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ZERO DISTRIBUTIONS VIA ORTHOGONALITY

by Laurent BARATCHART, Reinhold KÜSTNER & Vilmos TOTIK(*)

Dedicated to Zbigniew Ciesielski on his 70th birthday

1. Introduction.

Let μ be a finite positive Borel measure with infinite compact support $S \subset \mathbb{R}$, and consider the corresponding monic orthogonal polynomials $q_n(x) = x^n + \cdots$ satisfying

$$\int q_n(t)\, t^k \, d\mu(t) = 0, \qquad k = 0, 1, \ldots, n-1. \tag{1}$$

Under quite weak hypotheses on the measure μ, it is true [20] Theorem 2.2.1, that the zero distribution of the polynomials q_n is asymptotically equal to the equilibrium measure ω_S of S for the logarithmic potential (see the end of this introduction for a definition). The proofs in the literature usually go through estimates on the norm of the polynomials. In what follows, we present a new approach that uses directly the orthogonality relation (1). Under mild assumptions this approach generalizes to non-Hermitian orthogonality, that is to the case where μ gets replaced by some complex measure λ; this cannot be said of classical proofs. See further motivation for and connection with non-Hermitian orthogonality in Section 6.

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Another feature of the present approach, is that it applies to other kinds of orthogonality relations, like those arising from best rational or meromorphic approximation with free poles. For example, if the support of λ lies in the unit disk, the denominator of a best rational approximant of degree n to the Cauchy transform of λ in L^2 of the unit circle satisfies (see [7], [3]) the relation

$$\int \frac{q_n(t)}{Q_n^2(t)}\, t^k \, d\lambda(t) = 0, \qquad k = 0, 1, \ldots, n-1, \tag{2}$$

where Q_n is the reflected polynomial associated with q_n (see Section 4). Although (2) looks like ordinary orthogonality with varying weight $Q_n^{-2}(t)$, it is not so because this weight itself depends on q_n. When λ is a positive measure on $[a,b] \subset (-1,1)$ satisfying the Szegő condition, then the asymptotic zero distribution of the polynomials q_n in (2) is known (and much beyond, see [2], [5]), but still our result deals with less regular situations. And when λ is complex, the theorem we prove is first of this kind. The method also allows one to handle the case of an additional strongly convergent varying weight function which is relevant in meromorphic approximation (see [3]).

In this work we shall extensively use logarithmic potential theory. For fundamental notions like equilibrium measure, potential, capacity, balayage, as well as the basic theorems concerning them, the reader may want to consult some recent texts such as [15], [16] or the appendix of [20]. However, for the reader's convenience, we review below those concepts which are systematically refered to in the statements of the present paper.

Let $E \subset \mathbb{C}$ be a compact set. To support his intuition, one may view E as a plane conductor and imagine one puts a unit electric charge on it. Then, if a distribution of charge is described as being a Borel measure μ on E, the electrostatic equilibrium has to minimize the internal energy

$$I(\mu) = \iint \log \frac{1}{|z-\zeta|}\, d\mu(z) d\mu(\zeta)$$

among all probability measures supported on E. This is because on the plane the Coulomb force is proportional to the inverse of the distance between the particles, and therefore the potential is its logarithm. There are sets E (called polar sets or sets of zero logarithmic capacity) which are so thin that the energy $I(\mu)$ is infinite, no matter what the probability measure μ is on E; for those we do not define the equilibrium measure. But if E is such that $I(\mu)$ is finite for *some* probability measure μ on E, then there is a unique minimizer for $I(\mu)$ among all such probability measures. This minimizer is called the *equilibrium measure* (with respect to logarithmic potential) of E, and we denote it by ω_E. For example, the equilibrium measure of a disk or circle is the normalized arc measure on the circumference, while the equilibrium measure of a segment $[a,b]$ is given by

$$d\omega_{[a,b]}(t) = \frac{dt}{\pi\sqrt{(t-a)(b-t)}}.$$

That the equilibrium measure of a disk is supported on its circumference is no accident: the equilibrium measure of E is always supported on the outer boundary $\partial_e E$ of E.

Associated to a finite positive measure μ on a compact set E is its *logarithmic potential*

$$U^\mu(z) = \int \log \frac{1}{|z-\zeta|}\, d\mu(\zeta),$$

which is superharmonic on $\mathbb{C}$ with values in $(-\infty, +\infty]$. From the physical viewpoint, this is simply the electrostatic potential corresponding to the distribution of charge μ. Perhaps the nicest characterization of ω_E among all probability measures on E is that $U^{\omega_E}(z)$ is equal to some constant D on E, except possibly on a polar subset of E where it may be less than D. Of necessity then, we have that $D = I(\omega_E)$ because measures of finite energy like ω_E do not charge polar sets. Points at which U^{ω_E} is discontinuous are called *irregular* (of necessity they lie on $\partial_e E$), and E itself is said to be *regular* if it has no irregular points. All nice compact sets are regular, in particular all whose boundary has no connected component that reduces to a point. At the other extreme, a compact polar set consists solely of irregular points. The simplest example of a polar set is a countable set, but there also exist uncountable polar sets. The regularity of E is in fact equivalent to the *regularity of the Dirichlet problem* in the unbounded component $\mathcal{V}$ of $\overline{\mathbb{C}} \setminus E$, meaning that for any continuous function f on $\partial_e E$ there is a harmonic function in $\mathcal{V}$ which is continuous on $\overline{\mathcal{V}} = \mathcal{V} \cup \partial_e E$ and coincides with f on $\partial_e E$.

The number $\mathrm{cap}\,(E) = \exp(-I(\omega_E))$ is called the *logarithmic capacity* of E, and conventionally polar sets have capacity zero. A property that holds in the complement of a polar set is said to hold *quasi-everywhere*.

For $\mathcal{U} \subset \overline{\mathbb{C}}$ an open set whose boundary $\partial\mathcal{U}$ is non-polar, given $\zeta \in \mathcal{U}$ the *Green function of* $\mathcal{U}$ *with pole at* ζ is the unique real-valued function $z \mapsto g_{\mathcal{U}}(z,\zeta)$ which is harmonic in $\mathcal{U} \setminus \{\zeta\}$, bounded outside each neighbourhood of ζ, that tends to $+\infty$ logarithmically as z tends to ζ, and to 0 as z tends to w for quasi-every $w \in \partial\mathcal{U}$. For instance if $\infty \in \mathcal{U}$

then $g_{\mathcal{U}}(z,\infty) = I(\omega_{\partial\mathcal{U}}) - U^{\omega_{\partial\mathcal{U}}}(z)$; formulas for the general case can be obtained from this one *via* conformal mapping. Note that $g_{\mathcal{U}}(z,\zeta)$ is non-negative; actually, it is strictly positive on the connected component of $\mathcal{U}$ that contains ζ.

When $\mathcal{U}$ is a domain, one can introduce concepts similar to those of logarithmic potential theory upon replacing the kernel $\log\big(1/|z-\zeta|\big)$ by the kernel $g_{\mathcal{U}}(z,\zeta)$, the Green function of $\mathcal{U}$ with pole at ζ. This gives rise to the notions of *Green potential* and *Green energy* for finite positive compactly supported measures on $\mathcal{U}$, and to the notions of *Green equilibrium measure* and *Green capacity* for compact subsets of $\mathcal{U}$. We use them only when $\mathcal{U}$ is the unit disk, and they will be introduced as needed in Section 4.

2. Real orthogonal polynomials.

In this section, we let μ be a finite positive Borel measure with compact support $S \subset \mathbb{R}$. We shall assume that μ is sufficiently thick, namely that there exist two constants $c, L > 0$ such that

$$\mu([x-\delta, x+\delta]) \geqslant c\,\delta^L \qquad \text{for all } \ x \in S \text{ and for all } \delta \in (0,1). \tag{3}$$

Let ν_n be the normalized counting measure on the zeros of the polynomials q_n satisfying (1), namely the discrete probability measure having equal mass at each of the zeros (these are simple). The theorem below asserting the asymptotic behaviour of ν_n is not new (see [20], Theorems 2.2.1 and 4.2.5), but our method of proof will serve as a model for more general situations to come.

THEOREM 2.1. — *Suppose that the support S of μ is regular with respect to the Dirichlet problem in $\overline{\mathbb{C}}\setminus S$, and that (3) holds. Then ν_n tends to the equilibrium measure ω_S of S in the weak* topology as n tends to ∞.*

Proof. — Let ν be a weak* limit point of $\{\nu_n\}$, say $\nu_n \to \nu$ as $n \to \infty$, $n \in \mathbb{N}_1$. It is well-known (and elementary to check, see [20], Lemma 1.1.3), that the zeros of the orthogonal polynomials lie in the convex hull of S, that they are simple, and that each component of $\mathbb{R}\setminus S$ can contain at most one of them. Hence ν is supported on S and it has total mass 1. *We claim:* it is enough to prove that there exists a real constant D such that the logarithmic potential U^ν of ν equals D quasi-everywhere on S. Indeed, the lower semicontinuity of U^ν implies then that $U^\nu(x) \leqslant D$ for all $x \in S$.

Thus, ν has finite logarithmic energy, and, in particular, every polar set has measure zero with respect to ν and ω_S. Now, integrating the equality "$U^\nu(x) = D$ for quasi-every $x \in S$" against ω_S and interchanging the order of integration, we get that

$$D = \int U^\nu(t)\, d\omega_S(t) = \int U^{\omega_S}(t)\, d\nu(t) = I(\omega_S),$$

and then integrating the same equality against ν yields that ν has the same logarithmic energy as ω_S. By uniqueness of the equilibrium measure (see [16], Theorem I.3.1) it follows that $\nu = \omega_S$ and, since this is true of every weak* limit point, it follows by means of the Helly selection theorem that the whole sequence $\{\nu_n\}$ converges to ω_S, *as claimed.*

Thus, it has left to prove that there exists a real constant D such that U^ν equals D quasi-everywhere on S. Suppose to the contrary that there exist two constants $d \in \mathbb{R}$, $\tau > 0$ and two non-polar Borel sets $E_-, E_+ \subset S$ such that

$$U^\nu(x) \leqslant d - 2\tau \quad \text{for } x \in E_- \qquad \text{and} \quad U^\nu(x) \geqslant d + \tau \quad \text{for } x \in E_+$$

(this is the only alternative for U^ν is finite quasi-everywhere on S since it is a superharmonic function on $\mathbb{C}$ which is clearly not identically $+\infty$).

We contend that there exists a point $y_0 \in \operatorname{supp}(\nu)$ such that $U^\nu(y_0) > d$. For if not $U^\nu(x) \leqslant d$ for all $x \in \operatorname{supp}(\nu)$, so by the maximum principle for potentials (see e.g. [16], Corollary II.3.3) the same inequality holds for all $z \in \mathbb{C}$, a contradiction since on E_+ we have bigger values for U^ν. *This proves our contention.*

According to the principle of descent (see e.g. [16], Theorem I.6.8), which is valid since the support of ν_n remains in a fixed compact set (namely the convex hull of S), we subsequently have

$$\liminf_{n\to\infty,\ n\in\mathbb{N}_1} U^{\nu_n}(z_n) \geqslant U^\nu(y_0) > d$$

for any sequence $z_n \to y_0$. Therefore there exist $\rho > 0$ and N_1 such that, for $y \in [y_0 - 2\rho, y_0 + 2\rho]$ and $n \geqslant N_1$, $n \in \mathbb{N}_1$, the inequality $U^{\nu_n}(y) \geqslant d$ holds. By the definition of ν_n and of the logarithmic potential, this means that

$$(4) \qquad |q_n(y)| \leqslant e^{-nd}, \qquad y \in [y_0 - 2\rho, y_0 + 2\rho], \qquad n \geqslant N_1, \quad n \in \mathbb{N}_1.$$

We shall use that this inequality remains true (at the cost perhaps of increasing N_1) if $\{q_n\}$ gets replaced by any sequence of monic polynomials $\{p_n\}$ having the same asymptotic zero distribution ν and whose zeros are uniformly bounded, for these were the only facts we used in deriving (4). This is true even if p_n does not have exact degree n (the counting measure of the zeros being still normalized with n), but merely degree $n + \mathrm{o}(n)$.

In another connection, the lower envelope theorem (see [16], Theorem I.6.9) implies that for quasi-every $x \in E_-$ we have

$$\liminf_{n\to\infty,\ n\in\mathbb{N}_1} U^{\nu_n}(x) = U^{\nu}(x) \leqslant d - 2\tau,$$

and since the logarithm of $|q_n(x)|^{-1/n}$ stands on the left, we deduce that for some subsequence $\mathbb{N}_2 \subset \mathbb{N}_1$ and for sufficiently large $n \in \mathbb{N}_2$, say for $n \geqslant N_2$,

$$M_n := \max_{x\in S} |q_n(x)| \geqslant e^{-n(d-\tau)}. \tag{5}$$

Our hypothesis that S is regular means: the Green function of $\overline{\mathbb{C}}\setminus S$ with pole at ∞ is such that

$$\lambda(\varepsilon) := \max_{\mathrm{dist}(z,S)\leqslant 2\varepsilon} g_{\overline{\mathbb{C}}\setminus S}(z,\infty)$$

tends to zero as $\varepsilon \to 0$. According to the Bernstein-Walsh lemma (see e.g. [21], p. 77 or [15], Theorem 5.5.7), we have that

$$|q_n(z)| \leqslant M_n \exp\bigl(n\, g_{\overline{\mathbb{C}}\setminus S}(z,\infty)\bigr) \leqslant M_n e^{n\lambda(\varepsilon)} \quad \text{if } z \in \mathbb{C}\setminus S,\ \mathrm{dist}(z,S) \leqslant 2\varepsilon.$$

From this it follows on differentiating the Cauchy formula that

$$|q_n'(z)| \leqslant \frac{M_n e^{n\lambda(\varepsilon)}}{\varepsilon} \qquad \text{if } \mathrm{dist}(z,S) \leqslant \varepsilon,$$

where $q_n'(z)$ indicates the derivative. Thus, if we let $x_n \in S$ be a point where $|q_n(x)|$ attains its maximum on S (i.e. $|q_n(x_n)| = M_n$) and if $x \in \mathbb{R}$ is such that

$$|x - x_n| \leqslant \frac{\varepsilon}{2e^{n\lambda(\varepsilon)}}, \tag{6}$$

we obtain from the mean-value theorem the estimate:

$$|q_n(x)| \geqslant |q_n(x_n)| - |q_n(x) - q_n(x_n)| \geqslant |q_n(x_n)| - M_n/2 = M_n/2.$$

For fixed $\varepsilon > 0$, the interval defined by (6) is contained in $[x_n - \rho, x_n + \rho]$ when n is sufficiently large since $\lambda(\varepsilon) > 0$, and therefore by (3) and (5) we get that

$$\begin{aligned}\int_{S\cap[x_n-\rho,x_n+\rho]} |q_n(t)|^2\, d\mu(t) &\geqslant \left(\frac{M_n}{2}\right)^2 c\left(\frac{\varepsilon}{2e^{n\lambda(\varepsilon)}}\right)^L \geqslant \frac{c}{4}\left(\frac{\varepsilon}{2e^{n\lambda(\varepsilon)}}\right)^L e^{-2n(d-\tau)}\\ &\geqslant e^{-2nd+n\tau}\end{aligned} \tag{7}$$

for $n \in \mathbb{N}_2$ and, say, $n \geqslant N_3 = N_3(\varepsilon)$, provided $\varepsilon > 0$ is so small that $\lambda(\varepsilon)L < \tau$.

