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THE NUMBER OF VERTICES OF A FANO POLYTOPE

by Cinzia CASAGRANDE

ABSTRACT. — Let X be a Gorenstein, $\mathbb{Q}$-factorial, toric Fano variety. We prove two conjectures on the maximal Picard number of X in terms of its dimension and its pseudo-index, and characterize the boundary cases. Equivalently, we determine the maximal number of vertices of a simplicial reflexive polytope.

RÉSUMÉ. — Soit X une variété de Fano torique, Gorenstein et $\mathbb{Q}$-factorielle. Nous démontrons deux conjectures sur le nombre de Picard maximal de X en fonction de sa dimension et de son pseudo-indice, et nous caractérisons les cas limites. De façon équivalente, nous déterminons le nombre maximal de sommets d'un polytope réflexif simplicial.

Let X be a normal, complex, projective variety of dimension n. Assume that X is Gorenstein and Fano, namely the anticanonical divisor $-K_X$ of X is Cartier and ample. The *pseudo-index* of X was introduced in [18] as

$$\iota_X := \min\{-K_X \cdot C \,|\, C \text{ rational curve in } X\} \in \mathbb{Z}_{>0}.$$

By Mori theory we know that $\iota_X \leqslant n+1$ when X is smooth, and $\iota_X \leqslant 2n$ in general (see for instance [8, Theorems 3.4 and 3.6]).

The object of this paper is to give some bounds on the Picard number ρ_X of X, in terms of n and ι_X, when X is *toric* and *$\mathbb{Q}$-factorial*[(1)]. More precisely, we will prove the following:

THEOREM 1. — *Let X be a $\mathbb{Q}$-factorial, Gorenstein, toric Fano variety of dimension n, Picard number ρ_X and pseudo-index ι_X. Then:*

(i) *$\rho_X \leqslant 2n$, with equality if and only if n is even and $X \cong (S_3)^{n/2}$, where S_3 is the blow-up of $\mathbb{P}^2$ at three non collinear points;*

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(1) For every Weil divisor D there exists $m \in \mathbb{Z}_{>0}$ such that mD is Cartier.

(ii) $\rho_X(\iota_X - 1) \leqslant n$, *with equality if and only if* $X \cong (\mathbb{P}^{\iota_X - 1})^{\rho_X}$.

Part (ii) of Theorem 1 has been conjectured in [5] for any smooth Fano variety X, generalizing a conjecture by S. Mukai. For such X, (ii) is known in the cases $\iota_X \geqslant \frac{1}{2}n + 1$ [18, 7, 14], $n \leqslant 4$ [5], $n = 5$ [1], and, provided that X admits an unsplit covering family of rational curves, $\iota_X \geqslant \frac{1}{3}n + 1$ [1]. For X smooth and toric, (ii) was already known in the cases $n \leqslant 7$ or $\iota_X \geqslant \frac{1}{3}n + 1$ [5].

For a smooth toric Fano X, (i) was conjectured by V. V. Batyrev (see [10, page 337]) and was already known to hold up to dimension 5 (for $n \leqslant 4$ thanks to the classifications [2, 17, 4, 15], and for $n = 5$ it is [6, Theorem 4.2]). Recently B. Nill [13] has extended this conjecture to the $\mathbb{Q}$-factorial Gorenstein case, and has shown (i) for a certain class of $\mathbb{Q}$-factorial, Gorenstein toric Fano varieties (see on page 124).

Observe that the bound in (i) does not hold for non toric Fano varieties, already in dimension two. It is remarkable that in the non toric case, there is no known bound for the Picard number of a smooth Fano variety in terms of its dimension (at least to our knowledge). If S is a surface obtained by blowing-up $\mathbb{P}^2$ at eight general points, for any even n the variety $S^{n/2}$ has Picard number $9n/2$, and one could conjecture this is the maximum for any dimension n (see [9, page 122]).

For X smooth, toric, and Fano, V. E. Voskresenskiĭ and A. Klyachko have shown that $\rho_X \leqslant n^2 - n + 1$ [16, Theorem 1]; O. Debarre has improved this bound in $\rho_X \leqslant 2 + \sqrt{(2n-1)(n^2-1)}$ [9, Theorem 8].

