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ZYGMUND'S PROGRAM: SOME PARTIAL SOLUTIONS

by Alexander STOKOLOS (*)

1. Introduction.

Let B be a differential basis in $\mathbb{R}^n$, which we can treat as a family of open bounded sets covering $\mathbb{R}^n$, and let $M_B f$ be the corresponding maximal function:

$$M_B f(x) = \sup_{B \ni I \ni x} |I|^{-1} \int_I |f(y)|\, dy.$$

In many areas of harmonic analysis a key role is played by the so-called *weak type estimates* for $M_B f$, in particular the weak type $(1,1)$ estimate

$$|\{M_B f > \lambda\}| \leqslant C \frac{\|f\|_1}{\lambda}, \qquad \lambda > 0, \tag{1}$$

and the weak type $L \log^+ L$ estimate

$$|\{M_B f > \lambda\}| \leqslant C \int \frac{|f|}{\lambda} \left(1 + \log^+ \frac{|f|}{\lambda}\right) dx, \qquad \lambda > 0. \tag{2}$$

If $M_B f$ satisfies (1) or (2), then we also say that the basis B has the corresponding weak type.

Roughly speaking, weak type estimates are quantitative forms of a.e. convergence statements. With different levels of specification, this theory is presented in [1, 9, 10, 11, 13, 16, 18, 21] *etc.*

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An excellent explanation of the place of weak type estimates in harmonic analysis is given in E. M. Stein's encyclopedic monograph [17].

Let us only mention that the $(1,1)$ estimate is the best possible estimate which may be required from a maximal function. It holds for the basis of all possible cubic intervals, but does not hold even for the rectangular intervals in $\mathbb{R}^n$, $n \geqslant 2$.

Therefore it is vital to characterize those bases whose maximal function has weak type $(1,1)$. For homothety invariant bases this was done in [9]. Namely, for such bases the maximal function has weak type $(1,1)$ if the elements of the basis are comparable with balls in measure, i.e. each $R \in B$ may be embedded in a ball D such that $|B| \geqslant c|D|$ with some universal constant $c > 0$.

Now let us consider the same question for translation invariant bases, which we will call *TI-bases* for brevity. In spite of similarity of statement, this problem is much more complicated. Therefore we restrict ourselves to TI-bases of *multidimensional intervals* (i.e. Cartesian products of one-dimensional intervals), which are quite important in applications (*cf.* [3, 5, 6, 7, 8, 19, 21]).

It is well known that a basic method of investigating the properties of maximal operators is the study of the covering properties of the corresponding bases. Moreover, properties of the maximal operator depend on the covering properties of a certain finite family $\{R_\alpha\}$. The idea is to choose a subfamily whose union has measure comparable to the measure of the original family, but with as limited overlap as possible. Obviously, the best one can hope for is no overlap at all. For bases consisting of cubes this can be achieved by application of a selection procedure going back to Vitali and described in detail e.g. in [16]. Namely, we say that a basis B has the *Vitali covering property* if there are constants c and C such that every finite family $\{R_\alpha \mid \alpha \in A\} \subset B$ has a subfamily $\{R_\alpha \mid \alpha \in A'\}$ such that

$$\Big|\bigcup_{\alpha\in A} R_\alpha\Big| \leqslant c\Big|\bigcup_{\alpha\in A'} R_\alpha\Big| \quad \text{and} \quad \Big\|\sum_{\alpha\in A'} \chi_{R_\alpha}\Big\|_\infty \leqslant C. \tag{3}$$

It is well known that (3) implies (1). P. Hagelstein [12] conjectured that (1) and (3) are equivalent.

In this article we show the equivalence for the TI-basis of intervals.

Families of multidimensional cubes satisfy (3), which can be easily shown by applying the Vitali selection procedure. However, the only property of cubes used in this proof is that for any two cubes, there is always a translation placing one of them inside the other. This justifies introducing the following terminology.