Observe now that, for sufficiently large $n \in \mathbb{N}_1$, there are arbitrarily many zeros of q_n in $[y_0 - \rho, y_0 + \rho]$ because y_0 lies in the support of the limit measure ν. In particular there will be two such zeros, say $\alpha_n < \beta_n$, as soon as $n \in \mathbb{N}_1$ is sufficiently large, say $n \geqslant N_4$. For $n \in \mathbb{N}_2$ with $n \geqslant \max(N_1, N_2, N_3, N_4)$, consider the polynomial $P_{n-2}(x) = q_n(x)/((x - \alpha_n)(x - \beta_n))$ of degree $n-2$. Obviously $q_n(x)P_{n-2}(x)$ is nonnegative for $x \in \mathbb{R}\backslash(\alpha_n, \beta_n)$. Moreover, since $\tau > 0$ and $|q_n(x_n)| = M_n$, it follows from (4) and (5) that $x_n \notin [y_0 - 2\rho, y_0 + 2\rho]$, and thus $[x_n - \rho, x_n + \rho] \cap (\alpha_n, \beta_n) = \emptyset$. Consequently, since α_n and β_n lie in the convex hull of S, we obtain on the one hand from (7) that for large $n \in \mathbb{N}_2$

$$\int_{S\backslash(\alpha_n,\beta_n)} q_n(t)P_{n-2}(t)\,d\mu(t) \geqslant \frac{1}{(\mathrm{diam}(S))^2} e^{-2nd+n\tau}, \tag{8}$$

where $\mathrm{diam}(S)$ was used to indicate the diameter of S.

On the other hand, the limit distribution of the zeros of the polynomials P_{n-2}, $n \in \mathbb{N}_2$, $n \geqslant N_4$, is again ν, and these zeros remain bounded as they lie in the convex hull of S so that, as pointed out after (4), we also have

$$|P_{n-2}(y)| \leqslant e^{-nd}, \qquad y \in [y_0 - 2\rho, y_0 + 2\rho],$$

for sufficiently large $n \in \mathbb{N}_2$. We thus obtain from this and (4) the estimate

$$\left| \int_{S\cap(\alpha_n,\beta_n)} q_n(t)P_{n-2}(t)\,d\mu(t) \right| \leqslant \mu\big(S \cap (\alpha_n, \beta_n)\big)e^{-2nd} \leqslant \mu(S)e^{-2nd}. \tag{9}$$

But the sum of the two integrals in the left-hand sides of (8) and (9) should be zero by orthogonality, which is clearly impossible because, as an inspection of the right-hand sides shows, the first integral is much bigger than the second one for large $n \in \mathbb{N}_2$ since $\tau > 0$.

This contradiction proves the theorem. □

3. Non-Hermitian orthogonal polynomials.

Let λ be a complex Borel measure having compact support $S \subset \mathbb{R}$, and consider an associated sequence of monic non-Hermitian orthogonal polynomials, i.e. a sequence of polynomials $q_n(x) = x^{d_n} + \cdots$ with $d_n \leqslant n$ such that

$$\int q_n(t)\, t^k \, d\lambda(t) = 0, \qquad k = 0, 1, \ldots, n-1. \tag{10}$$

Although the orthogonalization process can no longer be used here, the existence of such a sequence $\{q_n\}$ is guaranteed by elementary linear algebra since n linear homogeneous equations in $n+1$ unknowns always have a nontrivial solution. Therefore, there exist $a_{0,n}, a_{1,n}, \ldots, a_{n,n}$, not all zero, such that

$$\int \left(\sum_{j=0}^{n} a_{j,n}\, t^j \right) t^k d\lambda(t) = 0, \qquad k = 0, 1, \ldots, n-1,$$

and now if $a_{d_n,n}$ is the highest non-vanishing coefficient, then a normalization gives (10). This time, however, q_n need not be unique nor have exact degree n. In fact (10) has exactly one monic solution of minimal degree, which divides all other solutions, and when there exists another solution, say of degree m, then every monic polynomial of degree at most m which is a multiple of the minimal degree solution is in turn a solution (see [18], Lemma 1). Therefore, there is a certain inaccuracy in the words "a sequence of monic non-Hermitian orthogonal polynomials associated with λ". Hereafter, we simply assume that some monic polynomial q_n of degree $d_n \leqslant n$ satisfying (10) has been chosen for each $n \geqslant 1$, for the results will not depend on the precise choice of q_n meeting these requirements. Note that if the support of λ is infinite then d_n necessarily goes to infinity with n, as follows easily from the uniform density of polynomials in the space of continuous functions on S. The hypotheses on λ to come will in fact ensure much more, namely they imply that $n - d_n$ remains bounded (see Lemma 3.2). For that very reason, it does not matter in convergence issues whether the zero counting measure ν_n of q_n is formed by putting mass k/d_n or mass k/n at each zero of q_n of multiplicity k. For definiteness, we make the former definition so that ν_n is still a probability measure.

About the complex Borel measure λ we assume that it is of the form

$$d\lambda(t) = e^{i\varphi(t)} d\mu(t), \tag{11}$$

where φ is a real function of bounded variation on the support of μ, and μ is a finite positive Borel measure satisfying the properties set forth in Section 2, i.e. its support $S \subset \mathbb{R}$ is a regular compact set with respect to the Dirichlet problem in $\overline{\mathbb{C}} \setminus S$ and (3) holds. This is equivalent to require that the complex measure λ has regular compact support $S \subset \mathbb{R}$ and moreover, letting $\mu = |\lambda|$ be the total variation of λ, that the Radon-Nikodym derivative of λ with respect to μ is of bounded variation on S (more precisely that it coincides μ-a.e. with a function of bounded variation). Indeed, this derivative is necessarily unimodular μ-a.e. so that (11) holds for some argument function φ, and it is quite easy to see that $e^{i\varphi}$ is of bounded variation on S if, and only if, φ itself can be chosen of bounded variation there.

Note that, in the present setting, q_n may have nonreal zeros so that, in general, its zero counting measure ν_n is not supported on the convex hull of S. However, the next theorem entails that it tends to be so asymptotically, at least in proportion.

THEOREM 3.1. — *With the preceding assumptions on λ, the measure ν_n tends to the equilibrium measure ω_S of S in the weak* topology as n tends to ∞.*

For the proof of Theorem 3.1, we shall rely on a lemma which is of independent interest. This lemma does not require μ to satisfy (3), nor that S be regular but only that it is infinite. To state this result, we need some more pieces of notation as follows.

First, for $z \in \mathbb{C}$ we let $\mathrm{Arg}(z) \in (-\pi, \pi]$ denote the principal branch of the argument, making the agreement that $\mathrm{Arg}(0) = \pi$, so that Arg is left-continuous on $\mathbb{R}$. Next, for $a, b \in \mathbb{R}$, $a \leqslant b$, and $\xi \in \mathbb{C}$, we define the angle

$$\mathrm{Angle}(\xi, [a, b]) = |\,\mathrm{Arg}(a - \xi) - \mathrm{Arg}(b - \xi)| \in [0, \pi] \tag{12}$$

in which the interval $[a, b]$ is seen at ξ, so that $\mathrm{Angle}(\xi, [a, b]) = \pi$ if and only if $\xi \in [a, b]$.

Now, suppose that $S \subseteq \cup_{j=1}^{m}[a_j, b_j]$ where the $[a_j, b_j]$ are disjoint real closed intervals, with $a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_m \leqslant b_m$. We define

$$\theta(\xi) = \sum_{j=1}^{m} \mathrm{Angle}(\xi, [a_j, b_j]) \in [0, \pi]$$

which is the total angle in which $\cup_{j=1}^{m}[a_j, b_j]$ is seen at ξ. Note that $\theta(\xi) \leqslant \pi$, since the union is disjoint hence $\theta(\xi) \leqslant \mathrm{Angle}(\xi, [a_1, b_m]) \leqslant \pi$, and that $\theta(\xi) = \pi$ if, and only if, $\xi \in \cup_{j=1}^{m}[a_j, b_j]$.

Finally, we denote by

$$V(\varphi, S) = \sup\left\{\sum_{j=1}^{k} |\varphi(x_j) - \varphi(x_{j-1})| : k \in \mathbb{N},\ \{x_0 < x_1 < \ldots < x_k\} \subseteq S\right\} \tag{13}$$

the total variation of φ on S.

LEMMA 3.2. — *Let $q_n(z) = \prod_{k=1}^{d_n}(z - \xi_k)$, $d_n \leqslant n$, be an n-th orthogonal polynomial in the sense of (10). Under the sole hypothesis that the support S of λ is an infinite compact subset of the real line, it holds with the previous notations that*

$$\sum_{k=1}^{d_n}\big(\pi - \theta(\xi_k)\big) + (n - d_n)\pi \leqslant V(\varphi, S) + (m-1)\pi, \tag{14}$$

where φ is any argument for $d\lambda/d|\lambda|$ on S, i.e. $(d\lambda/d|\lambda|)(t) = e^{i\varphi(t)}$ for $t \in S$.

It follows from the lemma that the defect $n - d_n$ in the degree of q_n is at most $V(\varphi, S)/\pi$ and therefore remains uniformly bounded whenever φ can be chosen of bounded variation. To see it, apply the case $m = 1$ of Lemma 3.2 with $[a_1, b_1]$ being the convex hull of S, and recall that $\theta(\xi_k) \leqslant \pi$ for $k = 1, 2, \ldots, d_n$. In particular when $V(\varphi, S) = 0$, there can be no defect and all the zeros lie in the convex hull of S. This is for instance the case when $\lambda \geqslant 0$, and then it is a well-known result.

COROLLARY 3.3. — *Under the hypotheses of Lemma 3.2 and the additional assumption that $V(\varphi, S) < \infty$, then to every open neighbourhood U of S there is a constant c_U such that q_n has at most c_U zeros outside U.*

The corollary is immediate for $U \cap \mathbb{R}$ consists of disjoint open intervals, finitely many of which contain the compact S; call them $I_1, \ldots, I_m$, and pick $[a_j, b_j] \subset I_j$ such that $S \subseteq \cup_{j=1}^m [a_j, b_j] \subset U$ to apply Lemma 3.2, noting that a zero outside U contributes to the left-hand side of (14) by more than a fixed positive constant that depends only on $\cup_{j=1}^m [a_j, b_j]$ and U.

Proof of Lemma 3.2. — We may restrict to the case $V(\varphi, S) < \infty$ otherwise there is nothing to prove. We may also assume that φ is defined and of bounded variation on the whole real line upon extending it linearly to the finite complementary intervals of S while making it constant and equal to $\varphi(a)$ (resp. $\varphi(b)$) on $(-\infty, a)$ (resp. $(b, +\infty)$), where $[a, b]$ is the convex hull of S. This extension has the same total variation. Also, we can suppose that every jump of φ has size at most π, for by the finiteness of the variation there are only finitely many discontinuity points where it is not so, and then we can always add to φ a piecewise constant function with jumps in multiples of 2π to convert all jumps of size greater than π into jumps of size at most π. This modification can only decrease the variation, and therefore will not affect the validity of (14).

For the proof, we may further assume that q_n has no zeros on $\cup_{j=1}^m [a_j, b_j]$. Indeed, putting $d\mu_1(t) = (t-\xi_k)^2 d\mu(t)$ and $q_{n-1}(x) = q_n(x)/(x-\xi_k)$, we have

$$\int q_{n-1}(t) t^k\, e^{i\varphi(t)} d\mu_1(t) = 0, \qquad k = 0, 1, \ldots, n-2; \tag{15}$$

but if $\xi_k \in \cup_{j=1}^m [a_j, b_j]$ then μ_1 is again positive and $\theta(\xi_k) = \pi$, so we are back to prove the lemma with $n-1$ instead of n, with $d_n - 1$ instead of d_n, with q_{n-1} instead of q_n, and with μ_1 instead of μ. Proceeding recursively, we reach a situation where q_n has no zero on $\cup_{j=1}^m [a_j, b_j]$ or where $n = 1$ and either $d_n = 1$ with the only zero of q_1 lying on $\cup_{j=1}^m [a_j, b_j]$ or else $d_n = 0$, i.e. $q_1 = 1$. But if $d_n = n = 1$ and $q_1(x) = x - \xi_1$ with $\theta(\xi_1) = \pi$, then (14) is trivial for its left-hand side vanishes, while if $n = 1$, $d_n = 0$ and $q_1 = 1$, then (14) holds because the relation $\int e^{i\varphi(t)} d\mu(t) = 0$ entails $V(\varphi, S) \geqslant \pi$ (otherwise $e^{i\varphi(t)}$ lies in a sector of aperture strictly less than π so the previous integral cannot vanish by the positivity of μ).

Thus we can safely assume that q_n has no zero on $\cup_{j=1}^m [a_j, b_j]$, in which case $\psi(t) = \sum_{k=1}^{d_n} \operatorname{Arg}(t - \xi_k)$ is an argument function for $q_n(t)$ which is continuous on a neighbourhood of $\cup_{j=1}^m [a_j, b_j]$. Then $f(t) = \psi(t) + \varphi(t)$ is an argument function for $q_n(t) e^{i\varphi(t)}$, and it is of bounded variation on $\mathbb{R}$, has left and right limits $f(x-)$ and $f(x+)$ at every $x \in \cup_{j=1}^m [a_j, b_j]$, and it has the same discontinuities as φ there (at most countably many). In particular we have $f(x-) = f(x)$ if $x = a_j$ and $f(x+) = f(x)$ if $x = b_j$, due to the way we extended φ from S to $\mathbb{R}$ at the beginning of the proof. For every $x \in \cup_{j=1}^m [a_j, b_j]$, we let $I_f^{\pm}(x)$ denote the half-open (possibly empty) interval whose endpoints are $f(x)$ and $f(x\pm)$ with $f(x)$ excluded. By the *connected graph* of f over $[a_j, b_j]$, we mean the set

$$\Big\{ (x, y) \in \mathbb{R} \times \mathbb{R} \,:\, x \in [a_j, b_j],\, y \in \{f(x)\} \cup I_f^-(x) \cup I_f^+(x) \Big\}.$$

Thus, the connected graph of f over $[a_j, b_j]$ can be visualized as its ordinary graph plus all the vertical segments that represent the jumps of f on $[a_j, b_j]$.

We will show that

$$\sum_{j=1}^m V(f, [a_j, b_j]) \geqslant (n - m + 1)\pi, \tag{16}$$

and this will imply (14). Indeed, the monotonicity of $\vartheta_k(t) = \operatorname{Arg}(t - \xi_k)$ with respect to $t \in \mathbb{R}$ yields

$$\operatorname{Angle}(\xi_k, [a_j, b_j]) = |\vartheta_k(a_j) - \vartheta_k(b_j)| = V(\vartheta_k, [a_j, b_j])$$

and, since $f=\varphi+\psi=\varphi+\sum_{k=1}^{d_n}\vartheta_k$ while $V(\varphi,S)=V(\varphi,\mathbb{R})$, we get that

$$\begin{aligned}\sum_{j=1}^{m} V(f,[a_j,b_j]) &\leqslant \sum_{j=1}^{m} V(\varphi,[a_j,b_j]) + \sum_{j=1}^{m}\sum_{k=1}^{d_n} V\big(\vartheta_k,[a_j,b_j]\big)\\ &\leqslant V(\varphi,\mathbb{R}) + \sum_{k=1}^{d_n}\sum_{j=1}^{m} \mathrm{Angle}(\xi_k,[a_j,b_j])\\ &= V(\varphi,S) + \sum_{k=1}^{d_n}\theta(\xi_k),\end{aligned}$$

where we used for the second inequality that the union $\cup_{j=1}^{m}[a_j,b_j]$ is disjoint.

