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PDE'S FOR THE DYSON, AIRY AND SINE PROCESSES

by Mark ADLER(*)

1. Results.

The **Dyson Brownian motion** (see [4]) is defined as the motion of n particles $\lambda_i(t)$ diffusing according to Brownian motions and forced not to intersect one another. Then it is well-known that the transition density $p(t,\mu,\lambda)$ for this motion $(\lambda_1(t),\dots,\lambda_n(t)) \in \mathbb{R}^n$, satisfies the diffusion equation

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{1}{2}\sum_1^n \frac{\partial}{\partial \lambda_i}\Phi(\lambda)\frac{\partial}{\partial \lambda_i}\frac{1}{\Phi(\lambda)}p\\ &= \sum_1^n\left(\frac{1}{2}\frac{\partial^2}{\partial\lambda_i^2} - \frac{\partial}{\partial\lambda_i}\frac{\partial \log\sqrt{\Phi(\lambda)}}{\partial\lambda_i}\right)p\end{aligned}$$

with

$$\Phi(\lambda) = \Delta^2(\lambda)\prod_1^n e^{-\lambda_i^2}.$$

In other terms, this describes n Brownian motions repelling each other, but held together by the exponential term in $\Phi(\lambda)$. In other words, each of the particles (at λ_i) evolve locally according to Brownian motion, but at λ_i we have an additional external electric force

$$\frac{\partial}{\partial\lambda_i}\sqrt{\Phi(\lambda)}.$$

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According to Dyson [4], the motion also corresponds to the motion of the eigenvalues of an Hermitian matrix $B=(B_{ij})$ evolving according to n^2 independent Ornstein-Uhlenbeck processes

$$\frac{\partial P}{\partial t}=\sum_{i,j=1}^{n^2}\Big(\frac{1}{4}(1+\delta_{ij})\frac{\partial^2}{\partial B_{ij}^2}+\frac{\partial}{\partial B_{ij}}B_{ij}\Big)P\ ,$$

with transition density ($c=e^{-t}$)

$$P(t,\overline{B},B)=Z^{-1}\frac{1}{(1-c^2)^{n^2/2}}e^{-\frac{1}{(1-c^2)}\operatorname{Tr}(B-c\overline{B})^2}.$$

The B_{ij}'s denote the n^2 free real quantities in the Hermitian matrix B and in particular the B_{ii} are its diagonal elements; moreover, one may view $\overline{B}$ as parametrizing initial data. Here each of the B_{ij}'s execute independent uncoupled Brownian motions, subject to a harmonic force towards the origin. In the limit $t\to\infty$ we find the stationary distribution

$$Z^{-1}e^{-\operatorname{Tr}B^2}dB=Z^{-1}\Delta^2(\lambda)\prod_1^n e^{-\lambda_i^2}\,d\lambda_i$$

and taking this invariant measure as the initial condition, one finds for the joint distribution ($c=e^{-(t_2-t_1)}$)

$$\begin{aligned}P(B(t_1)\in dB_1,&B(t_2)\in dB_2)\\&=Z^{-1}\frac{dB_1dB_2}{(1-c^2)^{n^2/2}}e^{-\frac{1}{1-c^2}\operatorname{Tr}(B_1^2-2cB_1B_2+B_2^2)}\\&=\frac{\Delta_n(\mu)\Delta_n(\lambda)}{Z_n(1-c^2)^{n^2/2}}\det\left[e^{-\frac{2\mu_i\lambda_j}{1-c^2}}\right]_{1\leqslant i,j\leqslant n}\prod_1^n e^{-\frac{\mu_i^2+\lambda_i^2}{1-c^2}},\end{aligned}$$

and similarly for the joint distribution involving more times. The latter identity is obtained using the Harish-Chandra-Itzykson-Zuber formula.

