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IMMEDIATE AND VIRTUAL BASINS OF NEWTON'S METHOD FOR ENTIRE FUNCTIONS

by Sebastian MAYER & Dierk SCHLEICHER

Abstract. — We investigate the well known Newton method to find roots of entire holomorphic functions. Our main result is that the immediate basin of attraction for every root is simply connected and unbounded. We also introduce "virtual immediate basins" in which the dynamics converges to infinity; we prove that these are simply connected as well.

Résumé. — Nous étudions la méthode bien connue de Newton pour trouver les racines des applications holomorphes entières. Notre résultat principal est que le domaine d'attraction immédiat de chaque racine est simplement connexe et non borné. D'ailleurs, nous introduisons les "domaines immédiats virtuels" dans lesquels la dynamique converge vers l'infini ; nous démontrons aussi qu'ils sont simplement connexes.

1. Introduction

Newton's method is one of the preferred methods to find roots of differentiable maps: it often converges very fast and it is very easy to implement. But there are problems: for example, even for polynomials, there are open sets of initial conditions for which the Newton map does not converge to any root.

There has been substantial progress understanding the dynamics of Newton's method for finding roots of complex polynomials: Przytycki [9] has shown that all immediate basins are simply connected and unbounded; Shishikura [12] has shown more generally that if a rational map has a multiply connected Fatou component, then it must have two repelling or parabolic fixed points (which is impossible for Newton maps of polynomials). Hubbard, Schleicher and Sutherland [7] used this to find a rather

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small set of starting points which together find all roots of a complex polynomial; and in [11] there is a (not very efficient) bound on the number of iterations it takes to find all roots with given accuracy. In a different spirit, Smale [13] has shown that Newton's method is quite efficient from a probabilistic point of view.

Newton's method for transcendental entire functions is less understood. Bergweiler and Terglane [2] have shown that Newton maps have no multiply connected wandering domains, and in certain cases no wandering domains at all.

In this note, we extend Przytycki's result to entire holomorphic functions: *for every root ξ of a non-constant entire holomorphic function f, the immediate basin of attraction for the Newton map associated to f is simply connected and unbounded.* This result goes back to the Diploma thesis [8]. The result is in analogy to the polynomial case. It turns out that essential ideas of the polynomial proof apply in the entire case, even though it might appear a priori that the possibility of a dense set of singularities in $\mathbb{C}$ could cause serious problems. However, the dynamics of Newton's method for entire functions allows for an important kind of Fatou components which we call "virtual immediate basins" and which do not occur in the case of polynomials: virtual immediate basins are domains in which the dynamics converges to ∞ as if there was a root at ∞ (subject to further conditions; see Definition 3.2). It has recently been shown by Buff and Rückert [3] that in many (but not all) cases, virtual immediate basins are related to asymptotic values 0 of f similarly as immediate basins of roots are related to zeroes of f. We show also that *every virtual immediate basin is simply connected.* One area where virtual immediate basins come naturally into play is when considering the combinatorial restrictions on immediate basins; see Section 3 and [10].

It would be interesting to extend the ideas of [12] to the transcendental case, showing that all Fatou components are simply connected; the case of wandering domains is treated in [2].

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2. Immediate Basins

Throughout this paper, let $f : \mathbb{C} \to \mathbb{C}$ be a nonlinear entire holomorphic map and $N_f = \mathrm{id} - f/f'$ its associated Newton map. We will be concerned with the set of points which converge to any given root ξ of f. Clearly, the roots of f are exactly the fixed points of N_f in $\mathbb{C}$, and these are attracting.