Thus, it has left to prove (16). Suppose to the contrary that

$$\sum_{j=1}^{m} V(f,[a_j,b_j]) < (n-m+1)\pi. \tag{17}$$

For $u\in\mathbb{R}$ let $N(u,f,[a_j,b_j])$ be the (possibly infinite) number of intersections of the line $y=u$ with the connected graph of f over $[a_j,b_j]$. In this evaluation, an intersection at (x,u) is counted twice if u lies in both $I_f^-(x)$ and $I_f^+(x)$. As f is bounded on $[a_j,b_j]$, being of bounded variation there, we can define L_j to be the smallest integer such that $|f(x)|\leqslant L_j\pi$ for all $x\in[a_j,b_j]$, and then we have that $N(u,f,[a_j,b_j])=0$ for $|u|>L_j\pi$. Now, Kestelman's generalization (see [11] or [10], page 129) of Banach's indicatrix theorem implies that

$$\int_{-L_j\pi}^{L_j\pi} N(u,f,[a_j,b_j])\,du = V(f,[a_j,b_j]). \tag{18}$$

Thus, if for $u\in[0,\pi]$ we define

$$N_\pi(u,f,[a_j,b_j]) = \sum_{\ell=-L_j}^{L_j-1} N(u+\ell\pi,f,[a_j,b_j]),$$

then we deduce from (18) that

$$\int_0^\pi N_\pi(u,f,[a_j,b_j])\,du = V(f,[a_j,b_j]).$$

Adding up over $j=1,2\ldots,m$ and using that $N_\pi(u,f,[a_j,b_j])\in\{0,1,\ldots,+\infty\}$ we can infer from (17) that there exists a set $E\subset[0,\pi]$ of positive Lebesgue measure such that

$$\sum_{j=1}^{m} N_\pi(u,f,[a_j,b_j]) \leqslant n-m \qquad \text{for all } u\in E. \tag{19}$$

Pick $u_0 \in E$ which is not equal modulo π to any value $f(a_j)$, $f(a_j+)$, $f(b_j-)$ or $f(b_j)$ for $j = 1, 2, \ldots, m$, nor to any value $f(x-)$, $f(x)$ or $f(x+)$ for x a discontinuity point of f in $\cup_{j=1}^m [a_j, b_j]$. This is possible since these values form a countable set, whereas E has positive Lebesgue measure. We will show that there exists a monic polynomial P_{n-1} of degree at most $n-1$ such that either

$$\text{(20)} \qquad \operatorname{Im}\left(e^{-iu_0} q_n(t) P_{n-1}(t) e^{i\varphi(t)}\right) \geqslant 0, \qquad t \in \cup_{j=1}^m [a_j, b_j],$$

or

$$\text{(21)} \qquad \operatorname{Im}\left(e^{-iu_0} q_n(t) P_{n-1}(t) e^{i\varphi(t)}\right) \leqslant 0, \qquad t \in \cup_{j=1}^m [a_j, b_j],$$

where the inequality is strict except perhaps for finitely many values of t. However, (20) and (21) are then equally impossible, because by (10) and (11)

$$\int e^{-iu_0} q_n(t) P_{n-1}(t) e^{i\varphi(t)} d\mu(t) = 0,$$

whereas taking the imaginary part we get the integral, against the positive measure μ with infinite support S, of a function having constant sign which is non-zero at all but finitely many points. This contradiction will prove (16), pending the proof of either (20) or (21) with strict inequality at all but finitely many values of t for some polynomial P_{n-1} of degree at most $n-1$.

To construct the polynomial P_{n-1}, paint red on the plane all the horizontal strips $u_0 + 2l\pi < y < u_0 + (2l+1)\pi$ with $l \in \mathbb{Z}$, and let G be their union; and paint white the remaining strips $u_0 + (2l-1)\pi < y < u_0 + 2l\pi$ with $l \in \mathbb{Z}$. The connected graph of f over $[a_j, b_j]$ intersects the boundary ∂G of G (i.e. the union of the lines $y = u_0 + l\pi$ with $l \in \mathbb{Z}$) above those abscissa $x \in [a_j, b_j]$ which are either a continuity point of f such that $(x, f(x)) \in \partial G$ or else a discontinuity point where at least one of the two vertical segment $\{x\} \times I_f^-(x)$, $\{x\} \times I_f^+(x)$ is bicolour (remember that all jumps have size at most π and if x is a discontinuity point that neither $(x, f(x-))$ nor $(x, f(x))$ nor $(x, f(x+))$ can lie on ∂G, by our choice of u_0). In the first case the multiplicity of the intersection is 1, and in the second case it is equal to the number of bicolour segments, i.e. 1 or 2. By definition the sum of all multiplicities is $N_\pi(u_0, f, [a_j, b_j])$ which is bounded by $n-m$, according to (19), since $u_0 \in E$; in particular, there are only finitely many intersections and they occur at isolated abscissa that we call the *intersection abscissa* of f over $[a_j, b_j]$. Among those, we further distinguish the *crossing abscissa* where the connected graph of f actually crosses ∂G, meaning that either x is an intersection abscissa which is a continuity point of f and there exists $\varepsilon > 0$ such that $(t, f(t))$ is red for $t \in (x-\varepsilon, x)$ (resp. $t \in (x, x+\varepsilon)$) and white for $t \in (x, x+\varepsilon)$ (resp. $t \in (x-\varepsilon, x)$), or else x is an intersection abscissa where f is discontinuous (in this case it is automatically a crossing abscissa). We also say that the (ordinary) graph of f starts inside (resp. outside) G at a_j if $(a_j, f(a_j))$ is red (resp. white) (remember that $(a_j, f(a_j))$ cannot lie on ∂G, by our choice of u_0).

Now, if $x_{1,j}, x_{2,j}, \ldots, x_{n_j,j}$ denote the crossing abscissa of f over $[a_j, b_j]$ with respective multiplicities $m_{1,j}, m_{2,j}, \ldots, m_{n_j,j}$, let us define

$$R_j(x) = \prod_{k=1}^{n_j} (x - x_{k,j})^{m_{k,j}} \quad \text{and} \quad \psi_j(t) = \sum_{k=1}^{n_j} m_{k,j} \operatorname{Arg}(t - x_{k,j})$$

so that $R_j(x)$ is a real polynomial of the variable x and $\psi_j(t)$ is an argument function for $R_j(t)$. A moment's thinking will convince the reader that the (ordinary) graph of

$$f_j(t) = f(t) + \psi_j(t) = \psi(t) + \psi_j(t) + \varphi(t)$$

over $[a_j, b_j]$ is either completely red or completely white, except for those t that are intersection abscissa of f (whether the graph is red or white except at such intersection abscissa depends whether a_j is a crossing abscissa or not, and whether the graph starts inside or outside G at a_j). Therefore, as $f_j(t)$ is an argument function for $q_n(t)R_j(t)e^{i\varphi(t)}$ and since the latter is a real multiple of e^{iu_0} at intersection abscissa of f (because $R_j(t)$ is real and vanishes at crossing abscissa while $q_n(t)e^{i\varphi(t)}$ has argument u_0 modulo π at non-crossing intersection abscissa), we can conclude in any case that either

$$\operatorname{Im}\left(e^{-iu_0} q_n(t) R_j(t) e^{i\varphi(t)}\right) \geqslant 0, \qquad t \in [a_j, b_j], \tag{22}$$

or

$$\operatorname{Im}\left(e^{-iu_0} q_n(t) R_j(t) e^{i\varphi(t)}\right) \leqslant 0, \qquad t \in [a_j, b_j], \tag{23}$$

where the inequality is strict except at intersection abscissa of f.

Geometrically, the above procedure may be described as follows. Suppose to fix ideas that the graph of f starts at a_j inside G. Then, a simple zero of R_j placed at a crossing abscissa $x_{k,j}$ where the graph would leave a red strip pushes the graph down by π, so that the graph continues in a red strip, while ensuring that the left-hand side of (22) is equal to zero at the point $t = x_{k,j}$ (where the colour is not under control); and if the graph does not leave a red strip, except for the point $(x_{k,j}, f(x_{k,j}))$ which is white (in which case the intersection multiplicity $m_{k,j}$ is 2), we place a double zero at $x_{k,j}$ to push the graph down by 2π, so that it continues in the next red strip while ensuring that the left-hand side of (22) is again equal to zero at $t = x_{k,j}$. Finally, a non-crossing intersection abscissa requires no treatment, because the graph remains red locally while at the abscissa itself (where the colour is not defined) we have already seen that the left-hand side of (22) is zero.

Now, since R_j is monic and all its zeros lie in $[a_j, b_j]$, it holds that $R_j(t) > 0$ for all $t > b_j$, and either $R_j(t) > 0$ or $R_j(t) < 0$ for all $t < a_j$ (whether $R_j(t)$ is always strictly positive or always strictly negative for such t depends whether its degree is even or odd). Consequently, since the union $\cup_{j=1}^m [a_j, b_j]$ is disjoint and each R_j satisfies either (22) or (23), the monic polynomial $R(x) = \prod_{j=1}^m R_j(x)$ satisfies for *each* $j = 1, 2, \dots, m$ that either

$$\mathrm{Im}\left(e^{-iu_0} q_n(t) R(t) e^{i\varphi(t)}\right) \geqslant 0, \qquad t \in [a_j, b_j], \tag{24}$$

or

$$\mathrm{Im}\left(e^{-iu_0} q_n(t) R(t) e^{i\varphi(t)}\right) \leqslant 0, \qquad t \in [a_j, b_j], \tag{25}$$

and in any case the inequality is strict except at the intersection abscissas of f.

Let K denote the set of all indices $k \in \{1, 2, \dots, m-1\}$ such that either (24) holds for $j = k$ and (25) holds for $j = k+1$ or else (25) holds for $j = k$ and (24) holds for $j = k+1$. Subsequently, we define the monic polynomials:

$$Q(x) = \prod_{k \in K} \left(x - \frac{b_k + a_{k+1}}{2}\right) \text{ and } P_{n-1}(x) = Q(x)R(x) = Q(x) \prod_{j=1}^m R_j(x).$$

Because $Q(t)$ changes sign in between every pair of consecutive intervals $[a_j, b_j]$ meeting distinct inequalities from (24)-(25), it follows at once that P_{n-1} satisfies either (20) or (21), and in any case the inequality is strict except at the intersection abscissas of f in $\cup_{j=1}^m [a_j, b_j]$. In another connection, the degree of R_j is $\sum_{k=1}^{n_j} m_{k,j}$ which is the number of crossing abscissa of f over $[a_j, b_j]$ (counting multiplicity) hence it is *a fortiori* bounded by the number of intersections abscissa over $[a_j, b_j]$ (counting multiplicity), namely by $N_\pi(u_0, f, [a_j, b_j])$. Moreover, it is clear that Q has degree at most $m - 1$. Therefore, as $u_0 \in E$, we can conclude from (19) that the degree of P_{n-1} is at most $n - 1$, as required. This completes the proof of Lemma 3.2. □

To establish Theorem 3.1, we need another lemma that is reminiscent of the previous proof, but which is actually much simpler since we do not care about the degree of the polynomial involved. Recall that $\mathrm{Arg}(z) \in (-\pi, \pi]$ denotes the principal branch of the argument and we made the agreement that $\mathrm{Arg}(0) = \pi$.

LEMMA 3.4. — *Let φ be a real function of bounded variation on an interval $[a, b]$. Then there exists a polynomial $T \neq 0$ and a constant $\delta \in (0, \pi/16)$ such that*

$$\left|\mathrm{Arg}\big(e^{i\varphi(x)}T(x)\big)\right| < \pi/2 - 2\delta \qquad \text{for } x \in [a, b], T(x) \neq 0. \tag{26}$$

Proof. — Up to the addition of a piecewise constant function taking values in integer multiples of 2π, we may assume as in the proof of Lemma 3.2 that the jumps of φ all lie in the interval $(-\pi, \pi]$ because there are only finitely many jumps that are not already so.

We first suppose that φ is left continuous. Let $x_1, \ldots, x_k \in [a, b]$ denote the points where $\varphi(x_j+) - \varphi(x_j) = \pi$, and set $\psi(x) = \varphi(x) - \sum_{j=1}^{k} \mathrm{Arg}(x - x_j)$. The function ψ is of bounded variation and left continuous on $[a, b]$, and all its jumps are strictly less than π in absolute value. Let $\gamma = \sup_{x \in [a,b]} |\psi(x+) - \psi(x)|$ be the supremum of the absolute value of its jumps. Then $\gamma < \pi$ and there exists a continuous real function f such that $|\psi(x) - f(x)| < (\gamma + \pi)/4$ for all $x \in [a, b]$. Let $0 < \delta < (\pi - \gamma)/16$. By the Weierstrass approximation theorem, there exists a polynomial T_0 such that $|e^{-if(x)} - T_0(x)| < \delta$ for all $x \in [a, b]$. Now, for $x \in [a, b]$ we have that $|e^{if(x)}T_0(x) - 1| < \delta$ thus $|\mathrm{Arg}(e^{if(x)}T_0(x))| < \delta\pi/2 < 2\delta$, and therefore:

$$\left|\mathrm{Arg}\big(e^{i\psi(x)}T_0(x)\big)\right| < (\gamma + \pi)/4 + 2\delta = \pi/2 - (\pi - \gamma)/4 + 2\delta < \pi/2 - 2\delta.$$

Consequently the polynomial $T(x) = T_0(x) \prod_{j=1}^{k}(x - x_j)$ satisfies (26).

In the general case (when φ is not left continuous) φ can be written as the sum of a left continuous function of bounded variation, say φ_1, plus an absolutely convergent series $\sum_{j=1}^{\infty} \psi_j$ of functions ψ_j such that $\psi_j = 0$ on $[a, b]$ except in one point ξ_j where $\psi_j(\xi_j) = y_j$, say. We apply the previous part of the proof to φ_1, and we select finitely many y_j such that the sum of the absolute values of the remaining ones is less that δ. Putting an additional double zero of T at each ξ_j corresponding to a selected y_j yields (26) with δ replaced by $\delta/2$. □

Proof of Theorem 3.1. — Let ν be a weak* limit point of $\{\nu_n\}$. By Corollary 3.3 this ν is supported on S and it has total mass 1. Following the proof in the preceding section, it is again sufficient to show that there exists a real constant D such that the potential U^ν of ν equals D quasi-everywhere on S.

We first suppose that the zeros of the polynomials q_n remain in a bounded set. We argue by contradiction exactly as in Theorem 2.1, following the argument there up to the choice of P_{n-2}. In this connection, it must be stressed that the principle of descent that led to (4) is valid because we assume that no zero of q_n can go to infinity. In fact, up to (7) everything remains true word for word with the degree n replaced by $d_n = \deg(q_n)$. In particular, (7) takes the form

$$\int_{S\cap[x_n-\rho,x_n+\rho]} |q_n(t)|^2 d\mu(t) \geqslant e^{-2d_n d + d_n\tau}, \tag{27}$$

x_n being a point where $|q_n|$ attains its maximum on S.

By assumption the argument function φ of the measure λ is of bounded variation on the support S of λ. We extend φ from S to the convex hull $[a,b]$ of S, as we did it at the beginning of the proof of Lemma 3.2, and we pick $\delta \in (0, \pi/16)$ together with a polynomial $T(x)$, say, of degree k meeting (26). If c_k is the leading coefficient of T, we get from the Boutroux-Cartan lemma (see [14]) that

$$|T(z)| \geqslant |c_k| \left(\frac{\varepsilon}{8e^{n\lambda(\varepsilon)+1}}\right)^k$$

outside a union of at most k open disks, the sum of whose radii is at most $\varepsilon/(4e^{n\lambda(\varepsilon)})$. Using this it is easy to see from the proof of (7) that (27) is also true in the form

$$\int_{S\cap[x_n-\rho,x_n+\rho]} |q_n(t)|^2 |T(t)| d\mu(t) \geqslant e^{-2d_n d + d_n\tau} \tag{28}$$

when d_n is large enough, provided $\varepsilon > 0$ is so small that $\lambda(\varepsilon)(k+L) < \tau$.

