ANNALES

DE

L'INSTITUT FOURIER

Alexey A. GLUTSYUK

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Tome 56, nº 4 (2006), p. 887-917.

<http://aif.cedram.org/item?id=AIF_2006__56_4_887_0>

Ann. Inst. Fourier, Grenoble
56, 4 (2006) 887-917

AN EXPLICIT FORMULA FOR PERIOD DETERMINANT

by Alexey A. GLUTSYUK

Abstract. — We consider a generic complex polynomial in two variables and a basis in the first homology group of a nonsingular level curve. We take an arbitrary tuple of homogeneous polynomial 1-forms of appropriate degrees so that their integrals over the basic cycles form a square matrix (of multivalued analytic functions of the level value). We give an explicit formula for the determinant of this matrix.

Résumé. — Nous considérons un polynôme générique à deux variables complexes et une base de cycles dans le premier groupe d'homologie d'une courbe de niveau non singulière. Nous prenons une collection arbitraire de 1-formes polynomiales homogènes de degrés appropriés, de sorte que leurs intégrales le long des cycles de la base forment une matrice carrée (de fonctions multivaluées en la valeur du niveau). Nous calculons le déterminant de cette matrice.

1. Introduction and main results

Consider a complex polynomial $H(x,y)$ of degree $n+1 \geqslant 2$ in two variables. We assume it *generic* (see the next definition) and for each t denote

$$S_t = \{H = t\} \subset \mathbb{C}^2.$$

Then for any noncritical value t the homology group $H_1(S_t, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^\mu$, $\mu = n^2$ (see [1]). Let $\delta_1(t), \dots, \delta_\mu(t)$ be its generators.

Fix a set of μ complex polynomial 1-forms

$$\Omega = (\omega_1, \dots, \omega_\mu), \quad \omega_i = P_i(x,y)\mathrm{d}x + Q_i(x,y)\mathrm{d}y.$$

Keywords: Complex polynomial in two variables, homology of nonsingular level curve, monodromy, abelian integral, gradient ideal, period determinant.
Math. classification: 14D05, 32S10, 32S20.

The *period matrix* associated to the polynomial H and the form set Ω is the (multivalued) matrix function

$$\mathbb{I}(t) = \big(I_{ij}(t)\big), \quad I_{ij}(t) = \int_{\delta_j(t)} \omega_i. \tag{1.1}$$

Its determinant is called the *period determinant*:

$$\Delta(t) = \Delta_{H,\Omega}(t) = \det \mathbb{I}(t). \tag{1.2}$$

Remark 1.1. — The definition of the period determinant does not depend (up to sign) on the choice of the homology basis in the level curves, since the transition between two different bases is given by a matrix with determinant ± 1.

Remark 1.2. — The functions $I_{ij}(t)$ are holomorphic multivalued with branching points at the critical values of H. At the same time, as it will be shown, the period determinant is a single-valued function, and even polynomial.

In the present paper we give an explicit formula (see (1.18)) for the period determinant through the coefficients of the polynomial and the forms, provided that the latter forms are homogeneous and of appropriate degrees, see (1.4) below.

A lower bound of the period determinant was used in [7] (joint paper with Yu. S. Ilyashenko), where we have obtained an explicit upper bound of the number of zeros for a wide class of Abelian integrals. This lower bound is proved in a separate author's paper [6] by using formula (1.18).

Definition 1.3. — *We say that a homogeneous polynomial is* generic, *if has only simple zero lines. A (not necessarily homogeneous) polynomial H is said to be generic, if so is its highest homogeneous part.*

First we give (in the next subsection) an explicit formula for $\Delta(t)$ defined by arbitrary generic polynomial H and the following special form set. Namely, let $(\ell(i), m(i))$, $i = 1, \dots, \mu = n^2$ be the lexicographic ordered set of integer pairs (ℓ, m), $0 \leqslant \ell, m \leqslant n-1$. Put

$$e_i(x,y) = x^{\ell(i)} y^{m(i)}, \quad \omega_i = y e_i(x,y)\mathrm{d}x, \quad d(i) = \ell(i) + m(i). \tag{1.3}$$

Afterwards, in Subsection 1.2 we extend the above-mentioned formula for $\Delta(t)$ to the case of arbitrary form set of the type

$$\Omega = (\omega_1, \dots, \omega_{n^2}), \quad \omega_i \text{ are homogeneous of degrees } d(i) + 1. \tag{1.4}$$

The proof of the extended formula takes the rest of the paper.