The probability of the distribution of the eigenvalues for the GUE ensemble is expressible as a Fredholm determinant involving the well-known Hermite kernel [5]. P. Forrester, T. Nagao and G. Honner [5] showed that the Dyson process goes with a so-called "extended Hermite kernel", following K. Johansson [6,7], to wit the matrix kernel

$$K^{H,n}_{t_it_j}(x,y):=\begin{cases}\displaystyle\sum_{k=1}^{\infty}e^{-k(t_i-t_j)}\varphi_{n-k}(x)\varphi_{n-k}(y), & \text{if } \ t_i\geqslant t_j\\ \displaystyle-\sum_{k=-\infty}^{0}e^{k(t_j-t_i)}\varphi_{n-k}(x)\varphi_{n-k}(y), & \text{if } t_i<t_j\end{cases}\qquad(1.1)$$

where

$$\begin{aligned}\varphi_k(x) &= e^{-x^2/2}p_k(x), \quad \text{for } k \geqslant 0, \quad \text{with } p_k(x) = \tfrac{H_k(x)}{2^{k/2}\sqrt{k!}\pi^{1/4}},\\ &= 0, \qquad\qquad\qquad \text{for } k < 0,\end{aligned}$$

with $p_k(x)$ the normalized Hermite polynomials. Then we have

$$P(\text{all } \lambda_i(t_1) \in E_1^c,\ \text{all } \lambda_i(t_2) \in E_2^c) = \det\left(I - K^{H,E}\right), \tag{1.2}$$

with the matrix kernel

$$K^{H,E}(x,y) = \left(I_{E_i}(x)K^{H,n}_{t_it_j}(x,y)I_{E_j}(y)\right)_{1\leqslant i,j\leqslant 2}; \tag{1.3}$$

$I_E(x)$ is the indicator function of E.

The **Airy process** is defined by an appropriate rescaling of the largest eigenvalue λ_n in the Dyson process

$$A(t) = \lim_{n\to\infty} \sqrt{2}n^{1/6}\left(\lambda_n(n^{-1/3}t) - \sqrt{2n}\right),$$

in the sense of convergence of distributions for a finite number of t's. Prähofer and Spohn [8] introduced this process in the context of polynuclear growth models and showed it is a stationary process with continuous sample paths; hence the probability $P(A(t) \leqslant u)$ is actually independent of t and given by the Tracy-Widom distribution [9] Painlevé II equation,

$$P(A(t) \leqslant u) = F(u) := \exp\left(-\int_u^\infty (\alpha - u)q^2(\alpha)d\alpha\right), \tag{1.4}$$

with $q(\alpha)$ the solution of the **Painlevé II** equation,

$$q'' = \alpha q + 2q^3 \quad \text{with} \quad q(\alpha) \cong \begin{cases} -\dfrac{e^{-\frac{2}{3}\alpha^{\frac{3}{2}}}}{2\sqrt{\pi}\alpha^{1/4}} & \text{for } \alpha \nearrow \infty \\ \sqrt{-\alpha/2} & \text{for } \alpha \searrow -\infty. \end{cases} \tag{1.5}$$

Similarly the Sine process, introduced by Tracy-Widom [11], is an infinite collection of non-colliding processes $S_i(t)$, obtained by rescaling the bulk of the Dyson process, in the same way as the bulk of the spectrum of a large Gaussian random matrix; namely

$$S_i(t) := \lim_{n\to\infty} \frac{\sqrt{2n}}{\pi}\lambda_{\frac{n}{2}+i}\left(\frac{\pi^2 t}{2n}\right) \quad \text{for } -\infty < i < \infty,$$

in the sense of convergence of distributions for a finite number of t's. Now rescale the extended kernel by

$$\textbf{Airy process}: \ x = \sqrt{2n} + \frac{u}{\sqrt{2}n^{1/6}},\ y = \sqrt{2n} + \frac{v}{\sqrt{2}n^{1/6}},\ t = \frac{\tau}{n^{1/3}} \tag{1.7}$$

$$(1.8)\quad \textbf{Sine process}: \qquad x = \frac{u\pi}{\sqrt{2n}},\quad y = \frac{v\pi}{\sqrt{2n}},\quad t = \frac{\pi^2\tau}{2n}.$$

One is lead, upon letting $n \to \infty$, to the Airy and Sine kernels

$$(1.9)\quad K^A_{t_it_j}(u,v) := \begin{cases} \displaystyle\int_0^\infty e^{-z(\tau_i-\tau_j)}\mathrm{Ai}(u+z)\mathrm{Ai}(v+z)dz, & \text{if } \tau_i \geqslant \tau_j \\ \displaystyle -\int_{-\infty}^0 e^{z(\tau_j-\tau_i)}\mathrm{Ai}(u+z)\mathrm{Ai}(v+z)dz, & \text{if } \tau_i < \tau_j \end{cases}$$

$$(1.10)\quad K^S_{t_it_j}(u,v) := \begin{cases} \displaystyle\frac{1}{\pi}\int_0^\pi e^{z^2(\tau_i-\tau_j)/2}\cos z(u-v)dz, & \text{if } \tau_i \geqslant \tau_j \\ \displaystyle -\frac{1}{\pi}\int_\pi^\infty e^{-z^2(t_j-t_i)/2}\cos z(u-v)dz, & \text{if } \tau_i < \tau_j\ . \end{cases}$$