Define 2ℓ to be the smallest even integer which is strictly larger than k, and recall from (4) the definition of ρ and y_0. For $\eta > 0$, denote by $\mathcal{N}_\eta([y_0-\rho/2, y_0+\rho/2])$ the η-neighbourhood of $[y_0-\rho/2, y_0+\rho/2]$ in $\mathbb{C}$. By the continuity of $\mathrm{Arg}(z)$ in $\mathrm{Re}z > 0$ and since 2ℓ is even, we can choose $\eta \leqslant \rho/2$ so small that, whenever $z_1, \dots z_{2\ell} \in \mathcal{N}_\eta([y_0-\rho/2, y_0+\rho/2])$, one has:

$$\left|\mathrm{Arg}\left(\Pi_{j=1}^{2\ell}(x - z_j)\right)\right| < \delta \quad \text{for} \quad x \in \mathbb{R} \setminus [y_0-\rho, y_0+\rho]. \tag{29}$$

Now, for n sufficiently large, pick $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,2\ell}$ to be 2ℓ zeros of q_n in $\mathcal{N}_\eta([y_0-\rho/2, y_0+\rho/2])$, counting multiplicities. This is possible because for large n (hence large d_n since $n - d_n$ is bounded as we saw after the proof of Lemma 3.2), there are certainly 2ℓ zeros lying in that neighbourhood as y_0 belongs to the support of the limit measure ν. Consider the polynomial

$$P_{n-1}(z) = \frac{\overline{q_n(\overline{z})}T(z)}{\prod_{j=1}^{2\ell}(z - \overline{\alpha_{n,j}})} \tag{30}$$

which has degree at most $n-1$ since $k < 2\ell$. By Lemma 3.4 the absolute value of the argument of $e^{i\varphi(x)}T(x)$ is smaller than $\pi/2 - 2\delta$ for all $x \in [a,b]$ such that $T(x) \neq 0$. Thus, from (29) and (30), we get for all $x \in [a,b] \setminus [y_0 - \rho, y_0 + \rho]$ with $q_n(x)P_{n-1}(x) \neq 0$ that the absolute value of the argument of $q_n(x)P_{n-1}(x)e^{i\varphi(x)}$ is smaller than $\pi/2 - \delta$, hence its real part is at least $|q_n(x)P_{n-1}(x)|\sin(\delta) > 0$. Consequently, as $x_n \notin [y_0 - 2\rho, y_0 + 2\rho]$ so that $[y_0 - \rho, y_0 + \rho] \cap [x_n - \rho, x_n + \rho] = \emptyset$, we obtain from (28) and (30) that

$$\begin{aligned}
&\operatorname{Re}\int_{S\setminus[y_0-\rho,y_0+\rho]} q_n(t)P_{n-1}(t)\,d\lambda(t)\\
&\geqslant \int_{S\cap[x_n-\rho,x_n+\rho]} \operatorname{Re}\big(q_n(t)P_{n-1}(t)e^{i\varphi(x)}\big)\,d\mu(t)\\
&\geqslant \int_{S\cap[x_n-\rho,x_n+\rho]} \sin(\delta)\frac{|q_n(t)|^2|T(t)|}{(\operatorname{diam}(S)+\rho)^{2\ell}}d\mu(t)\\
&\geqslant \frac{\sin(\delta)}{(\operatorname{diam}(S)+\rho)^{2\ell}}e^{-2d_nd+d_n\tau}
\end{aligned} \tag{31}$$

as soon as $n \in \mathbb{N}_2$ is large enough, where we used in the next to last inequality that the distance from $\overline{\alpha_{n,j}}$ to $y_0 \in S$ is at most $\rho/2 + \eta \leqslant \rho$.

In another connection, the limit distribution of the zeros of the polynomials P_{n-1}, $n \in \mathbb{N}_2$, is again ν for T is fixed and we only discarded a fixed amount of 2ℓ zeros $\overline{\alpha_{n,1}}, \overline{\alpha_{n,2}}, \ldots, \overline{\alpha_{n,2l}}$ from $\overline{q_n(\overline{z})}$ whose asymptotic zero distribution is the same as q_n because ν is supported on $\mathbb{R}$. Therefore, as we already remarked after (4), we also have since c_k is the leading coefficient of P_{n-1} that

$$|P_{n-1}(y)| \leqslant |c_k|e^{-d_nd}, \qquad y \in [y_0 - 2\rho, y_0 + 2\rho],$$

for sufficiently large $n \in \mathbb{N}_2$. From this and (4) (with d_nd instead of nd) we obtain that

$$\left|\int_{S\cap[y_0-\rho,y_0+\rho]} q_n(t)P_{n-1}(t)\,d\lambda(t)\right| \leqslant \mu\big(S\cap[y_0-\rho,y_0+\rho]\big)|c_k|e^{-2d_nd} \tag{32}$$

for sufficiently large $n \in \mathbb{N}_2$. But the sum of the two integrals on the left-hand sides of (31) and (32) should be zero for q_n satisfies the orthogonality relation (10). This is clearly not the case when n (thus d_n) is sufficiently large, in view of the estimates (31)-(32) and the fact that $\tau > 0$, because the first integral is larger in modulus that its real part which is itself much bigger than the modulus of the second integral. This contradiction proves the theorem when the zeros of the polynomials q_n remain bounded.

In the general case, we see from Corollary 3.3 that at most a fixed number of zeros of q_n, say N, can leave every compact subset of $\mathbb{C}$ as n goes large. Refining the subsequence $\mathbb{N}_1$ if necessary, we may therefore assume that exactly m zeros of q_n actually go to infinity while the remaining $d_n - m$ zeros remain bounded, where m is some integer which is no larger than N. We denote the zeros that go to infinity by

$$\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_m^{(n)},$$

and we put

$$p_m^{(n)}(t) = \Pi_{j=1}^m \left(t - \xi_j^{(n)}\right), \qquad q_n^*(t) = \frac{q_n(t)}{p_m^{(n)}(t)}.$$

Note that q_n^* has degree at most $n-m$, that its zeros remain bounded, and that it satisfies the orthogonality relations:

$$\int q_n^*(t) t^k e^{i\varphi(t)} \left|p_m^{(n)}(t)\right|^2 d\mu(t) = 0, \qquad k = 0, \ldots, n-m-1. \tag{33}$$

For n large we certainly have that $\xi_j^{(n)} \neq 0$, so we can define a sequence of positive measures by setting

$$d\mu^{(n)}(t) = \left| \frac{p_m^{(n)}(t)}{\Pi_{j=1}^m \xi_j^{(n)}} \right|^2 d\mu(t), \qquad n \in \mathbb{N}_1,\ n \text{ large enough},$$

and upon renormalizing (33) we get:

$$\int q_n^*(t) t^k e^{i\varphi(t)} d\mu^{(n)}(t) = 0, \qquad k = 0, \ldots, n-m-1.$$

Because the asymptotic zero distribution of q_n and q_n^* is the same, we would be done if only we could apply the first part of the proof to q_n^* with $n-m$ instead of n and $\mu^{(n)}$ instead of μ. Although $\mu^{(n)}$ depends on n, this is indeed possible provided that the constant c in (3) can be made uniform with respect to n, and provided also that the total mass remains bounded independently of n, for these are the only facts that were used on μ beyond positivity. But it is trivial that $\mu^{(n)}$ meets these requirements because

$$\frac{\left|p_m^{(n)}(t)\right|^2}{\Pi_{j=1}^m |\xi_j^{(n)}|^2} \longrightarrow 1, \qquad n \to \infty,\ n \in \mathbb{N}_1$$

uniformly on S as n goes to infinity, thereby proving Theorem 3.1. □

4. Rational approximants to Markov functions.

Let μ be a finite positive Borel measure with infinite compact support $S \subset (-1,1)$, and consider the associated Markov function

$$M(z) = \int \frac{1}{z-t} d\mu(t). \tag{34}$$

If we form the diagonal Padé approximants to $M(z)$ at ∞, it is well-known that their denominators are the orthogonal polynomials with respect to μ considered in Section 2. If however we form a best rational L^2 approximant of degree at most n to $M(z)$ on the unit circle $\mathbb{T}$, say p_{n-1}/q_n (by Parseval's theorem it must vanish at infinity) then its monic denominator q_n has exact degree n and all its roots lie in the open unit disk:

$$q_n(t) = \prod_{j=1}^{n} (t - \alpha_j), \qquad |\alpha_j| < 1 \text{ for } j \in \{1, \ldots, n\}. \tag{35}$$

Moreover, q_n satisfies the orthogonality conditions:

$$\int \frac{q_n(t)}{Q_n^2(t)} t^k d\mu = 0, \qquad 0 \leqslant k < n, \tag{36}$$

where

$$Q_n(t) = \prod_{j=1}^{n} (t - 1/\overline{\alpha_j})$$

is the polynomial whose zeros are reflected from those of q_n across the unit circle (if $\alpha_j = 0$, then the corresponding factor is missing from Q_n). In fact, that q_n must be real is not immediate but true [1], and then (36) follows from [7] (which deals with real approximants only). In particular q_n is the n-th orthogonal polynomial associated to the positive measure $Q_n^{-2}(t) d\mu(t)$.

However, a best approximant of degree n need not be unique, hence (36) may have several solutions [4]. Moreover, equation (36) is met not only by the denominator of a best approximant, but more generally by the denominator of each critical point of the L^2-error in degree n, including local minima, saddles and so on. Sufficient condition for uniqueness can be found in [7], [6], but the theorem we shall prove is valid for any sequence of polynomials satisfying (35)-(36). Therefore we simply assume that we are given such a sequence, which is consistently denoted by $\{q_n\}$. From Lemma 5.2 to come (put $m = 1$ and $\varphi = 0$ in that lemma) q_n necessarily has real roots so that, being the n-th orthogonal polynomial associated to the positive measure $Q_n^{-2}(t)d\mu(t)$, it has n simple zeros lying in the convex hull of S.

We are interested in the limit distribution of the zeros of q_n, so we let ν_n be their normalized counting measure.

About μ we assume exactly as in Section 2 that S is regular and that (3) holds.

To formulate our result we need to introduce the Green equilibrium measure of S with respect to the unit disk. Let

$$g(z,a) = \log\left|\frac{1-\overline{a}z}{z-a}\right|$$

be the Green function of the unit disk with pole at a. To each probability measure σ with support in S we associate the Green energy:

$$\iint g(z,a)d\sigma(z)d\sigma(a).$$

Now, among all probability measures supported on S, there is one and only one measure Ω_S minimizing the Green energy, which is called the Green equilibrium measure of S associated with the unit disk. It is the only probability measure on S whose *Green potential*

$$G^{\Omega_S}(z) = \int g(z,a)\, d\Omega_S(a)$$

is equal to a constant quasi-everywhere on S and less or equal to that constant everywhere (see e.g. [16]).

THEOREM 4.1. — *Suppose that the support S of μ is regular with respect to the Dirichlet problem in $\overline{\mathbb{C}}\setminus S$, and that* (3) *holds. Then ν_n tends to the Green equilibrium measure Ω_S of S in the weak* topology as n tends to ∞.*

Proof. — We start the proof with the observation that the problem is invariant under Möbius transformations $z \mapsto (z-a)/(1-az)$, $a \in (-1,1)$. It is known that the Green equilibrium measure is invariant under Möbius transformation, and we claim that the statement is also invariant. In fact, let $w = (z-a)/(1-az)$ be a new complex variable, reserving the special notation $\tau = (t-a)/(1-at)$ when $t \in (-1,1)$, and define a new measure ν by setting:

$$\frac{d\nu(\tau)}{1+a\tau} = d\mu(t).$$

Observe that the support of ν, being the image of S under the conformal map $z \mapsto w$, is again a regular compact subset of $(-1,1)$, and that (3) will hold for ν (with a different c).

Note also that for any bounded measurable function h on $(-1,1)$ we have:

$$\int h\left(\frac{t-a}{1-at}\right) d\mu(t) = \int h(\tau)\frac{d\nu(\tau)}{1+a\tau}.$$

If λ_j is the image of α_j under the mapping $z \mapsto w$, then $1/\overline{\lambda_j}$ is the image of $1/\overline{\alpha_j}$. Therefore, the orthogonality relations

$$\int t^k \left(\prod_{j=1}^n \frac{t-\alpha_j}{(t-1/\overline{\alpha_j})^2}\right) d\mu(t) = 0, \qquad k = 0, 1, \dots n-1$$

take (modulo a multiplicative constant) the form:

$$\int \left(\frac{\tau+a}{1+a\tau}\right)^k \left(\prod_{j=1}^n \frac{(\tau-\lambda_j)(1+a\tau)}{(\tau-1/\overline{\lambda_j})^2}\right) \frac{d\nu(\tau)}{1+a\tau} = 0, \qquad k = 0, 1, \dots n-1.$$

Here

$$\left(\frac{\tau+a}{1+a\tau}\right)^k (1+a\tau)^{n-1}, \qquad k = 0, 1, \dots, n-1$$

generate all polynomials in τ of degree at most $n-1$, hence the preceding relations are the same as the orthogonality relations:

$$\int \tau^k \left(\prod_{j=1}^n \frac{\tau-\lambda_j}{(t-1/\overline{\lambda_j})^2}\right) d\nu(\tau) = 0, \qquad k = 0, 1, \dots n-1. \tag{37}$$

If, on the basis of (37), we are now able to prove the theorem for ν, it will hold for μ as well by invariance of the Green equilibrium measure under Möbius transforms, as announced.

Back to our problem, having seen that it is invariant under Möbius transformations $z \mapsto (z-a)/(1-az)$ where $a \in (-1,1)$ can be chosen at will, we pick it to the left of S and make this preliminary transformation so as to be able to assume in what follows that $S \subset (0,1)$.

After these we prove the promised asymptotic behavior of the zeros, closely following the proof in Section 2. Let ν be a weak* limit point of $\{\nu_n\}$, say $\nu_n \to \nu$ as $n \to \infty$, $n \in \mathbb{N}_1$. Since, we saw at the beginning of this section, q_n is the n-th orthogonal polynomial with respect to the positive measure $Q_n^{-2}(t)\, d\mu(t)$, each subinterval of $\mathbb{R} \setminus S$ contains at most one zero of q_n which is simple hence ν is a probability measure supported on S.

Together with ν_n we also consider the normalized zero counting measure of Q_n, let it be σ_n. It is immediate that if $\nu_n \to \nu$, then $\sigma_n \to \sigma$, where σ is the reflection of ν across the unit circle. In particular, σ is supported outside the unit disk on the compact subset S^{-1} of $\mathbb{R}$ (remember that $S \subset (0,1)$), and its potential is continuous in the unit disk. Likewise, taking into account that the zeros of Q_n remain in the convex hull of S^{-1}, it is easy to see that the potentials U^{σ_n} (i.e. $\log|Q_n|^{-1/n}$) are uniformly equicontinuous and bounded on S. In particular, up to refining the subsequence $\mathbb{N}_1$, we may assume that U^{σ_n} converges uniformly on S and the limit is necessarily U^σ because for $t \in S$ $z \mapsto \log(1/|z-t|)$ is continuous in a neighbourhood of S^{-1}.