Let us call two intervals I and I' *comparable* and write $I \sim I'$ if there exists a translation placing one of them inside the other. In the opposite case we call them *incomparable* and write $I \not\sim I'$. In other words, if $I = I_1 \times \cdots \times I_n$ and $I' = I_1' \times \cdots \times I_n'$ then $I \sim I'$ means that either $|I_j| \leqslant |I_j'|$ or $|I_j| \geqslant |I_j'|$ for all $j = 1, \ldots, n$. We call a family of intervals *monotonic* if it consists of pairwise comparable intervals. The above considerations can be summarized in

Claim 1. — *If a basis B has the property that for some fixed natural m, any finite subfamily of B decomposes into at most m monotonic subfamilies, then B has weak type* $(1,1)$.

For a long time, the Vitali selection had been the only selection procedure (up to inessential modifications) used to investigate maximal operators. As shown by H. Bohr (see Note 1 in [2]), this procedure does not apply to the basis of multidimensional intervals. However, if we majorize the n-dimensional *strong maximal function* $\mathcal{M}f$, i.e. the maximal function associated with the basis of multidimensional intervals, by the composition of one-dimensional maximal functions in the directions of the coordinate axes, then iterative arguments yield the desired result also in this case. Such considerations led to the proof of the well known Jessen–Marcinkiewicz–Zygmund theorem, stating the differentiation of integrals of functions in $L(\log^+ L)^{n-1}(\mathbb{R}^n)$. A quantitative form of this theorem, the weak type estimate

$$|\{\mathcal{M}f > \lambda\}| \leqslant C \int \frac{|f|}{\lambda}\left(1 + \log^+ \frac{|f|}{\lambda}\right)^{n-1} dx, \qquad \lambda > 0, \tag{4}$$

was obtained much later by M. de Guzmán (see [9, 10]), again by iterative considerations.

A geometric proof of the JMZ theorem was finally given due to deep studies of A. Córdoba and R. Fefferman. In [4] they proved that the basis of all n-dimensional dyadic intervals has the exponential covering property of type $n-1$ (see also [5]).

Due to the important role played by maximal functions in the theory of singular integrals and in harmonic analysis in general, A. Zygmund initiated a program of investigating maximal functions which cannot be reduced to the case of product operators.

A classical example of a product operator is the Poisson integral for biharmonic functions on the product of two halfplanes. One can majorize the maximal Poisson operators by the strong maximal function and prove that $\mathcal{P}_y \star f(x) \to f(x)$ a.e. as $|y| \to 0+$, $y = (y_1, y_2)$, where $P_t(u)$ is the standard Poisson kernel and $\mathcal{P}_y(x) = P_{y_1}(x_1)P_{y_2}(x_2)$. The proof is based on splitting the kernel into pieces according to $|y_1| \sim 2^{-n}$, $|y_2| \sim 2^{-m}$ (see e.g. [18]).

This is a very special case of other naturally occurring examples, such as Poisson integrals for symmetric spaces. In those spaces the above approach leads to maximal functions of special types. For example, in the simplest case of a Siegel domain with boundary space of real symmetric 2×2 matrices, the maximal function associated with the basis of 3D intervals with side lengths $t, s, \sqrt{ts}$ appears.

Since this set of intervals is a 2-parameter collection, A. Zygmund conjectured that the corresponding maximal function has the same weak type as the 2D strong maximal function. This conjecture has been proved by A. Córdoba [3]. Actually, A. Córdoba solved Zygmund's problem in a more general setting corresponding to the case of intervals with side lengths $t, s, \phi(t, s)$ with ϕ monotonic in t and in s.

This justifies the consideration of the following bases. Let $\phi_1, \dots, \phi_n$ be n positive real functions of k variables, increasing in each variable separately and assuming arbitrarily small values. Consider the basis $\mathcal{B}$ of all intervals in $\mathbb{R}^n$ whose side length in the i-th direction is determined by the function ϕ_i. It has been conjectured that the differentiability properties of such a basis should only depend upon the number of parameters, degrees of freedom in some sense, involved. In particular, the basis should behave not worse than the basis of all intervals in $\mathbb{R}^k$; that is, it should differentiate the Orlicz class $L(\log^+ L)^{k-1}$, independently of the choice of the functions $\phi_1, \dots, \phi_n$.

Córdoba's result gives some support for this conjecture. However, it turned out that in such general setting the conjecture is false. F. Soria[15] gave a counterexample in the simplest possible case, when $n = 3$ and $k = 2$. He also explored some positive results, the possibility of some aspects of the conjecture remaining true. In all these examples we are forced to assume extra conditions on the ϕ_i's.