Just as in the Dyson process, we find the joint probabilities for both the Airy and Since processes, can be expressed in terms of a Fredholm determinant involving the above kernels, to wit:

$$P(A(\tau_1) \in F_1^c, A(\tau_2) \in F_2^c) = \det(I - K^{A,F})$$

$$(1.11)\qquad P(\text{all } S_i(\tau_1) \in F_1^c,\ \text{all } S_2(\tau_2) \in F_2^c) = \det(I - K^{S,F}),$$

with the matrix kernels

$$K^{A,F}(u,v) = \big(I_{F_i}(u)K_{\tau_i\tau_j}(u,v)I_{F_j}(v)\big)_{1\leqslant i,j\leqslant 2}$$
$$K^{S,F}_{ij}(u,v) = \big(I_{F_i}(u)K_{\tau_i\tau_j}(u,v)I_{F_j}(v)\big)_{1\leqslant i,j\leqslant 2}\,.$$

For the Sine process, F_1 and F_2 must be compact. Natural choices for the F_i are

$$F_i = (u_i, \infty) \ \text{ for the Airy process}$$
$$F_i = (u_i, v_i) \ \text{ for the Sine process.}$$

Indeed, it turns out that when F_1 and F_2 are the union of a finite number of intervals, then all three Fredholm determinants going with the Dyson, Airy and Sine processes satisfy a third order partial differential equations in the time $t = t_2 - t_1$ and the end points of the intervals making up E_1 and E_2.

In order to state the results, the disjoint union of intervals in $\mathbb{R}$,

$$(1.12)\qquad E_1 := \bigcup_{i=1}^r [a_{2i-1}, a_{2i}] \text{ and } E_2 := \bigcup_{i=1}^s [b_{2i-1}, b_{2i}] \subseteq \mathbb{R},$$

and $t = t_2 - t_1$, $c = e^{-t}$ define an associated set of linear operators

$$\mathcal{A}_1 = \sum_1^{2r} \frac{\partial}{\partial a_j} + c \sum_1^{2s} \frac{\partial}{\partial b_j}$$

$$\mathcal{A}_2 = \sum_1^{2r} a_j \frac{\partial}{\partial a_j} + c^2 \sum_1^{2s} b_j \frac{\partial}{\partial b_j} + (1-c^2)\frac{\partial}{\partial t} - c^2$$

$$\mathcal{B}_1 = \mathcal{A}_1\Big|_{a\longleftrightarrow b}, \mathcal{B}_2 = \mathcal{A}_2\Big|_{a\longleftrightarrow b}. \tag{1.13}$$

We now state three Theorems due to [1,2,3]:

Theorem 1.1 (Dyson process). — *Given $t_1 < t_2$ and $t = t_2 - t_1$, the logarithm of the joint distribution for the Dyson Brownian motion $(\lambda_1(t), \dots, \lambda_n(t))$,*

(1.14)

$$G_n(t; a_1, \dots, a_{2r}; b_1, \dots, b_{2s}) := \log P(\textit{all } \lambda_i(t_1) \in E_1,\ \textit{all } \lambda_i(t_2) \in E_2)$$

satisfies a third order non-linear PDE in the boundary points of E_1 and E_2, which takes on the simple form, setting $c = e^{-t}$,

$$\mathcal{A}_1 \frac{\mathcal{B}_2 \mathcal{A}_1 G_n}{\mathcal{B}_1 \mathcal{A}_1 G_n + 2nc} = \mathcal{B}_1 \frac{\mathcal{A}_2 \mathcal{B}_1 G_n}{\mathcal{A}_1 \mathcal{B}_1 G_n + 2nc}. \tag{1.15}$$

