivalence (5.17) of the metrics. Using (5.22), we obtain the claim (5.21). □

Let us now consider the third region where $-\varepsilon < \rho < -k^{-1/2}$.

Proposition 5.7. — *Let*

$$III_\varepsilon := \int_{k^{-1/2} < -\rho < \varepsilon} B_X^{q,k}(\omega_k)_n.$$

Then

$$III_\varepsilon = O(\varepsilon).$$

Proof. — We just have to observe that

$$k^{-n} B_X^{q,k} \leqslant C, \tag{5.23}$$

when $\rho < -k^{-1/2}$. This follows from inner elliptic estimates as in the proof of the previous proposition, now using $s = 1/2$ (compare [4]). □

5.3. End of the proof of Theorem 2.1 (the weak Morse inequalities).

First observe that

$$\int_{0 < -\rho < \varepsilon} B_X^{q,k}(\omega_k)_n \lesssim (-1)^q \left(\frac{1}{2\pi}\right)^n \cdot \int_{\partial X} \int_{T(q)} dt (\partial\overline{\partial}\phi + t\partial\overline{\partial}\rho)^{n-1} \wedge \partial\rho + o(\varepsilon). \tag{5.24}$$

Indeed, for a fixed R we may write the limit of integrals above as the sum $I_R + II_R + III_\varepsilon$. Letting R tend to infinity and using the previous three propositions we get the estimate above. Moreover, we have that

$$\int_{X_\varepsilon} B_X^{q,k} \lesssim \int_{X_\varepsilon} (\partial\overline{\partial}\phi)^n, \tag{5.25}$$

where X_ε denotes the set where $-\rho$ is larger than ε. This follows from the estimates

$$B_X^{q,k}\omega_n \lesssim \left(\frac{i}{2\pi}\right)^n (-1)^q 1_{X(q)} (\partial\overline{\partial}\phi)_n \quad \text{and} \quad B_X^{q,k} \leqslant \mathrm{C} \text{ in } X_\varepsilon,$$

proved in [4] (compare (5.23) for the uniform estimate). Finally, writing $\dim_{\mathbb{C}} \mathcal{H}^{0,q}(X, L^k)$ as the sum of the integrals in (5.24) and (5.25) and letting ε tend to zero, yields the dimension bound in Theorem 2.1 for the space of harmonic forms. By the Hodge theorem we are then done.

5.4. The sequence of metrics ω_k.

In this section the metrics ω_k will be defined and their scaling properties, that were used above, will be verified. Recall that we have to define a sequence of smooth functions a_k such that the metrics

$$\omega_k = \omega_T + a_k(\rho)^{-1} 2i\partial\rho \wedge \overline{\partial}\rho,$$

have the scaling properties of Lemma 5.1 in the first region and satisfy (5.17) in the second region. First observe that the tangential part ω_T clearly scales the right way, *i.e.* that $F_k^* k\omega_T$ tends to $\frac{i}{2}\partial\overline{\partial}\left|z\right|^2$. Indeed, since the coordinates (z, w) are orthonormal at 0 the forms ω_T and $\frac{i}{2}\partial\overline{\partial}\left|z\right|^2$ coincide at 0. Since $\frac{i}{2}\partial\overline{\partial}\left|z\right|^2$ is invariant under $F_k^* k$ the convergence then follows immediately. We now consider the normal part of ω_k and show how to define the functions a_k. Consider first the piecewise smooth functions $\widetilde{a_k}$ where $\widetilde{a_k}$ is defined as R^2/k in the first region, as $k\rho^2$, in the second region and as 1 in the third region and on the rest of X. Then it is not hard to check that $\widetilde{a_k}$ satisfies our demands, except at the two middle boundaries between the three regions. We will now construct a_k as a regularization of $\widetilde{a_k}$. To this end we write $\widetilde{a_k} = k\widetilde{b_k}^2$, where $\widetilde{b_k}$ is defined by

$$\begin{cases} Rk^{-1}, & -\rho \leqslant Rk^{-1} \\ -\rho, & Rk^{-1} \leqslant -\rho \leqslant k^{-1/2} \\ k^{-1/2}, & -\rho \leqslant k^{-1/2} \end{cases}$$

in the three regions. It will be enough to regularize the sequence of continuous piecewise linear functions $\widetilde{b}_k$. Decompose $\widetilde{b}_k$ as a sum of continuous piecewise linear functions

$$\widetilde{b}_k(-\rho) = \frac{R}{k}\widetilde{b}_{1,k}\Big(-\frac{k}{R}\rho\Big) + \frac{1}{k^{1/2}}\widetilde{b}_{2,k}(-k^{1/2}\rho),$$

where $\widetilde{b}_{1,k}$ is determined by linearly interpolating between

$$\widetilde{b}_{1,k}(0) = 1, \;\; \widetilde{b}_{1,k}(1) = 1, \;\; \widetilde{b}_{1,k}(k^{1/2}/R) = 0, \;\; \widetilde{b}_{1,k}(\infty) = 0$$

and $\widetilde{b}_{2,k}$ is determined by

$$\widetilde{b}_{2,k}(0) = 0 \;\; \widetilde{b}_{2,k}(Rk^{-1/2}) = 0 \;\; \widetilde{b}_{2,k}(1) = 1 \;\; \widetilde{b}_{2,k}(\infty) = 1.$$

Now consider the function b_k obtained by replacing $\widetilde{b}_{1,k}$ and $\widetilde{b}_{2,k}$ with the continuous piecewise linear functions b_1 and b_2, where b_1 is determined by

$$b_1(0) = 1, \;\; b_1(1/2) = 1, \;\; b_1(1) = 0, \;\; b_1(\infty) = 0,$$

and b_2 is determined by

$$b_2(0) = 0, \;\; b_2(1) = 1, \;\; b(\infty) = 1.$$

Finally, we smooth the corners of the two functions b_1 and b_2. Let us now show that the sequence of regularized functions b_k scales in the right way. In the first region we have to prove that Lemma 5.1 is valid, which is equivalent to showing that there is a function b_0 such that

$$kb_k(t/k) \to b_0 \tag{5.26}$$

with all derivatives, for t such that $0 \leqslant t \leqslant \mathrm{R}\ln k$. From the definition we have that

$$kb_k(t/k) = Rb_1(t/R) + t, \tag{5.27}$$

which is even independent of k, so (5.26) is trivial then. Next, consider the second region. To show that (5.17) holds we have to show that, for parameters s such that $1/2 \leqslant s < 1$, the t-dependent functions $k^s b_k(1/k^s + t/2k^s)$ (where $|t| \leqslant 1$) are uniformly bounded from above and below by positive constants and have uniformly bounded derivatives. First observe that in the second region the sequence of functions may be written as

$$k^{s-1/2} b_2(1/k^{1/2-s} + t/2k^{1/2-s}),$$

and it is not hard to see that it is bounded from above and below by positive constants, independently of s and k. Moreover, differentiating with respect to t shows that all derivatives are bounded, independently of s and k. All in all this means that we have constructed a sequence of metrics ω_k with the right scaling properties. In particular, (5.27) shows that the factor $a^{-1}(\rho_0)$ in the model metric ω_0 (5.2) satisfies

$$C_R^{-1}(1-\rho_0)^{-2} \leqslant a^{-1}(\rho_0) \leqslant C_R(1-\rho_0)^{-2}$$

for some constant C_R depending on R.