The common approach to these questions in the toric case, is to associate a so-called “reflexive” polytope to a Gorenstein toric Fano variety, and to combine the techniques coming from geometry with the ones coming from the theory of polytopes.

We recall some basic notions on reflexive polytopes and toric Fano varieties; we refer the reader to [10], [9] and references therein for more details. Let $N \cong \mathbb{Z}^n$ be a lattice and let $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice. Set $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$; for $x \in N_{\mathbb{Q}}$ and $y \in M_{\mathbb{Q}}$ we denote by $\langle x, y \rangle$ the standard pairing. For any set of points $x_1, \dots, x_r \in N_{\mathbb{Q}}$, we denote by $\operatorname{Conv}(x_1, \dots, x_r) \subset N_{\mathbb{Q}}$ their convex hull.

Let $P \subset N_{\mathbb{Q}}$ be a lattice polytope of dimension n containing the origin in its interior. We denote by $V(P)$ the set of vertices of P. The *dual polytope* of P is defined as

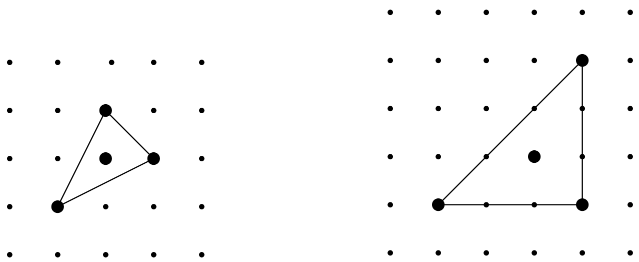
$$P^* := \{ y \in M_{\mathbb{Q}} \mid \langle x, y \rangle \geqslant -1 \text{ for all } x \in P \}.$$

P is called a *reflexive polytope* if P^* is a lattice polytope; if so, also P^* is reflexive and $(P^*)^* = P$. Reflexive polytopes were introduced in [3]; their isomorphism classes are in bijection with isomorphism classes of Gorenstein toric Fano varieties.

Let P be a reflexive polytope of dimension n; we denote by X_P the associated n-dimensional Gorenstein toric Fano variety. The fan of X_P is given by the cones over the faces of P in $N_{\mathbb{Q}}$.

Many geometric properties of X_P can be read from P. In particular, X_P is $\mathbb{Q}$-factorial if and only if P is simplicial[(2)], while X_P is smooth if and only if the vertices of every facet of P are a basis of the lattice. In this last case, we say that P is smooth.

Example. — In dimension two, let e_1, e_2 be a basis of N, and e_1^*, e_2^* the dual basis of M. Consider $P := \mathrm{Conv}(e_1, e_2, -e_1 - e_2)$; its dual polytope is $P^* = \mathrm{Conv}(2e_1^* - e_2^*, 2e_2^* - e_1^*, -e_1^* - e_2^*)$.



P $\qquad$ P^*

Both polytopes are reflexive, the associated surfaces are $X_P = \mathbb{P}^2$ and $X_{P^*} = \{xyz = w^3\} \subset \mathbb{P}^3$. The surface X_{P^*} has three singular points of type A_2. We have $\iota_{X_P} = 3$, while $\iota_{X_{P^*}} = 1$.

We denote by $|A|$ the cardinality of a finite set A. When P is simplicial, the number of vertices $|V(P)|$ of P is equal to $\rho_{X_P} + n$.

Recall that there is a bijection between the vertices of P^* and the facets of P; if $u \in V(P^*)$, we denote by F_u the corresponding facet of P, namely $F_u := \{x \in P \mid \langle x, u\rangle = -1\}$. We define

$$\delta_P := \min\{\, \langle v, u\rangle \mid v \in V(P),\, u \in V(P^*),\, v \notin F_u\} \in \mathbb{Z}_{\geqslant 0}.$$

The pseudoindex ι_{X_P} is related to δ_P as follows.

(2) A polytope is simplicial if the vertices of every facet are linearly independent, where a facet is a proper face of maximal dimension.