Similarly, the disjoint union of intervals in $\mathbb{R}$

$$F_1 := \bigcup_{i=1}^{r} [u_{2i-1}, u_{2i}] \text{ and } F_2 := \bigcup_{i=1}^{s} [v_{2i-1}, v_{2i}] \subseteq \mathbb{R},$$

and $t = t_2 - t_1$ define an associated set of linear operators

$$\begin{aligned} L_u &:= \sum_1^{2r} \frac{\partial}{\partial u_i}, \qquad L_v := \sum_1^{2s} \frac{\partial}{\partial v_i}, \\ L_v &:= L_u\Big|_{u\longleftrightarrow v}, \qquad E_v := E_u\Big|_{u\longleftrightarrow v}. \end{aligned} \tag{1.16}$$

We now state the analogous equations for the Airy and Sine processes.

Theorem 1.2 (Airy process). — *Given $t_1 < t_2$ and $t = t_2 - t_1$, the joint distribution for the Airy process $A(t)$,*

$$G(t; u_1, \dots, u_{2r}; v_1, \dots, v_{2s}) := \log P\left(A(t_1) \in F_1, A(t_2) \in F_2\right), \tag{1.17}$$

satisfies a third order non-linear PDE [1] in the u_i, v_i and t,

$$\begin{aligned} &\Big((L_u + L_v)(L_u E_v - L_v E_u) + t^2 (L_u - L_v) L_u L_v\Big) G \\ &\qquad = \frac{1}{2} \Big\{ (L_u^2 - L_v^2) G\ ,\ (L_u + L_v)^2 G \Big\}_{L_u + L_v}. \end{aligned} \tag{1.18}$$

[1] in terms of the Wronskian $\{f(y), g(y)\}_y := f'(y)g(y) - f(y)g'(y)$.

When

$$F_1 := \left(-\infty, \frac{y+x}{2}\right), \quad F_2 := \left(-\infty, \frac{y-x}{2}\right), \tag{1.19}$$

the Airy joint probability

$$H(t;x,y) := \log P\left(A(t_1) \leqslant \frac{y+x}{2}, A(t_2) \leqslant \frac{y-x}{2}\right), \tag{1.20}$$

satisfies the simple PDE in x, y and t^2:

$$2t\frac{\partial^3 H}{\partial t\partial x\partial y} = \left(t^2\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial y^2}\right) + 8\left\{\frac{\partial^2 H}{\partial x\partial y}, \frac{\partial^2 H}{\partial y^2}\right\}_y \tag{1.21}$$

with initial condition

$$\lim_{t\searrow 0} H\left(t;x,y\right) = \log F\left(\min(\frac{y+x}{2}, \frac{y-x}{2})\right),$$

with F as in (1.4).

Theorem 1.3 (Sine process). — *For $t_1 < t_2$, and compact E_1 and $E_2 \subset \mathbb{R}$, the log of the joint probability for the sine processes $S_i(t)$,*

$$G(t;u_1,\ldots,u_{2r};v_1,\ldots,v_{2s}) := \log P\left(\text{all } S_i(t_1)\in E_1^c, \text{all } S_i(t_2)\in E_2^c\right), \tag{1.22}$$

satisfies the third order non-linear PDE,

$$L_u\frac{\big(2E_vL_u + (E_v - E_u - 1)L_v\big)G}{(L_u+L_v)^2G+\pi^2} = L_v\frac{\big(2E_uL_v + (E_u - E_v - 1)L_u\big)G}{(L_u+L_v)^2G+\pi^2}. \tag{1.23}$$

In the case of single intervals, the joint probability for the Sine process

$$\begin{aligned} H(t;x,y) = \log P\Big(&\text{all } S_i(t_1)\notin [x_1+x_2, x_1-x_2]\,, \\ &\text{all } S_i(t_2)\notin [y_1+y_2, y_1-y_2]\Big) \end{aligned} \tag{1.24}$$

satisfies the PDE

$$\begin{aligned} \frac{\partial}{\partial x_1}&\frac{\left(2E_y\frac{\partial}{\partial x_1} + (E_y - E_x - 1)\frac{\partial}{\partial y_1}\right)H}{\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial y_1}\right)^2 H + \pi^2} \\ &= \frac{\partial}{\partial y_1}\frac{\left(2E_x\frac{\partial}{\partial y_1} + (E_x - E_y - 1)\frac{\partial}{\partial x_1}\right)H}{\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial y_1}\right)^2 H + \pi^2}. \end{aligned} \tag{1.25}$$

The PDE's are an effective tool to compute large time asymptotics for these processes, if one accepts an assumption on the interchange of sum and limits. We now illustrate these asymptotics in the case of the Airy process, as done in [1].