Part II. THE STRONG MORSE INEQUALITIES AND SHARP EXAMPLES.

6. The strong Morse inequalities.

We will assume that the boundary of X satisfies condition $Z(q)$ (compare Remark 1.2) and use the same notation as in Section 1. Let μ_k be a sequence tending to zero. Denote by $\mathcal{H}^{0,q}_{\leqslant\mu_k}(X)$ the space spanned by the $(0,q)$-eigen forms of the Laplacian Δ, with eigenvalues bounded by μ_k. The forms are assumed to satisfy $\overline{\partial}$-Neumann boundary conditions and they will be called *low energy forms.* Since we have assumed that condition $Z(q)$ holds, this space is finite dimensional for each k [15]. Recall that the Laplacian is defined with respect to the metric $k\omega_k$, so that the eigenvalues corresponding to μ_k are multiplied with k if the metric ω_k is used instead.

We will first show that the weak holomorphic Morse inequalities are *equalities* for the space $\mathcal{H}^{0,q}_{\leqslant\mu_k}(X)$ of low energy forms. When X has no boundary this yields *strong* Morse inequalities for the truncated Euler characteristics of the Dolbeault complex with values in L^k. However, when X has a boundary one has to assume that the boundary of X has either concave or convexity properties to ensure that the corresponding cohomology groups are finite dimensional, in order to obtain strong Morse inequalities.

The Bergman form for the space $\mathcal{H}^{0,q}_{\leqslant\mu_k}(X)$ defined as in Section 3 will be denoted by $\mathbb{B}^q_{\leqslant\mu_k}$. By $L^2_m(X)$ we will denote the Sobolev space with m derivatives in $L^2(X)$ and a subscript m on a norm will indicate the corresponding Sobolev norm. The essential part in proving that we now get equality in the weak Morse inequalities is to show that the estimate on the Bergman form (5.12) in the proof of Proposition 5.5 becomes an asymptotic equality, when considering low energy forms. The rest of the argument is more or less as before.

Let us first prove the upper bound, *i.e.* that the low energy Bergman form $\mathbb{B}^q_{\leqslant\mu_k}$ is asymptotically bounded by the model harmonic Bergman form.

Proposition 6.1. — *We have that*

$$B^q_{\leqslant\mu_k,\theta_k}(0, iv/k) \lesssim B_{X_0,\theta}(0, v)$$

and the sequence $B^q_{\leqslant\mu_k}(0, iv/k)$ is uniformly bounded in the first region.

Proof. — Let α_k be a sequence of normalized forms, such that α_k is an extremal for the Hilbert space $\mathcal{H}^{0,q}_{\leqslant\mu_k}(X)$ at the point $(0, iv/k)$ in the direction θ_k. In the following all norms will be taken over $F_k^{-1}(X)$. Observe that by the invariance property in Lemma 5.2 of the Laplacian, the scaled form $\alpha^{(k)}$ satisfies

$$\left\|(\Delta^{(k)})^p\alpha^{(k)}\right\|^2 \leqslant \mu_k^{2p} \to 0 \tag{6.1}$$

for all positive integers p. Let us now show that

$$\left\|\Delta^{(k)}\alpha^{(k)}\right\|_m^2 \to 0 \tag{6.2}$$

for all non-negative integers m. First observe that $(\Delta^{(k)})^p\alpha^{(k)}$ satisfies $\overline{\partial}$-Neumann boundary conditions for all p. Indeed, by definition all forms in the space $\mathcal{H}_{\leqslant\mu_k}(X_0)$ satisfy $\overline{\partial}$-Neumann boundary conditions and since Δ preserves this space, the forms $(\Delta)^p\alpha_k$ also satisfy $\overline{\partial}$-Neumann boundary conditions for all p. By the scaling of the Laplacian this means that the forms $(\Delta^{(k)})^p\alpha^{(k)}$ satisfy $\overline{\partial}$-Neumann boundary conditions with respect to the scaled metrics. Now applying the subelliptic estimates (5.6) to forms of the type $(\Delta^{(k)})^p\alpha^{(k)}$ one gets , using induction, that

$$\left\|\Delta^{(k)}\alpha^{(k)}\right\|_m^2 \leqslant C\sum_{j=1}^{m+1}\left\|(\Delta^{(k)})^j\alpha^{(k)}\right\|^2.$$

Combining this with (6.1) proves (6.2). Now the rest of the argument proceeds almost word for word as in the proof of the claim (5.12) in the proof of Proposition 5.5. The point is that the limit form β will still be in $\mathcal{H}(X_0)$, thanks to (6.1). □

Let us now show how to get the corresponding reverse bound for $\mathbb{B}^{q,k}_{\leqslant\mu_k}$. We will have use for the following lemma.

Lemma 6.2. — *Suppose that β is a normalized extremal form for $\mathcal{H}^{0,q}(X_0, \phi_0)$ at the point $(0, iv_0)$ in the direction θ. Then*

$$\left|\beta_\theta(0, iv_0)\right|^2 = \frac{1}{4\pi}\int_{T(q)} B_{t,\theta}(z, z)e^{v_0 t}b(t)^{-1}dt \tag{6.3}$$

with notation as in Lemma 4.4. Moreover, β is in $L^2_m(X_0)$ for all m.

Proof. — Let x_0 be the point $(0, iv_0)$ in X_0. Since β is extremal we have, according to Section 3, that $\left|\beta_\theta(0, iv_0)\right|^2 = B_{X_0,\theta}(0, iv_0)$, which in turn gives (6.3) according to Lemma 4.4. To prove that β is in $L^2_m(X_0)$ for all m, we write β as

$$\beta(z,w) = \int_{T(q)} \widehat{\beta}_t(z) e^{\frac{i}{2}wt} dt,$$

in terms of its Fourier transform as in Section 4. Recall that we have assumed that condition $Z(q)$ holds on the boundary of X, so that $T(q)$ is finite. Using Proposition 4.3 we can write

$$\Big\| \frac{\partial^l}{\partial^l w} \partial^{I\bar{J}} \beta \Big\|^2_{X_0} = 4\pi \int_{T(q)} \big\| \partial^{I\bar{J}} \widehat{\beta}_t(z) \big\|^2_t t^{2l} b(t) dt,$$

where $\partial^{I\bar{J}}$ denotes the complex partial derivatives taken with respect to $z_i$and $\bar{z}_j$ for i and j in the multi index set I and J, respectively. Since, by assumption, β is in $L^2(X_0)$ the integral converges for $l = 0$ with I and J empty. Now it is enough to show that $\widehat{\beta}_t$ is in $L^2_m(\mathbb{C}^n, t\psi + \phi)$ for all t and positive integers m. To this end we will use the following generalization of (3.3):

$$|\mathbb{K}_{x,\theta}(y)|^2 = |\beta(y)|^2 B_\theta(x) \tag{6.4}$$

if β is an extremal form at the point x, in the direction θ (compare [5]). By Lemma 4.4 the Fourier transform of $\mathbb{K}_{x,\theta}$ evaluated at t is proportional to $\mathbb{K}_{z,\theta,t}$ where $\mathbb{K}_{z,\theta,t}$ is the Bergman kernel form for the space $\mathcal{H}^{0,q}(\mathbb{C}^{n-1}, t\psi+\phi)$ at the point z (and $x = (z,w)$) in the direction θ. In [4] it was essentially shown that $\mathbb{K}_{z,\theta,t}$ is in $L^2_m(\mathbb{C}^n, t\psi + \phi)$ (more precisely: the property was shown to hold for the corresponding extremal form). Hence, the same thing holds for $\widehat{\beta}_t$, according to (6.4), which finishes the proof of the lemma. □

Now we can construct a sequence α_k of approximate extremals for the space $\mathcal{H}_{\leqslant \mu_k k}(X)$ of low energy forms.

