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LIMIT SETS OF FOLIATIONS

by Richard SACKSTEDER and Arthur J. SCHWARTZ

1. Introduction.

Let V be an n-manifold with a foliated structure of co-dimension one. A leaf of the foliation is called *proper* if its topology as an $(n-1)$-manifold agrees with its topology as a subset of V. One type of theorem which is proved here asserts that proper leaves behave much like compact leaves with respect to certain stability properties. For example, Theorem 1 can be viewed as an extension of a theorem of Reeb on the behavior of leaves in a neighborhood of compact leaf to non-compact, proper leaves. Other theorems show that leaves whose holonomy is finite in a certain sense have properties like those of non-periodic solutions of differential equations on 2-manifolds. For instance, Theorems 2, 3, 4 are closely related to results concerning differential equations on 2-manifolds contained implicitly, at least, in the papers of Haas [1], [2] (however, see [8]).

Some examples are given in Section 9 which illustrate how the hypotheses of Theorem 1-4 can be satisfied. Finally, in Section 10 some sharper results, which hold only if $n = 2$, are obtained.

2. Statements of the theorems.

In all of the theorems stated in this section, V denotes a (connected, paracompact, Hausdorff) n-manifold ($n \geqslant 2$) with a foliated structure of co-dimension one. If x is a point of V, the leaf containing x is denoted by F_x, its closure by C_x. D_x, called the *limit set of* x, is defined to be the intersection of the closures of the sets $F_x - K$ where K is any compact subset of F_x. The concept of a locally infinite holonomy pseudogroup, which appears in the conclusion of the theorems is defined in Section 3. Its meaning is elucidated by Propositions 3.4 and 3.5.

THEOREM 1. — *Let x be a point of* V *such that F_x is proper, C_x is compact, and $F_x \subset D_y$ for some y in* V. *Then F_x has a locally infinite holonomy pseudogroup.*

THEOREM 2. — *Let y be a point of* V *such that C_y is compact, $C_y \neq V$, and C_y contains an open subset of* V. *Then there is a leaf in the boundary of C_y with a locally infinite holonomy pseudogroup.*

THEOREM 3. — *Let* V *contain a dense leaf, F_y, and a non-dense leaf F_z such that C_z is compact. Then C_z contains a leaf with a locally infinite holonomy pseudogroup.*

THEOREM 4. — *Let $S^1 \subset V$ be a closed curve intersecting each leaf transversally. Suppose that the subset* A *of* V *consisting of the leaves which intersect S^1 is relatively compact. Then either $A = V$ or there is a leaf, F_x, in the boundary of* A *which has a locally infinite holonomy pseudogroup.*

It will be clear from the proofs that very little smoothness need be assumed in these theorems in contrast to those obtained in [7] and [8]. A sufficient condition is that there exist a continuous vector field on V which is never tangent to a leaf and is such that the solution curve through each point is unique. We shall always suppose that such a vector field has been given. The solution curve through a point x of V will be denoted by T_x and called the *transversal through* x.

3. The holonomy group and pseudogroup.

The material in this section is, to a large extent, well known, but we include it here in order to make our presentation as self contained as possible. For the sake of simplicity we consider only foliations of co-dimension one and we assume that there is a metric defined on each transversal curve.

We introduce first a notion that may be roughly described as lifting a path from one leaf to another, continuously, along transversals.

DEFINITION. — *Let* $P : [a, b] \times [0, S] \to V$ *be a continuous function satisfying for each* (t, s).

1) $P(t, s)$ *is in* $F_{P(0,s)} \cap T_{P(t,0)}$.

2) $s \to P(0, s)$ *is an isometry.*

Thus the maps g_s, defined by $g_s(t) = P(t, s)$ determine a family of paths, each contained in a single leaf, with initial points in $T_{P(0,0)}$. On the other hand, the maps P_t, defined by $P_t(s) = P(t, s)$ determine a family of transversals with initial points in $g_0([a, b])$.

We call P a *projector.* P is said to *project* g_0 *onto* g_S, and g_S is called a *projection* of g_0 by P along $P_0([0, S])$.