These results can be included among the cornerstones of the theory of differentiation of integrals.

We state Córdoba's theorem in the form suitable for further generalizations. For this purpose let us introduce the following binary property for three-dimensional intervals:

$$\mathrm{pr}(R_1) \sim \mathrm{pr}(R_2) \;\Rightarrow\; R_1 \sim R_2, \tag{5}$$

where $\mathrm{pr}(R)$ denotes the projection onto the (x,y) plane. By Córdoba's lemma [3, p. 30], every family of dyadic intervals satisfying (5) (i.e. such that each pair in the family satisfies (5)) has the exponential covering property, which yields weak type $L\log^+ L$ of the corresponding maximal operator, while the three-dimensional strong maximal function only has weak type $L\log^2 L$. A direct generalization is the following

Claim 2. — *If a basis B of three-dimensional intervals has the property that for some fixed natural m, any finite subfamily of B decomposes into at most m subfamilies satisfying* (5), *then B has weak type $L\log^+ L$.*

Before turning to generalizations of Claims 1 and 2, let us make a remark concerning exposition. Obviously, when working in two and three dimensions, it is convenient and advisable to use graphical representations. However, having in mind multidimensional generalizations, it is preferable to have proofs without drawings. We have managed to describe the geometric objects we use in visible but formal terms, and thus avoid having recourse to any drawings.

2. Main results.

For every interval $I \in B$ we denote by I^* the concentric interval of minimal measure containing I with side lengths of the form 2^k, $k \in \mathbb{Z}$. Thus to every basis B of intervals we attach, in a natural way, another basis $B^* = \{I^* \mid I \in B\}$, called the basis *associated with* B. Informally, B^* is a dyadic skeleton of the basis B. It is clear that

$$M_B f(x) \leqslant 2^n M_{B^*} f(x)$$

so the $(1,1)$ estimate for B^* implies the estimate for B, generally with another constant.

Theorem 1. — *Let B be a TI-basis of multidimensional intervals. If*

(w) *B^* does not contain arbitrarily long sequences of pairwise incomparable elements,*

then $M_B f$ is of weak type $(1,1)$.

Proof. — First, we prove the two-dimensional version, and then reduce the general case to considerations of two-dimensional projections. For simplicity we call two-dimensional intervals *rectangles*.

The proof consists in showing that property (w) is equivalent to the possibility of representing the associated basis as a finite union of monotonic families.

It is clear that the associated basis is generated by translations of rectangles from some basic family, all elements of which have a common lower left vertex. It is enough to show that this basic family may be split into a finite union of monotonic families. Thus, till the end of the proof we assume that *all the rectangles considered have a common lower left vertex*; as will be seen from the proof, this does not restrict generality.

It is clear that property (w) can be written in the form

$$\text{(w)} \qquad \exists k > 1 \ \forall R_1, \dots, R_k \in B^* \ \exists i \neq j, \qquad R_i \sim R_j.$$

Suppose that a finite family $\widetilde{B}$ of rectangles $R = H \times V$ with dyadic side lengths has this property. We will show that this family is a union of at most k monotonic subfamilies. Choose an $R^1 \in \widetilde{B}$ of maximal height and let it be the first element of $\widetilde{B}^1$. Put in $\widetilde{B}^1$ all R of the same height as R^1, i.e. with $|V| = |V^1|$. From the remaining ones, take those with height $2^{-1}|V^1|$ and arrange them arbitrarily in a sequence. Then step by step add to $\widetilde{B}^1$ all R which are comparable with those already selected. Then from the remaining ones pick those with height $2^{-2}|V^1|$, arrange them in a sequence etc. This gives a family $\widetilde{B}^1$.

If there are $R \notin \widetilde{B}^1$, pick a highest one, say R^2, and repeat the above procedure for the remaining ones. This yields a family $\widetilde{B}^2$.

If there are $R \notin \widetilde{B}^2$, pick a highest one, say R^3, and so on.