LEMMA 6.3. — *For any point $x_{0,k} = (0, iv_0/k)$ and direction form θ in the first region there is a sequence $\{\alpha_k\}$ and direction forms $\theta_k$such that α_k is in $\Omega^{0,q}(X, L^k)$ and*

(i) $|\alpha_{k,\theta_k}(0, iv_0/k)|^2 \sim B_{X_0,\theta}(0, iv_0)$

(ii) $\|\alpha_k\|^2_X \sim 1$

(iii) $\|(\overline{\partial} + \overline{\partial}^{*(k)})\alpha^{(k)}\|^2_m \sim 0$

(iv) $(\Delta\alpha_k, \alpha_k)_X \leqslant \delta_k \|\alpha_k\|^2$

where δ_k is a sequence, independent of $x_{0,k}$, tending to zero, when k tends to infinity. Moreover, α_k satisfies $\overline{\partial}$-Neumann boundary conditions on $\partial X \cap F_k^{-1}(D)$, where D is a polydisc in $\mathbb{C}^n$centered at 0.

Proof. — Consider a sequence of points $x_{0,k}$ that can be written as $(0, iv_0/k)$ in local coordinates as in Section 5.1. Let us first construct a form α_k with the properties (i) to (iv). It is defined by

$$\alpha_k := (F_k^{-1})^*(\chi_k\beta)$$

where $\chi_k(z,w) = \chi(z/\ln k, w/\ln k)$ for χ a smooth function in $\mathbb{C}^n$ that is equal to one on the polydisc D of radius one centered at 0, vanishing outside the polydisc of radius two and where β is the extremal form at the point $(0, v_0)$ in the direction θ from the previous lemma. The definition of α_k is made so that

$$\alpha^{(k)} = \chi_k\beta.$$

We have used the fact that the form β extends naturally as a smooth form to the domain X_δ with defining function $\rho_0 - \delta$, to make sure that α_k is defined on all of X. The extension is obtained by writing β in terms of its Fourier transform with respect to t as in the proof of the previous lemma:

$$\beta(z,w) = \int_{T(q)} \widehat{\beta}_t(z)e^{-\frac{i}{2}wt}dt.$$

In fact, the right hand side is defined for all w since we have assumed that condition $Z(q)$ holds on the boundary so that $T(q)$ is finite. Note that the L^2_m-norm of β over X_δ tends to the L^2_m-norm of β over X_0 when δ tends to zero, as can be seen from the analog of Proposition 4.3 on the domain X_δ. Indeed, the dependence on δ only appears in the definition of $b(t)$, where the upper integration limit is shifted to δ. Now the statements (i) and (ii) follow from the corresponding statements in the previous lemma. To see that (iii) holds, first observe that

$$\overline{\partial}^{*(k)} = \overline{\partial}^{*0} + \varepsilon_k\mathcal{D},$$

where $\mathcal{D}$ is a first order differential operator with bounded coefficients on the ball $B_{lnk}(0)$ and ε_k is a sequence tending to zero. Indeed, this is a simple modification of the statement (5.3). Moreover, by construction $(\overline{\partial} + \overline{\partial}^{*0})\beta = 0$. Hence, Leibniz rule gives

$$\left\|(\overline{\partial} + \overline{\partial}^{*(k)})\alpha^{(k)}\right\| \leqslant \delta_k \left\|\beta\right\|_1 + \left\||d\chi_k|\,\beta\right\|.$$

The first term tends to zero since β is in $L^2_1(X_0)$ and the second terms tends to zero, since it can be dominated by the "tail" of a convergent integral. The estimates for $m \geqslant 1$ are proved in a similar way (compare [4]). Finally, to prove (iv) observe that by the scaling property (5.5) for the Laplacian

$$(\Delta\alpha_k, \alpha_k)_X = \left\|(\overline{\partial} + \overline{\partial}^*)\alpha_k\right\|_X^2 = \left\|(\overline{\partial} + \overline{\partial}^{*(k)})\alpha^{(k)}\right\|.$$

By (ii), the norm of $\|\alpha_k\|_X^2$ tends to one and the norm in the right hand side above can be estimated as above. To see that δ_k can be taken to be independent of the point $x_{0,k}$ in the first region, one just observes that the constants in the estimates depend continuously on the eigenvalues of the curvature forms (compare [4]). Finally, consider a polydisc D in $\mathbb{C}^n$ with small radius. We will perturb α_k slightly so that it satisfies $\overline{\partial}$-Neumann boundary conditions on $\partial X \cap F_k^{-1}(D)$ while preserving the properties (i) to (iv). Recall that a form $\overline{\partial}$-closed form η_k satisfies $\overline{\partial}$-Neumann boundary conditions on ∂X if

$$\overline{\partial}\rho^* \eta_k = 0, \tag{6.5}$$

where $\overline{\partial\rho}^*$ is the fiber-wise adjoint of the operator obtained by wedging with the form $\overline{\partial}\rho$, and where the adjoint is taken with respect to the metric ω_k on X. Equivalently,

$$\overline{\partial}k\rho^{(k)*}\eta^{(k)} = 0,$$

where the adjoint is taken with respect to the scaled metrics. By construction we have that $\alpha^{(k)}$ is $\overline{\partial}$-closed on $F_k^{-1}(D)$ and

$$\overline{\partial}\rho_0^{*0}\alpha^{(k)} = 0, \tag{6.6}$$

where now the adjoint is taken with respect to the model metrics. Let

$$u^{(k)} := -\overline{\partial}(k\rho^{(k)}\overline{\partial}k\rho^{(k)*}\alpha^{(k)})$$

and let $\widetilde{\alpha}^{(k)} := \alpha^{(k)} + \chi u^{(k)}$, where χ is the cut-off function defined above. Then, using that ρ vanishes on ∂X, we get that the $\overline{\partial}$-closed form $\widetilde{\alpha}^{(k)}$ satisfies the scaled $\overline{\partial}$-Neumann boundary conditions, *i.e.* the relation (6.5) on ∂X. Moreover, using that $k\rho^{(k)}$ converges to ρ with all derivatives on a fixed polydisc centered at 0 (Lemma 5.1) and (6.6) one can check that $u^{(k)}$ tends to zero with all derivatives in X_δ. Finally, since $\chi u^{(k)}$ is supported on a bounded set in X_δ and converges to zero with all derivatives it is not hard to see that $\widetilde{\alpha_k}$ also satisfies the properties (i) to (iv), where $\widetilde{\alpha_k} := F_k^{-1*}(\widetilde{\alpha^{(k)}})$. ☐

By projecting the sequence α_k of approximate extremals, from the previous lemma, on the space of low energy forms we will now obtain the following lower bound on $\mathbb{B}^q_{\leqslant \mu_k}$.