DEFINITION 2.1 (Immediate basin). — *Let ξ be an attracting fixed point of N_f. The* basin of attraction *of ξ is the open set of all points z such that $(N_f^{\circ m}(z))_{m\in\mathbb{N}}$ converges to ξ. The connected component containing ξ of the basin is called the* immediate basin *of ξ.*

Throughout this paper, we will fix a root ξ of f and denote its immediate basin by U. In order to show that U is both simply connected and unbounded, we will construct a curve $\gamma\colon \mathbb{R}_0^+ \to U$ with $N_f(\gamma(t)) = \gamma(t-1)$ for $t \geqslant 1$. If U is multiply connected, then we can arrange things so that $\gamma(\mathbb{R}_0^+)$ is bounded (and the same is obviously true if U itself is bounded). In this case, we show that $\lim_{t\to+\infty}\gamma(t)$ is a fixed point of N_f in $\partial U \cap \mathbb{C}$, and this will lead to a contradiction. Note that U cannot contain a punctured neighborhood of ∞: otherwise, ∞ would be in the Fatou set, but for Newton maps of entire functions, ∞ is either an essential singulary or a parabolic or repelling fixed point (compare Subsection 3.2). Sinc U is invariant, this implies that U also cannot contain a pole of N_f.

2.1. An Exhaustion of Immediate Basins

Let $\mathrm{Crit}(N_f)$ be the set of critical values of N_f and

$$P_U := \bigcup_{m\geqslant 0} N_f^{\circ m}(\mathrm{Crit}(N_f)\cap U)$$

be the postcritical set restricted to critical values in U. Since $\mathrm{Crit}(N_f)$ is countable, the set P_U is countable as well (but in general not closed). There is thus an open disk $S_0 = B_r(\xi) \subset U$ centered at ξ such that $\partial S_0 \cap P_U = \emptyset$, and small enough such that $N_f(\overline{S_0}) \subset S_0$. For every $k \in \mathbb{N}_0$ define S_{k+1} to be the connected component of $N_f^{-1}(S_k)$ containing S_0; then $\overline{S_{k+1}}$ is the connected component of $N_f^{-1}(\overline{S_k})$ containing S_0.

LEMMA 2.2. — *The immediate basin satisfies $U = \bigcup_{k\in\mathbb{N}} S_k$.*

Proof. — Clearly, U is open and $U' := \bigcup_{k\in\mathbb{N}} S_k$ is an open subset of U. Suppose there is a $z \in U \setminus U'$. Then there is an $M \in \mathbb{N}$ with $N_f^{\circ M}(z) \in S_0$, so there is a connected neighborhood $V \subset U$ of z with $N_f^{\circ M}(V) \subset S_0$. For all $m \geqslant M$, z is by assumption in a component of $N_f^{-m}(S_0)$ different from S_m, and so is V. Hence $U \setminus U'$ is open in contradiction to the fact that U is connected. □

Clearly, if U is multiply connected, then some of the S_k are multiply connected (if U is multiply connected, then it contains a non-contractible loop which is compact and thus contained in finitely many S_k, hence in a single S_k).

Lemma 2.3. — *If S_M is mutiply connected but S_{M-1} is not, then all S_m with $m < M$ are bounded and homeomorphic to open disks.*

Proof. — There is a bounded connected component B of $\mathbb{C} \setminus S_M$. Its boundary ∂B is a compact subset of $\overline{S_M}$, so $N_f(\partial B)$ is a compact subset of $\overline{S_{M-1}}$. There are no postcritical points in ∂S_0, so $N_f^{\circ M}$ restricted to a neighborhood of ∂B is a local injection and $N_f^{\circ M}\colon \partial B \to \partial S_0$ is a covering map. This implies that ∂B and $N_f(\partial B)$ are homeomorphic to circles. Since $N_f(\partial B)$ is a boundary component of S_{M-1} and S_{M-1} is simply connected, it follows that S_{M-1} is contained in the bounded complementary component of $N_f(\partial B)$, so S_{M-1} is bounded and homeomorphic to an open disk.

Clearly, all S_m with $m < M$ are contained in S_{M-1}, hence also bounded and simply connected. □

2.2. Extending Paths Invariantly to Infinity

The goal of this section is the construction of curves $\delta\colon \mathbb{R}_0^+ \to U$ such that $N_f(\delta(t)) = \delta(t-1)$ for $t \geqslant 1$.