LEMMA 2. — *Let P be a simplicial reflexive polytope and X_P the associated Fano variety. Then $\iota_{X_P} \leqslant \delta_P + 1$. If moreover P is smooth, then $\iota_{X_P} = \delta_P + 1$.*

Theorem 1 is then a consequence of Lemma 2 and of the following:

THEOREM 3. — *Let P be a simplicial reflexive polytope of dimension n. Then:*

(i) *$|V(P)| \leqslant 3n$, with equality if and only if n is even and $X_P \cong (S_3)^{n/2}$;*

(ii) *if $\delta_P > 0$, then $|V(P)| \leqslant n + \frac{n}{\delta_P}$, with equality if and only if the following two conditions hold:*

  (a) *$P = \mathrm{Conv}(Q_1, \ldots, Q_r)$ with $r = \frac{n}{\delta_P}$ and each $Q_j \subset N_{\mathbb{Q}}$ a reflexive lattice simplex of dimension δ_P, the sum of whose vertices is zero;*

  (b) *if H_j is the linear span of Q_j in $N_{\mathbb{Q}}$, we have $N_{\mathbb{Q}} = H_1 \oplus \cdots \oplus H_r$.*

In [13], Theorem 3 (i) is proven for a simplicial reflexive polytope P for which there exists a vertex u of P^* such that $-u \in P^*$ [13, Theorem 5.8]. We refer the reader to [13] for a discussion on the number of vertices of a reflexive polytope in the non simplicial case.

Example. — The polytopes P and P^* of the previous example have $\delta_P = \delta_{P^*} = 2$, and both satisfy equality in (ii). Notice that $\iota_{X_{P^*}} < \delta_{P^*} + 1$.

We will first prove Theorem 3, then Lemma 2 and Theorem 1. For the proof, we will use the same technique as [16] and [9], and also some results from [13].

First of all, we need a property of pairs of vertices $v \in V(P)$ and $u \in V(P^*)$ with $\langle v, u \rangle = \delta_P$.

Let P be a simplicial polytope of dimension n. We say that a vertex v is *adjacent* to a facet $F = \mathrm{Conv}(e_1, \ldots, e_n)$ if $\mathrm{Conv}(v, e_1 \ldots, \check{e}_i, \ldots, e_n)$ is a facet of P for some $i = 1, \ldots, n$.

LEMMA 4. — *Let P be a simplicial reflexive polytope and $v \in V(P)$, $u \in V(P^*)$ such that $\langle v, u \rangle = \delta_P$. Then v is adjacent to F_u.*

Proof. — This property is shown in [9, Remark 5(2)] and [13, Lemma 5.5] in the case $\delta_P = 0$. The same proof works for the general case. □

Proof of Theorem 3. — First of all, observe that for any $u \in V(P^*)$ we have

$$(1) \quad |\{v \in V(P) \mid \langle v, u \rangle = -1\}| = n \quad \text{and} \quad |\{v \in V(P) \mid \langle v, u \rangle = 0\}| \leqslant n.$$

In fact, since P is simplicial, the facet F_u contains n vertices. Moreover, if $\langle v,u\rangle = 0$, then $\delta_P = 0$, and by Lemma 4 we know that v is adjacent to F_u. Again, since P is simplicial, F_u has at most n adjacent vertices, and we get (1).

The origin lies in the interior of P^*, so we can write a relation

$$m_1 u_1 + \cdots + m_h u_h = 0 \tag{2}$$

where $h > 0$, $u_1,\dots,u_h$ are vertices of P^*, and $m_1,\dots,m_h$ are positive integers. Set $I := \{1,\dots,h\}$ and $M := \sum_{i\in I} m_i$. For any vertex v of P define

$$A(v) := \{i \in I \mid \langle v,u_i\rangle = -1\} \quad \text{and} \quad B(v) := \{i\in I \mid \langle v,u_i\rangle = 0\}.$$

Then observe that $\langle v,u_i\rangle \geqslant 1$ for any $i \notin A(v)\cup B(v)$. So for every $v\in V(P)$ we have

$$\begin{aligned} 0 = \sum_{i\in I} m_i\langle v,u_i\rangle &= -\sum_{i\in A(v)} m_i + \sum_{i\notin A(v)\cup B(v)} m_i\langle v,u_i\rangle \\ &\geqslant -\sum_{i\in A(v)} m_i + \sum_{i\notin A(v)\cup B(v)} m_i = M - 2\sum_{i\in A(v)} m_i - \sum_{i\in B(v)} m_i. \end{aligned}$$