PROPOSITION 3.1. — *If* $P, Q : [a, b] \times [0, S] \to V$ *are projectors and* $P(t, s) = Q(t, s)$ *for* (t, s) *in* $0 \times [0, S] \cup [a, b] \times 0$, *then* $P = Q$.

The proof is straightforward.

Thus, if $g_0 : [a, b] \to V$ is a path contained in a single leaf, and P_0 is a transversal interval with $g_0(a)$ as one end point, there exists at most one projection of g_0 along P_0.

PROPOSITION 3.2. — Suppose that $0 < \lambda < 1$. — *If* $g : [a, b] \to V$ *is a path contained in a single leaf, there exists a projector* $P : [a, b] \times [0, S] \to V$, *with* $S > 0$, *such that* $g(t) = P(t, \lambda S)$.

The proof is straightforward.

PROPOSITION 3.3. — *Let* $g : [a, b] \to V$ *be a path contained in a single leaf such that* $g(t)$ *is the midpoint of a transversal interval of length* $2L > 0$, *for each* t *in* $[a, b]$. *Let* J *be a transversal interval with length* $M < L$ *with one end point at* $g(a)$. *Let* c *be the largest number in* $[a, b]$ *such that there exists a projector* $P : [a, c] \times [0, M] \to V$ *satisfying.*

1) $P(t, 0) = g(t)$.

2) $P(0 \times [0, M]) = J$.

3) $P(t \times [0, M])$ *has length less than or equal to* L. *Then either* $c = b$ *or* $P(c \times [0, M])$ *is of length* L.

Proof. — If $c < b$ and $P(c \times [0, M]) < L$ it would be possible to extend P to $P: [a, c + \varepsilon] \times [0, M] \to V$ for some positive ε.

Let x be a point of V and $g: [0, 1] \to V$ a path in F_x such that $g(0) = g(1) = x$. Let $P: [0, 1] \times [0, S] \to V$ be the projector satisfying the conclusion of Proposition 3.2. Define the map $H_g: [-\lambda S, (1 - \lambda)S] \to R^1$ by $H_g(u) = h(P_1(u + \lambda S))$, where $P_1(s) = P(1, s)$ and $h: P_1([0, S]) \to R^1$ is the isometry such that $h(P(0, s)) = s - \lambda S$ for s near λS. Then H_g is a diffeomorphism of neighborhoods of zero in R^1. The pseudogroup generated (cf. [7] sect. 3) by the diffeomorphisms H_g for any path g in F_x with $g(0) = g(1) = x$ is called the *holonomy pseudogroup* of F_x at x. This pseudogroup depends, of course, on the metric on T_x, but only up to an inner automorphism. The set of germs of elements of the holonomy pseudogroup at x forms a group called *the holonomy group* of F_x at x. It depends similarly on the metric on T_x. The element of the holonomy group of F_x corresponding to a path g depends only on the homotopy class of g in F_x, hence, there is a homomorphism from the fundamental group of F_x onto the holonomy group.

We say that a leaf F_x has a *locally infinite holonomy pseudogroup* if for *every* neighborhood N of x on T_x there is an orientation-preserving element of the holonomy pseudogroup of F_x which is not a restriction of the identity and whose domain corresponds to a subset of N. We do not know if a leaf can have a locally infinite holonomy pseudogroup if its holonomy group is finite; however, the following propositions show a relationship between a locally infinite holonomy pseudogroup and the holonomy group.

Proposition 3.4. — *Let* F_x *have a locally infinite holonomy pseudogroup. Then given any neighborhood* N *of* x, *there is a leaf which intersects* N *and has an infinite holonomy group.*

Proof. — Let $S > 0$ be arbitrarily small and suppose that $g: [0, 1] \to F_x$ satisfies $g(0) = g(1) = x$ and that the element

$H_g : \left[\frac{-S}{2}, \frac{+S}{2}\right] \to R^1$ of the holonomy pseudogroup corresponding to g is such that for some t, $0 < H_g(t) < t < \frac{S}{2}$.

The hypothesis implies that g and H_g with these properties exist for any $S > 0$. Let H_g^p denote p-fold composition and define $c = \lim_{p \to \infty} H_g^p(t) \geq 0$ and $y = P_0(c)$. One easily verifies that $H_g(c) = c$ and if $c < u < t$, $c < H_g(u) < u < t$. From this it easily follows that H_g represents a germ of infinite order in the holonomy group of F_y. This proves Proposition 3.4.