Definition 2.4 (Extension of a curve). — *If $\delta : [0,1] \to \mathbb{C} \setminus \mathrm{Crit}(N_f)$ is a curve with $N_f(\delta(1)) = \delta(0)$, then define its* (maximal) extension $\mathrm{ext}_\delta :$ $[0, M_\delta) \to \mathbb{C} \setminus \mathrm{Crit}(N_f)$ *to be the curve with*

$$\begin{aligned} \forall\, t \in [0, M_\delta - 1): \quad N_f(\mathrm{ext}_\delta(t+1)) &= \mathrm{ext}_\delta(t) \\ \mathrm{ext}_\delta|_{[0,1]} &\equiv \delta \end{aligned}$$

where M_δ is chosen maximal in $[1, \infty) \cup \{\infty\}$. By analytic continuation, this defines both M_δ and the curve ext_δ *uniquely.*

Lemma 2.5 (Possibilities for the extension of a curve). — *Given any curve $\delta\colon [0,1] \to \mathbb{C} \setminus \mathrm{Crit}(N_f)$, then exactly one of three cases occurs:*

(i) $M_\delta = \infty$,

(ii) $M_\delta < \infty$ *and* $\lim_{t \to M_\delta} \mathrm{ext}_\delta(t)$ *is a critical point of N_f, or*

(iii) $M_\delta < \infty$, $\mathrm{ext}_\delta(M_\delta - 1)$ *is an asymptotic value, and* $\mathrm{ext}_\delta(t) \to \infty$ *as* $t \to M_\delta$.

Proof. — Choose $T \in (0, M_\delta]$ with $T < \infty$ and suppose there is a sequence $t_n \nearrow T$ with $\mathrm{ext}_\delta(t_n) \to a \in \mathbb{C}$. If a is not a critical point, then there is a neighborhood V of a such that $N_f|_V$ is univalent, and it follows that $T < M_\delta$. Therefore, if $M_\delta < \infty$, then ext_δ converges either to a critical point or to infinity along an asymptotic path. □

The following lemma is related to typical proofs of "landing of periodic dynamic rays" for iterated polynomials.

LEMMA 2.6 (A homotopy class of unbounded curves). — *Let $W, W' \subset U \setminus S_0$ be two simply connected domains such that $N_f(W') \subset W$ and $W' \subset W$. Let*

$$S_{W',W} := \{\sigma : [0,1] \to W \textit{ continuous}, \ \sigma(1) \in W', \ \sigma(0) = N_f(\sigma(1))\} .$$

If there is an extension $\mathrm{ext}_\sigma : [0, \infty) \to \mathbb{C} \setminus \mathrm{Crit}(N_f)$ for all $\sigma \in S_{W,W'}$, then every $\sigma \in S_{W,W'}$ has $\lim_{t\to\infty} \mathrm{ext}_\sigma(t) = \infty$.

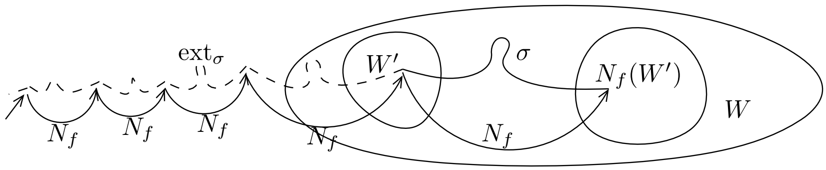

Figure 2.1. W and W' and an extended curve σ.

Proof. — Set $Y := \bigcup_{\sigma \in S_{W,W'}} \mathrm{image}\,(\mathrm{ext}_\sigma)$. We use $S_{W',W}$ to construct a sequence of local inverse mappings of $N_f^{\circ m}$ on W. Note that the hypothesis implies that $W \cap \mathrm{Crit}(N_f) = \emptyset$.