Suppose we have chosen subfamilies $\widetilde{B}^1, \dots, \widetilde{B}^k$, but there still is an $R \notin \widetilde{B}^j$, $j = 1, \dots, k$. As $R \notin \widetilde{B}^k$, it was dropped when compared with higher rectangles, which means that there is a rectangle $R_k \in \widetilde{B}^k$ higher than R, but narrower. In its turn, $R_k \notin \widetilde{B}^{k-1}$, and repeating the above considerations we find $R_{k-1} \in \widetilde{B}^{k-1}$ which is higher than R_k, but narrower. Then it is easy to see that R, R_k and R_{k-1} are pairwise incomparable. Next, $R_{k-1} \notin \widetilde{B}^{k-2}$, and arguing similarly we find, this time, four pairwise incomparable rectangles R, R_k, R_{k-1} and R_{k-2}, and so on. Eventually, we get $k+1$ pairwise incomparable intervals $R, R_k, R_{k-1}, \dots, R_1$, contrary to property (w).

Accordingly, if a two-dimensional basis B has property (w), then by Claim 1 it is of weak type $(1,1)$.

The next lemma will be used in the proof of the general case.

LEMMA 1. — *A family of n-dimensional rectangles is monotonic if and only if its projections on the $(x_1,x_2),(x_1,x_3),\dots,(x_1,x_n)$ planes are all monotonic.*

Proof. — The "only if" part is obvious. Let us prove the "if" part. Let B be a family of n-dimensional intervals whose relevant projections are monotonic. This means that if $I\in B$, $I=I_1\times\cdots\times I_n$ and $I'\in B$, $I'=I'_1\times\cdots\times I'_n$, and $|I_1|\leqslant|I'_1|$, then $|I_2|\leqslant|I'_2|$ since $I_1\times I_2\sim I'_1\times I'_2$. Also, $|I_3|\leqslant|I'_3|$ since $I_1\times I_3\sim I'_1\times I'_3$, etc. Thus, $|I_j|\leqslant|I'_j|$ for all $j=1,\dots,n$, which means that $I\sim I'$.

This completes the proof.

Now, let us introduce another selection procedure, which we call *filtration*. Namely, if B has property (w), then it is clear that its (x_1,x_2) projection also satisfies (w). Thus, applying the above two-dimensional argument we may split every finite subfamily of B^* into at most k families whose (x_1,x_2) projections are monotonic families. Next, each of these families may be split into at most k families whose (x_1,x_3) projections are monotonic, etc. Finally, we get a bounded number of families whose projections onto the $(x_1,x_2),(x_1,x_3),\dots,(x_1,x_n)$ planes are all monotonic. According to Lemma 1, these families are themselves monotonic and according to Claim 1, M_Bf is of weak type $(1,1)$.

This completes the proof of Theorem 1.

The following theorem demonstrates the sharpness of property (w).

THEOREM 2. — *Let B be a TI-basis of multidimensional intervals which fails property* (w)*, i.e.*

(s) $$\forall k>1\ \exists I_1,\dots,I_k\in B^*\ \forall i\neq j,\qquad I_i\not\sim I_j.$$

Then M_Bf is not of weak type $(1,1)$ and moreover, it satisfies an inequality reverse to (2)*: for any $0<\lambda<1$ there exists a set E such that*

$$|\{M_B\chi_E>\lambda\}|\geqslant C\int\frac{\chi_E}{\lambda}\log\frac{\chi_E}{\lambda}\,dx$$

with some constant C independent of E and λ.

Theorems 1 and 2 together state that $M_B f$ is of weak type $(1,1)$ if and only if the number of pairwise incomparable intervals in the associated basis is bounded.

For the basis of all rectangles, the proof of the inequality of Theorem 2 is very simple, it uses the so-called "Bohr staircase" (*cf.* [13], [9]). For natural k and $j = 0, 1, \ldots, k$ let I_j denote the rectangle $[0, 2^{k-j}] \times [0, 2^j]$. Set $Y = I_0 \cup \cdots \cup I_k$ and let Θ be the unit square. The set Y resembles a staircase, and it is called the *Bohr staircase*; H. Bohr noticed some extremal properties of such sets and used them to prove that the Vitali lemma does not hold for families of rectangles.