THEOREM 1.4 (Large time asymptotics for the Airy process). — *For large $t = t_2 - t_1$, the joint probability admits the following asymptotic series*

$$P(A(t_1) \leqslant u, A(t_2) \leqslant v) = F(u)F(v) + \frac{F'(u)F'(v)}{t^2} + \frac{\Phi(u,v)+\Phi(v,u)}{t^4} + O\left(\frac{1}{t^6}\right), \tag{1.26}$$

in terms of the Tracy-Widom distribution (see (1.4), (1.5)*)*

$$F(u) = \exp\left(-\int_u^\infty (\alpha - u) q^2(\alpha) d\alpha\right),$$

with

$$\Phi(u,v) := F_2(u)F_2(v) \begin{pmatrix} \frac{1}{4}\left(\int_u^\infty q^2 d\alpha\right)^2 \left(\int_v^\infty q^2 d\alpha\right)^2 \\ + q^2(u)\left(\frac{1}{4}q^2(v) - \frac{1}{2}\left(\int_v^\infty q^2 d\alpha\right)^2\right) \\ + \int_v^\infty d\alpha \left(2(v-\alpha)q^2 + q'^2 - q^4\right) \int_u^\infty q^2 d\alpha \end{pmatrix}. \tag{1.27}$$

Moreover, the covariance for large $t = t_2 - t_1$ behaves as

$$E(A(t_2)A(t_1)) - E(A(t_2))E(A(t_1)) = \frac{1}{t^2} + \frac{c}{t^4} + \cdots \quad , \tag{1.28}$$

where

$$c := 2 \iint_{\mathbb{R}^2} \Phi(u,v) du\, dv.$$

Section 2 of the present paper merely gives a sketch of the proof of these results. For a complete and detailed account, see M. Adler and P. van Moerbeke [1]. In a recent paper, Tracy and Widom [10] express the joint distribution for several times $t_1, \ldots, t_m$, in terms of an augmented system of auxiliary variables, which satisfy an implicit closed system of non-linear PDE's. In [11], Tracy and Widom define the Sine process and find an implicit PDE for this process, with methods analogous to the Airy process. Their methods are function-theoretical and the quantities involved seem quite different from ours; the connection between the two sets of results is not transparent. Later H. Widom gave a rigorous proof to the expansion (1.27), based on the Fredholm determinant (1.9) giving the joint distribution.

2. Sketch of Proofs.

In this section we briefly sketch proofs of the results in the previous section, referring the reader to [1] and [2] for the full story. The point being what the results really depend on, which requires discussions whose depth varies case by case! Consider a product ensemble

$$(M_1, M_2) \in \mathcal{H}_n^2 := \mathcal{H}_n \times \mathcal{H}_n$$

of $n \times n$ Hermitian matrices, equipped with a Gaussian probability measure

$$c_n dM_1 dM_2\, e^{-\frac{1}{2}\mathrm{Tr}(M_1^2+M_2^2-2cM_1M_2)},$$

with Haar measure given by

$$dM_j = \Delta_n^2(x)\prod_1^n dx_i dU_j, \qquad j = 1, 2.$$

The disjoint union

$$E = E_1 \times E_2 := \bigcup_{i=1}^r [a_{2i-1}, a_{2i}] \times \bigcup_{i=1}^s [b_{2i-1}, b_{2i}] \subset \mathbb{R}^2$$

specifies linear operators

$$\begin{aligned} \widetilde{\mathcal{A}}_1 &= \frac{1}{c^2-1}\left(\sum_1^{2r}\frac{\partial}{\partial a_j} + c\sum_1^{2s}\frac{\partial}{\partial b_j}\right) & \widetilde{\mathcal{A}}_2 &= \sum_{j=1}^{2r} a_j\frac{\partial}{\partial a_j} - c\frac{\partial}{\partial c} \\ \widetilde{\mathcal{B}}_1 &= \mathcal{A}_1\Big|_{a\longleftrightarrow b} & \widetilde{\mathcal{B}}_2 &= \widetilde{\mathcal{A}}_2\Big|_{a\longleftrightarrow b}. \end{aligned}$$