Another implication of the condition that a leaf have a locally infinite holonomy group is given by:

PROPOSITION 3.5. — *Suppose that $n = 2$ and F_x is homeomorphic to R^1. Then the holonomy pseudogroup of F_x consists entirely of maps which are restrictions of the identity map. In particular, it is not locally infinite.*

The idea of the proof is simple, but the details are tedious, hence we only give a rough indication of the proof. Let $g : [0, 1] \to F_x$ be a path with $g(0) = g(1) = x$ and P a projector as in Proposition 3.2. If the image of P lies entirely within a distinguished neighborhood the conclusion is almost obvious. In general, the image of P is covered by a finite number of distinguished neighborhoods and the proposition can be proved by induction on the number of them.

4. The main Lemma.

The following lemma is used in the proof of Theorems 1-4.

LEMMA 4.1. — *Let x in V be such that C_x is compact. Let $P_i : [0, b_i] \times [0, S_i] \to V$ be a sequence of projectors such that $P_i(0, 0) = x$ for all i, $P_i(0 \times (0, S_i)) \cap F_x = \emptyset$, S_i tends to 0, and the length of $P_i(b_i \times [0, S_i])$ tends to $L > 0$.*

Then F_x has a locally infinite holonomy pseudogroup.

Proof. — Since C_x is compact, we may assume that $P(b_i, 0) \to z$ and $P(b_i \times [0, S_i]) \to T$, a transversal interval with end points z and y. Let N be a distinguished neighborhood containing T such that $N = \varphi((-\varepsilon, L+\varepsilon) \times (-\varepsilon, \varepsilon)^{n-1})$ for some ε, $0 < \varepsilon < \frac{L}{3}$ where φ satisfies.

1) $\varphi(0, \ldots, 0) = z$.
2) $\varphi(x_1, x_2, \ldots, x_n)$ is in $T_{\varphi(0, x_2, \ldots, x_n)}$
3) $\varphi(x_1, x_2, \ldots, x_n)$ is in the plaque through $\varphi(x_1, 0, \ldots, 0)$.
4) $\varphi(L, 0, \ldots, 0) = y$.

For sufficiently large K, we may modify P_i without changing P_i at any point whose image is outside N so that $P(b_i, [0, S_i])$ belongs to $\varphi((-\varepsilon, L+\varepsilon), 0, \ldots, 0)$ if $i \geqslant K$.

Now observe that for each i and t, $P_i(t, s)$ does not belong to F_x unless $s = 0$, since $P_i(0 \times [0, S_i]) \cap F_x = \emptyset$ by assumption. Using this fact we shall show that there exists M sufficiently large so that $P_i(b_i, 0) = z$ if $i \geqslant M$. In fact, choose $M \geqslant K$ such that for $i \geqslant M$, $P_i(b_i, 0)$ is in

$$\varphi\left(\left(-\varepsilon, \frac{1}{3} L\right), 0, \ldots, 0\right)$$

and $P_i(b_i, S_i)$ is in $\varphi\left(\left(\frac{2}{3} L, L+\varepsilon\right), 0, \ldots, 0\right)$. If for some $i, j \geqslant M$, $P_i(b_i, 0) = \varphi(r_i, 0, \ldots, 0)$ and $P_j(b_j, 0) = \varphi(r_j, 0, \ldots, 0)$ with $r_j > r_i$, $P_j(b_j, 0)$ is in $P_i(b_i, (0, S_i))$. But since $P_j(b_j, 0)$ is in F_x this contradicts the assumption that

$$P_i(t \times [0, S_i]) \cap F_x = \emptyset;$$

thus $P_i(b_i, 0) = P_j(b_j, 0) = z$ if $i, j \geqslant M$. Thus z is in F_x.

Choose s_0 so that $P_M(b_M, s_0) = \varphi\left(\frac{1}{2} L, 0, \ldots, 0\right)$.

Choose $R > M$ so that $P_R(0, S_R)$ is in $P_M(0, [0, s_0])$.

Choose s_1 so that $P_R(b_R, s_1) = \varphi\left(\frac{1}{2} L, 0, \ldots, 0\right)$.