For $n \in \mathbb{N}$, we define maps $\eta_n : W \longrightarrow Y$ as follows: given $z \in W$, choose a curve $\sigma \in S_{W',W}$ and $0 \leqslant t \leqslant 1$ such that $\sigma(t) = z$ and define $\eta_n(z) = \mathrm{ext}_\sigma(t + n)$. It is easy to check that this is well-defined, i.e. independent of the choice of curve and its parametrization. The maps η_n are continuous for all $n \in \mathbb{N}$ with $\eta_n^{-1} = N_f^{\circ n}$, so all η_n are holomorphic. The sequence $(\eta_n)_{n\in\mathbb{N}}$ clearly forms a normal family, so there is a locally uniformly convergent subsequence $(\eta_{n_l})_{l\in\mathbb{N}}$; its limit function $\eta : W \to \overline{Y}$ is holomorphic by the theorem of Weierstraß.

Suppose there is a $y_0 \in \mathrm{image}\,(\eta) \cap \mathbb{C}$. We have $W \cap S_0 = \emptyset$, so for all $m \in \mathbb{N}$ and sufficiently big n (depending on m), $\mathrm{image}\,(\eta_n) \cap S_m = \emptyset$. Furthermore $U = \bigcup_{m\geqslant 0} S_m$ is open, so $y_0 \in \partial U$ and by Hurwitz' theorem,

$\eta \equiv y_0$ is constant. If $y_0 \neq \infty$, then choose some $w \in W'$ and define $z := N_f(w)$. Then

$$N_f(y_0) = N_f(\eta(w)) = \lim_{l\to\infty} N_f(\eta_{n_l}(w)) = \lim_{l\to\infty} \eta_{n_l}(z) = \eta(z) = y_0 \ .$$

Thus y_0 is a fixed point of N_f on $\partial U \cap \mathbb{C}$, while the only fixed points of N_f in $\mathbb{C}$ are the zeros of f, and these are not on the boundary of U. This contradiction shows that $y_0 = \infty$.

Suppose there is a single curve $\sigma \in S_{W,W'}$ for which $\lim_{t\to\infty} \mathrm{ext}_\sigma(t) = \infty$ is false. Then there is a subsequence (η_{n_l}) for which the limit function $\eta\colon W \to \overline{Y}$ could not be identically equal to ∞, and this is a contradiction which proves the lemma. □

Remark 2.7. — A different way to prove this lemma is to use contraction properties of the hyperbolic metric in $U \setminus \overline{P_U}$.

2.3. Immediate Basins are Simply Connected and Unbounded

Now it is time for our first main result.

Theorem 2.8 (Immediate basins). — *Let $f : \mathbb{C} \to \mathbb{C}$ be a nonlinear entire map, $N_f := id - \frac{f}{f'}$ its Newton map and ξ a root of f. Then the immediate basin U of ξ is simply connected and unbounded.*

Proof. — Choose an open disk S_0 around ξ with $N_f(\overline{S_0}) \subset S_0$, $\partial S_0 \cap P_U = \emptyset$ and define S_{k+1} as the component of $N_f^{-1}(S_k)$ containing S_0. Note that N_f has no postcritical points on ∂S_0, so there are no critical points on any ∂S_k and N_f is locally biholomorphic in a neighborhood of ∂S_k.

By Lemma 2.2, $U = \bigcup_{n\geqslant 0} S_n$. If U is not simply connected, then there is a minimal M such that S_M is mutiply connected. Choose a bounded component B_0 of $\mathbb{C} \setminus S_M$; then there is a bounded component B of $U \setminus S_M$ as well. If U is bounded, let $B := U \setminus S_M$ for an arbitrary M. In both cases, B is a bounded component of $U \setminus S_M$, and this will lead to a contradiction (compare Figure 2.2).

Define $P_B := \bigcup_{m\geqslant 0} N_f^{\circ m}(\mathrm{Crit}(N_f) \cap B)$; since B is bounded, $\mathrm{Crit}(N_f) \cap B$ is finite, and the only accumulation point of P_B is ξ. Choose $w \in \left(S_{M+1} \setminus \overline{S_M}\right) \cap B \setminus P_B$ (compare Figure 2.3) and set $z := N_f(w) \in S_M \setminus \overline{S_{M-1}}$. There is an injective path $\gamma^0 : [0,1] \to \left(S_{M+1} \setminus \overline{S_{M-1}}\right) \setminus P_B$ with $\gamma^0(0) = z$ and $\gamma^0(1) = w$.