1.1. Formula for the period determinant: case of special form set (1.3)

Let $H(x, y)$ be a generic polynomial of degree $n+1 \geqslant 2$, h be its highest homogeneous part. Let a_i, $i = 1, \ldots, n^2$, be the critical values of H. Let e_i, ω_i be the monomials and the forms from (1.3), $\Delta(t)$ be the corresponding determinant (1.2).

As it will be shown below,

$$\Delta(t) = C(h) \prod_{i=1}^{n^2} (t - a_i), \tag{1.5}$$

$C(h)$ depends only on h so that:

- $C(h)$ is a meromorphic function on the double cover over the space of generic homogeneous polynomials h with branching at the "discriminant" hypersurface of the nongeneric polynomials (this hypersurface, which consists of the polynomials with multiple zero lines, will be denoted by S),
- $C(h)$ tends to infinity, as the discriminant of h tends to zero,
- $C(h) = 0$ if and only if there exists a $d = n, \ldots, 2n-2$ such that a nontrivial linear combination

$$\sum_{d(i)=d} c_i e_i \tag{1.6}$$

belongs to the gradient ideal of h, which is generated by its partial derivatives.

In particular, this implies the following

Corollary 1.4. — *Let $H(x, y)$ be a polynomial with the highest homogeneous part*

$$h(x, y) = x^{n+1} + y^{n+1}.$$

Then the corresponding constant $C(h)$ from (1.5) *does not vanish.*

The corollary follows from the statement that the monomials e_i form a basis in the quotient ring of all the polynomials in two variables modulo the gradient ideal of the polynomial $x^{n+1} + y^{n+1}$.

To state the formula for $C(h)$, let us recall the definition of the discriminant $\Sigma(h)$ of a homogeneous polynomial h, which vanishes on the nongeneric polynomials. Consider the decomposition

$$h(x, y) = h_0 \prod_{i=0}^{n} (y - c_i x)$$

of h into a product of linear factors. Put

$$\Sigma(h) = h_0^{2n} \prod_{0\leqslant j<i\leqslant n} (c_i - c_j)^2. \tag{1.7}$$

Remark 1.5. — The discriminant $\Sigma(h)$ is a degree $2n$ homogeneous irreducible polynomial in the coefficients of h:

$$h(x,y) = \sum_{i=0}^{n+1} h_i x^i y^{n+1-i}. \tag{1.8}$$

THEOREM 1.6. — *Let $n \geqslant 1$. There exists a homogeneous polynomial $P(h)$ of degree $n(n-1)$ in the coefficients* (1.8) *of the homogeneous polynomial h that satisfies the following statements:*

1) *for a generic h $P(h) = 0$, if and only if condition* (1.6) *holds;*

2) *let H, h, a_i, ω_i, $\Delta(t)$ be as at the beginning of the subsection. Then formula* (1.5) *holds with*

$$C(h) = C_n\big(\Sigma(h)\big)^{\frac{1}{2}-n} P(h), \quad C_n \in \mathbb{C} \text{ depends only on } n. \tag{1.9}$$

The theorem is proved at the end of 1.2. The formulas for the corresponding polynomial $P(h)$ and the constant C_n are given below.

The polynomial $P(h)$ from the theorem is the product

$$P(h) = \prod_{d=n}^{2n-2} P_d(h), \tag{1.10}$$

where $P_d(h)$ are the polynomials defined as follows.

DEFINITION 1.7. — *Let $n \geqslant 1$, $d \in \{n, \dots, 2n-2\}$, h be a homogeneous polynomial of degree $n+1$. Consider the following ordered $2(d-n+1)$ polynomials of degree d:*

$$x^\ell y^{d-n-\ell}\frac{\partial h}{\partial y}, \ (\ell = 0, \dots, d-n), \quad x^\ell y^{d-n-\ell}\frac{\partial h}{\partial x}, \ (\ell = 0, \dots, d-n).$$

Let $A_d(h)$ be the matrix whose columns are numerated by the monomials of degree d distinct from all the e_i' s (with $d(i) = d$); the lines are numerated by the previous $2(d-n+1)$ polynomials and consists of their corresponding coefficients.