Proposition 6.4. — *There is a sequence μ_k tending to zero such that*

$$\liminf_k B^q_{\leqslant \mu_k, \theta_k}(0, iv/k) \geqslant B_{X_0}(0, v)_\theta.$$

Proof. — The proof is a simple modification of the proof of Proposition 5.3 in [4]. Let $\{\alpha_k\}$ be the sequence that the previous lemma provides and decompose it with respect to the orthogonal decomposition $\Omega^{0,q}(X,L^k) = \mathcal{H}^q_{\leqslant\mu_k}(X,L^k) \oplus \mathcal{H}^q_{>\mu_k}(X,L^k)$, induced by the spectral decomposition of the subelliptic operator Δ [15]:

$$\alpha_k = \alpha_{1,k} + \alpha_{2,k}.$$

First, we prove that

$$\lim_k \left|\alpha_2^{(k)}(0,iv)\right|^2 = 0. \tag{6.7}$$

Since $\alpha_2^{(k)} = \alpha^{(k)} - \alpha_1^{(k)}$ the form $\alpha_2^{(k)}$ satisfies $\overline{\partial}$-Neumann boundary conditions on the intersection of the polydisc D with $F_k^{-1}(\partial X)$, using Lemma 6.3. Subelliptic estimates as in the proof of Lemma 6.1 then show that

$$\left|\alpha_2^{(k)}(0)\right|^2 \leqslant C\left(\left\|\alpha_2^{(k)}\right\|^2_{D\cap F_k^{-1}(X)} + \left\|(\Delta^{(k)})\alpha_2^{(k)}\right\|^2_{D\cap F_k^{-1}(\partial X),m}\right) \tag{6.8}$$

for some large integer m. To see that the first term in the right hand side tends to zero, we first estimate $\left\|\alpha_2^{(k)}\right\|^2_{D\cap F_k^{-1}(X)}$ with $\|\alpha_{k,2}\|^2_X$ using the norm localization in Lemma 5.3. Next, by the spectral decomposition of Δ_k:

$$\|\alpha_{2,k}\|^2_X \leqslant \frac{1}{\mu_k}\left\langle \Delta_k\alpha_{2,k},\alpha_{2,k}\right\rangle_X \leqslant \frac{1}{\mu_k}\left\langle \Delta_k\alpha_k,\alpha_k\right\rangle_X \leqslant \frac{\delta_k}{\mu_k}\|\alpha_k\|^2_X\,,$$

using property (iv) in the previous lemma in the last step. By property (ii) in the same lemma $\|\alpha_k\|^2_X$ is asymptotically 1, which shows that the first term in (6.8) tends to zero if the sequence μ_k is chosen as $\delta_k^{1/2}$, for example. To see that the second term tends to zero as well, we estimate

$$\left\|(\Delta^{(k)})\alpha_2^{(k)}\right\|_m \leqslant \left\|(\Delta^{(k)})\alpha^{(k)}\right\|_m + \left\|(\Delta^{(k)})\alpha_1^{(k)}\right\|_m.$$

The first term in the right hand side tends to zero by (iii) in the previous lemma and so does the second term, using (6.2) (that holds for any scaled sequence of forms in $\mathcal{H}^q_{\leqslant\mu_k}(X,L^k)$). This finishes the proof of the claim (6.7). Finally,

$$\liminf_k B^q_{\leqslant\mu_k}(0,iv/k)_{\theta_k} \geqslant |\alpha_{1,k}(0)|^2_{\theta_k}$$

and

$$\liminf_k |\alpha_{k,1,\theta_k}(0)|^2 \geqslant B_{X_0,\theta}(0,v) + 0,$$

when k tends to infinity, using (6.7) and (i) in the previous lemma. □

Now we can prove that the Morse inequalities are essentially equalities for the space $\mathcal{H}^{0,q}_{\leqslant\mu_k}(X,L^k)$ of low-energy forms. But first recall that $\mathcal{H}^{0,q}_{\leqslant\mu_k}(X,L^k)$ depend on a large parameter R, since the metrics ω_k depend on R.

THEOREM 6.5. — *Suppose that X is a compact complex manifold with boundary satisfying condition $Z(q)$. Then there is a sequence μ_k tending to zero such that the limit of $k^{-n}\dim\mathcal{H}^{0,q}_{\leqslant\mu_k}(X,L^k)$ when k tends to infinity is equal to*

$$(-1)^q\Big(\frac{1}{2\pi}\Big)^n\Big(\int_{X(q)}\Theta_n+\int_{\partial X}\int_{T(q)_{\rho,x}}(\Theta+t\mathcal{L})_{n-1}\wedge\partial\rho\wedge dt\Big)+\varepsilon_R$$

where the sequence ε_R tends to zero when R tends to infinity.

Proof. — The proof is completely analogous to the proof of Theorem 6.5. In the first region one just replaces the claim (5.12) in the proof of Proposition 5.5 by the asymptotic equality for $B^q_{\leqslant\mu_k}(0,iv/k)_{\theta_k}$ obtained by combing the Propositions 6.1 and 6.4. Moreover, a simple modification of the proof of Proposition 6.1 shows that there is no contribution from the integrals over the second and third region, when R tends to infinity, as before. Finally, the convergence on the inner part of X was shown in [4].□

Recall that the Dolbeault cohomology group $H^{0,q}(X,L^k)$ is isomorphic to the space of harmonic forms, which is a subspace of $\mathcal{H}^{0,q}_{\leqslant\mu_k}(X,L^k)$. Hence, the previous theorem is stronger than the weak Morse inequalities for the dimensions $h^q(L^k)$ of $H^{0,q}(X,L^k)$ (Theorem 2.1). When X has no boundary Demailly showed that, by combining a version of Theorem 6.5 with some homological algebra, one gets strong Morse inequalities for the Dolbeault cohomology groups. These are inequalities for an alternating sum of all $h^i(L^k)$ when the degree i varies between 0 and a fixed degree q [9]. In fact, a variation of the homological algebra argument yields inequalities for alternating sums when the degree i varies between a fixed degree q and the complex dimension n of X (the two versions are related by Serre duality). However, when X has a boundary one has to impose certain curvature conditions on ∂X to obtain strong Morse inequalities from Theorem 6.5. Indeed, to apply the theorem one has to assume that ∂X satisfies condition $Z(i)$ for all degrees i in the corresponding range. In particular the corresponding dimensions will then be finite dimensional so that the alternating sum makes sense. Now, to state the strong holomorphic Morse inequalities for a manifold with boundary, recall that the boundary of a compact complex manifold is called q-convex if the Levi form $\mathcal{L}$ has at least $n-q$ positive eigenvalues along $T^{1,0}(\partial X)$ and it is called q-concave if the Levi form has at least $n-q$ negative eigenvalues along $T^{1,0}(\partial X)$ (*i.e.* ∂X is q-convex "from the inside" precisely when it is q-concave "from the outside"). We will denote by $X(\geqslant q)$ the union of all sets $X(i)$ with $i \geqslant q$ and $T(\geqslant q)_{\rho,x}$ is defined similarly. The sets $X(\leqslant q)$ and $T(\leqslant q)_{\rho,x}$ are defined by putting $i \leqslant q$ in the previous definitions. Finally, we set

$$I_{\geqslant q} := \left(\frac{1}{2\pi}\right)^n \left(\int_{X(\geqslant q)} \Theta_n + \int_{\partial X} \int_{T(\geqslant q)_{\rho,x}} (\Theta + t\mathcal{L})_{n-1} \wedge \partial\rho \wedge dt\right)$$

and define $I_{\leqslant n-1-q}$ similarly.