Summing over all vertices of P we get

$$\begin{aligned} M|V(P)| &\leqslant 2\sum_{v\in V(P)}\sum_{i\in A(v)} m_i + \sum_{v\in V(P)}\sum_{i\in B(v)} m_i \\ &= 2\sum_{i\in I} m_i\,|\{v\in V(P)\mid \langle v,u_i\rangle = -1\}| \\ &\qquad + \sum_{i\in I} m_i|\{v\in V(P)\mid \langle v,u_i\rangle = 0\}| \end{aligned}$$

and using (1) this gives $|V(P)| \leqslant 3n$.

Assume that $|V(P)| = 3n$. Then all inequalities above are equalities; in particular, for any v and u_i such that $\langle v,u_i\rangle > 0$, we must have $\langle v,u_i\rangle = 1$. Observe now that we can choose a relation as (2) involving all vertices of P^*, namely with $h = |V(P^*)|$ (see Remark 5), so $\langle v,u\rangle \in \{-1,0,1\}$ for every $v\in V(P)$ and $u\in V(P^*)$. Then P and P^* are centrally symmetric.

Smooth centrally symmetric reflexive polytopes are classified in [16, Theorem 6], and the only case with $3n$ vertices is for n even and $X_P \cong (S_3)^{n/2}$. For the general case, we apply [13, Theorem 5.8].

Now assume that $\delta_P > 0$ and let's prove (ii). For every vertex v of P we have $\langle v, u_i\rangle \geqslant \delta_P$ if $i \notin A(v)$; similarly to what precedes, we get

$$\begin{aligned} 0 &= \sum_{i\in I} m_i\langle v,u_i\rangle = -\sum_{i\in A(v)} m_i + \sum_{i\notin A(v)} m_i\langle v,u_i\rangle \\ &\geqslant -\sum_{i\in A(v)} m_i + \delta_P \sum_{i\notin A(v)} m_i = (-\delta_P - 1)\sum_{i\in A(v)} m_i + \delta_P M, \end{aligned} \tag{3}$$

namely $\frac{\delta_P}{\delta_P+1} M \leqslant \sum_{i\in A(v)} m_i$. This gives

$$|V(P)|\frac{\delta_P}{\delta_P+1}M \leqslant \sum_{v\in V(P)}\sum_{i\in A(v)} m_i = nM$$

and

$$|V(P)| \leqslant \frac{n(\delta_P+1)}{\delta_P} = n + \frac{n}{\delta_P}.$$

Suppose that $|V(P)| = n + n/\delta_P$. Again, we can choose a relation as (2) involving all vertices of P^*. Then, since we must have all equalities in (3), we conclude that $\langle v, u\rangle \in \{-1, \delta_P\}$ for any $v \in V(P)$ and $u \in V(P^*)$.

Set $r := n/\delta_P$. Fix $u \in V(P^*)$ and call $e_1, \dots, e_n$ the vertices of F_u, and $f_1, \dots, f_r$ the remaining vertices. Set $K := \{1, \dots, n\}$ and $J := \{1, \dots, r\}$. For any $k \in K$, the face $\mathrm{Conv}(e_1, \dots, \check{e}_k, \dots, e_n)$ lies on exactly two facets, one of which is F_u. Hence there exists a unique $\varphi(k) \in J$ such that

$$F_k := \mathrm{Conv}(f_{\varphi(k)}, e_1, \dots, \check{e}_k, \dots, e_n)$$

is a facet of P. This defines a function $\varphi\colon K \to J$.

Since P is simplicial, $e_1, \dots, e_n$ is a basis of $N_{\mathbb{Q}}$; if $e_1^*, \dots, e_n^*$ is the dual basis in $M_{\mathbb{Q}}$, we have $u = -e_1^* - \dots - e_n^*$. Fix $k \in K$ and let u_k be the vertex of P^* such that $F_k = F_{u_k}$. We have

$$\langle e_i, u_k\rangle = -1 \text{ for all } i \in K \smallsetminus \{k\}, \text{ and } \langle e_k, u_k\rangle = \delta_P,$$

so $u_k = u + (\delta_P + 1)e_k^*$.