Let $g : [0, b_R + b_M] \to V$ be defined by

$$g(t) \begin{cases} = P_M(t, 0) & \text{if} \quad 0 \leqslant t \leqslant b_M. \\ = P_R(b_M + b_R - t, 0) & \text{if} \quad b_M \leqslant t \leqslant b_R + b_M. \end{cases}$$

Let $P : [0, b_R + b_M] \times [0, s_0] \to V$ be the projector such that $P(0, s) = P_M(0, s)$ and $P(t, 0) = g(t)$, which exists by Proposition 3.3. According to Proposition 3.1, $P(t, s) = P_M(t, s)$

for $t \leqslant b_M$ and $P(t, s) = P_R(b_M + b_R - t, h(s))$ for $t \geqslant b_M$ where $P_R(b_R, h(s)) = P_M(b_M, s)$ defines h.

Thus $H_g(s_0) = s_1 < s_0$, which proves that the holonomy pseudogroup of F_x contains an orientation-preserving element which is not a restriction of the identity. Such an element can be chosen to have an arbitrarily small domain by the condition $S_i \to 0$. This proves the Lemma.

5. Proof of Theorem 1.

Let $L > 0$ be so small that no point on the subinterval of T_x of length $2L$ centered at x is in F_x other than x itself. Such an L exists because F_x is proper. Moreover, it can be assumed that L is so small that each point of C_x is at the center of a subinterval of an orthogonal trajectory of length $2L$.

Let $x_1, x_2, \ldots$ be a sequence of points in $F_y \cap T_x$ such that the distance between x_i and x along T_x is less than L, and decreases to zero in a strictly monotone fashion.

Let $g_i : [0, 1] \to F_y (i = 1, 2, \ldots)$ be a sequence of paths satisfying $g_i(0) = x_i$, $g_i(1) = x_{i+1}$. According to Proposition 3.3, there are projectors $P_1, P_2, \ldots$ such that:

(5.1) $P_i : [0, t_i] \times [0, s_i] \to V, \qquad (0 \leqq t_i < 1),$

(5.2) $P_i(t, s_i) = g_i(t) \quad \text{for} \quad 0 \leqq t \leqq t_i,$

(5.3) $P_i(0, 0) = x,$

(5.4) $d_i(t) < L \quad \text{for} \quad 0 \leqq t < t_i,$

(5.5) $d_i(t_i) = L \quad \text{if} \quad t_i < 1,$

where $d_i(t)$ denotes the length of the transversal $P_i(t \times [0, s_i])$.

The following two cases exhaust all possibilities:

1) For infinitely many i, $t_i = 1$; 2) for all but a finite number of i, $t_i < 1$. In the case 1), it is clear that, since $0 < s_{i+1} < s_i$, there is an element in holonomy pseudogroup of F_x at x corresponding to the path $P_i(t, 0)$ $(0 \leqq t \leqq 1)$ which is orientation preserving and not a restriction of the identity map and since $x_i \to x$ it can be chosen to have an arbitrarily small domain. Thus F_x has a locally infinite holonomy pseudogroup and Theorem 1 is valid in this case. In the case 2), the hypotheses of Lemma 4.1 apply to the sequence $P_1, P_2, \ldots$ Thus Theorem 1 is proved.

6. Proof of Theorem 2.

Let x be a point on the boundary of the interior of C_y such that for some $L > 0$, T_x contains an open interval of length $2L$ centered at x and such that one half of it, say I_1, is contained in C_y, while the other half, I_2, contains points of $V - C_y$ arbitrarily close to x. Such a point is easily seen to exist. It can be supposed that L is so small that each point of C_x is the midpoint of a subinterval of an orthogonal trajectory of length 2L. Let $x_1, x_2, \ldots$ be a sequence of points in $I_1 \cap F_y$ such that x_{i+1} lies between x_i and x and $x_i \to x$. Let $g_i : [0, 1] \to F_y$ be a sequence of curves satisfying $g_i(0) = x_i$, $g_i(1) = x_{i+1} \neq x_i$. Proposition 3.3 implies that there exists a sequence $P_1, P_2, \ldots$ of projectors satisfying (5.1)-(5.5). One considers two cases exactly as in the proof of Theorem 1, and the treatment of case 1) here is exactly the same as for Theorem 1. In case 2) Lemma 4.1 applies as for Theorem 1, with the minor difference that here $P_i(0 \times (0, s_i)) \cap F_x = \emptyset$ holds because $I_1 \subset C_y$ implies that $I_1 \cap F_x = \{x\}$ since every point of F_x is on the boundary of C_y. This proves Theorem 2.