Since for any interval forming Y, say I, we have

$$\frac{|\Theta \cap I|}{|I|} = \frac{|\Theta|}{|I|} = \frac{1}{2^k}$$

it follows that $\{\mathcal{M}\chi_\Theta \geqslant 2^{-k}\} \supset Y$ and so

$$|\{\mathcal{M}\chi_\Theta \geqslant 2^{-k}\}| \geqslant |Y| \geqslant \frac{1}{2}(k+1)2^k|\Theta| = \frac{1}{2}\int \frac{\chi_\Theta}{2^{-k}}\left(1 + \log_2^+ \frac{\chi_\Theta}{2^{-k}}\right) dx.$$

However, if we consider a basis of rectangles with a rare ratio of side lengths, e.g. 2^{2^j}, then the measure of the Bohr staircase becomes $k2^{2^k}$, and instead of the logarithm we only get a double logarithm.

In the next lemma we present a construction which replaces the Bohr staircase for rare rectangles. The resulting set looks more like a garden grating than a staircase, but if we move all the slats to the lower left corner close to each other, then we get exactly the Bohr staircase.

LEMMA 2. — *Let $I_1, \ldots, I_k$ be pairwise incomparable two-dimensional intervals with dyadic side lengths. Then for each positive integer k there are two sets Θ and Y in the plane such that*

$$|Y| \geqslant k2^{k-2}|\Theta|$$

and for every $x \in Y$ there is a shift τ such that for some j,

$$x \in \tau(I_j) \quad \textit{and} \quad |\tau(I_j) \cap \Theta| \geqslant 2^{1-k}|\tau(I_j)|.$$

Proof. — As usual, let $x = (x_1, x_2)$. Without loss of generality assume that $I_1, \ldots, I_k$ have a common lower left vertex. Let $I_j = I_j^1 \times I_j^2$, $|I_j^1| = 2^{-m_j}$ and $|I_j^2| = 2^{-n_j}$. Assume $I_1^1 \subset I_2^1 \subset \cdots \subset I_k^1$ while $I_1^2 \supset I_2^2 \supset \cdots \supset I_k^2$; in other words, $m_1 > \cdots > m_k$ and $n_1 < \cdots < n_k$.

Define

$$\Theta^1 = \Big\{x_1 \in I_k^1 : \prod_{j=1}^{k-1} \sum_{s=0}^{2^{m_j - m_{k-1}}-1} \chi_{I_j^1}(x_1 - 2s|I_j^1|) = 1\Big\},$$

$$\Theta^2 = \Big\{x_2 \in I_1^2 : \prod_{j=2}^{k} \sum_{s=0}^{2^{n_j - n_{1-1}}-1} \chi_{I_j^2}(x_2 - 2s|I_j^2|) = 1\Big\}.$$

By geometric reasons it is clear that $|\Theta^1| = 2^{1-k}|I_k^1|$ and $|\Theta^2| = 2^{1-k}|I_1^2|$. Set $\Theta = \Theta^1 \times \Theta^2$. Then $|\Theta| = 2^{2-2k}|I_k^1| \cdot |I_1^2|$.

Now set $Y_k^1 = I_k^1$, $Y_1^2 = I_1^2$ and

$$Y_i^1 = \Big\{x_1 \in I_k^1 : \prod_{j=i}^{k-1} \sum_{s=0}^{2^{m_j - m_{k-1}}-1} \chi_{I_j^1}(x_1 - 2s|I_j^1|) = 1\Big\},$$

$$Y_i^2 = \Big\{x_2 \in I_1^2 : \prod_{j=2}^{i} \sum_{s=0}^{2^{n_j - n_{1-1}}-1} \chi_{I_j^2}(x_2 - 2s|I_j^2|) = 1\Big\}$$

for $i = 1, \dots, k-1$ and $i = 2, \dots, k$ respectively. By geometric reasons, $|Y_i^1| = 2^{-(k-i)}|I_k^1|$ and $|Y_i^2| = 2^{1-i}|I_1^2|$, so if $Y_i = Y_i^1 \times Y_i^2$, then $|Y_i| = 2^{1-k}|I_k^1| \cdot |I_1^2|$.