In the case, when $d = n$, all the monomials of degree d are e_i except for x^n and y^n (see Fig. 1 in the case, when $n = d = 3$: the monomials e_i of degree 3 are xy^2 and x^2y), so, we take the coefficients at x^n and y^n only.

Put

$$P_d(h) = \det A_d(h) \quad \text{if } n \geqslant 2, \quad P_d(h) \equiv 1 \quad \text{if } n = 1. \tag{1.11}$$

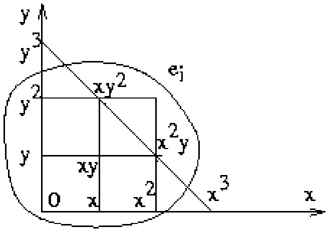

Figure 1.1. Case $n = 3$; the monomials e_i

Remark 1.8. — The matrix $A_d(h)$ is square of the size $2(d-n+1)$: the number of the monomials of degree d distinct from e_i is equal to $2(d-n+1)$. Therefore, the polynomials $P_d(h)$ are well defined. Let $P(h)$ be their product (1.10). As it will be shown in 2.1 (Proposition 2.4), for generic h, $P(h) = 0$ if and only if condition (1.6) holds.

Example 1.9. — Let us calculate the polynomial $P(h)$ in the case, when h is a general homogeneous cubic polynomial

$$h(x,y) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3.$$

Then $n = 2$. The corresponding set of the values of d from the previous definition consists of the unique value $d = 2$, since $n = 2n - 2 = 2$. The corresponding matrix $A_2(h)$ is the 2×2- matrix whose lines consist of the coefficients of the partial derivatives of h at the monomials x^2 and y^2 (these are the only quadratic monomials distinct from the e_i's, see the same definition). Hence,

$$(1.12) \qquad A_2(h) = \begin{pmatrix} 3a_0 & a_2 \\ a_1 & 3a_3 \end{pmatrix}, \quad \text{thus,} \;\; P_2(h) = P(h) = 9a_0a_3 - a_1a_2.$$

The latter equality $P_2(h) = P(h)$ holds true since there are no other values $d \neq 2$ for which the matrices A_d are well defined.

THEOREM 1.10. — *Let $P_d(h)$ be the polynomials defined by* (1.11), *$P(h)$ be their product* (1.10). *Then* (1.9) *holds for*

$$(1.13) \qquad C_n = (-1)^{\frac{1}{4}n(3n-1)} \frac{(2\pi)^{\frac{1}{2}n(n+1)}(n+1)^{\frac{1}{2}(n^2+n-4)}((n+1)!)^n}{\prod_{m=1}^{n-1}(m+n+1)!}.$$

Theorem 1.10 is proved in Section 3.

Theorem 1.6 will follow from its generalization (Theorem 1.17 stated in 1.2), which deals with a generalized form set (1.4). Extending the form set has also the following independent motivation. A direct proof of Theorem 1.6 is done via the study of the divisor of $C(h)$. The first step is to prove that the divisor of zeros is exactly the zero locus of $P(h)$.

This divisor is simple. To prove that we need to generalize the problem and to extend the set of forms ω_i used in the definition of the period determinant.

In this way we get that $C(h) = \widetilde{S}P(h)$, where $\widetilde{S}$ is a "polar" term. It appears that $\widetilde{S} = \infty$ if and only if $\Sigma = 0$. Hence, $\widetilde{S} = C_n\Sigma^s$ for some negative s. The homogeneity arguments imply that $s = \frac{1}{2} - n$. Thus, Theorem 1.6 holds.

In order to find the factor C_n, and to prove Theorem 1.10, it is sufficient to find the period determinant for some specific h, as well as $\Sigma(h)$ and $P(h)$ via a straightforward calculation. This is done for $h(x,y) = x^{n+1} + y^{n+1}$ in Section 3.