We want to show that ext_{γ^0} converges to ∞ within B, which would be a contradiction. Since $\gamma^0([0,1]) \cap P_B = \emptyset$, there is a maximal curve

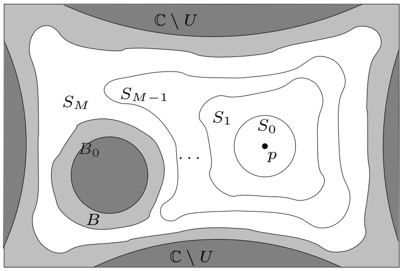

Figure 2.2. $U=\bigcup_{n\geqslant 0} S_n$ and B

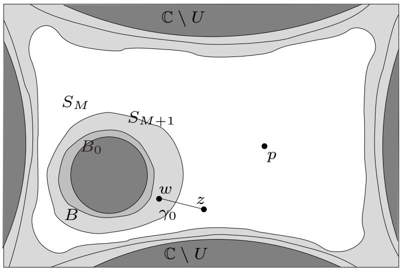

Figure 2.3. How to choose γ^0.

$\gamma := \mathrm{ext}_{\gamma^0} : [0, M_\gamma) \to \mathbb{C}$ with:

$$\begin{aligned} \forall t \in [1, M_\gamma): \quad N_f(\gamma(t)) &= \gamma(t-1) \\ \gamma|_{[0,1]} &\equiv \gamma^0 . \end{aligned}$$

By induction, it follows that $\gamma([n, n+1]) \subset S_{M+n+1} \setminus \overline{S_{M+n-1}}$ for every $n \in \mathbb{N}$, and in particular we have $\gamma([1, M_\gamma)) \subset U \setminus \overline{S_M}$. In fact, even $\gamma([1, M_\gamma)) \subset B$ (because B is the component of $U \setminus S_M$ containing $\gamma(1) = w$). Since B is bounded, Lemma 2.5 implies that $M_\gamma = \infty$.

Choose an open, bounded and simply connected neighborhood $W \subset S_{M+1} \setminus \overline{S_{M-1}}$ of image (γ^0) disjoint from P_B. This can be done because image (γ^0) and $\overline{P}_B = P_B \cup \{\xi\}$ are compact and disjoint.

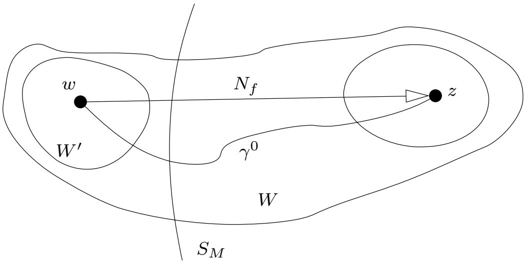

Figure 2.4. W and W'.

Let $W' \subset W$ be a simply connected neighborhood of w with $N_f(W') \subset W$ (compare Figure 2.4). By Lemma 2.5, every curve

$$\sigma \in S := \{\sigma : [0,1] \to W \text{ continuous}, \ \sigma(1) \in W', \ \sigma(0) = N_f(\sigma(1))\}$$

has an extension $\mathrm{ext}_\sigma : [0, \infty) \to B \cup W$. By Lemma 2.6, every curve $\sigma \in S$ satisfies $\lim_{t\to\infty} \mathrm{ext}_\sigma(t) = \infty$, so B is unbounded: a contradiction. $\square$

Remark 2.9. — In many cases, it even follows that ∞ is accessible within U. The case in which we cannot prove this is if U contains infinitely many critical points of N_f such that P_U is dense in U.