We claim that it is enough to prove there is a constant D such that the potential $U^{\nu-\sigma}$ of $\nu - \sigma$ equals D quasi-everywhere on S. In fact, then the lower semi-continuity of $U^{\nu-\sigma}$ implies that $U^{\nu-\sigma}(x) \leqslant D$ for all $x \in S$. Thus, ν has finite logarithmic energy. Let $\widehat{\sigma}$ be the balayage of the measure σ onto S. The regularity of S with respect to the Dirichlet problem implies that $U^{\widehat{\sigma}}(x) = U^\sigma(x) + c_1$ for some constant c_1 and all $x \in S$. In particular $\widehat{\sigma}$ has finite logarithmic energy, and the potential $U^{\nu-\widehat{\sigma}}(x)$ is equal to $D - c_1$ for quasi-every $x \in S$. Now, integrating the equality "$U^{\nu-\widehat{\sigma}}(x) = D - c_1$ for quasi-every $x \in S$" against $\nu - \widehat{\sigma}$ we get from the fact that $(\nu - \widehat{\sigma})(S) = 0$ the equality:

$$0 = \int U^{\nu-\widehat{\sigma}} d(\nu - \widehat{\sigma}) = \int\int \log\frac{1}{|z-t|} d(\nu - \widehat{\sigma})(t) d(\nu - \widehat{\sigma})(z),$$

and it is known for a signed measure with finite logarithmic energy and of total mass zero that the logarithmic energy can be zero only if the measure is zero (see [16], Lemma I.1.8). Thus, $\nu = \widehat{\sigma}$. Let now $\tilde{\sigma}$ and $\tilde{\nu}$ denote the respective balayages of σ and ν onto the unit circle. Seeing that the balayage of the Dirac delta δ_α at α onto the unit circle is the same as the balayage of the Dirac delta $\delta_{1/\overline{\alpha}}$ at the reflected point $1/\overline{\alpha}$, it follows that $\tilde{\sigma} = \tilde{\nu}$. However, forming the balayage of σ onto S can be done in two steps: first form the balayage onto the unit circle, then form the balayage of the so obtained measure $\tilde{\sigma}$ onto S. Thus, using as before a "hat" to denote the balayage onto S and a "tilde" to denote the balayage on the unit circle, we get that $\widehat{\sigma} = \widehat{\tilde{\sigma}}$, and so we can write

$$\nu = \widehat{\sigma} = \widehat{\tilde{\sigma}} = \widehat{\tilde{\nu}},$$

i.e. ν has the property that if we form its balayage onto the unit circle and then form the balayage of that measure back onto S, we obtain ν again. This, however, characterizes the Green equilibrium measure Ω_S (see [16], Theorem VIII.2.6), hence $\nu = \Omega_S$. Since this is true for all weak* limit points of $\{\nu_n\}$, the whole sequence converges to Ω_S, *as claimed.*

Thus, it has left to prove there is a constant D such that the potential of $\nu-\sigma$ equals D quasi-everywhere on S. We closely follow the reasoning in the proof of Theorem 2.1, but there are some minor changes that we have to indicate.

Suppose to the contrary that the claim is not true, and there are d, $\tau > 0$ and two non-polar sets $E_- \subset S$ and $E_+ \subset S$ such that

$$U^{\nu-\sigma}(x) \leqslant d - 2\tau \quad \text{for } x \in E_- \quad \text{and} \quad U^{\nu-\sigma}(x) \geqslant d + \tau \quad \text{for } x \in E_+$$

(note that $U^{\nu-\sigma}$ is certainly finite quasi-everywhere on S since both U^ν and U^σ are). We claim that there is $y_0 \in \operatorname{supp}(\nu)$ such that $U^{\nu-\sigma}(y_0) > d$. In fact, in the opposite case $U^\nu(x) \leqslant U^\sigma(x) + d$ for all $x \in S$, which implies first of all that ν has finite logarithmic energy and then by the principle of domination that the same inequality holds for all $z \in \mathbb{C}$, which is a contradition, for on E_+ we have bigger values for $U^{\nu-\sigma}$.

According to the principle of descent (remember that the support of ν_n remains in the convex hull of S and that U^{σ_n} converges uniformly to U^σ on the disk), we have that

$$\liminf_{n\to\infty,\ n\in\mathbb{N}_1} U^{\nu_n-\sigma_n}(y_n) \geqslant U^{\nu-\sigma}(y_0) > d$$

for any sequence $y_n \to y_0$. Therefore, there is $\rho > 0$ and N_1 such that, for $y \in [y_0 - 2\rho, y_0 + 2\rho]$ and $n \geqslant N_1$, $n \in \mathbb{N}_1$, the inequality $U^{\nu_n-\sigma_n}(y) \geqslant d$ holds which is the same as

$$|q_n(y)/Q_n(y)| \leqslant e^{-nd}, \qquad y \in [y_0 - 2\rho, y_0 + 2\rho]. \tag{38}$$

As before, this inequality remains true (for sufficiently large n) if we replace q_n by some monic polynomial q_n^* with $\deg(q_n^*) = n + \mathrm{o}(n)$, provided that the zeros of q_n^* remain in a compact set of the unit disk and their asymptotic distribution is still ν.

In another connection, the lower envelope theorem implies that, for quasi-every $x \in E_-$, we have

$$\liminf_{n\to\infty,\ n\in\mathbb{N}_1} U^{\nu_n-\sigma_n}(x) = U^{\nu-\sigma}(x) \leqslant d - 2\tau,$$

hence for some subsequence $\mathbb{N}_2 \subset \mathbb{N}_1$ and sufficiently large $n \in \mathbb{N}_2$, we get:

$$M_n := \max_{x\in S} |q_n(x)/Q_n(x)| \geqslant e^{-n(d-\tau)}. \tag{39}$$

We let x_n be a point where the maximum is attained.

As in the proof of Theorem 2.1, we now need an estimate for q_n away from S in terms of M_n. To this effect, we set $w(x) = 1/|Q_n(x)|^{1/n}$, $x \in S$, and we consider the weighted energy problem for w on S (see [16], Theorem I.1.3). If we denote by $\widehat{\sigma_n}$ the balayage of σ_n onto S, we deduce from the definition of balayage and the regularity of S that $w(x) = \exp(U^{\widehat{\sigma_n}}(x) - c_n)$ for every $x \in S$ and some constant c_n. Then it follows from [16], Theorem I.3.3, that the equilibrium measure associated with this weighted energy problem is precisely $\widehat{\sigma_n}$. So, from the generalized Bernstein–Walsh lemma [16], Theorem III.2.1, we obtain:

$$|q_n(z)| \leqslant M_n \exp(-nU^{\widehat{\sigma_n}}(z) + nc_n). \tag{40}$$

Note that for $z = x_n$ the exponential factor on the right is just $|Q_n(x_n)|$. Since $U^{\widehat{\sigma_n}}$ is continuous on S, it follows from the continuity theorem for logarithmic potentials (Maria's theorem) [16], Theorem II.3.5, that $U^{\widehat{\sigma_n}}$ is continuous on the whole plane, and actually the continuity on S is uniform in n (this follows from the equicontinuity of the potentials U^{σ_n} on S, and from the very proof of the continuity theorem). Hence, for every $\theta > 0$, there is $\varepsilon > 0$ such that for $|z - x_n| \leqslant 2\varepsilon$ we have

$$|U^{\widehat{\sigma_n}}(z) - U^{\widehat{\sigma_n}}(x_n)| \leqslant \theta. \tag{41}$$

All these imply in view of (40) that, for $|z - x_n| \leqslant 2\varepsilon$,

$$|q_n(z)| \leqslant |Q_n(x_n)| M_n e^{n\theta}.$$

Then, by Cauchy's formula,

$$|q_n'(z)| \leqslant \frac{|Q_n(x_n)| M_n e^{n\theta}}{\varepsilon}$$

for $|z - x_n| \leqslant \varepsilon$ and, as in the proof of Theorem 2.1, we obtain that

$$|q_n(z)| \geqslant |Q_n(x_n)| M_n/2 \tag{42}$$

for $|z - x_n| \leqslant \varepsilon/(2e^{n\theta})$.

From the elementary identity:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e,$$

it easily follows that for $|z - x_n| = o(1/n)$ we have $|Q_n(z)| \leqslant 2|Q_n(x_n)|$ for sufficiently large n. On dividing (42) by the latter inequality, we see that

$$|q_n(z)/Q_n(z)| \geqslant M_n/4, \quad |z - x_n| \leqslant \varepsilon/(2e^{n\theta})$$

when $n \in \mathbb{N}_2$ is large enough.

From here, (3), and (39), we deduce as in the proof of Theorem 2.1 that

$$\begin{aligned}\int_{[x_n-\rho,x_n+\rho]}\frac{|q_n(t)|^2}{|Q_n(t)|^2}d\mu(t) &\geqslant \left(\frac{M_n}{4}\right)^2 c\left(\frac{\varepsilon}{2e^{n\theta}}\right)^L \geqslant \frac{c}{16}\left(\frac{\varepsilon}{2e^{n\theta}}\right)^L e^{-2n(d-\tau)}\\ &\geqslant e^{-2nd+n\tau} \qquad (43)\end{aligned}$$

for sufficiently large $n \in \mathbb{N}_2$, say for $n \geqslant N_2$, provided $\theta > 0$ is so small that $\theta L < \tau$.

Now, for $n \in \mathbb{N}_2$, $n \geqslant \max(N_1, N_2)$ let us choose two zeros $\alpha_{n,1} < \alpha_{n,2}$ of q_n in the interval $[y_0-\rho, y_0+\rho]$, and consider the polynomial $P_{n-2}(t) = q_n(t)/((t-\alpha_{n,1})(t-\alpha_{n,2}))$ of degree at most $n-2$ (we can find such zeros because y_0 is in the support of ν). As in the proof of Theorem 2.1 we get on the one hand from (43) and the definition of P_{n-2}, together with the disjointness of $[x_n-\rho, x_n+\rho]$ and $[y_0-\rho, y_0+\rho]$, that

$$\int_{S\setminus(\alpha_{n,1},\alpha_{n,2})}\frac{q_n(t)}{Q_n^2(t)}P_{n-2}(t)d\mu(t) \geqslant \frac{1}{(\mathrm{diam}(S))^2}e^{-2nd+n\tau}. \qquad (44)$$

On the other hand, the limit distribution of the zeros of the polynomials P_{n-2}, $n \in \mathbb{N}_2$, is again ν, and these zeros remain in the convex hull of S, hence as we have already remarked after (38) we also have

$$|P_{n-2}(y)/Q_n(y)| \leqslant e^{-nd}, \qquad y \in [y_0-2\rho, y_0+2\rho]$$

for sufficiently large n. This and (38) together lead to the estimate:

$$\left|\int_{S\cap(\alpha_{n,1},\alpha_{n,2})}\frac{q_n(t)}{Q_n^2(t)}P_{n-2}(t)d\mu(t)\right| \leqslant Ce^{-2nd} \qquad (45)$$

for some constant C. Now clearly, the sum of the two integrals on left of (44) and (45) cannot be zero for large n since $\tau > 0$. But this contradicts the orthogonality relation (36), and this contradiction proves the theorem. $\square$

5. Rational approximants to Cauchy transforms.

In this section, we let λ be a complex measure with infinite compact support $S \subset (-1,1)$, and we consider its Cauchy transform

$$M(z) = \int \frac{1}{z-t}d\lambda(t). \qquad (46)$$

When λ is positive, then this is just the Markov function discussed in the preceding section. If we form again a best-L^2 rational approximant of degree at most n to $M(z)$ on the unit circle, say p_{n-1}/q_n, then the denominators

$$q_n(t) = \prod_{j=1}^{n}(t - \alpha_j) \tag{47}$$

have exact degree n, their zeros lie in the open unit disk, and they satisfy the orthogonality relations [3]:

$$\int \frac{q_n(t)}{Q_n^2(t)} t^k \, d\lambda(t) = 0, \qquad 0 \leqslant k < n, \tag{48}$$

where

$$Q_n(t) = \prod_{j=1}^{n}(t - 1/\overline{\alpha_j}) \tag{49}$$

is the polynomial that has zeros at the reflected zeros of q_n across the unit circle (if $\alpha_j = 0$, then the corresponding factor is missing from Q_n). We emphasize that in the present case the zeros of q_n need not be real and so, besides λ that is in general complex-valued, the factor $q_n(t)/Q_n^2(t)$ in the orthogonality relation is also complex-valued.

Of course, even less than in the real case does (48) characterize q_n uniquely in general. There may be many solutions of degree n with zeros inside the disk, induced by several best approximants, local minima, saddles and so on. There may also be solutions of lower degree, but these have no special interpretation with respect to approximation and we shall not consider them. It is not known whether there may be solutions having zeros outside the disk.

For our purposes, we shall simply assume that we are given a sequence of solutions $\{q_n\}$ of exact degree n whose zeros lie in the unit disk. In particular, $\{q_n\}$ can be a sequence of denominators of best-$L^2(\mathbb{T})$ rational approximants to M.

Again, we are interested in the limit distribution of the zeros of q_n, so we let ν_n be their normalized counting measure.

About λ and its support S we assume the same conditions as in Section 3, namely that S is a regular set with respect to the Dirichlet problem in $\overline{\mathbb{C}} \setminus S$, and that λ can be written as

$$d\lambda(t) = e^{i\varphi(t)} d\mu(t) \tag{50}$$

where φ is of bounded variation on S and μ satisfies the density relation (3).

Theorem 5.1. — *With the preceding assumptions ν_n tends to the Green equilibrium measure Ω_S of S in the weak* topology as n tends to ∞.*

Proof. — By now it should be clear how the proof proceeds: in fact Theorem 5.1 is related to Theorem 4.1 exactly as Theorem 3.1 is related to Theorem 2.1. We follow the proof of Theorems 4.1 and 3.1.

First we apply a preliminary Möbius transformation $z \mapsto (z–a)/(1–az)$, $a \in (-1,1)$ so as to ensure that $S \subset (0,1)$ (see the proof of Theorem 4.1).

Next, we consider any collection of m disjoint intervals $[a_j, b_j]$ such that

(51) $S \subseteq \cup_{j=1}^m [a_j, b_j] \subset (0,1)$ with $a_1 \leqslant b_1 < a_2 \leqslant b_2 < \ldots < a_m \leqslant b_m$.

Recalling the notation $\mathrm{Angle}(\xi, [a,b]) = |\,\mathrm{Arg}(a-\xi) - \mathrm{Arg}(b-\xi)|$ for the angle in which an interval $[a,b]$ is seen at ξ, we set as in Lemma 3.2

$$\theta(\xi) = \sum_{j=1}^m \mathrm{Angle}(\xi, [a_j, b_j]), \tag{52}$$

which is the the total angle in which $\cup_j [a_j, b_j])$ is seen at ξ.

We shall need an analog to that lemma:

Lemma 5.2. — *Let $q_n(z) = \prod_{k=1}^n (z - \xi_k)$ be a n-th orthogonal polynomial in the sense of (48), where $|\xi_k| < 1$ for $1 \leqslant k \leqslant n$. Under the sole hypothesis that the support S of λ is an infinite compact subset of $(-1,1)$, it holds with the previous notations that*

$$\sum_{k=1}^n (\pi - \theta(\xi_k)) \leqslant V(\varphi, S) + (m-1)\pi. \tag{53}$$

Corollary 5.3. — *Under the assumptions of Theorem 5.1, for every neighbourhood U of S there is a constant K_U such that q_n has at most K_U zeros outside U.*

This corollary immediately follows from Lemma 5.2 exactly as we deduced Corollary 3.3 from Lemma 3.2.

We will have to take care of the argument of $1/Q_n^2(t)$, as well. This will be done via the next lemma.

Lemma 5.4. — *Let $[a,b] \subset (0,1)$ be an interval containing the support S of λ. To every $\delta > 0$ there exists an integer l such that, for each*

n large enough, there is a polynomial $T_{l,n}$ of degree at most l satisfying:

$$\left|\frac{Q_n(t)}{|Q_n(t)|} - T_{l,n}(t)\right| < \delta, \qquad t \in [a,b]. \tag{54}$$

In particular, the argument of $T_{l,n}(t)/Q_n(t)$ lies in the interval $(-2\delta, 2\delta)$ when n is large enough.

Taking these lemmas for granted, we complete the proof of Theorem 5.1 and return to the lemmas afterwards.