Example 1.11. — Let us check the statement of Theorem 1.10 in the simplest case, when $n = 1$, $h(x,y) = H(x,y) = x^2 + y^2$. Then

$$\Delta(t) = \int_{x^2+y^2=t} y\mathrm{d}x = \pi t.$$

On the other hand, Theorem 1.6 claims that

$$\Delta(t) = C(h)t, \quad C(h) = C_1\Sigma^s P(h), \quad s = \frac{1}{2} - n = -\frac{1}{2}\cdot$$

For our $H(x,y) = (x - iy)(x + iy)$ one has

$$\Sigma = (2i)^2 = -4, \quad \Sigma^{-\frac{1}{2}} = \pm(2i)^{-1}, \quad P(h) = \prod_{d=n}^{2n-2} P_d(h) = 1,$$

since $P_d \equiv 1$, see (1.11). Therefore,

$$C_1 = \pm\Delta(t)2i/t = \pm 2\pi i.$$

The substitution of $n = 1$ to (1.13) gives the same result up to sign (the sign of the first factor $(-1)^{\frac{1}{4}n(3n-1)} = (-1)^{\frac{1}{2}} = \pm i$ in (1.13) is not uniquely defined). This deduces Theorem 1.10 from Theorem 1.6 for $n = 1$.

Example 1.12. — Let us calculate the period determinant in the simplest nontrivial case of a general cubic polynomial:

$$h(x,y) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3, \quad n = 2. \tag{1.14}$$

To do this, we calculate the terms in formula (1.9) for $C(h)$. The polynomial $P(h)$ was already calculated in (1.12). The formula for the discriminant $\Sigma(h)$ is given in [10, p. 141, Exercise 11]:

$$\Sigma(h) = a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3.$$

Let us calculate the constant $C_n = C_2$. By (1.13), one has

$$C_2 = (-1)^{\frac{1}{2}} \frac{(2\pi)^3 3^{\frac{1}{2}(2^2+2-4)}(3!)^2}{4!} = \pm i \frac{(2\pi)^3 \times 3 \times 36}{24} = \pm 36\pi^3 i.$$

Substituting the two latter formulas and (1.12) to (1.9) yields

$$C(h) = \pm 36\pi^3 i (a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3)^{-\frac{3}{2}}(9a_0a_3 - a_1a_2).$$

Thus, for any cubic polynomial with highest homogeneous part (1.14) and critical values $a_1, \dots, a_4$ one has

$$\Delta(t) = C(h) \prod_{j=1}^{4} (t - a_j), \quad C(h) \text{ is as above.}$$

1.2. Formula for the period determinant: general case.

Everywhere below in the present subsection we consider that ω_i are arbitrary 1-forms of the type (1.4). Let $H(x, y)$ be a generic polynomial of degree $n + 1 \geqslant 2$, h be its highest homogeneous part. Let a_i, $i = 1, \dots, n^2$, be the critical values of H. Let $\Delta(t)$ be the corresponding period determinant (1.2).

We state and prove a generalization of formulas (1.5), (1.9) for $\Delta(t)$ given by Theorem 1.6 to the case of arbitrary forms ω_i as in (1.4). The generalized formulas (1.5), (1.9) coincide with their previous versions, but now the constant $C(h) = C(h, \Omega)$ from (1.5) depends on Ω, and the polynomial $P(h)$ in (1.9) should be replaced by its appropriate extension up to a polynomial $P(h, \Omega)$ with variable Ω. To define the extension of P, let us introduce some notations.

For a polynomial 1-form ω by $D\omega$ denote the polynomial defined by the equality

$$\mathrm{d}\omega = D\omega \mathrm{d}y \wedge \mathrm{d}x. \tag{1.15}$$

Example 1.13. — Let e_j, $m(j)$, ω_j be as in (1.3). Then

$$D\omega_j = \big(m(j) + 1\big)e_j. \tag{1.16}$$

Definition 1.14. — *Let h be a generic homogeneous polynomial of degree $n+1$, D be the operator defined by (1.15). A set Ω of forms* (1.4) *is said to be h-degenerate, if either the polynomials $(D\omega_i)_{d(i)<n}$ are linearly dependent, or condition* (1.6) *holds with e_i replaced by $D\omega_i$, for some d, $n \leqslant d \leqslant 2n-2$. Otherwise Ω is said to be h-nondegenerate.*

The extended polynomial $P(h,\Omega)$ we are looking for is constructed as follows. As it will be shown in 2.1 (Proposition 2.4), it vanishes, if and only if Ω is h-degenerate.