Using integrable systems and Virasoro theory, it is established in [1] that :

Theorem 2.1. — *Given the joint distribution*

$$P_n(E) := P(\mathit{all}(M_1\text{-}\mathit{eigenvalues}) \in E_1,\ \mathit{all}(M_2\text{-}\mathit{eigenvalues}) \in E_2), \tag{2.1}$$

the function $F_n(c; a_1, \ldots, a_{2r}, b_1, \ldots, b_{2s}) := \log P_n(E)$ satisfies the non-linear third-order partial differential equation:

$$\widetilde{\mathcal{A}}_1 \frac{\widetilde{\mathcal{B}}_2\widetilde{\mathcal{A}}_1 F_n}{\widetilde{\mathcal{B}}_1\widetilde{\mathcal{A}}_1 F_n + \frac{nc}{1-c^2}} = \widetilde{\mathcal{B}}_1 \frac{\widetilde{\mathcal{A}}_2\widetilde{\mathcal{B}}_1 F_n}{\widetilde{\mathcal{A}}_1\widetilde{\mathcal{B}}_1 F_n + \frac{nc}{1-c^2}}. \tag{2.2}$$

Remark. — Note that both $P_n(E_1 \times E_2)$ and $P_n(E_1^c \times E_2^c)$ satisfy the same equation.

Theorem 1.1 is easily derived from Theorem 2.1 via the identity (see (1.14) and (2.1) for the definition of G_n and F_n)

$$\begin{aligned} &G_n(t; a_1, \ldots, a_{2r}; b_1, \ldots, b_{2s}) \\ &= F_n\Big(c; \frac{a_1}{\sqrt{(1-c^2)/2}}, \ldots, \frac{a_{2r}}{\sqrt{(1-c^2)/2}}; \frac{b_1}{\sqrt{(1-c^2)/2}}, \ldots, \frac{b_{2s}}{\sqrt{(1-c^2)/2}}\Big), \end{aligned}$$

where $c = e^{-t}$, $t = t_2 - t_1$, which just results from a simple change of variables in the matrix integrals.

Theorem 1.2 is derived from Theorem 1.1 by first doing an asymptotic analysis, upon substituting the Airy scaling (1.7) into G_n, as in (1.14):

$$\begin{aligned} G_n\Big(\frac{\tau}{n^{1/3}}\ ;\sqrt{2n}+\frac{u_1}{\sqrt{2}n^{1/6}},\dots,\sqrt{2n}+\frac{u_{2r}}{\sqrt{2}n^{1/6}};\sqrt{2n}+\frac{v_1}{\sqrt{2}n^{1/6}},\\ \dots,\sqrt{2n}+\frac{v_{2s}}{\sqrt{2}n^{1/6}}\Big)\\ := H_n(\tau;u,v) = G(\tau;u,v)+O(1/n^{1/6}),\ \text{for } n\to\infty. \end{aligned} \tag{2.3}$$

with, as in (1.17)

$$G(\tau,u,v) := \log P(A(\tau_1)\in F_1, A(\tau_2)\in F_2),$$

which is gotten by analyzing the limit of the extended Hermite kernel to the Airy kernel. Then one observes upon making the following substitution in the operators $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$,

$$a_i \mapsto \sqrt{2n}+\frac{u_i}{\sqrt{2}n^{1/6}},\quad b_i \mapsto \sqrt{2n}+\frac{v_i}{\sqrt{2}n^{1/6}},\quad t\mapsto \frac{\tau}{n^{1/3}} \tag{2.4}$$

and setting $k := n^{1/6}$,

$$L := L_u + L_v,\quad E := E_u + E_v, \tag{2.5}$$

that for large k:

$$\mathcal{A}_1 = \sqrt{2}k\left(L-(\frac{\tau}{k^2}-\frac{\tau^2}{2k^4}+\frac{\tau^3}{6k^6})L_v + O(\frac{1}{k^8})\right)$$

$$\mathcal{A}_2 = 2k^4\left(L-\frac{2\tau}{k^2}L_v+\frac{1}{2k^4}(E-1+4\tau^2L_v)-\frac{\tau}{k^6}(E_v-1+\frac{4}{3}\tau^2L_v)+O(\frac{1}{k^8})\right)$$

$$\mathcal{B}_1 = \mathcal{A}_1|_{u\longleftrightarrow v},\qquad \mathcal{B}_2 = \mathcal{A}_2|_{u\longleftrightarrow v}. \tag{2.6}$$