THEOREM 6.6. — *Suppose that X is an n-dimensional compact manifold with boundary. If the boundary is strongly q-convex, then*

$$k^{-n} \sum_{i=q}^{n} (-1)^{q-j} h^j(L^k) \leqslant I_{\geqslant q} k^n + o(k^n).$$

If X has strongly q-concave boundary, then

$$\sum_{i=0}^{n-1-q} (-1)^{q-j} h^j(L^k) \leqslant I_{\leqslant n-1-q} k^n + o(k^n).$$

Proof. — First note that if ∂X is q-convex, then ∂X satisfies condition $Z(i)$ for i such that $n-q \leqslant i \leqslant n$. Similarly, if ∂X is q-concave, then ∂X satisfies condition $Z(i)$ for i such that $0 \leqslant i \leqslant n-q-1$. The proof then follows from Theorem 6.5 and the homological algebra argument in [9], [10]. See also [7] and [22]. □

6.1. Strong Morse inequalities on open manifolds.

One can also define q-convexity and q-concavity on open manifolds following Andreotti and Grauert [2]. First, one says that a function ρ is q-convex if $i\partial\overline{\partial}\rho$ has at least $n-q+1$ positive eigenvalues. Next, an open manifold Y is said to be q-convex if it has an exhaustion function ρ that is q-convex outside some compact subset K of Y. The point is that the regular sublevel sets of ρ are then q-convex considered as compact manifolds with boundary. The extra positive eigenvalue occurring in the definition of q-convexity for an open manifold is needed to make sure that $i\partial\overline{\partial}\rho$ still has at least $n-q$ positive eigenvalues along a regular level surface of ρ. Finally, an open manifold Y is said to be q-concave if it has an exhaustion function ρ such that $-\rho$ is q-convex outside some compact subset K of Y.

Now, by Remark 1.3, Theorem 6.6 extends to any q-convex open manifold Y with a line bundle L if one uses the usual Dolbeault cohomology $H^{0,*}(Y, L^k)$ (or equivalently the sheaf cohomology $H^*(Y, \mathcal{O}(L^k))$ and the curvature integrals are taken over a regular level surface of ρ in the complement of the compact set K. However, for a q-concave open manifold Y one only gets the corresponding result if $n-q-1$ is replaced with $n-q-2$. Indeed, by Remark 1.3 one has to make sure that condition $Z(i+1)$ holds for the highest degree i occurring in the alternating sum. In this form the q-convex case and q-concave case was obtained by Bouche [7] and Marinescu [22], respectively, under the assumption that the curvature of the line bundle L is adapted to the curvature of the boundary of X in a certain way. Comparing with Theorem 6.6 their assumptions imply that the boundary integral vanishes. There is also a very recent preprint [23] of Marinescu where strong Morse inequalities on a q-concave manifold with an arbitrary line bundle L are obtained. However, the corresponding boundary term is not as precise as the one in Theorem 6.6 and in Section 7 we will show that Theorem 6.6 is sharp.

Note that since the curvature integrals are taken over *any* regular level surface of ρ in Y one expects that $I(\geqslant q)$ and $I(\leqslant n-1-q)$ are independent of the level surface. This is indeed the case (see Remark 7.4).

6.2. Application to the volume of semi-positive line bundles.

Now assume that X is a strongly pseudoconcave manifold X with a semi-positive line bundle L (*i.e.* the Levi form $\mathcal{L}$ is negative along $T^{1,0}(\partial X)$ and the curvature form of L is semi-positive in X). The case of pseudoconcave surfaces has been recently studied in [18, 13], by different methods. When the dimension of X is at least three, the strong Morse inequalities give a lower bound on the dimension of the space of holomorphic sections with values in L^k. Namely, $h^0(L^k)$ is asymptotically bounded from below by

$$(6.9)\quad \Big(\frac{1}{2\pi}\Big)^n\Big(\int_{X(0)}\Theta_n+\int_{\partial X}\int_{T(\leqslant 1)}(\Theta+t\mathcal{L})_{n-1}\wedge\partial\rho\wedge dt\Big)k^n+h^1(L^k)+o(k^n).$$

In particular, if the curvatures are such that the coefficient in front of k^n is positive, then the dimension of $H^0(X, L^k)$ grows as k^n. In other words, the line bundle L is *big* then. For example, this happens when the curvature forms are conformally equivalent along the complex tangential directions, *i.e.* if there is a function f on ∂X such that

$$\mathcal{L} = -f\Theta \tag{6.10}$$

when restricted to $T^{1,0}(\partial X) \otimes T^{0,1}(\partial X)$. In fact, by multiplying the original ρ by f^{-1} we may and will assume that $f = 1$. The lower bound (6.9) combined with the upper bound from the weak Morse inequalities (Theorem 2.1) then gives the following corollary.

Corollary 6.7. — *Suppose that X is a strongly pseudoconcave manifold X of dimension $n \geqslant 3$ with a semi-positive line bundle L. Then if the curvature forms are conformally equivalent at the boundary*

$$h^0(L^k) = k^n \Big(\frac{1}{2\pi}\Big)^n \Big(\int_X \Theta_n + \frac{1}{n}\int_{\partial X} (i\partial\overline{\partial}\rho)_{n-1} \wedge i\partial\rho\Big) + o(k^n).$$

When L is positive, the conformal equivalence in the previous corollary says that the symplectic structure on X determined by L is compatible with the contact structure of ∂X determined by the complex structure, in a strong sense (compare [13]) and the conclusion of the corollary may be expressed by the formula

$$\mathrm{Vol}(L) = \mathrm{Vol}(X) + \frac{1}{n}\mathrm{Vol}(\partial X) \tag{6.11}$$

in terms of the symplectic and contact volume of X and ∂X, respectively (where the volume of a line bundle L is defined as the lim sup of $(2\pi)^n k^{-n} h^0(L^k)$ [21]). The factor $\frac{1}{n}$ in the formula is related to the fact that if $(X_+, d\alpha)$ is a $2n$-dimensional real symplectic manifold with boundary, such that α is a contact form for ∂X_+, then, by Stokes theorem, the contact volume of ∂X_+ divided by n is equal to the symplectic volume of X_+. In fact, this is how we will show that (6.11) is compatible with hole filling in Section 7.1.

7. Sharp examples and hole filling.

In this section we will show that the leading constant in the Morse inequalities 6.5 is sharp. When X is a compact manifold without boundary, this is well-known. Indeed, let X be the n-dimensional flat complex torus $\mathbb{C}^n/\mathbb{Z}^n + i\mathbb{Z}^n$ and consider the Hermitian holomorphic line bundle L_λ over T^n determined by the constant curvature form

$$\Theta = \sum_{i=1}^{n} \frac{i}{2}\lambda_i dz_i \wedge \overline{dz}_i,$$

where λ_i are given non-zero integers [17]. Then one can show (see the remark at the end of the section) that

$$B^q(x) \equiv \frac{1}{\pi^n} 1_{X(q)} \left|\det_\omega \Theta\right|, \tag{7.1}$$

where $1_{X(q)}$ is identically equal to one if exactly q of the eigenvalues λ_i are negative and equal to zero otherwise. This shows that the leading constant in the Morse inequalities on a compact manifold is sharp.