As in the proof of Lemma 3.2, we extend φ to the convex hull $[a,b]$ of S without increasing its variation and then, using Lemma 3.4, we fix a polynomial T of degree, say, k such that, for some $0 < \delta < \pi/16$ we have $\operatorname{Arg}(e^{i\varphi(t)}T(t)) \in [-\pi/2+2\delta, \pi/2-2\delta]$ for $t \in [a,b]$ provided that $T(t) \neq 0$.

Let ν be a weak* limit point of $\{\nu_n\}$, say $\nu_n \to \nu$ as $n \to \infty$, $n \in \mathbb{N}_1$. By Corollary 5.3 this ν is supported on S. Together with ν_n we also consider σ_n, the normalized zero counting measure of Q_n, that converges weak* to the reflexion σ of ν across the unit circle for $n \in \mathbb{N}_1$. By Lemma 5.4 there is a fixed integer l and a polynomial $T_{l,n}$ of degree at most l such that (54) holds with $\delta/8$ instead of δ for all $t \in [a,b]$ and all n sufficiently large. Since the sequence $T_{l,n}$ is uniformly bounded on $[a,b]$ of degree at most l, we may assume up to refining $\mathbb{N}_1$ that $T_{n,l}$ converges uniformly on $[a,b]$ to some polynomial T_l of degree at most l. Then, for n large enough, (54) will hold with T_l in place of $T_{l,n}$ and still $\delta/8$ instead of δ. Consequently, for $t \in [a,b]$, we shall have that $\operatorname{Arg}(T_l^2(t)/Q_n^2(t)) \in [-\delta/2, \delta/2]$ for all large $n \in \mathbb{N}_1$ and also that, say, $1/2 < |T_l(t)| < 2$.

Exactly as in the proof of Theorem 4.1 it is enough to verify that there is a constant D such that the potential $U^{\nu-\sigma}$ of $\nu - \sigma$ equals D quasi-everywhere on S. *We first assume that no zero of Q_n goes to ∞, or equivalently that no zero of q_n goes to zero.* Then, if there is no constant D as above, we obtain as in the preceding section that (38) holds when $n \in \mathbb{N}_1$ is sufficiently large, for some y_0 lying in the support of ν and some $\rho > 0$, and also (cf. (43)) that

$$\int_{[x_n-\rho, x_n+\rho]} \frac{|q_n(t)|^2}{|Q_n(t)|^2} d\mu(t) \geqslant e^{-2nd+n\tau}$$

for large $n \in \mathbb{N}_2 \subset \mathbb{N}_1$ with x_n a maximum point in (39). This time we actually rather need the estimate

$$\int_{[x_n-\rho, x_n+\rho]} \frac{|q_n(t)|^2}{|Q_n(t)|^2} \, |T(t)||T_l^2(t)| d\mu(t) \geqslant e^{-2nd+n\tau}, \tag{55}$$

which can be obtained by the same method upon choosing θ so small (cf. (41)) that $\theta(L+k)<\tau$ and then appealing to the Boutroux-Cartan lemma for T, as we did to obtain (28), while taking into account that $|T_l(t)|>1/2$.

Next, we let 2ℓ be the smallest even integer which is strictly greater than $k+2l$ and, like in the proof of Theorem 3.1 (cf. (29)), we pick $0<\eta\leqslant\rho/2$ so small that, whenever $z_1,\dots z_{2\ell}\in\mathcal{N}_\eta([y_0-\rho/2,y_0+\rho/2])$, one has

$$\left|\operatorname{Arg}\left(\Pi_{j=1}^{2\ell}(x-z_j)\right)\right|<\delta/2 \quad \text{for} \ \ x\in\mathbb{R}\setminus[y_0-\rho,y_0+\rho]. \tag{56}$$

Select $\alpha_{n,1},\alpha_{n,2},\dots,\alpha_{n,2\ell}$ to be 2ℓ zeros of $\overline{q_n(t)}$ lying in $\mathcal{N}_\eta([y_0-\rho/2,y_0+\rho/2])$; there exist such zeros because y_0 lies in the support of ν, and the latter is included in $(0,1)$ and therefore is invariant under complex conjugation. Consider the polynomial

$$P_{n-1}(t)=\overline{q_n(t)}T(t)T_l^2(t)\Big/\prod_{j=1}^{2\ell}(t-\alpha_{n,j}) \tag{57}$$

of degree at most $n-1$. From (56) it follows that

$$\left|\operatorname{Arg}\left(1/\prod_{j=1}^{2\ell}(t-\alpha_{n,j})\right)\right|<\delta/2 \qquad \text{for} \ \ t\in\mathbb{R}\setminus[y_0-\rho,y_0+\rho], \tag{58}$$

and since on $[a,b]$

$$\operatorname{Arg}\left(\frac{q_n(t)}{Q_n^2(t)}\,\overline{q_n(t)}\,T(t)T_l^2(t)e^{i\varphi(t)}\right)$$

lies in the interval $[-\pi/2+3\delta/2,\pi/2-3\delta/2]$ by our choice of T and T_l, provided that $T(t)\neq 0$, it follows from (58) that

$$\operatorname{Arg}\left(\frac{q_n(t)}{Q_n(t)^2}\,P_{n-1}(t)e^{i\varphi(t)}\right)=\operatorname{Arg}\left(\frac{q_n(t)}{Q_n(t)^2}\,\frac{\overline{q_n(t)}T(t)T_l^2(t)}{\prod_{j=1}^{2\ell}(t-\alpha_{n,j})}e^{i\varphi(t)}\right)$$

lies in $[-\pi/2+\delta,\pi/2-\delta]$ on $S\setminus[y_0-\rho,y_0+\rho]$, except when $T(t)=0$. But as $\eta<\rho/2$, no $\alpha_{n,j}$ can lie in $S\setminus[y_0-\rho,y_0+\rho]$ by definition and therefore, in view of (57), a zero of T on $S\setminus[y_0-\rho,y_0+\rho]$ is necessarily a zero of P_{n-1}. Thus the inequality

$$\operatorname{Re}\left(\frac{q_n(t)}{Q_n(t)^2}\,P_{n-1}(t)e^{i\varphi(t)}\right)\geqslant\sin\delta\left|\frac{q_n(t)}{Q_n(t)^2}\,P_{n-1}(t)e^{i\varphi(t)}\right|$$

holds for each $t\in S\setminus[y_0-\rho,y_0+\rho]$, implying by (55), (57), and (11) that

$$\begin{aligned}\operatorname{Re}&\left(\int_{S\setminus[y_0-\rho,y_0+\rho]}\frac{q_n(t)}{Q_n(t)^2}\,P_{n-1}(t)d\lambda(t)\right)\\ &\geqslant\frac{\sin\delta}{(\operatorname{diam}(S)+\rho)^{2\ell}}\int_{[x_n-\rho,x_n+\rho]}\frac{|q_n(t)|^2}{|Q_n(t)|^2}\,|T(t)||T_l^2(t)|d\mu(t)\\ &\geqslant\frac{\sin\delta}{(\operatorname{diam}(S)+\rho)^{2\ell}}\,e^{-2nd+n\tau},\end{aligned} \tag{59}$$

where we used in the first inequality that the distance from $\alpha_{n,j}$ to $y_0 \in S$ is at most $\rho/2 + \eta \leqslant \rho$ and also that $[y_0 - \rho, y_0 + \rho] \cap [x_n - \rho, x_n + \rho] = \emptyset$.

On the other hand, the limit distribution of the zeros of the polynomials P_{n-1} for $n \in \mathbb{N}_2$ is again ν, and since their leading coefficient is independent of n (equal to the leading coefficient of TT_l^2), we get exactly as in (38) that

$$|P_{n-1}(y)/Q_n(y)| \leqslant C_1 e^{-nd}, \qquad y \in [y_0 - 2\rho, y_0 + 2\rho],$$

for some constant C_1 and sufficiently large $n \in \mathbb{N}_2$. Here, we should stress that the principle of descent that led to (38) is valid because the zeros of P_{n-1} remain bounded, being either zeros of T, T_l which are fixed or zeros of $\overline{q_n(\overline{z})}$ which lie in the unit disk.

From this and (38) we obtain for some constant C_2 the estimate:

$$\left| \int_{[y_0 - \rho, y_0 + \rho]} \frac{q_n(t)}{Q_n^2(t)} P_{n-1}(t) d\lambda(t) \right| \leqslant C_2\, e^{-2nd}, \tag{60}$$

and again the contradiction comes from the fact that the sum of the two integrals on the left of (60) and (59) cannot be zero for large n as $\tau > 0$.

We need finally to handle the case where some zeros of Q_n go to infinity as $n \to \infty$, $n \in \mathbb{N}_1$. The number of such zeros is necessarily bounded, uniformly with n, because only a bounded number of zeros of q_n can tend to 0 by Corollary 5.3 and the fact that $S \subset (0,1)$. Refining $\mathbb{N}_1$ if necessary, we may assume that exactly, say, m zeros of q_n go to zero, where m is some fixed integer. Call these zeros $\alpha_{n,1}, \ldots, \alpha_{n,m}$ for $n \in \mathbb{N}_1$ and let us define

$$Q_n^*(t) = \frac{Q_n(t)}{\Pi_{j=1}^m (t - 1/\overline{\alpha}_{n,j})},$$

$$d\mu_n(t) = \left|\Pi_{j=1}^m (1 - \overline{\alpha}_{n,j} t)\right|^{-2} d\mu(t),$$

$$\varphi_n(t) = \varphi(t) - 2 \sum_{j=1}^m \operatorname{Arg}(1 - \overline{\alpha}_{n,j} t),$$

noting that all the quantities above are well defined for n large enough. Then, upon renormalizing (48), we can rewrite the orthogonality relations as:

$$\int \frac{q_n(t)}{\left(Q_n^*\right)^2(t)}\, t^k\, e^{i\varphi_n(t)}\, d\mu_n(t) = 0, \qquad 0 \leqslant k < n. \tag{61}$$

Because Q_n^* has asymptotic zero-distribution σ and its roots are now bounded, we seek to apply the first part of the proof to μ_n and φ_n instead of μ and φ even though these depend on n. As for μ_n this is possible because

$$\left|\Pi_{j=1}^m(1-\overline{\alpha}_{n,j}t)\right|^2 \longrightarrow 1$$

uniformly on S as $n\to\infty$, hence the constant c in (3) can be made uniform with respect to n, and the total mass remains bounded independently of n (compare the end of the proof of Theorem 3.1). As for φ_n, we notice that $\varphi_n-\varphi$ converges uniformly to zero on the convex hull of S together with all its derivatives (compare (71) in the proof of Lemma 5.2 below), thus $V(\varphi-\varphi_1,S)$ goes to zero as well. Therefore Corollary 5.3 and Lemma 3.4 remain true with the same T when φ gets replaced by φ_n and n is large enough. Because these were the only facts we used on φ, this demonstrates Theorem 5.1. □

In the previous proof we used Lemmas 5.2 and 5.4, and we need now to establish them.

Proof of Lemma 5.2. — We follow the proof of Lemma 3.2. Exactly as in that lemma, we may assume that $V(\varphi,S)<\infty$ and that φ is defined on the whole real line with jumps of size at most π. Next, an induction argument similar to (15) reduces the proof to the case where q_n has no zero on $\cup_j[a_j,b_j]$ (in that argument, set $d\mu_1(t)=t^2d\mu(t)$ if $\xi_k=0$ and $d\mu_1(t)=(t-\xi_k)^2(t-1/\xi_k)^{-2}d\mu(t)$ otherwise). Then, the reasoning that led us to (16) yields:

$$V(\varphi,S)+\sum_{k=1}^{n}\sum_{j=1}^{m}V\left(\arg\left(\frac{t-\xi_k}{(t-1/\overline{\xi}_k)^2}\right),[a_j,b_j]\right)\geqslant(n-m+1)\pi, \tag{62}$$

where $\arg\left((t-\xi_k)/(t-1/\overline{\xi}_k)^2\right)$ is *any* argument function for $(t-\xi_k)/(t-1/\overline{\xi}_k)^2$ on $(-1,1)$ and it is understood that the factor $(t-1/\overline{\xi}_k)^{-2}$ is not present if $\xi_k=0$. Indeed, if (62) did not hold, the construction of Lemma 3.2 would provide us with a polynomial P_{n-1} of degree at most $n-1$ such that, for some $u_0\in\mathbb{R}$, the function

$$t\mapsto \operatorname{Im}\left(e^{-iu_0}\frac{q_n(t)}{Q_n^2(t)}e^{i\varphi(t)}P_{n-1}(t)\right) \tag{63}$$

has constant sign on S except at finitely many points where it vanishes. But this contradicts the fact that, by (48) and (50), we have:

$$\int e^{-iu_0}\frac{q_n(t)}{Q_n^2(t)}e^{i\varphi(t)}P_{n-1}(t)d\mu(t)=0,$$

whereas the measure μ is positive with infinite support.

We shall invoke (62) with a choice of the argument given by

$$\arg\left(\frac{t-\xi}{(t-1/\overline{\xi})^2}\right) = \mathrm{Arg}\,(t-\xi) - 2\,\mathrm{Arg}\left(t-1/\overline{\xi}\right), \tag{64}$$

where Arg is the normalized argument function used in (12). This will imply what we want once we have shown that if $[a,b] \subset (-1,1)$ is any interval and if we let for simplicity

$$g_\zeta(t) = \mathrm{Arg}\,(t-\zeta) - 2\,\mathrm{Arg}\left(t-1/\overline{\zeta}\right), \qquad \zeta \in \mathbb{C}\setminus[a,b], \tag{65}$$

where the summand $-2\,\mathrm{Arg}(t-1/\overline{\zeta})$ is omitted if $\zeta = 0$, then

$$V(g_\xi,[a,b]) \leqslant \mathrm{Angle}(\xi,[a,b]) \quad \text{for } |\xi|<1,\ \xi \notin [a,b]. \tag{66}$$

Indeed, plugging (66) into (62) for each $[a,b] = [a_j,b_j]$ we get the statement in Lemma 5.2.

Note that the desired inequality certainly holds with equality when $\xi \in (-1,1)\setminus[a,b]$, because then (64) is independent of $t \in [a,b]$ (it is either 0, $-\pi$, or π depending whether $\xi < a$ or $\xi > b$ and also whether $\xi = 0$ or not) hence both sides of (66) are equal to 0 in that case. Thus we assume that ξ is imaginary, and we suppose first that it lies in the upper half Δ_+ of the open unit disk with $(-1,1)$ deleted. Because $g_\zeta(t)$ is a smooth function of (ζ,t) on $\mathbb{C}\setminus\mathbb{R}\times[a,b]$, the map $\xi \mapsto V(g_\xi,[a,b])$ is continuous for $\xi \notin \mathbb{R}$ (it is the integral over $[a,b]$ of $|dg_\xi/dt|$). Moreover, by definition, $V(g_\xi,[a,b])$ is the supremum of all sums of the form

$$\Sigma(\xi) = \sum_{k=1}^{M-1} \varepsilon_k\left(g_\xi(t_{k+1}) - g_\xi(t_k)\right)$$

where $t_1 < t_2 < \cdots < t_M$ is any finite sequence in $[a,b]$ and $\varepsilon_k = \pm 1$ any sequence of signs. Since g_ξ is clearly a harmonic function of ξ in Δ_+ by (65), so is every $\Sigma(\xi)$ and thus $V(g_\xi,[a,b])$, being the supremum of a family of harmonic functions and being continuous, is subharmonic. In another connection $\mathrm{Arg}(t-\xi)$ increases with t since $\mathrm{Im}\xi > 0$, and therefore, by (12),

$$\mathrm{Angle}(\xi,[a,b]) = \mathrm{Arg}(b-\xi) - \mathrm{Arg}(a-\xi) \tag{67}$$

is a positive harmonic function of ξ on Δ_+. Thus, by the extended maximum principle for subharmonic functions [15], Theorem 3.6.9, it is sufficient for (66) to hold when $\xi \in \Delta_+$ that $V(g_\xi,[a,b])$ be bounded from above and that, for quasi-every z on the boundary $\partial\Delta_+$ of Δ_+, we have

$$\limsup_{\xi\to z} V(g_\xi,[a,b]) \leqslant \lim_{\xi\to z} \mathrm{Angle}(\xi,[a,b]). \tag{68}$$

To show that $\xi \mapsto V(g_\xi, [a,b])$ is bounded on Δ_+, we observe first from (67) and the monotonicity of $t \mapsto \operatorname{Arg}(t-\xi)$ that

$$V(\operatorname{Arg}(t-\xi),[a,b]) = \operatorname{Arg}(b-\xi) - \operatorname{Arg}(a-\xi) = \operatorname{Angle}(\xi,[a,b]) \leqslant \pi. \tag{69}$$

Next, we put $\xi = \rho e^{i\theta}$ with $0<\rho<1$ and $0<\theta<\pi$, and we notice from elementary geometry that

$$\tan\big(\operatorname{Arg}(t-1/\overline{\xi}) + \pi/2\big) = \frac{\rho t - \cos\theta}{\sin\theta} \tag{70}$$

hence

$$\frac{d}{dt}\operatorname{Arg}(t-1/\overline{\xi}) = \frac{\rho\sin\theta}{\sin^2\theta + (\rho t-\cos\theta)^2}, \quad t\in[a,b]. \tag{71}$$

Setting $\delta>0$ for the distance from $[a,b]$ to the unit circle, the majorization

$$\sin^2\theta + (\rho t - \cos\theta)^2 = 1 + \rho^2 t^2 - 2\rho t\cos\theta > \delta^2 \tag{72}$$

shows together with (71) that

$$V(\operatorname{Arg}(t-1/\overline{\xi}),[a,b]) < (b-a)/\delta^2. \tag{73}$$

Then, the desired boundedness follows from (65), (69), (73), and the triangle inequality.