3. Virtual Immediate Basins

3.1. A Motivating Example

The dynamics of Newton's map for transcendental entire functions has a class of Fatou components which we want to call *virtual immediate basins*. Let $f(z) := z\exp(-\frac{1}{n}z^n)$; its Newton map $N_f(z) = z\left(1 - \frac{1}{1-z^n}\right)$ is a rational function. The involution $\iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, z \mapsto \frac{1}{z}$ conjugates N_f to the polynomial $\iota \circ N_f \circ \iota(\zeta) = \zeta - \zeta^{n+1}$. In this case, the Leau-Fatou "Flower Theorem" shows that there are exactly n attracting and n repelling petals at $\zeta = 0$, so N_f has exactly n unbounded Fatou components with convergence to $z = \infty$; moreover, the immediate basin of the root 0 has exactly n accesses to ∞ (these accesses are called "channels to ∞" [7]); compare Figure 3.1. We call the attracting petals at infinity *virtual immediate basins*: their dynamics is similar as if there was a root at ∞ in each of these n directions. Note that the n channels of the root 0 separate all these n virtual basins. The immediate basin of 0 has finite area if $n \geqslant 3$ [6] but not if $n \in \{1, 2\}$ [4].

For Newton's method of polynomials, it is known [10] that any pair of channels to ∞ of the same root must enclose a different root of the polynomial. As this example shows, an analogous statement for transcendental entire functions would be false if virtual immediate basins were not taken into account. An investigation of the combinatorial possibilities between channels to ∞ and immediate basins (including virtual basins) can be found in [10].

3.2. An Exhaustion of Virtual Immediate Basins

In order to define virtual immediate basins, we need the following definition.

Definition 3.1 (Absorbing set). — *If U is an N_f-invariant domain in $\mathbb{C}$, then an open set $A \subsetneq U$ is called* absorbing set *(of U) if the following conditions hold:*

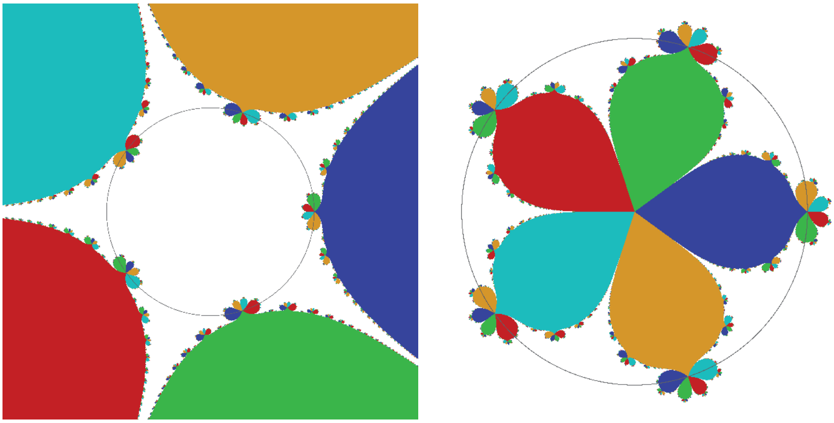

Figure 3.1. Dynamics of Newton's map for $f(z) = ze^{-\frac{1}{5}z^5}$. *The immediate basin of the root* 0 *is white, the other colors correspond to virtual immediate basins and their backward images. The unit circle* $\mathbb{S}^1$ *is marked in grey.* Right: *the same situation in* $\zeta = \iota(z) = 1/z$ *coordinates:* $\iota \circ N_f \circ \iota^{-1}(\zeta) = \zeta - \zeta^{n+1}$ *is a polynomial.*

(1) *A is simply connected;*
(2) $N_f(\overline{A} \cap \mathbb{C}) \subset A$;
(3) *for every* $z \in U$ *there is a* k *such that* $N_f^{\circ k}(z) \in A$.

Definition 3.2 (Virtual immediate basin). — *A domain* $U \subset \mathbb{C}$ *is called a* virtual immediate basin *if it is maximal (among domains in* $\mathbb{C}$*) with respect to the following properties:*

(1) $\lim_{n\to\infty} N_f^{\circ n}(z) = \infty$ *for all* $z \in U$;
(2) *there is an absorbing set* $A \subset U$.

Clearly, every virtual immediate basin, and every absorbing set there of, must be unbounded with ∞ as accessible boundary point.