Let us now return to the case of a manifold with boundary. We let X be the manifold obtained as the total space of the unit disc bundle in the dual of the line bundle L_μ (where L_μ is defined as above) over the torus T^{n-1}, where μ_i are $n-1$ given non-zero integers. Next, we define a Hermitian holomorphic line bundle over X. Denote by π the natural projection from X onto the torus T^{n-1}. Then the pulled back line bundle $\pi^* L_\lambda$ is a line bundle over X. The construction is summarized by the following commuting diagram

$$\begin{array}{ccccc} \pi^* L_\lambda & & & & L_\lambda \\ \downarrow & & & & \downarrow \\ X & \hookrightarrow & L_\mu^* & \to & T^{n-1}. \end{array}$$

Let h be the positive real-valued function on X, defined as the restriction to X of the squared fiber norm on L_μ^*. Then $\rho := \ln h$ is a defining function for X close to the boundary and we define a Hermitian metric ω on X by

$$\omega = \frac{i}{2}\partial\overline{\partial}\left|z\right|^2 + \frac{i}{2} h^{-1}\partial h \wedge \overline{\partial} h$$

extended smoothly to the base T^{n-1} of X. The following local description of the situation is useful. The part of X that lies over a fundamental domain of T^{n-1} can be represented in local holomorphic coordinates (z, w), where w is the fiber coordinate, as the set of all (z, w) such that

$$h(z,w) = \left|w\right|^2 \exp\Big(+ \sum_{i=1}^{n-1} \mu_i \left|z_i\right|^2 \Big) \leqslant 1$$

and the fiber metric ϕ for the line bundle $\pi^* L_\lambda$ over X may be written as

$$\phi(z,w) = \sum_{i=1}^{n-1} \lambda_i \left|z_i\right|^2.$$

The proof of the following proposition is very similar to the proof of Theorem 4.5, but instead of Fourier transforms we will use Fourier series, since the $\mathbb{R}$-symmetry is replaced by an S^1-symmetry (the model domain X_0 in Section 4 is the universal cover of X defined above).

Theorem 7.1. — *Let $J(q)$ be the set of all integers j such that the form $\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho$ has exactly q negative eigenvalues. Then*

$$B_X^q = \left(\frac{1}{2\pi}\right)^n \sum_{j\in J(q)} \det{}_\omega\left(i(\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho)\right)\frac{1}{2}(j+1)h^j. \tag{7.2}$$

In particular, the dimension of $H^{0,q}(X, \pi^* L_\lambda)$ is given by

$$\left(\frac{i}{2\pi}\right)^n \int_{\partial X} \sum_{j\in J(q)} (\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho)_{n-1} \wedge \partial\rho \tag{7.3}$$

and the limit of the dimensions of $H^{0,q}(X, (\pi^* L_\lambda)^k)$ divided by k^n is

$$\left(\frac{i}{2\pi}\right)^n \int_{\partial X} \int_{T_{x,\rho}(q)} (\partial\overline{\partial}\phi + t\partial\overline{\partial}\rho)_{n-1} \wedge \partial\rho \wedge dt. \tag{7.4}$$

Proof. — First note that if α_j is a form on T^{n-1} with values in $L_\mu^j \otimes L_\lambda$, then

$$\alpha(z, w) := \sum_{j\geqslant 0} \alpha_j(z) w^j$$

defines a global form on X with values in $\pi^* L_\lambda$. The proof of Proposition 4.1 can be adapted to the present situation to show that any form α in $\mathcal{H}^{0,q}(X, \phi)$ is of this form with α_j in $\mathcal{H}^{0,q}(T^{n-1}, L_\mu^j \otimes L_\lambda)$. Actually, since X is a fiber bundle over T^{n-1} with *compact* fibers one can also give a somewhat simpler proof. For example, to show that α is tangential one solves the $\overline{\partial}$- equation along the fibers of closed discs in order to replace the normal part α_N with an exact form. Then using the assumption the α is coclosed one shows that the exact form must vanish. The details are omitted. We now have the following analog of Proposition 4.3 for any α in $\mathcal{H}^{0,q}(X, \phi)$

$$(\alpha, \alpha) = 2\pi \sum_j (\alpha_j, \alpha_j) b_j, \;\; b_j = \int_0^1 (r^2)^j r dr = 1/2(j+1)^{-1} \tag{7.5}$$

in terms of the induced norms. To see this, one proceed as in the proof of Proposition 4.3, now using the Taylor expansion of α. Writing $\psi(z) = \sum_{i=1}^{n-1} \mu_i |z_i|^2$ and restricting z to the fundamental region of T^{n-1} we get that (α, α) is given by

$$\int_{|w|^2 < e^{-\psi(z)}} \left|\sum_j \alpha_j(z) w^j\right|^2 e^{-\phi(z)} \left(\frac{i}{2}\partial\overline{\partial}|z|^2\right)_{n-1} e^{\psi(z)} r dr d\theta.$$

Now using Parceval's formula for Fourier series in the integration over θ this can be written as

$$2\pi \sum_j \int_z |\widehat{\alpha}_j(z)|^2 e^{-\phi(z)} \left(\frac{i}{2}\partial\overline{\partial}|z|^2\right)_{n-1} e^{\psi(z)} \int_0^{e^{-\psi(z)/2}} (r^2)^j r dr.$$

Finally, the change of variables $r' = e^{\psi(z)/2}r$ in the integral over r gives a factor $e^{-j\psi(z)}$ and the upper integration limit becomes 1. This proves (7.5).

As in the proof of Theorem 4.5 we infer that B_X may be expanded as

$$B_X(z,w) = \frac{1}{2\pi}\sum_j B_j(z)h^j b_j^{-1},$$

where B_j is the Bergman function of the space $\mathcal{H}^{0,q}(T^{n-1}, L_\mu^j \otimes L_\lambda)$. According to (7.1), we have that

$$B_j(z) \equiv \left(\frac{1}{2\pi}\right)^{n-1}\delta_{j,J(q)}\left|\det_\omega(i(\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho))\right|,$$

where $J(q) = \{j : \operatorname{index}(\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho) = q\}$and where the sequence $\delta_{j,J(q)}$ is equal to 1 if $j \in J(q)$ and zero otherwise. Thus, (7.2) is obtained. Integrating (7.2) over X gives

$$\int_X B_X\omega_n = \frac{1}{2\pi}\int_{T^{n-1}}\sum_j B_j\left(\frac{i}{2}\partial\overline{\partial}\,|z|^2\right)_{n-1}\int_0^{2\pi}d\theta\int_0^1 (r^2)^j r dr b_j^{-1}.$$

The integral over the radial coordinate r is cancelled by b_j^{-1} and we may write the resulting integral as

$$\frac{1}{2\pi}\int_{\partial X}\sum_j B_j\omega_{n-1}\wedge i\partial\rho = \left(\frac{i}{2\pi}\right)^n\int_{\partial X}\sum_{j\in J(q)}(\partial\overline{\partial}\phi + j\partial\overline{\partial}\rho)_{n-1}\wedge\partial\rho.$$