7. Proof of Theorem 3.

Let x be a point of C_z which is such that for some $L > 0$, a subinterval I_1 of T_x with one endpoint at x contains no points of F_x other than x. It can also be supposed that L is so small that every point of C_x is the midpoint of a subinterval of an orthogonal trajectory of length 2L, and there are points of V at a distance greater than L from C_x. Let $x_1, x_2, \ldots$ be a sequence of points of $I_1 \cap F_y$ which approach x. Let $g_i : [0, 1] \to F_y$ be a sequence of curves such that $g_i(0) = x_i$ and $g_i(1)$ is a point of V at a distance greater than L from C_x. Such a sequence exists by the choice of L and the assumption that $C_y = V$. Let $P_1, P_2, \ldots$ be a corresponding sequence of projectors which exists and satisfies (5.1)-(5.5) by Proposition 3.3. Note that the condition on $g_i(1)$ implies that $t_i < 1$. Now applying Lemma 4.1 gives the desired conclusion.

8. Proof of Theorem 4.

If $A \neq V$, there is a leaf F_x in the boundary of the interior of A such that C_x is at a distance greater than $L > 0$ from S^1 and an open subinterval I_1 of T_x of length L with one end point at x is in the interior of A. Again, it can be supposed that L is so small that each point of C_x is the midpoint of a subinterval of an orthogonal trajectory of length 2L. Let $x_1, x_2, \ldots$ be a sequence of points of $A \cap I_1$ which approach x and are at a distance less than L from x. Denote the leaf containing x_i by F_i. Let $g_i : [0, 1] \to F_i$ be a sequence of curves such that $g_i(0) = x_i$ and $g_i(1) \in S^1$. Let P_1, P_2 be the corresponding sequence of projectors satisfying (5.1)-(5.5) which exists by Proposition 3.3. The condition on $g_i(1)$ implies that $t_i < 1$, and $I_1 \subset \mathrm{Int}\ A$ implies $P_i(0 \times [0, s_i]) \cap F_x = \emptyset$. Therefore, Lemma 4.1 again gives the desired conclusion.

9. Examples.

The examples 1-4 below illustrate, respectively, Theorems 1-4. Similar examples have been given in articles by G. Reeb in the Annales de l'Institut Fourier, vol. 6 and vol. 11.

Example 1. — It is very easy to construct examples in which a compact (hence proper) leaf is in the limit set of another leaf. It is a little more difficult to construct an example in which a non-compact proper leaf is in the limit set of another leaf. To construct such an example, let $V = S^1 \times S^1 \times S^1$. Let (x, y, z) denote a typical point of V, where x, y, and z are real numbers mod π. Let $f : (0, \pi) \to R^1$ be a C^∞ diffeomorphism. Then $\omega = d(f(x) + y) + \sin^2(f(x) + y)\, dz$ is a completely integrable 1-form on $V_0 = (0, \pi) \times S^1 \times S^1$. The desired foliation of V is obtained by completing this foliation of $V_0 \subset V$ by adding the leaf $F_0 = \{0\} \times S^1 \times S^1$. Then one easily sees that the leaf passing through any point $(x, y, z) \in V_0$ such that $f(x) + y = 0 \bmod \pi$ is non-compact, proper, and in the limit set of every leaf through a point (x', y', z') in V_0 such that $f(x') + y' \neq 0 \bmod \pi$. The leaf F_0 is in the limit set of every other leaf.

It is perhaps also worth noting that V_0 can be imbedded in S^3 in such a way that its complement consists of two disjoint solid tori, which can be given a foliation which fits together with the foliation of V_0 to give a foliation of S^3.