Definition 1.15. — *Let $n \geqslant 2$, $d \in \mathbb{N}$, $0 \leqslant d \leqslant 2n-2$, h be a homogeneous polynomial of degree $n+1$. Let $\Omega(d) = (\omega_1', \dots, \omega_s')$ be an ordered tuple of homogeneous 1-forms of degree $d+1$, the number s of the forms being equal to $s = d+1$ in the case, when $d \leqslant n-1$, and $s = 2n-d-1$ otherwise. The matrix $A_d(h,\Omega(d))$ associated to the form tuple $\Omega(d)$ is the $(d+1)\times(d+1)$ matrix whose columns are numerated by all the monomials $y^d, y^{d-1}x, \dots, x^d$ of degree d and the lines consist of the corresponding coefficients of the following polynomials:*

- *Case $d \leqslant n-1$. — Take the $d+1$ polynomials $D\omega_r'/(d-r+2)$.*
- *Case $d \geqslant n$. — Take the $d-n+1$ polynomials $x^j y^{d-n-j}\, \partial h/\partial y$, $0 \leqslant j \leqslant d-n$; the $2n-d-1$ polynomials $D\omega_r'/(n-r+1)$; the $d-n+1$ polynomials $x^j y^{d-n-j}\, \partial h/\partial x$, $0 \leqslant j \leqslant d-n$.*

Let Ω be as in (1.4), $n \leqslant d \leqslant 2n-2$, $\Omega(d)$ be the tuple of the forms in Ω of degree $d+1$ (numerated in the same order, as in Ω). The number s of forms in $\Omega(d)$ is equal to $2d-n-1$. Indeed, by definition, it is equal to the number of monomials e_i of degree d; the latter number is equal to $2d-n-1$ by Remark 1.8. Put

$$\begin{gathered} A_d(h,\Omega) = A_d\big(h,\Omega(d)\big), \quad P_d(h,\Omega) = \det A_d(h,\Omega), \\ P(h,\Omega) = \prod_{d=0}^{2n-2} P_d(h,\Omega). \end{gathered} \tag{1.17}$$

Proposition 1.16. — *Let $\Omega = (\omega_1, \dots, \omega_{n^2})$, ω_i be the same, as in* (1.3). *Let $A_d(h)$, $A_d(h,\Omega)$ be the matrix functions from Definition* 1.7 *and* (1.17) *respectively, $P_d(h)$, $P_d(h,\Omega)$ be their determinants. Then for any $d < n$ $P_d(h,\Omega) \equiv 1$. For any $d \geqslant n$ $P_d(h,\Omega) = P_d(h)$ (thus, $P(h) = P(h,\Omega)$).*

Proof. — Let us prove the statement of the Proposition for $d < n$. Then the matrix $A_d(h,\Omega)$ is unit. Indeed, by definition, its lines consist of the coefficients of the monomials $D\omega_i/(m(i)+1) = e_i$, $d(i) = d$, see (1.16). The columns are numerated by all the monomials of degree d. The latters coincide with e_i and are ordered lexicographically, so, $A_d(h,\Omega)$ is unit. Hence, $P_d(h,\Omega)\equiv 1$. Now let us prove the statement of the proposition in the case, when $d\geqslant n$. In this case the matrix $A_d(h)$ is obtained from the matrix $A_d(h,\Omega)$ by deleting its central $2n-d-1$ lines (which consist of the coefficients of the monomials e_i) and the central $2n-d-1$ columns, which are numerated by e_i. The matrix formed by the deleted lines and columns is identity: its lines correspond to the monomials $D\omega_i/(m(i)+1)=e_i$, as before. The elements of the deleted lines outside the deleted columns are zeros. Therefore, $P_d(h,\Omega)=\det A_d(h)=P_d(h)$. Proposition 1.16 is proved. $\square$

THEOREM 1.17. — *Let ω_i, $\Omega=(\omega_1,\dots,\omega_{n^2})$, H, h, a_i, $\Delta(t)$ be as at the beginning of the Subsection, $P(h,\Omega)$ be as in* (1.17). *Then*

$$\Delta(t) = C(h,\Omega)\prod_{i=1}^{n^2}(t-a_i), \tag{1.18}$$

$$C(h,\Omega) = C_n\big(\Sigma(h)\big)^{\frac12-n}P(h,\Omega), \tag{1.19}$$

C_n is the same, as in (1.13).