Further, let $Y = Y_1 \cup \dots \cup Y_k$. The side lengths of the intervals involved being dyadic rationals, we have

$$|Y| \geqslant \frac{1}{2}\sum_{i=1}^{k} |Y_i| = k2^{-k}|I_k^1| \cdot |I_1^2| = k2^{k-2}|\Theta|.$$

It is clear that each Y_i is a disjoint union of rectangles, each of which is a shift of I_i, say $\tau(I_i)$, and

$$\frac{|\tau(I_i) \cap \Theta|}{|\tau(I_i)|} = \frac{|I_i \cap \Theta|}{|I_i|} = \frac{|Y_i \cap \Theta|}{|Y_i|} = \frac{|\Theta|}{|Y_i|} = \frac{2^{2-2k}|I_k^1| \cdot |I_1^2|}{2^{1-k}|I_k^1| \cdot |I_1^2|} = 2^{1-k},$$

which completes the proof of Lemma 2.

Lemma 3. — *If a family of multidimensional intervals has property* (s), *then so does its projection onto some two-dimensional plane.*

Proof. — Indeed, if every two-dimensional projection fails (s), then each such projection satisfies (w). An application of the filtration procedure now splits B^* into a finite number of monotonic families. But then it would be impossible to choose arbitrary many pairwise incomparable intervals from B^*, contrary to (s). This proves the lemma.

Now, let us turn to the proof of Theorem 2. Define the basis B_* in the same manner as B^* but replacing "circumscribed" dyadic intervals of minimal measure with "inscribed" dyadic intervals of maximal measure. It is easy to see that properties (w) and (s) may be written either in terms of B^* or B_*, whichever is preferable. So, we assume that

$$\text{(s)} \qquad \forall k \geqslant 1\ \exists R_1, \dots, R_k \in B_*\ \forall i \neq j, \quad R_i \not\sim R_j.$$

Without loss of generality we may assume that B has property (s) when projected onto the (x_1, x_2) plane, so if $R_j = I_j \times Q_j$ where I_j and $Q_j = Q_j^3 \times \dots \times Q_j^n$ are two-dimensional and $(n-2)$-dimensional intervals respectively, then $I_1, \dots, I_k$ satisfy the assumptions of Lemma 2. Choose Θ and Y as in that lemma and let

$$J^i = Q_1^i \cup \dots \cup Q_k^i \quad (i = 1, \dots, n),$$
$$U = \Theta \times J^3 \times \dots \times J^n, \quad W = Y \times J^3 \times \dots \times J^n.$$

Now, let $x = (x_1, \dots, x_n) \in W$. Since $(x_1, x_2) \in Y$, by Lemma 2 for some I_j there is a translation τ such that $(x_1, x_2) \in \tau(I_j)$. As $x_s \in J^s$ and $Q_j^s \subset J^s$ for $s = 3, \dots, n$, there are translations τ_s in the direction of the x_s axes such that $x_s \in \tau_s(Q_j^s) \subset J^s$, $s = 3, \dots, n$. We now define the final translation as the composition $\bar\tau \equiv \tau \circ \tau_3 \circ \dots \circ \tau_n$.

Then $\bar\tau(R_j) = \tau(I_j) \times \tau_1(Q_j^3) \times \dots \times \tau_n(Q_j^n)$, and so $\bar\tau(R_j) \subset W$, and $\bar\tau(R_j) \cap U = (\tau(I_j) \cap \Theta) \times \tau_1(Q_j^3) \times \dots \times \tau_n(Q_j^n)$. Thus

$$\frac{|\bar\tau(R_j) \cap U|}{|\bar\tau(R_j)|} = \frac{|\tau(I_j) \cap \Theta|}{\tau(I_j)} \geqslant 2^{1-k},$$

which implies that

$$W \subset \{x : M_{B_*}\chi_U(x) \geqslant 2^{1-k}\}.$$

But

$$|W| = |Y| \cdot |J^3| \cdot \ldots \cdot |J^n| \geqslant k2^{k-2}|\Theta| \cdot |J^3| \cdot \ldots \cdot |J^n| = k2^{k-2}|U|.$$

So

$$|\{x : M_{B_*}\chi_U(x) \geqslant 2^{1-k}\}| \geqslant |W| \geqslant k2^{k-2}|U| \geqslant \frac{1}{2}\int \frac{\chi_U}{2^{1-k}} \log_2 \frac{\chi_U}{2^{1-k}}\, dx.$$

Since it is clear that $2^{-n}M_{B_*}f(x) \leqslant M_B f(x)$, for sufficiently small $0 < \lambda < 1$ we get

$$|\{x : M_B\chi_U(x) \geqslant \lambda\}| \geqslant \frac{1}{2^{n+3}}\int \frac{\chi_U}{\lambda} \log_2 \frac{\chi_U}{\lambda}\, dx.$$

This completes the proof of Theorem 2.