Example 2. — Let V_0 be as in example 1 and let the form $\omega = dx - (\text{sine } x)(dy + \alpha\, dz)$, where α is irrational define a foliation of V_0. As above, V_0 can be imbedded in S^3 in such a way that its foliation can be extended to a foliation of all of S^3. The only leaves of this foliation with non-trivial holonomy groups will be two disjoint tori $S^1 \times S^1$ which are the boundary of V_0 as a subset of S^3. One easily verifies that the closure of every leaf containing a point of V_0 is the closure of V_0 in S^3. Thus there are leaves which are dense in an open subset of S^3, but not in S^3 itself.

Example 3. — Let V_0 be foliated as in example 2, but imbed it in $V = S^1 \times S^1 \times S^1$, as in example 1. Complete the foliation induced on $V_0 \subset V$ be adding the leaf $F_0 = \{0\} \times S^1 \times S^1$. This leaf will be compact, hence nowhere dense, but every other leaf will be dense in V. The leaf F_0 is the only leaf with a non-trivial holonomy group.

Example 4. — It is easy to see that the Reeb foliation of S^3, cf. [3], [7], admits a transversal curve S^1 which does not intersect every leaf. The torus which bounds the set of leaves which intersect S^1 has, of course, a non-trivial holonomy group.

10. Application to differential equations on surfaces.

Let V be a surface and $\alpha : R^1 \times V \to V$ a flow on V determined by a continuous vector field X on V. If $V_0 \subset V$ is the submanifold of V consisting of those points where X does not vanish, X determines a foliation of V_0 in which the leaves are the integral curves of X. If x is a point of V_0, we define the sets $A_x = \bigcap_{N \leq 0} \text{closure } \{\alpha(t, x) : t \leq N\}$ $\Omega_x = \bigcap_{N \geq 0} \text{closure } \{\alpha(t, x) : t \geq N\}$ called, respectively, the *alpha* and *omega* limit sets of x. We use the notation, D_x, C_x, F_x, etc., for the foliated manifold V_0 just as in previous sections.

The main goal of this section and the following one is to prove:

THEOREM 5. — *Let* V *be a surface* (= 2-*manifold*) *and* $\alpha : R^1 \times V \to V$ *a flow generated by a continuous vector field,* X, *on* V. *Let* $V_0 \subset V$ *be the foliated manifold consisting of all points of* V *where* X *does not vanish. Suppose that* x *is in* V_0 *and* C_x *is compact. Then every leaf in* A_x (*or* Ω_x) *is everywhere dense in* A_x (*or* Ω_x) *and either*

a) $C_x = A_x = \Omega_x = V$ *or,*

b) C_x *is nowhere dense.*

This can be viewed as a strengthening of Theorem 1 for the case $n = 2$ in the sense that, in the terminology of topological dynamics, Theorem 1 (or more precisely, Lemma 10.4) is easily seen to imply that A_x and Ω_x are Poisson stable, while Theorem 5 asserts the stronger condition that these sets are minimal.

We need to recall some preliminary results and definitions before proceeding to the proof of Theorem 5. An orbit (= leaf) $F_x \subset V_0 \subset V$ is called *periodic* if $\alpha(t, x) = x$ for some $t > 0$ and x is then called a *periodic point.* F_x is compact if and only if F_x is periodic, in which case F_x is proper, $D_x = \emptyset$, and $F_x = \Omega_x = A_x$. If F_x is not periodic, $D_x = A_x \cup \Omega_x$, F_x is homeomorphic to R^1, and if C_x is compact, both A_x and Ω_x are non-empty.

LEMMA 10.1. — *If* A_x (*resp.* Ω_x) *contains a periodic point,* y, *then* $A_x = F_y$ (*resp.* $\Omega_x = F_y$).

Proof. — See [1].

The next two lemmas are partially due to Haas [1].

LEMMA 10.2. — *If for some* x *in* V_0, $C_x = V_0$, *is compact, then* $V = V_0 = S^1 \times S^1$ *and every leaf is dense in* V.

Proof. — Since F_x is dense in $V = V_0$, F_x is not proper and $F_x \subset C_x = A_x \cup \Omega_x$.

We now claim that V contains no periodic leaf. Suppose, on the contrary, that V contains a periodic leaf, F_y. Then $F_y \subset A_x$ or $F_y \subset \Omega_x$. Say $F_y \subset A_x$, then according to Lemma 10.1, $F_y = A_x$. But then $\Omega_x = V$, $F_y \subset \Omega_x$ and $F_y = \Omega_x$ which yields $V = F_y$ which is absurd.