Hence, (7.3) is obtained. Finally, applying the formula (7.3) to the line bundle $(\pi^*L_\lambda)^k = \pi^*(L_\lambda^k)$ shows, since the curvature form of $\pi^*(L_\lambda^k)$ is equal to $k\partial\overline{\partial}\phi$, that

$$k^{-n}\int_X B_X^{q,k}\omega_n = \left(\frac{i}{2\pi}\right)^n\int_{\partial X}\sum_j(\partial\overline{\partial}\phi + \frac{j}{k}\partial\overline{\partial}\rho)^{n-1}\frac{1}{k}\wedge\partial\rho,$$

where the sum is over all integers j such that $\partial\overline{\partial}\phi + \frac{j}{k}\partial\overline{\partial}\rho$ has exactly q negative eigenvalues. Observe that the sum is a Riemann sum and when k tends to infinity we obtain (7.4). □

Note that since the line bundle π^*L_λ over X is flat in the fiber direction the integral over X in (1.2) vanishes. Hence, the theorem above shows that the holomorphic Morse inequalities are sharp. The most interesting case covered by the theorem above is when the line bundle π^*L_λ (simply denoted by L) over X is semi-positive, and positive along the tangential directions, and X is strongly pseudoconcave. This happens precisely when all λ_i are positive and all μ_i are negative. Then, for $n \geqslant 3$, the theorem above shows that the dimension of $H^{0,1}(X, L^k)$ grows as k^n unless the curvature of L is a multiple of the Levi curvature of the boundary, *i.e.* unless

λ and μ are parallel as vectors. This is in contrast to the case of a manifold without boundary, where the corresponding growth is of the order $o(k^n)$ for a semi-positive line bundle. Note that the bundle L above always admits a metric of *positive* curvature. Indeed, the fiber metric $\phi + \varepsilon h$ on L can be seen to have positive curvature, if the positive number ε is taken sufficiently small. However, if λ and μ are not parallel as vectors, there is no metric of positive curvature which is conformally equivalent to the Levi curvature at the boundary. This follows from the weak holomorphic Morse inequalities, Theorem 2.1, since the growth of the dimensions of $H^{0,1}(X, L^k)$ would be of the order $o(k^n)$ then.

Remark 7.2. — To get examples of open manifold Y as described in Remark 1.3 one may take the total space of the line bundle L_μ^* over T^{n-1}, as defined in the beginning of the section. Then ρ is an exhaustion function, exhausting L_μ^* by disc bundles. Furthermore, to get examples of manifolds with boundary X where the index of the Levi curvature form is non-constant one may take X to be an annulus bundle in L_μ^*. Such a manifold is neither q-convex or q-concave for any q. Theorem 7.1 extends to such manifolds X if one uses Laurent expansions of sections instead of Taylor expansions. A concrete example is given by the hyper plane bundle $O(1)$ over $\mathbb{P}^{n-1}$. Then the corresponding annulus bundle is biholomorphic to a spherical shell in $\mathbb{C}^n$, *i.e.* all z in $\mathbb{C}^n$ such that $r \leqslant |z| \leqslant r'$ for some given numbers r and r'. It has one pseudoconvex and one pseudoconcave boundary component.

Finally, a remark about the proof of formula (7.1).

Remark 7.3. — To prove formula (7.1) one can for example reduce the problem to holomorphic sections, *i.e.* when $q = 0$ (compare [4]). One could also use symmetry to first show that the Bergman kernel is constant and then compute the dimension of $H^q(T^n, L_\lambda)$ by standard methods. To compute the dimension one writes the line bundle L_λ as $L_\lambda = \pi_1^* L^{\lambda_1} \otimes \pi_2^* L^{\lambda_2} \otimes \cdots \otimes \pi_n^* L^{\lambda_n}$, using projections on the factors of T^n, where L is the classical line bundle over the elliptic curve $T^1 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$, such that $H^0(\mathbb{C}/\mathbb{Z}+i\mathbb{Z}, L)$ is generated by the Riemann theta function [17]. Now, using Kunneth's theorem one gets that $H^q(T^n, L_\lambda)$ is isomorphic to the direct sum of all tensor products of the form

$$H^1(T^1, L^{\lambda_{i_1}}) \otimes \cdots \otimes H^1(T^1, L^{\lambda_{i_q}}) \otimes H^0(T^1, L^{\lambda_{i_{q+1}}}) \otimes \cdots \otimes H^0(T^1, L^{\lambda_{i_n}}).$$

Observe that this product vanishes unless the index $I = (i_1, \ldots, i_n)$ is such that the first q indices are negative while the others are positive.

Indeed, first observe that if m is a positive integer, the dimension of $H^0(T^1, L^{-m})$ vanishes, since L^{-m} is a negative line bundle. Next, by Serre duality $H^1(T^1, L^m) \approxeq H^0(T^1, L^{-m})$, since the canonical line bundle on T^1 is trivial. So the dimension of $H^1(T^1, L^m)$ vanishes as well. In particular, the dimension of $H^q(T^n, L_\lambda)$ vanishes unless exactly q of the numbers λ_i are negative, *i.e.* unless the index of the curvature of L_λ is equal to q. Finally, if the index is equal to q, then, using that $H^0(T^1, L)$ is one-dimensional, combined with Serre duality and Kunneth's formula again, one gets that the dimension of $H^0(T^1, L^{-1})$ is equal to the absolute value of the product of all eigenvalues λ_i. This proves (7.1).

7.1. Relation to hole filling and contact geometry.

Consider a compact strongly pseudoconcave manifold X with a semi-positive line bundle L. We will say that the pair (X, L) may be *filled* if there is a compact complex manifold $\widetilde{X}$, without boundary, with a semi-positive line bundle $\widetilde{L}$ such that there is a holomorphic line bundle injection of L into $\widetilde{L}$. (2) The simplest situation is as follows. Start with a compact complex manifold $\widetilde{X}$ with a positive line bundle $\widetilde{L}$ (by the Kodaira embedding theorem $\widetilde{X}$ is then automatically a projective variety [17]). We then obtain a pseudoconcave manifold X by making a small hole in $\widetilde{X}$ in the following way. Consider a small neighborhood of a fixed point x in $\widetilde{X}$, holomorphically equivalent to a ball in $\mathbb{C}^n$, where $\widetilde{L}$ is holomorphically trivial and let ϕ be the local fiber metric. We may assume that $\phi(x) = 0$ and that ϕ is non-negative close to x. Then for a sufficiently small ε the set where ϕ is strictly less than ε is a strongly pseudoconvex domain of $\widetilde{X}$ and its complement is then a strongly pseudoconcave manifold that we take to be our manifold X. We let L be the restriction of $\widetilde{L}$ to X. A defining function of the boundary of X can be obtained as $\rho = -\phi$. Now, since $\widetilde{L}$ is a positive line bundle it is well-known that

$$\lim_k k^{-n} \dim_{\mathbb{C}} H^0(\widetilde{X}, \widetilde{L}^k) = \Big(\frac{i}{2\pi}\Big)^n \frac{1}{n!} \int_{\widetilde{X}} (\partial\overline{\partial}\widetilde{\phi})^n.$$

In fact, this holds for any semi-positive line bundle, as can be seen by combining Demailly's holomorphic Morse inequalities 1.1 with the Riemann-Roch theorem (this was first proved by different methods in [27]).