Now for any $j \in J$ we have

$$\langle f_j, e_k^*\rangle = \frac{1}{\delta_P+1}\left(\langle f_j, u_k\rangle - \delta_P\right) = \begin{cases} -1 & \text{if } \varphi(k) = j, \\ 0 & \text{otherwise.} \end{cases}$$

This means $f_j + \sum_{k\in\varphi^{-1}(j)} e_k = 0$. Finally, we have $\delta_P = \langle f_j, u\rangle = |\varphi^{-1}(j)|$, so as j varies in J, the $\varphi^{-1}(j)$'s give a partition of K in r subsets of cardinality δ_P. Setting $Q_j := \mathrm{Conv}\{f_j, e_k \,|\, k \in \varphi^{-1}(j)\}$, we see that $Q_1, \dots, Q_r$ satisfy the properties claimed in (ii). □

Remark 5. — Consider any polytope $Q \subset M_{\mathbb{Q}}$, of dimension n, containing the origin in its interior. Let u be a vertex of Q and let F be the minimal face of Q such that $-u$ is contained in the cone over F in $M_{\mathbb{Q}}$. Then writing $-u$ as a linear combination of the vertices of F, we get a relation as (2) containing u. Summing enough relations of this type, one easily finds a relation $\sum_{u \in V(Q)} m_u u = 0$ with all m_u's positive integers.

It is interesting to observe that when Q is a reflexive polytope, there is a special relation:

$$\sum_{u \in V(Q)} \mathrm{Vol}(F_u)\, u = 0, \tag{4}$$

where F_u is the facet of Q^* corresponding to u, and $\mathrm{Vol}(F_u)$ is the lattice volume of F_u. This follows from a theorem by Minkowski, see [11, page 332] and [12, Lemma 4.9].

If P is a smooth reflexive polytope, then all facets of P are standard simplices, so (4) yields that the sum of all vertices of P^* is zero. This remarkable fact can also be proven using the recent results on the factorization of birational maps between smooth toric varieties, and it was used for the proof of Theorem 3 in a previous version of this work.

Let P be a simplicial reflexive polytope. We denote by $\mathcal{N}_1(X_P)$ the $\mathbb{Q}$-vector space of 1-cycles in X_P, with rational coefficients, modulo numerical equivalence. It is a well known fact in toric geometry (see for instance [19]) that there is an exact sequence

$$0 \longrightarrow \mathcal{N}_1(X_P) \longrightarrow \mathbb{Q}^{V(P)} \longrightarrow N_{\mathbb{Q}} \longrightarrow 0,$$

so that $\mathcal{N}_1(X_P)$ is canonically identified with the group of linear rational relations among the vertices of P.

Moreover, if a class $\gamma \in \mathcal{N}_1(X_P)$ corresponds to a relation

$$\sum_{v \in V(P)} m_v v = 0, \qquad m_v \in \mathbb{Q},$$

then the anticanonical degree of γ is $-K_{X_P} \cdot \gamma = \sum_{v \in V(P)} m_v$.

Proof of Lemma 2. — To show that $\iota_{X_P} \leqslant \delta_P + 1$, we exhibit a rational curve in X_P whose anticanonical degree is less or equal than $\delta_P + 1$. Fix $v \in V(P)$ and $u \in V(P^*)$ such that $\langle v, u \rangle = \delta_P$. By Lemma 4, v is adjacent to F_u. Let $e_1, \dots, e_n$ be the vertices of F_u; up to reordering, we can assume that $\mathrm{Conv}(v, e_2, \dots, e_n)$ is a facet of P. Since P is simplicial, $e_1, \dots, e_n$ is a basis of $N_{\mathbb{Q}}$; if $e_1^*, \dots, e_n^*$ is the dual basis of $M_{\mathbb{Q}}$, we have $u = -e_1^* - \cdots - e_n^*$.