We are now going to show that (68) in fact holds for each $z\in\partial\Delta_+\setminus\{a,b,0\}$. Indeed, $\xi\mapsto\operatorname{Angle}(\xi,[a,b])$ is continuous on $\mathbb{C}\setminus\{a,b\}$ hence the right-hand side of (68) is just $\operatorname{Angle}(z,[a,b])$ for $z\neq a,b$. Besides, we see from (72) that (71) tends to 0 when $\rho\to\rho_0\in[0,1]$ and $\theta\to 0$ or π, uniformly with respect to $t\in[a,b]$. Therefore

$$\lim_{\xi\to z} V(\operatorname{Arg}(t-1/\overline{\xi}),[a,b]) = 0, \quad z\in[-1,1],$$

implying by (65), (69), the triangle inequality, and the continuity of $\operatorname{Angle}(\xi,[a,b])$ just mentioned that for $z\in[-1,1]$ with $z\neq a,b,0$:

$$\limsup_{\xi\to z} V(g_\xi,[a,b]) \leqslant \lim_{\xi\to z} V(\operatorname{Arg}(t-\xi),[a,b]) = \operatorname{Angle}(z,[a,b]).$$

It remains to consider the case where $z\in\partial\Delta_+$ is imaginary of modulus 1. Then $1/\overline{z} = z$ so that $g_z(t) = -\operatorname{Arg}(t-z)$ is monotonic. Therefore, by the continuity of $V(g_\xi,[a,b])$ on $\mathbb{C}\setminus\mathbb{R}$ already pointed out, we get

$$\lim_{\xi\to z} V(g_\xi,[a,b]) = V(g_z,[a,b]) = |\operatorname{Arg}(b-z) - \operatorname{Arg}(a-z)| = \operatorname{Angle}(z,[a,b]).$$

Thus we have proven that (68) holds at each boundary point of Δ_+ except perhaps a, b, and 0. The reasoning when ξ lies in the lower half of the unit disk is entirely similar, the only difference being that $\operatorname{Arg}(t-\xi)$ decreases with t (so there is a change of sign in (67)) and also that $\operatorname{Arg}(t-1/\overline{\xi})+\pi/2$

gets replaced by $\operatorname{Arg}(t - 1/\overline{\xi}) - \pi/2$ in (70) where, this time, $-\pi < \theta < 0$. Altogether, we find that (66) holds, as was to be shown. □

Proof of Lemma 5.4. — The geometry is here simplified by our assumption that the support $S \subseteq [a, b]$ lies within $(0, 1)$.

It is sufficient to prove that the functions $Q_n(t)/|Q_n(t)|$ have uniformly bounded derivatives on $[a, b]$. Indeed, each of them in this case can be extended to a function $\Phi_n(t)$, defined on $[0, 1]$, whose derivative Φ'_n is uniformly bounded independently of n, say $|\Phi'_n| \leqslant L$ for $t \in [0, 1]$. Then, by Jackson's theorem [8], Theorem 6.2, there is a constant C and there are polynomials $T_{l,n}$ of degree at most l such that

$$|\Phi_n(t) - T_{n,l}(t)| \leqslant \frac{CL}{l}, \qquad t \in [0, 1],$$

and the right hand side will be smaller than δ if $l > CL/\delta$.

In turn, it is clearly enough to verify that $\arg(Q_n(t))$ has uniformly bounded derivative on $[a, b]$, where $\arg(Q_n(t))$ is any argument function for $Q_n(t)$ (note that Q_n can have no zero on $[a, b]$). Notations being as in (47)-(49), we choose to set

$$\arg(Q_n(t)) = \sum_{j=1}^{n} \operatorname{Arg}(t - 1/\overline{\alpha}_j)$$

where, as usual, the j-th summand is omitted if $\alpha_j = 0$. For $1/\overline{\alpha}_j = x + iy$ a zero of Q_n, let $\theta_j = \operatorname{Arg}(1/\overline{\alpha}_j)$ which is the same as $\operatorname{Arg}(\alpha_j)$ and put for simplicity $\varphi_j = \operatorname{Angle}(\alpha_j, [a, b])$, the angle in which $[a, b]$ is seen at α_j. Consider first the case where $\theta_j \in [0, \pi/4]$ and $|\alpha_j| \geqslant a/2$. It is elementary that θ_j is at most $\pi - \varphi_j$, and so if $\pi - \varphi_j \leqslant \pi/4$ we get:

$$y \leqslant (2/a) \tan \theta_j \leqslant (2/a) \tan(\pi - \varphi_j) \leqslant 2(2/a)(\pi - \varphi_j),$$

where we used that $\tan \gamma \leqslant 2\gamma$ for $\gamma \in [0, \pi/4]$. If however $\pi - \varphi_j \geqslant \pi/4$, then

$$y \leqslant (2/a) \leqslant 2(2/a)(\pi - \varphi_j), \tag{74}$$

so this inequality holds regardless how large $\pi - \varphi_j$ is. From (71) we see that, for $t \in [a, b]$,

$$\frac{d(\operatorname{Arg}(t - 1/\overline{\alpha}_j))}{dt} = \frac{y}{(x - t)^2}, \tag{75}$$

and on using (74) we conclude that

$$\left| \frac{d(\operatorname{Arg}(t - 1/\overline{\alpha}_j))}{dt} \right| \leqslant \frac{4}{a(1 - b)^2} (\pi - \varphi_j).$$

A similar estimate holds if $\theta_j \in [-\pi/4, 0]$, $|\alpha_j| \geqslant a/2$. However, by Corollary 5.3, the number of those α_j for which either $|\alpha_j| \leqslant a/2$, or $\mathrm{Arg}(\alpha_j) \notin [-\pi/4, \pi/4]$ is less than a fixed constant K_1, and since by (75) $d(\mathrm{Arg}(t - 1/\overline{\alpha}_j))/dt$ is clearly bounded on $[a,b]$ for each such zero with an absolute bound (remember $[a,b] \subset (0,1)$), the contribution to the derivative of $\arg(Q_n(t))$ of all these exceptional zeros is less than a fixed constant K_2.

So far we have proved that

$$\left|\frac{d(\arg(Q_n(t)))}{dt}\right| \leqslant \sum_{j=1}^{n}\left|\frac{d(\mathrm{Arg}(t-1/\overline{\alpha}_j))}{dt}\right| \leqslant \frac{4}{a(1-b)^2}\sum_{j=1}^{n}(\pi-\varphi_j)+K_2.$$

Now the lemma follows from here and Lemma 5.2. □

6. Remarks and further results.

In this section we point out to some works that motivated the preceding results, as well as we make further remarks in connection with the method used in this paper and its extensions.

1. Generally speaking, the results of the paper would hold under the hypothesis that $d\lambda/d|\lambda|$ has bounded variation while $\mu = |\lambda|$ satisfies, on its support S, the so-called Λ condition introduced in [20]:

(76)
$$\mathrm{cap}\left(\left\{t \in S:\ \limsup_{r\to 0+}\frac{\mathrm{Log}\big(1/\mu([t-r,t+r])\big)}{\mathrm{Log}\big(1/r\big)} < +\infty\right\}\right) = \mathrm{cap}(S),$$

or its hyperbolic analog (replacing the logarithmic capacity by the Green capacity when we deal with Green equilibrium distributions). Above in all sections, we made the stronger assumption that S is regular and that μ is sufficiently thick in the sense of (3). This makes for a clearer exposition that displays already all the interesting features of the method. For the more general version, we refer the reader to [12] where the potential-theoretic arguments are kept a bit more elementary, in that only the principle of descent and the general properties of equilibrium measures are used.

2. In recent past, non-Hermitian orthogonality (see Section 3) has received a lot of interest in connection with rational approximation, for the minimal degree solution of (10) is the monic denominator of the n-th diagonal Padé approximant to the Cauchy transform of λ. In this context, the possibility that $d_n < n$ accounts for cases of non-normality which are well-known to occur with such approximants. Also, the main objective of this paper, namely of proving asymptotic zero distribution, is tantamount to showing the convergence in capacity of these approximants [17]. As to non-Hermitian orthogonality proper, we refer the reader to [18] for a survey on the segment while [19] already deals with a more general situation where orthogonality holds over an arbitrary symmetric contour for the logarithmic potential; in this setting, [9] treats the case of a varying weight in relation to multipoint Padé approximants. Let us point out that the method in Section 3, although restricted to the segment so far, applies to measures with considerably more general support at the cost of assuming a little more on their argument, namely that it has bounded variation. It is moreover interesting in that it provides *non-asymptotic* information on the zeros of the orthogonal polynomials, cf. Lemma 3.2.

From the point of view of Approximation Theory, a natural sequel to the results in Section 5 would be to establish the convergence in capacity of best rational and meromorphic approximants on the circle to Cauchy transforms of complex measures on a segment. However, the primary purpose of this paper is to present a specific technique to handle orthogonality equations, and including such applications would make the paper unbalanced. These we left here for further study.

3. From the strong convergence of Padé approximants established in [13], it follows that *all* the zeros of the minimal degree solution to (10) cluster on S when the latter is a real segment and λ is absolutely continuous with respect to Lebesgue measure with continuous and nowhere vanishing (complex) density. This result is not contained in, nor contains our Lemma 3.2, and it would be interesting to understand better the relations between them.

4. In relation to the Möbius transformation $\mu \to \nu$ in Section 4 (see the proof of Theorem 4.1) it is worth stressing that if M^* denotes the Markov function (34) associated to ν, then

$$M(z) = \frac{1+aw}{1-a^2} M^*(w),$$

and since $|dz| = |1-a^2|\,|1+aw|^{-2}\,|dw|$, the identity

$$\int_{\mathbb{T}} |M(z)|^2 |dz| = (1-a^2)^{-1} \int_{\mathbb{T}} |M^*(w)|^2 |dw|$$

holds. Likewise, if r_n is a rational function with numerator degree $\leqslant n-1$ and denominator degree n and with its poles in the unit disk and if we define a new rational function r_n^* by setting

$$r_n(z) = \frac{1+aw}{1-a^2} r_n^*(w),$$

we deduce in the same way upon regarding r_n and r_n^* as Cauchy transforms of discrete measures that

$$\int_{\mathbb{T}} |r_n(z)|^2 |dz| = (1-a^2)^{-1} \int_{\mathbb{T}} |r_n^*(w)|^2 |dw|.$$

Now, if we fix the denominator χ_n of r_n and if we adjust the numerator in such a way that $M - r_n$ has minimal $L^2(\mathbb{T})$-norm, we get by the characteristic property of orthogonal projections that

$$\int_{\mathbb{T}} |M(z) - r_n(z)|^2 |dz| = \int_{\mathbb{T}} |M(z)|^2 |dz| - \int_{\mathbb{T}} |r_n(z)|^2 |dz|,$$

which is easily seen [7] to be equivalent to the vanishing of $M - r_n$ at the reflected zeros of χ_n across the unit circle. But then $M^* - r_n^*$ vanishes at the reflected zeros of the denominator of r_n^*, so that

$$\int_{\mathbb{T}} |M(z) - r_n(z)|^2 |dz| = (1-a^2)^{-1} \int_{\mathbb{T}} |M^*(w) - r_n^*(w)|^2 |dw|$$

and we see that best approximants also transform in a natural way.

5. Lemma 3.2 and Theorem 3.1 remain valid, under the same assumptions, when λ is supported on an arbitrary line segment $\mathcal{L}$ (not necessarily real). In fact, such a segment is the image of a real segment $[\alpha, \beta]$ under some affine map $A(z) = az + b$ with $a \neq 0$, and if we let $p_n(z) = a^{-d_n} q_n(A(z))$ we deduce from (10) that

$$\int p_n(z)\, z^k \, d\lambda_1(z) = 0, \qquad k = 0, \ldots, n-1,$$

where $\lambda_1(E) = \lambda(A(E))$ for each Borel set E. Since λ_1 is supported on $[\alpha, \beta]$ where it satisfies the hypotheses of either the lemma or the theorem if λ did on $\mathcal{L}$, the conclusion follows from the invariance of equilibrium measures and polynomials of a given degree under affine maps.

6. It is a little less transparent but true that Theorem 5.1 remains valid if λ is supported on a *hyperbolic geodesic segment* of the disk; these are simply the closed subarcs contained in the open unit disk of those circles orthogonal to the unit circle (a diameter is a circular arc centered at infinity). In fact, a hyperbolic geodesic segment $\mathcal{H}$ is simply the image of a real segment $[\alpha, \beta]$ under some conformal automorphism of the unit disk, i.e. under some Möbius transform of the type:

$$M_{a,\theta_0}(z) = e^{i\theta_0} \frac{z-a}{1-\bar{a}z}, \quad |a| < 1. \tag{77}$$

Now, if we start from (48) where λ is supported on $\mathcal{H}$ which is the image of $[\alpha,\beta]$ under (77), and if we let:

$$p_n(z) = (1-\bar{a}z)^n \,\frac{q_n(M_{a,\theta_0}(z))}{\bar{a}^n q_n(-e^{i\theta_0}/\bar{a})}, \quad P_n(z) = z^n\overline{p_n(1/\bar{z})},$$

we get two polynomials, the first of which is monic of degree n with roots in the unit disk (they are the preimages of the roots of q_n under M_{a,θ_0}) and the second of which has roots reflected from those of the first across the unit circle. Then, the computation made at the begining of the proof of Theorem 4.1 shows that p_n and P_n satisfy the orthogonality relations:

$$\int \frac{p_n(z)}{P_n^2(z)}\, z^k \, d\lambda_2(z) = 0, \qquad 0 \leqslant k < n, \tag{78}$$

where $d\lambda_2(z) = (1-\bar{a}z)\, d\lambda_1(z)$ with $\lambda_1(E) = \lambda(M_{a,\theta_0}(E))$ for each Borel set E. Since λ_2 is supported on $[\alpha,\beta]$ where it satisfies the hypotheses of Theorem 5.1 if λ did on $\mathcal{H}$, the conclusion follows from the invariance of Green equilibrium measures under conformal automorphisms of the disk. Note that, by (78) and Lemma 5.2, all the zeros of q_n must lie *on* $\mathcal{H}$ whenever $(1-\bar{a}\xi)\, d\lambda(\xi)$ is real of constant sign.