Theorem 1.17 is proved in Section 2 (modulo the calculation of the constant C_n). The latter constant will be calculated in Section 3. Together with the previous proposition, Theorem 1.17 implies Theorem 1.6.

DEFINITION 1.18 (see [5]). — *Let $w=(w_1,w_2)\in\mathbb{N}^2$, $d\in\mathbb{N}$, $w_1,w_2\leqslant\frac12 d$. A polynomial $P(x,y)$ is said to be weighted homogeneous of type w and weighted degree d, if*

$$P(\tau^{w_1}x,\tau^{w_2}y)=\tau^dP(x,y)\quad\textit{for any}\ \tau,x,y\in\mathbb{C}.$$

A polynomial H is said to be semiweighted homogeneous of type w and weighted degree d, if

$$H=\sum_{i=0}^{d}H_i, H_i\quad \textit{are weighted homogeneous of type}\ w\ \textit{and degrees}\ i,$$

and the highest weighted homogeneous part H_d has an isolated critical point at 0.

Remark 1.19. — As it was shown in [5], formula (1.18) holds true (with a certain (unknown) constant $C(h,\Omega)$), if the polynomial H under consideration is semiweighted homogeneous, and Ω is a collection of monomial 1-forms of appropriate weighted homogeneous degrees. The corresponding constant $C(h,\Omega)$ is nonzero, if no nontrivial linear combination of the monomials $D\omega_j$ belongs to the gradient ideal of the highest weighted homogeneous part of H. The converse is also true (see Lemma 2.3 below: its statement and proof extend (with minor modifications) to the case, when H is a semiweighted homogeneous polynomial).

Question. — *Find an explicit formula for the period determinant in the case, when H is a semiweighted homogeneous polynomial. In other words, find an explicit formula for the constant $C(h,\Omega)$ from* (1.18) *in the case, when h is a weighted homogeneous polynomial with isolated critical point at* 0.

The author thinks that the method of calculation of $C(h,\Omega)$ presented below can be extended to the above-mentioned weighted homogeneous case.

1.3. Historical remarks

An explicit formula for the period determinant up to a constant factor depending on n was obtained by A.N. Varchenko [14]. In the present paper the constant factor is calculated.

2. Proof of formula for the period determinant up to constant C_n

Here we prove the formula from Theorem 1.17 for the period determinant, without calculation of the constant C_n.

2.1. The plan of the proof of Theorem 1.17

Definition 2.1. — *A generic complex polynomial H is said to be ultra-Morse, if it has distinct critical values (then their number is equal to $\mu = n^2$, $n = \deg H - 1$).*

It suffices to prove Theorem 1.17 for any ultra-Morse polynomial H (passing to a non ultra-Morse limit while the highest form h remains unchanged does not change the right-hand side of (1.18)). Everywhere below we consider that H is ultra-Morse (whenever the contrary is not specified). We denote

$$H_1(t) = H_1(S_t,\mathbb{Z}), \quad B = \mathbb{C} \setminus \{a_1, \dots, a_{n^2}\}.$$

We consider the period determinant as defined for a special basis in $H_1(t)$ called *marked basis of vanishing cycles*, see [1] (whenever the contrary is not specified). The definitions and some basic properties of vanishing cycles are recalled in 2.2.

Given a noncritical value $t \in \mathbb{C}$ and a loop $\gamma : [0,1] \to B$ with a base point $t = \gamma(0) = \gamma(1)$. Any cycle $\delta \in H_1(t)$ extends continuously along γ up to a family of cycles $\delta(\tau) \in H_1(\gamma(\tau))$, $\tau \in [0,1]$. The result $\delta(1) \in H_1(t)$ of extension is different from $\delta = \delta(0)$ in general. The mapping $M_\gamma : H_1(t) \to H_1(t)$ sending δ to $\delta(1)$ (which is a linear operator) is called the *monodromy operator* along γ.