Corollary 1. — *A TI-basis B of multidimensional intervals has weak type* $(1,1)$ *if and only if it has property* (w).

3. The two-dimensional case.

COROLLARY 2. — *A TI-basis of two-dimensional rectangles with sides parallel to the coordinate axes has either weak type* $(1,1)$ *or weak type* $L\log^+ L$*, but never an intermediate weak type* $L\varphi(L)$*.*

We now make a few comments on differentiation of integrals in $\mathbb{R}^2$. It is well known that a direct consequence of the $(1,1)$ estimate is the differentiation of integrals of locally summable functions (for details see [9, 10, 16]). The basis of all cubes differentiates integrals of all summable functions (Lebesgue theorem), while the basis of all n-dimensional rectangles differentiates integrals of functions from $L(\log^+ L)^{n-1}$. From these results it follows that the differentiation properties of a basis can be improved by making it sufficiently rare. Zygmund proposed the following rarefaction of the basis of all two-dimensional intervals (see [21, Ch. 6, § 4]). Let B be the TI-basis consisting of the two-dimensional intervals whose side lengths s, S satisfy $S^2 \leqslant s \leqslant S \leqslant 1$. Is it then true that B differentiates $L\sqrt{\log^+ L}$? R. Moriyón proved (see [9, App. IV]) that this is not the case: B does not differentiate $o(L\log^+ L)$. This shows that a rarefaction of this kind does not improve the differentiation properties of the basis.

Our results indicate that no rarefaction within the class of TI-bases permits the differentiation properties of bases to be improved in a continuous way. More precisely, *if* B *is a TI-basis then either* B *differentiates* L*, or* B *does not differentiate* $o(L\log^+ L)$ (for details see [20]).

4. The three-dimensional case.

In contrast to the two-dimensional situation, a complete description of the three-dimensional case is still an open problem, even for homothety invariant bases. However, some information can be deduced from Theorems 1 and 2. Let the *Córdoba basis* be the TI-basis of three-dimensional intervals with side lengths s, t and st. According to Córdoba's solution of Zygmund's conjecture [3] the corresponding maximal operator has $L\log^+ L$ weak type. Having in mind Theorems 1 and 2 we come to the conclusion that Corollary 2 still holds for any TI-subbasis of the Córdoba basis.

For the general case, every TI-basis of three-dimensional intervals has weak type $L\log^2 L$. Clearly, if a basis has property (w), then it is of weak type $(1,1)$. If we want to have weak type $L\log^+ L$, we have to require a condition weaker than (w). Since the two-dimensional situation is fairly clear, it is natural to look for a condition concerning two-dimensional projections of the basis.

Strictly speaking, we can weaken property (w),

$$\exists k>1\ \forall R_1,\dots,R_k\in B^*\ \exists i\neq j,\qquad R_i\sim R_j \tag{w}$$

by introducing the following condition:

$$\exists k>1,\ \forall R_1,\dots,R_k\in B^* \text{ with } \mathrm{pr}(R_i) \text{ pairwise comparable,}\quad \exists i\neq j,\qquad R_i\sim R_j, \tag{C}$$

or the following property, in a sense conjugate to (C):

$$\exists k>1,\ \forall R_1,\dots,R_k\in B^*,\ \exists i\neq j,\qquad \mathrm{pr}(R_i)\sim\mathrm{pr}(R_j), \tag{Z}$$

where $\mathrm{pr}(R)$ denotes the projection of the three-dimensional interval R onto the (x,y) plane.

It turns out that such a formal approach is most fruitful.

Theorem 3. — *If a basis B satisfies* (C) *or* (Z), *then it has weak type $L\log^+ L$.*

Proof. — In what follows, we assume $R=\mathrm{pr}(R)\times I$. We show that every finite family with property (C) is a union of at most k families satisfying (5).

Choose any $R^1\in B^*$ and let it be the first element of a basis B^1. We next consider only those $R\in B^*$ which are inside R^1.