Thus every leaf has a trivial holonomy group and it follows from Theorem 4 and Proposition 3.4 that every leaf is dense.

Now, $C_x = V$ is a compact surface. It carries a non-vanishing vector field and therefore the Euler characteristic of V must vanish. Thus it follows that V must be homeomorphic to $S^1 \times S^1$ or a Klein bottle. V cannot be a Klein bottle for, according to Kneser [4], V would then contain a periodic leaf.

LEMMA 10.3. — *Suppose that for some x in V_0, C_x is compact. Then either V is homeomorphic to $S^1 \times S^1$ and each leaf is dense in V, or C_x is nowhere dense.*

Proof. — If $C_x = V$, the desired result follows from Lemma 10.2. Suppose that $C_x \neq V$, but C_x contains an open subset of V. Then F_x is not proper, hence $D_x = A_x \cup \Omega_x = C_x$. D_x does not contain a periodic point y in this case, because if $F_y \subset D_x$ is periodic, $F_y = A_x$ (or Ω_x) by Lemma 10.1. But then $D_x = \Omega_x$ (or A_x) and $F_y = \Omega_x$ (or A_x), which is impossible if D_x contains an open set. This shows that D_x contains no periodic points. Now Theorem 2 implies that there is a leaf with a locally infinite holonomy pseudogroup in C_x. Proposition 3.5 then implies that this leaf is periodic. But this contradicts what has been proved and completes the proof of the lemma.

LEMMA 10.4. — *Let C_x be compact. Then if $x \in \Omega_y$ (or A_y) then $x \in \Omega_x$.*

The proof is clear if x is a periodic point. Otherwise the argument goes almost exactly like proof of Theorem 1. We omit the details.

11. Proof of Theorem 5.

Lemma 10.3 gives all of the assertions of Theorem 5 except that « if C_x is nowhere dense, every leaf in A_x (or Ω_x) is dense in A_x (or Ω_x) ». To verify this, suppose that C_x is nowhere dense.

Let y be a point in Ω_x. Let $\varphi : [-\varepsilon, \varepsilon] \to V$ be an isometry into T_y such that $\varphi(0) = y$, $\varphi(-\varepsilon)$ and $\varphi(\varepsilon)$ are not in Ω_x. For each $z = \varphi(s)$, there exists $h_z > 0$ such that $\alpha(t, z)$ is not in $\varphi([-\varepsilon, \varepsilon])$ if $0 < t < h_z$. Thus according to Lemma 10.4

for each z in $J = \varphi([-\varepsilon, \varepsilon]) \cap \Omega_x$, there exists a smallest positive number t_z such that $\alpha(t_z, z)$ is in J. Since J is compact there exists a number, M, such that $t_z \leqslant M$ for all z in J, hence there is an M′ such that if $t \geqslant 0$, $\alpha([t, t + M'], x) \cap J \neq \emptyset$. Let $N = \{\alpha(t, w) : w\varepsilon\varphi((-\varepsilon, \varepsilon)), |t| < 1\}$. Let $d > 0$ be such that q is in N if dist $(q, z) < d$ for some z in J. Let $\delta > 0$ be sufficiently small so that dist $(\alpha(t, q), \alpha(t, p)) < d$ if

$$0 \leqslant t \leqslant M' + 1,$$

q is in C_x, and dist$(q, p) \leqslant \delta$.

Now, for any p in Ω_x, there exists $t_0 \geqslant 0$ such that dist $(\alpha(t_0, x), p) < \delta$ and for some m, $1 \leqq m \leqq M' + 1$, $\alpha(t_0 + m, x)$ is in J. Then $\alpha(m, p)$ is in N, hence $\alpha(m', p)$ is in $\varphi((-\varepsilon, \varepsilon))$ for some m', where $0 \leqslant m - 1 \leqslant m'$.

Thus for every $p\varepsilon\Omega_x$, $\alpha([0, \infty), p)$ intersects $\varphi((-\varepsilon, \varepsilon))$. Since y and p were arbitrary points in Ω_x and ε may be arbitrarily small, the theorem is proved.

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