7. In connection with **6**, it is worth pointing out a hyperbolic version of Lemma 5.2. To state it, let $\mathrm{HAngle}(\xi,\mathcal{H}) \in [0,\pi]$ denote the *hyperbolic angle* under which the geodesic segment $\mathcal{H}$ is seen at ξ, that is to say the angle at ξ between the two geodesics through ξ and the endpoints of $\mathcal{H}$. To parallel (51), we assume that the support S of λ is covered by finitely many disjoint geodesic segments $\mathcal{H}_j$ contained in $\mathcal{H}$:

$$S \subseteq \cup_{j=1}^m \mathcal{H}_j \subset \mathcal{H}. \tag{79}$$

A hyperbolic analog to (52) is now given by

$$\theta_H(\xi) = \sum_{j=1}^m \mathrm{HAngle}(\xi,\mathcal{H}_j), \tag{80}$$

which is the total hyperbolic angle in which $\cup_j\mathcal{H}_j$ is seen at ξ. We may also define the total variation $V(\varphi,S)$ of the function φ on S in a manner similar to (13), since the parametrization $M_{a,\theta_0} : [\alpha,\beta] \to \mathcal{H}$ induces an ordering on $\mathcal{H}$ and any other continuous parametrization induces the same ordering or its opposite. Finally, we set $\Theta(\mathcal{H}) \in [0,\pi)$ to mean the *aperture* of the arc of circle $\mathcal{H}$, that is the ratio of its length and its radius.

LEMMA 6.1. — *Let $q_n(z) = \prod_{k=1}^n (z-\xi_k)$ be a n-th orthogonal polynomial in the sense of (48) with all its roots in the open unit disk, and* *let (50) be the polar decomposition of λ. If the support S of λ is infinite and satisfies(79), then*

$$\sum_{k=1}^{n}(\pi - \theta_H(\xi_k)) \leqslant 2V(\varphi, S) + (m-1)\pi + \Theta(\mathcal{H}). \tag{81}$$

When $S \subset \mathbb{R}$ so that $\mathcal{H}$ is a real segment, we get $\Theta(\mathcal{H}) = 0$ but still the right-hand side of (81) is less favorable than the right-hand side of (53) by a factor 2 in front of $V(f)$. Nevertheless, Lemma 6.1 is in other respects sharper than Lemma 5.2, even in this case, because if $[\alpha, \beta] \subset (0,1)$ and $|\xi| < 1$ then $\mathrm{HAngle}(\xi, [\alpha, \beta]) \leqslant \mathrm{Angle}(\xi, [\alpha, \beta])$ with equality only when $\xi \in (-1, 1)$, and $\mathrm{HAngle}(\xi, [\alpha, \beta])$ tends to zero when $|\xi|$ tends to 1 whereas $\mathrm{Angle}(\xi, [\alpha, \beta])$ does not if $\alpha \neq \beta$.

Proof of Lemma 6.1. — Parametrizing $\mathcal{H}$ by (77), we see from (78) and the invariance of hyperbolic angles under automorphisms of the disk that it is enough to prove the Lemma when $\mathcal{H}$ is a real segment, for the variation of the argument of λ_1 and λ is the same and the variation of $\mathrm{Arg}(1 - \bar{a}z)$ on S is less than its variation on $\mathcal{H}$ which is $\Theta(\mathcal{H})/2$ (to see this last point, observe that the oriented tangent to $\mathcal{H}$ at z has direction $1/(1-\bar{a}z)^2$ whose argument is monotonic as z traverses $\mathcal{H}$, thus $2\,\mathrm{Arg}(1-\bar{a}z)$ has total variation equal to the modulus of the difference between its extreme values which is indeed the aperture of $\mathcal{H}$). Thus we assume that $\mathcal{H}_j = [a_j, b_j]$, a real segment, and accordingly that $\Theta(\mathcal{H}) = 0$.

Let us show for $1 \leqslant j \leqslant m$ and $1 \leqslant k \leqslant n$ that

$$V\left(\mathrm{Arg}\left(\frac{t - \xi_k}{t - 1/\overline{\xi_k}}\right), [a_j, b_j]\right) = \mathrm{HAngle}(\xi_k, [a_j, b_j]), \tag{82}$$

with the usual convention that $t - 1/\overline{\xi}_k$ is to be replaced by 1 if $\xi_k = 0$. This certainly holds when $\xi_k \in (-1, 1)$, because then $\mathrm{HAngle}(\xi_k, [a_j, b_j]) = \mathrm{Angle}(\xi_k, [a_j, b_j])$ and both sides of (82) are equal to π or to 0 according whether ξ_k lies in $[a_j, b_j]$ or not. Therefore we may assume that $\xi_k \notin \mathbb{R}$ in which case $(t - \xi_k)/(t - 1/\overline{\xi}_k)$ is never negative real so that $\mathrm{Arg}((t - \xi_k)/(t - 1/\overline{\xi}_k))$ is a smooth function of t.

Then, the set

$$\mathcal{C}_t = \left\{ z \in \mathbf{C} : |z| < 1 \text{ and } \mathrm{Arg}\left(\frac{z - \xi_k}{z - 1/\overline{\xi}_k}\right) = \mathrm{Arg}\left(\frac{t - \xi_k}{t - 1/\overline{\xi}_k}\right) \right\}$$

is an arc of circle orthogonal to the unit circle (the preimage of a diameter under $M_{\xi_k, 0}$) passing through t and ξ_k, so it supports the geodesic segment linking these points. If we orient $\mathcal{C}_t$ from t to ξ_k, a little geometry shows that $\mathrm{Arg}(t-\xi_k)/(t-1/\overline{\xi}_k)$ is equal to the angle between the chord $[\xi_k, 1/\overline{\xi}_k]$ and the tangent to $\mathcal{C}_t$ at ξ_k. But since $1/\overline{\xi}_k$ is a real multiple of ξ_k this chord lies on a radius, so that $\mathrm{Arg}((t-\xi_k)/(t-1/\overline{\xi}_k))$ is in fact the oriented hyperbolic angle $\mathrm{OHAngle}(\xi_k, [0,t]) \in (-\pi, \pi)$ in which the segment $[0,t]$, oriented[1] from 0 to t, is seen at ξ_k. Because

$$\mathrm{OHAngle}(\xi_k, [0,t]) = \mathrm{OHAngle}(\xi_k, [0,a_j]) + \mathrm{OHAngle}(\xi_k, [a_j,t]),$$

it follows upon differentiating that

$$\frac{d}{dt}\,\mathrm{Arg}\left(\frac{t-\xi_k}{t-1/\overline{\xi}_k}\right) = \frac{d}{dt}\,\mathrm{OHAngle}(\xi_k, [a_j,t]). \tag{83}$$

Taking absolute values in (83) and integrating over $[a_j, b_j]$, we obtain (82) from the monotonicity of $\mathrm{OHAngle}(\xi_k, [a_j,t])$ with respect to t.

Next, *we claim* that for any interval $[a,b] \subset (0,1)$ one has

$$\mathrm{Angle}(1/\bar{\xi}, [a,b]) + \mathrm{Angle}(\xi, [a,b]) \leqslant \pi, \qquad |\xi| \leqslant 1, \tag{84}$$

where the first term is interpreted as being 0 if $\xi = 0$. Indeed, (84) is obvious when $\xi \in [-1,1]$ since the first term is zero while the second is either π or 0 according whether ξ belongs to $[a,b]$ or not. The inequality also holds when $|\xi| = 1$, $\xi \notin \mathbb{R}$, because then $1/\bar{\xi} = \xi$ while

$$\mathrm{Angle}(\xi, [a,b]) \leqslant \mathrm{Angle}(\xi, [-1,1]) = \pi/2.$$

As the left-hand side of (84) is a bounded harmonic function of ξ for $|\xi| < 1$, $\xi \notin \mathbb{R}$, and since this function extends continuously to the closed unit circle and to $(-1,1) \setminus \{a,b\}$, *the claim follows* from the extended maximum principle.

We are now in position to conclude the proof. We start from (62) where we set this time

$$\arg\left(\frac{t-\xi}{(t-1/\overline{\xi})^2}\right) = \mathrm{Arg}\left(\frac{t-\xi}{t-1/\overline{\xi}}\right) - \mathrm{Arg}\left(t-1/\overline{\xi}\right), \tag{85}$$

and we majorize termwise the double sum there using the elementary inequality:

$$V\left(\arg\left(\frac{t-\xi_k}{(t-1/\overline{\xi_k})^2}\right), [a_j,b_j]\right) \leqslant V\left(\mathrm{Arg}\left(\frac{t-\xi_k}{t-1/\overline{\xi_k}}\right), [a_j,b_j]\right) + V\left(\mathrm{Arg}\left(\frac{1}{t-1/\overline{\xi_k}}\right), [a_j,b_j]\right).$$

[1] To emphasize this orientation we write $[0,t]$ even if $t < 0$.

The first term in the right-hand side was identified in (82) and the second is easily computed to be

$$V\left(\operatorname{Arg}\left(t-1/\overline{\xi_k}\right),[a_j,b_j]\right)=\operatorname{Angle}(1/\overline{\xi_k},[a_j,b_j])$$

by the monotonicity of $\operatorname{Arg}(t-1/\overline{\xi_k})$ with respect to t. Thereby (62) yields:

$$V(\varphi,S)+\sum_{k=1}^{n}\left(\theta_H(\xi_k)+\theta(1/\overline{\xi_k})\right)\geqslant(n-m+1)\pi, \tag{86}$$

and since obviously $\theta(1/\overline{\xi_k})\leqslant\operatorname{Angle}(1/\overline{\xi_k},[a_1,b_m])$, it follows *a fortiori* that

$$V(\varphi,S)+\sum_{k=1}^{n}\left(\theta_H(\xi_k)+\operatorname{Angle}(1/\overline{\xi_k},[a_1,b_m])\right)\geqslant(n-m+1)\pi. \tag{87}$$

Applying (84) with $[a,b]=[a_1,b_m]$, we get

$$V(\varphi,S)+\sum_{k=1}^{n}\theta_H(\xi_k)+\sum_{k=1}^{n}(\pi-\operatorname{Angle}(\xi_k,[a_1,b_m]))\geqslant(n-m+1)\pi, \tag{88}$$

thus, in view (53) applied with $m=1$ to the interval $[a_1,b_m]$, we conclude that

$$2V(\varphi,S)+\sum_{k=1}^{n}\theta_H(\xi_k)\geqslant(n-m+1)\pi$$

which is (81) with $\Theta(\mathcal{H})=0$, as desired. □

8. As a corollary to the proof of Theorem 5.1, we get at little extra-cost a slightly extended version of this result featuring an additional weight varying with n:

COROLLARY 6.2. — *The conclusion of Theorem 5.1 remains true if $d\lambda$ gets replaced by $w_n d\lambda$, where w_n is a sequence of complex measurable functions on the convex hull of S whose moduli are uniformly bounded from above and below, and whose arguments are smooth with uniformly bounded derivatives.*

The corollary has special significance with respect to meromorphic approximation, as we now explain. If for $2<p\leqslant\infty$ we let h/q_n be a best approximant to $M(z)$ in (46) out of $H^p/\mathcal{Q}_n$, where H^p is the Hardy space of the disk and $\mathcal{Q}_n$ the space of polynomials of degree at most n, the (monic) denominator q_n in the irreducible form of the approximant has degree $d_n\leqslant n$ (in fact $d_n=n$ except perhaps when $p=\infty$), it has all its zeros in the unit disk, and it satisfies the orthogonality relations:

$$\int\frac{q_n(t)}{Q_n^2(t)}t^k\,w_n(t)\,d\lambda(t)=0,\qquad 0\leqslant k\leqslant d_n-1, \tag{90}$$

where w_n is analytic without zeros in the disk; in fact, w_n is the outer factor of a n-th singular vector of the (generalized) Hankel operator with symbol $M(z)$ [3]. In that work it is shown, provided λ is analytic, that w_n is a normal family of functions which does not have the null function as an accumulation point. Hence the hypotheses of Corollary 6.2 are met and (90) implies that the asymptotic distribution of the zeros of q_n is Ω_S. Minor modifications in the arguments of [3] prove that w_n is still normal if λ is not analytic but merely satisfies the hypotheses of Theorem 5.1, hence Corollary 6.2 actually settles the asymptotic behaviour of the poles of the best-L^p meromorphic approximants to $M(z)$ when $2 < p \leqslant \infty$. It is worth noting that when $p = 2$, the best meromorphic approximants to $M(z)$ are nothing but the best-L^2 rational approximants considered in Sections 4-5. By point **6** above, all these results are valid if the support of λ is a hyperbolic geodesic arc in the disk instead of a real segment. The situation when $p < 2$ is still not well understood.

Proof of Corollary 6.2. — Follow the reasoning in the proof of Theorem 5.1. We can extend the definition of w_n to $[-1, 1]$ without increasing the variation nor the bound on the derivative of the argument. Upon setting $d\mu_n = |w_n| d\mu$ and $\varphi_n = \varphi + \arg(w_n)$, we find ourselves in a situation like at the end of the proof of Theorem 5.1 when we had to extend the previous argument to varying φ and μ. Here again, dealing with μ_n is no problem because with our assumptions the constant c in (3) can be made uniform with respect to n, and the total mass is uniformly bounded. To handle φ_n, we notice first that it has uniformly bounded variation by the boundedness of $d\arg(w_n(t))/dt$, hence Corollary 5.3 continues to hold. Second, using Jackson's theorem as in the proof of Lemma 5.4 instead of the Weierstrass theorem, we find that Lemma 3.4 will hold with φ_n instead of φ and T_n instead of T, where the degree of T_n is uniformly bounded. From the way T was constructed in that lemma, $|T_n|$ will also be uniformly bounded (note that the factors needed to handle the discontinuities of φ will not depend on n). We can thus assume, up to extracting a subsequence, that T_n converges and thus can be made independent of n when the latter is large enough (see the end of the proof of Theorem 5.1). This is all we need to carry out the proof as in Theorem 5.1. □

A close look at the preceding proof would show (cf. [12]) that the assumptions on $|w_n|$ made in Corollary 6.2 can be weakened: in fact it is sufficient that $\inf_{x\in S}|w_n(x)|>0$ for each n, and that

$$\lim_{n\to\infty}\Big(\int_S |w_n(t)|d\mu(t)/\inf_{x\in S}|w_n(x)|\Big)^{1/n}=1.$$

The extend to which the assumptions on the argument of w_n can be relaxed is less clear. The authors suspect it might be sufficient that its variation be $\mathrm{o}\,(n)$, but they could not prove it so far. Actually, they would have a proof if only they could answer in the positive the following question which is of independent interest:

Given $[a,b]\subset\mathbb{R}$ and $\alpha\in\big(\pi/4\,,\pi/2\big)$, does there exist a constant C_α such that, to every real function ϕ of bounded variation on $[a,b]$, there is a polynomial $P\neq 0$ satisfying

$$\begin{cases}\deg(P)\leqslant C_\alpha\,V(\phi,[a,b]),\\ |\operatorname{Arg}(e^{i\phi(x)}P(x))|\leqslant\alpha,\quad\text{for } x\in[a,b],\ \ P(x)\neq 0\end{cases}\qquad ?$$

For $\alpha=\pi/2$ this holds with $C_{\pi/2}=1/\pi$, as a consequence of Kestelman's theorem (see [11] or [10], page 129) that we used in the proof of Lemma 3.2. The whole point here is that we need this property for $\alpha<\pi/2$. Note that an affirmative answer to this question would yield an improved version of Lemma 3.4, with a bound for the degree of the polynomial.

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