Put in B^1 all R which are the same height as R^1, i.e. $|I|=|I^1|$. From the remaining ones take those with height $2^{-1}|I^1|$ and arrange them arbitrarily in a sequence. Then step by step add to B^1 those R which satisfy (5) together with all those selected earlier.

From the remaining ones, consider those with height $2^{-2}|I^1|$, arrange them in a sequence, add to B^1 those R which satisfy (5) together with those selected earlier, etc. We thus get a basis B^1.

If there are $R\notin B^1$, then choose a highest one, say R^2, and repeat the above procedure for the remaining ones, which gives a basis B^2.

If there are $R\notin B^2$, then choose a highest one, say R^3, and so on.

Suppose we have thus obtained bases $B^1,\dots,B^k$ but there is an $R\notin B^j$, $j=1,\dots,k$. As $R\notin B^k$, there is $R_k\in B^k$ failing (5), i.e.

$\mathrm{pr}(R) \sim \mathrm{pr}(R_k)$ but $R \not\sim R_k$. However R was compared with the previous ones, which are no lower, and it was dropped, so R_k can be considered no lower, and since the definition of comparability implies that (5) always holds for rectangles of equal height we can assume that R_k is higher than R, which together with $R \not\sim R_k$ yields

$$\mathrm{pr}(R_k) \subset \mathrm{pr}(R), \qquad I \subset I_k, \qquad I_k \not\subset I.$$

In turn, $R_k \notin B^{k-1}$. Repeating the above considerations, we deduce that there exists $R_{k-1} \in B^{k-1}$ such that $R_k \not\sim R_{k-1}$ and

$$\mathrm{pr}(R_{k-1}) \subset \mathrm{pr}(R_k), \qquad I_k \subset I_{k-1}, \qquad I_{k-1} \not\subset I_k.$$

Hence

$$\mathrm{pr}(R_{k-1}) \subset \mathrm{pr}(R), \qquad I \subset I_{k-1}, \qquad I_{k-1} \not\subset I,$$

which means that R, R_k and R_{k-1} are pairwise incomparable, but there projections are pairwise comparable.

Continuing this process, we obtain $k+1$ intervals $R, R_k, R_{k-1}, \ldots, R_1$ which are pairwise incomparable, but their projections are pairwise comparable. But this contradicts property (C).

Thus, we have shown that every finite family with property (C) can be represented as a union of at most k subfamilies satisfying (5), which yields the weak type $L \log^+ L$ of the maximal operator, with a constant linear in k.

Consider now property (Z). Recall that the Jessen–Marcinkiewicz–Zygmund theorem was generalized by Zygmund [22], who proved that the basis of n-dimensional intervals with k sides of equal length differentiates integrals of functions from $L(\log^+ L)^{n-k}$. The proof is based on iterating estimates for the k-dimensional maximal function and $n-k$ one-dimensional ones. In the three-dimensional case, Zygmund's basis consists of cartesian products of elements of a basis of squares and one-dimensional intervals. In view of Theorem 1, the definition of this basis can be naturally generalized. Namely, we define a *Zygmund basis* to consist of cartesian products of monotonic bases and one-dimensional intervals. Such a basis has weak type $L \log^+ L$, which can be proved using Zygmund's scheme. However, if a basis has property (Z), then by Theorem 1, every finite subfamily decomposes into at most k families from the Zygmund basis, which gives weak type $L \log^+ L$ of the maximal operator, with a constant again depending linearly on k.

This completes the proof of Theorem 3.

Corollary 3. — *If a basis B fails property* (w) *but enjoys property* (C) *or* (Z), *then B has weak type $L\log^+ L$, and this estimate is sharp.*

To prove the corollary it is sufficient to note that the sharpness of the estimate follows from Theorem 2, since the basis fails (w).

5. Multidimensional conjecture.

Theorems 1, 2 and 3 justify the introduction of the following conjecture which can be regarded as specification of Zygmund's program.

Conjecture. — *If s is a minimum of exponents r so that TI-basis of multidimensional d-intervals has weak type $L(\log^+ L)^r$, then s is an integer ($0 \leqslant s \leqslant d-1$) and the basis is of weak type $L(\log^+ L)^s$.*

This conjecture looks so unusual that it is even hard to say whether confirming or rejecting it would be more surprising. At the moment we have a positive evidence but only for the two-dimensional case or more generally for the case of products of two cubic intervals.

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