Consider the relation

$$v + \sum_{i=1}^{n} a_i e_i = 0,$$

and the corresponding class $\gamma \in \mathcal{N}_1(X_P)$. We have

$$-K_{X_P} \cdot \gamma = 1 + \sum_{i=1}^{n} a_i = 1 + \langle v, u \rangle = 1 + \delta_P.$$

Now consider the invariant curve $C_0 \subset X_P$ corresponding to the cone over the face $\mathrm{Conv}(e_2, \dots, e_n)$. There exists $b \in \mathbb{Q}$, $b \in (0,1]$, such that the numerical class of C_0 is $b\gamma$ (see [19, § 2]). Then

$$-K_{X_P} \cdot C_0 = b(\delta_P + 1) \leqslant \delta_P + 1.$$

Assume now that X_P is smooth, and let C be an invariant curve having minimal anticanonical degree ι_{X_P}. The numerical class of C corresponds to a relation $f_0 + \sum_{i=1}^{n} b_i f_i = 0$, where $b_i \in \mathbb{Z}$ and $\mathrm{Conv}(f_1, \dots, f_n)$ is a facet F of P (see [19, § 2]). The vertices $f_1, \dots, f_n$ are a basis of N; if $f_1^*, \dots, f_n^*$ is the dual basis of M, then $F = F_{u_0}$ with $u_0 = -f_1^* - \dots - f_n^*$. So

$$\iota_{X_P} = -K_{X_P} \cdot C = 1 + \sum_{i=1}^{n} b_i = 1 + \langle f_0, u_0 \rangle \geqslant 1 + \delta_P \geqslant \iota_{X_P},$$

and $\iota_{X_P} = \delta_P + 1$. □

Proof of Theorem 1. — Part (i) and the inequality in (ii) are straightforward consequences of Theorem 3 and Lemma 2.

Assume that $\rho_{X_P}(\iota_{X_P} - 1) = n$. Again using Lemma 2 and Theorem 3, we get $\delta_P \geqslant \iota_{X_P} - 1 > 0$ and

$$|V(P)| \leqslant n + \frac{n}{\delta_P} \leqslant n + \frac{n}{\iota_{X_P} - 1} = n + \rho_{X_P} = |V(P)|,$$

so $\iota_{X_P} = \delta_P + 1$, $|V(P)| = n + \frac{n}{\delta_P}$ and the characterization in Theorem 3 (ii) holds. This means that X_P is the quotient of $(\mathbb{P}^{\delta_P})^{\rho_{X_P}}$ by a finite subgroup G of the big torus; it is then enough to show that X_P is smooth.

We keep the same notation as in the proof of Theorem 3. In order to show the smoothness of X_P, we have to show that $e_1, \dots, e_n$ is a basis of N. Let $w \in N$ and write $w = \sum_{i=1}^{n} \frac{t_i}{s_i} e_i$ with $t_i, s_i \in \mathbb{Z}$, $s_i \neq 0$, and t_i, s_i with no common factors for all i.

Suppose that $t_1 \neq 0$, we show that $s_1 = 1$ (for the other indices the proof is analogous). Consider the relation

$$f_{\varphi(1)} + \sum_{k \in \varphi^{-1}(\varphi(1))} e_k = 0,$$

and the corresponding class $\gamma \in \mathcal{N}_1(X_P)$. Now consider the invariant curve $C \subset X_P$ corresponding to the cone over the face $\mathrm{Conv}(e_2, \ldots, e_n)$, and recall that $\mathrm{Conv}(e_1, \ldots, e_n)$ and $\mathrm{Conv}(f_{\varphi(1)}, e_2, \ldots, e_n)$ are faces of P. Then there exists $b \in \mathbb{Q}$, $b \in (0, 1]$, such that the numerical class of C is $b\gamma$ (see [19, § 2]). So we have

$$\iota_{X_P} \leqslant -K_{X_P} \cdot C = b(-K_{X_P} \cdot \gamma) = b(\delta_P + 1) = b\iota_{X_P} \leqslant \iota_{X_P},$$

which gives $b = 1$. This is equivalent to saying that in the quotient lattice $N/N \cap (\mathbb{Q}e_2 \oplus \cdots \oplus \mathbb{Q}e_n)$, the image $\overline{e}_1$ of e_1 is a generator. Now if $\overline{w}$ is the image of w, we have $s_1\overline{w} = t_1\overline{e}_1$ with s_1, t_1 non zero and with no common factors, so $s_1 = 1$. □

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