If f is a polynomial, then N_f is a rational function with a repelling fixed point at ∞. If f is an entire function of the form Pe^Q where P and Q are polynomials (such that Q is non-constant and P does not identically vanish), then N_f is a rational function for which ∞ is a parabolic fixed point, as in our example above; these two statements are proved by a simple calculation. For all other transcendental entire functions f, the Newton map N_f is a transcendental meromorphic function with essential singularity at ∞ [1, Sec. 6.1]; a proof can be found in [10]. In the case of transcendental N_f, virtual basins can still occur (but they are no longer Leau-Fatou petals).

In [8], the following example is given: the function $f(z) = ze^{e^z}$ has a single zero, N_f has infinitely many virtual immediate basins, and the immediate basin of the unique zero has infinitely many channels; moreover, no two channels of the zero of f enclose further roots of f (because there aren't any), but they always enclose a virtual immediate basin. In this example $f(z) = ze^{e^z}$, as well as in the trivial example $f(z) = e^z$ with $N_f(z) = z-1$, virtual immediate basins of N_f are related to asymptotic values 0 of f. This is true in many (but not all) cases; see [3].

Let U be a virtual immediate basin with absorbing set S_0. Similarly as for immediate basins, define S_{k+1} to be the connected component of $N^{-1}(S_k)$ containing S_0, for all $k \geqslant 0$. As before, we have the following:

Lemma 3.3. — *U is open and $U = \bigcup_{k\in\mathbb{N}} S_k$. If U is mutiply connected, then one of the S_k is mutiply connected.*

3.3. Simple Connectivity

In a number of ways, virtual immediate basins have similar properties as immediate basins of roots; here is one such result.

Theorem 3.4. — *Virtual immediate basins are simply connected.*

Proof. — Let U be a virtual immediate basin with absorbing set S_0. By Lemma 3.3 there is an exhaustion $U = \bigcup_{m\in\mathbb{N}} S_m$ of U. If U is multiply connected, then there is a minimal M such that S_M is multiply connected.

There is a bounded connected component B of $\mathbb{C} \setminus S_M$. Its boundary ∂B is a compact subset of ∂S_M. We claim that there is a point $b \in \partial B \cap S_{M+1}$. If not, then $\partial B \subset S_M \cap (\partial U \cup U \backslash S_{M+1})$, and it then follows that every point in ∂B will either iterate to ∞ during the first M iterates of N_f, or it will map to $\partial S_0 \cup (\partial U \cup U \backslash S_1) \cap \mathbb{C}$. The first case can occur only for countably many points, while the second case is excluded by definition of an absorbing set: $\partial S_0 \cap \mathbb{C} \subset S_1$.

Therefore, every neighborhood of b contains a point $w \in B \cap (S_{M+1} \backslash \overline{S}_M)$ which can be joined to $z : N_f(w)$ by a curve $\gamma^0 : [0,1] \to S_{M+1} \backslash \overline{S}_{M-1})$, and we can assume that γ^0 avoids the countable set of all postcritical points.

Now the same argument as in the proof of Theorem 2.8 shows that under iterated pull-backs, this curve must converge to a fixed point of N_f within B, and this is again a contradiction. □

We cannot show in general that every virutal immediate basin equals an entire Fatou component; however, we have the following.

Remark 3.5. — Every virtual immediate basin U is contained in an invariant Fatou component F, which is necessarily a Baker domain if N_f is transcendental and a parabolic domain otherwise. If F is simply connected, then F is a virtual immediate basin.

This follows by using a Riemann map $\varphi : F \to \mathbb{D}$ to transport the dynamics of F into the unit disk $\mathbb{D}$, and using the main result of Cowen [5] which assures the existence of an absorbing set (except that Cowen does not explicitly state the condition $N_f(\overline{A} \cap \mathbb{C}) \subset A$; but he gives three dynamical models for the absorbing sets, up to conformal conjugacy, and in each of them the absorbing set can be deformed so as to satisfy this condition, and even be bounded by a smooth curve). It might be possible to extend Shishikura's results [12] to the transcendental case, to show the more general result that every Fatou component of Newton's map for entire functions is simply connected. In this case, every virtual immediate basin would be an entire Fatou component.

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