On the other hand we have by Harthog's phenomena (assuming that $n \geqslant 2$), that $H^0(\widetilde{X}, \widetilde{L}^k)$ is isomorphic to $H^0(X, L^k)$. So decomposing the

(2) By a theorem of Rossi [24], the pair (X, L) may always be filled if L is trivial close to the boundary and the dimension of X is at least 3.

integral above with respect to

$$\widetilde{X} = X \bigsqcup X^c \tag{7.6}$$

and using Stokes theorem gives that

$$\lim_k k^{-n} \dim_{\mathbb{C}} H^0(X, L^k) = \Big(\frac{i}{2\pi}\Big)^n \Big(\frac{1}{n!}\int_X (\partial\overline{\partial}\phi)_n - \frac{1}{n!}\int_{\partial X} (\partial\overline{\partial}\phi)^{n-1} \wedge \partial\phi\Big). \tag{7.7}$$

Let us now compare the boundary integral above with the curvature integral in the holomorphic Morse inequalities (1.2). Since $\rho = -\phi$ this integral equals

$$-\frac{1}{(n-1)!}\int_{\partial X\times[0,1]} ((1-t)\partial\overline{\partial}\phi)^{n-1} \wedge \partial\phi \wedge dt,$$

which coincides with the boundary integral in (7.7) since $\int_o^1 (1-t)^{n-1} dt = 1/n$. This shows that the holomorphic Morse inequalities, Theorem 6.5 are sharp for the line bundle L over X. To show that the Morse inequalities are sharp as soon as a pair (X, L) may be filled by a Stein manifold it is useful to reformulate the boundary term in (1.2) in terms of the contact geometry of the boundary ∂X.

Let us first recall some basic notions of contact geometry [3]. The distribution $T^{1,0}(\partial X)$ can be obtained as $\ker(-i\partial\rho)$ and since, by assumption, the restriction of $d(-i\partial\rho)$ is non-degenerate it defines a so-called *contact* distribution and ∂X is hence called a *contact manifold*. By duality $T^{1,0}(\partial X)$ determines a real line bundle in the real cotangent bundle $T^*(\partial X)$ that can be globally trivialized by the form $-i\partial\rho$. Denote by X_+ the associated fiber bundle over ∂X of "positive" rays and denote by α the tautological one form on $T^*(\partial X)$, so that $d\alpha$ is the standard symplectic form on $T^*(\partial X)$. The pair $(X_+, d\alpha)$ is called the *symplectification* of the contact manifold ∂X in the literature [3]. More concretely,

$$X_+ = \big\{t(-i\partial\rho_x) : x \in \partial X, t \geqslant 0\big\},$$

i.e. X_+ is isomorphic to $\partial X \times [0, \infty[$ and $\alpha = -it\partial\rho$ so that $d\alpha = i(t\partial\overline{\partial}\rho + \partial\rho \wedge dt)$. The boundary integral in (1.2) may now be compactly written as

$$\int_{X_+(q)} (\Theta + d\alpha)_n,$$

where $X_+(q)$ denotes the part of X_+ where the pushdown of $d\alpha$ to ∂X has exactly q negative eigenvalues along the contact distribution $T^{1,0}(\partial X)$.

Let us now assume that X is strongly pseudoconcave and that (X, L) is filled by $(\widetilde{X}, L)$ (abusing notation slightly). We will also assume that the strongly pseudoconvex manifold Y, in $\widetilde{X}$, obtained as the closure of the complement of X in $\widetilde{X}$, has a defining function that we write as $-\rho$ which is plurisubharmonic on Y. We may assume that the set of critical points of $-\rho$ on Y is finite and to simplify the notation in the argument we assume that there is exactly one critical point x_0 in Y and we assume that $\rho(x_0) = 1$ (the general argument is the same). For a regular value c of ρ we let $X_+(0)_c$ be the subset of the symplectification of $\rho^{-1}(c)$ defined as above, thinking of $\rho^{-1}(c)$ as a strictly pseudoconcave boundary. Now consider the following manifold with boundary:

$$\mathcal{X}_\varepsilon(0) = \bigcup_{c\in[0,1-\varepsilon]} X_+(0)_c.$$

More concretely, $\mathcal{X}_\varepsilon(0)$ can be identified with a subset of the positive closed cone in $T^*(Y, \mathbb{C})$ determined by $\partial\rho$:

$$\big\{t(-i\partial\rho_x) : x \in Y, t \geqslant 0\big\}.$$

Hence, $\mathcal{X}_\varepsilon(0)$ is a fiber bundle over a subset of Y and when ε tends to zero, the base of $\mathcal{X}_\varepsilon(0)$ tends to Y. Note that the fibers of $\mathcal{X}_\varepsilon(0)$ are a finite number of intervals and the induced function t on $\mathcal{X}_\varepsilon$ is uniformly bounded with respect to ε (*i.e.* the "height" of the fiber is uniformly bounded). Indeed, we have assumed that $i\partial\overline{\partial}\rho$ is strictly negative. This forces $\Theta + ti\partial\overline{\partial}\rho$ to be negative on all of Y for all t larger then some fixed number t_0. In particular such a t is not in $T_x(0)$ for any x in Y, *i.e.* not in any fiber of $\mathcal{X}_\varepsilon(0)$. Now observe that the form $\Theta + d\alpha$ on $X_+(0)$ extends to a closed form in $\mathcal{X}_\varepsilon(0)$ and the restriction of the form to Y coincides with Θ. Let us now integrate the form $(\Theta + d\alpha)_n$ over the boundary of $\mathcal{X}_\varepsilon(0)$. The boundary can be written as

$$\partial(\mathcal{X}_\varepsilon(0)) = X_+(0) \cup \Big(\bigcup_{c\in]0,1-\varepsilon[} \partial(X_+(0)_c)\Big) \cup X_+(0)_\varepsilon.$$

Since the form is closed, the integral over $\partial(\mathcal{X}_\varepsilon(0))$ vanishes according to Stokes theorem, giving

$$0 = \int_{X_+(0)} (\Theta + d\alpha)_n - \int_Y \Theta_n + 0 + O(\varepsilon)$$

where the zero contribution comes from the fact that the form $(\Theta + d\alpha)_n$ vanishes along $\big(\bigcup_{c\in]0,1-\varepsilon[} \partial(X_+(0)_c)\big) - Y$. The term $O(\varepsilon)$ comes from the integral (of a uniformly bounded function) over the "cylinder" $X_+(0)_\varepsilon$ around the point x_0. Finally, by letting ε tend to zero we see that the Morse inequalities for L over X are sharp in this situation as well.

Remark 7.4. — The preceding argument also shows that if ρ is a function on an open manifold Y with regular values c and c' (where c is less than c'), then

$$\int_{X_+(i)_c} (\Theta + d\alpha)_n = \int_{\rho^{-1}]c,c']} \Theta_n + \int_{X_+(i)_{c'}} (\Theta + d\alpha)_n$$

for all i such that $i \geqslant q$, if ρ is q-convex on $\rho^{-1}]c, c']$. In other words, the right hand sides in the weak Morse inequalities for $\rho^{-1}(\leqslant c)$ and $\rho^{-1}(\leqslant c')$ coincide. The analogous statement also holds in the q-concave case.

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