Substituting the asymptotics (2.3)–(2.6) into the Wronskian form of (1.15), one finds:

$$\begin{aligned} 0 &= \left\{\mathcal{B}_2\mathcal{A}_1H_n, \left(\mathcal{B}_1\mathcal{A}_1H_n + 2k^6e^{-\tau/k^2}\right)\right\}_{\mathcal{A}_1}\\ &\quad -\left\{\mathcal{A}_2\mathcal{B}_1H_n, \left(\mathcal{A}_1\mathcal{B}_1H_n + 2k^6e^{-\tau/k^2}\right)\right\}_{\mathcal{B}_1}\\ &= 16\tau\, k^6\left[\left((L_u+L_v)(L_uE_v-L_vE_u)+\tau^2(L_u-L_v)L_uL_v\right)G\right.\\ &\quad\left. -\frac{1}{2}\left\{(L_u^2-L_v^2)G, (L_u+L_v)^2G\right\}_{L_u+L_v} + O(\frac{1}{k})\right], \end{aligned}$$

upon using the linearity of the Wronskian $\{X,Y\}_Z$ in the three arguments and the following commutation relations

$$[L_u,E_u]=L_u,\ [L_u,E_v]=[L_u,L_v]=[L_u,\tau]=0 \text{ and } [E_u,\tau]=\tau,$$

including their dual relations by $u\leftrightarrow v$; also $\{L^2G,1\}_{L_u-L_v}=\{L(L_u-L_v)G,1\}_L$. It is also useful to note that the two Wronskians in the first expression are dual to each other by $u\leftrightarrow v$. The point of the computation is to preserve the Wronskian structure up to the end. This yields the first part of Theorem 1.2, the second part following from the first part by specialization.

Theorem 1.3 is derived from Theorem 1.1 in the style of Theorem 1.2, by first doing an asymptotic analysis, upon substituting the bulk scaling (1.8) into (1.14) G_n:

$$\begin{aligned}(2.7)\quad & G_n\left(\frac{\tau}{n};\frac{u_1}{2\sqrt{n}},\dots,\frac{u_{2r}}{2\sqrt{n}};\frac{v_1}{2\sqrt{n}},\dots,\frac{v_{2s}}{2\sqrt{n}}\right)\\ &\qquad := \widetilde{H}_n(\tau;u,v)=\widetilde{G}(\tau;u,v)+O(1/\sqrt{n}),\ \text{for } n\to\infty,\end{aligned}$$

with, as in (1.22)

$$\widetilde{G}(\tau,u,v):=\log P\left(\text{all } \sqrt{2}\pi S_i(\frac{2\tau_1}{\pi^2})\in F_1^c,\ \text{all } \sqrt{2}\pi S_i(\frac{2\tau_2}{\pi^2})\in F_2^c\right).$$

Then substitute

$$(2.8)\qquad a_i:=\frac{u_i}{2\sqrt{n}},\quad b_i:=\frac{v_i}{2\sqrt{n}},\quad t:=\frac{\tau}{n}$$

into the operators $\mathcal{A}_1,\mathcal{A}_2,\mathcal{B}_1,\mathcal{B}_2$, and setting $k=\sqrt{n}$ compute

$$(2.9)\qquad \begin{array}{ll}\mathcal{A}_1=2k\left(L-\frac{\tau}{k^2}L_v+O\left(\frac{1}{k^4}\right)\right) & \mathcal{A}_2=E-1-\frac{2\tau}{k^2}(E_v-1)+O\left(\frac{1}{k^4}\right)\\ \mathcal{B}_1=\mathcal{A}_1\big|_{u\longleftrightarrow v}, & \mathcal{B}_2=\mathcal{B}_1\big|_{u\longleftrightarrow v}.\end{array}$$

Finally, substituting all these asymptotics, (2.7)–(2.9) into the Wronskian form of (1.15), yields after some effort:

$$\begin{aligned}0 &= \Big(\{\mathcal{B}_2\mathcal{A}_1\widetilde{H}_n,\mathcal{B}_1\mathcal{A}_1\widetilde{H}_n+2k^2e^{-\tau/k^2}\}_{\mathcal{A}_1}\\ &\qquad\qquad -\{\mathcal{A}_2\mathcal{B}_1\widetilde{H}_n,\mathcal{A}_1\mathcal{B}_1\widetilde{H}_n+2k^2e^{-\tau/k^2}\}_{\mathcal{B}_1}\Big)\\ &=16\tau k^2\left[\left\{(E-1)L\widetilde{G},L^2\widetilde{G}+\frac{1}{2}\right\}_{L_u-L_v}\right.\\ &\qquad -\left\{(2(E_u-1)L+(E+1)L_v)\widetilde{G},L\widetilde{G}+\frac{1}{2}\right\}_L\\ &\qquad \left.+\ \left\{(2(E_v-1)L+(E+1)L_u)\widetilde{G},L\widetilde{G}+\frac{1}{2}\right\}_L+O\left(\frac{1}{k^2}\right)\right],\end{aligned}$$

which implies the first part of Theorem 1.3; the second part following by specialization.

In order to prove Theorem 1.4, we need to show first the following *a priori* asymptotic expansion:

$$\begin{aligned}\frac{P(A(t_1)\leqslant u, A(t_2)\leqslant v)}{P(A(t_1)\leqslant u)P(A(t_2)\leqslant v)} &= \frac{\det\left(I-(\hat{K}^A_{t_it_j})_{1\leqslant i,j\leqslant 2}\right)}{\det\left(I-\hat{K}^A_{t_1t_1}\right)\det\left(I-\hat{K}^A_{t_2t_2}\right)},\\ &= 1+\sum_{i\geqslant 1}\frac{f_i(u,v)}{t^i}\end{aligned}$$

with $t = t_2 - t_1$, following Widom [12], and then assume the following plausible conjecture:

$$\begin{aligned}\lim_{u\to\infty} f_i(u,v) &= 0, \text{ for fixed } v\in\mathbb{R}\\ \lim_{z\to\infty} f_i(-z,z+x) &= 0, \text{ for fixed } x\in\mathbb{R}\end{aligned}\tag{2.10}$$

being essentially equivalent, respectively, to the following natural conditions:

$$\lim_{v\to\infty} P(A(t)\leqslant v \mid A(0)\leqslant u) = 1$$

$$\lim_{z\to\infty} P(A(t)\leqslant z+x \mid A(0)\leqslant -z) = 1 \qquad \text{(non-explosion)}.$$

Remembering that $P(A(t)\leqslant u) = F(u)$ and noting $f_i(u,v) = f_i(v,u)$, the result is proven by substituting equation (2.10):

$$\begin{aligned}G(t;u,v) := H(t;u-v,u+v) &= \log P(A(0)\leqslant u, A(t) < v)\\ &= \log F_2(u) + \log F_2(v) + \sum_{i\geqslant 1}\frac{h_i(u,v)}{t^i}\\ &= \log F_2(u) + \log F_2(v) + \frac{f_1(u,v)}{t} + \frac{f_2(u,v) - f_1^2(u,v)/2}{t^2} + \dots,\end{aligned}$$

into the equation (1.21), now written in the t,u,v variables:

$$\begin{aligned}t\frac{\partial}{\partial t}\Big(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\Big)G = \frac{\partial^3 G}{\partial u^2\partial v}\Big(2\frac{\partial^2 G}{\partial v^2} + \frac{\partial^2 G}{\partial u\partial v} - \frac{\partial^2 G}{\partial u^2} + u - v - \tau^2\Big)\\ - \frac{\partial^3 G}{\partial v^2\partial u}\Big(2\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial u\partial v} - \frac{\partial^2 G}{\partial v^2} - u + v - \tau^2\Big)\\ + \Big(\frac{\partial^3 G}{\partial u^3}\frac{\partial}{\partial v} - \frac{\partial^3 G}{\partial v^3}\frac{\partial}{\partial u}\Big)\Big(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\Big)G\end{aligned}$$

yielding equations of the form

$$\mathcal{L}h_i = \text{ function}(h_1,\dots,h_{i-1}),$$

with

$$\mathcal{L} := \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\frac{\partial^2}{\partial u \partial v}$$

with null-space of the form $r_1(u) + r_3(v) + r_2(u + v)$. It precisely the conditions (2.10), (1.4) and (1.5) which enable us to kill off the unwanted null-space and deduce the first part of Theorem 1.4, the second part being an immediate consequence of the first part.

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