PROPOSITION 2.2 (see [5]). — *Let H be an ultra-Morse polynomial of degree $n+1 \geqslant 2$, ω_i, $i = 1, \dots, \mu = n^2$ be arbitrary polynomial* 1*-forms, $\Delta(t)$ be the corresponding period determinant* (1.2)*. The function $\Delta(t)$ is always polynomial.*

Proof. — The Picard-Lefschetz theorem [1] implies that the monodromy operator along a circuit around one critical value is always unipotent. Hence, the function $\Delta(t)$ does not change after the extension along the previous circuit. This implies that it is a single-valued function. It follows from definition that it is bounded in the neighborhood of the critical values. Hence, it is an entire function $\mathbb{C} \to \mathbb{C}$ (singularity erasing theorem). Simple estimates whose improved version is checked in 2.4 imply that Δ has at most polynomial growth at infinity. Hence, $\Delta(t)$ is a polynomial (its degree is calculated at the same place). □

On the first step of the proof we prove formula (1.18) with a $C(h,\Omega)$ depending only on h and Ω (Lemma 2.3) and we show (in Proposition 2.4 and Lemma 2.3) that for any fixed generic h the functions $C(h,\Omega)$ and $P(h,\Omega)$ have the same zeros: they vanish exactly on those pairs (h,Ω) where Ω is h-degenerate. On the second step we show that

$$C(h,\Omega) = C_n\big(\Sigma(h)\big)^s P(h,\Omega) \tag{2.1}$$

with some $s \in \mathbb{R}$, $C_n \in \mathbb{C}$. To do this, we prove that for any fixed generic h the functions $P(h,\Omega)$ and $C(h,\Omega)$ in Ω have simple zero at the hypersurface of h-degenerate tuples Ω (Lemma 2.5). After this the power s will be found by straightforward calculation of the homogeneity degrees in h of C and P (at the end of the present subsection).

LEMMA 2.3. — *Let $H(x,y)$ be an ultra-Morse polynomial of degree $n+1 \geqslant 2$, h be its highest homogeneous part. Let a_i, $i = 1, \dots, n^2$, be the critical values of H. Let Ω be as in* (1.4)*, $\Delta(t)$ be the corresponding period* *determinant* (1.2). *Let* $D\omega_j$ *be the polynomials from* (1.15) *corresponding to the forms* ω_j. *Then formula* (1.18) *holds with* $C(h,\Omega)$ *depending only on* h *and* Ω *such that* $C(h,\Omega) = 0$, *if and only if* Ω *is* h*-degenerate* (*see Definition* 1.14).

Lemma 2.3 is proved in 2.3–2.4. The statements of the Lemma saying that $\Delta(t)$ is a polynomial (1.18) and h-nondegeneracy implies $C(h,\Omega) \neq 0$) were proved in [5]. Elementary proofs of (1.18) and of the latter implication were obtained separately by Yu. S. Ilyashenko (his proof is represented in 2.4) and D. Novikov [12].

The *theorem on determinant* [1] implies that Δ is a polynomial nonzero for *"typical"* H and Ω. It does not specify concrete H and Ω with this property. Lemma 2.3 provides this specification.

In the proof of Lemma 2.3 we use a criterium (due to Yu. S. Ilyashenko [8], [9] and L. Gavrilov [5]) for identical vanishing of an Abelian integral over vanishing cycle (Theorem 2.14 and Corollary 2.16). We represent the statement and the proof of this criterium in 2.3.

Proposition 2.4. — *Let* $n \geqslant 2$, h *be a generic homogeneous polynomial of degree* $n+1$ (*see Definition* 1.3), $P(h,\Omega)$ *be the polynomial defined by* (1.17). *Then* $P(h,\Omega) = 0$, *if and only if* Ω *is* h*-degenerate.*

Proof. — If Ω is h-degenerate, then there is a d such that a nontrivial linear combination $\sum_{d(i)=d} c_i D\omega_i$ either vanishes, or belongs to the gradient ideal. Then this combination (which is homogeneous of degree d) is equal to a linear combination of the partial derivatives of h with (may be zero) homogeneous polynomial coefficients of degree $d-n$. This statement is equivalent to vanishing of the polynomial $P_d(h,\Omega) = \det A_d(h,\Omega)$ by definition, so, $P(h,\Omega) = 0$. Conversely, let h be generic and $P(h,\Omega) = 0$. Let us prove that Ω is h-degenerate. There exists a d such that $P_d(h,\Omega) = 0$ (fix such a d). By definition, this means that a nontrivial linear combination of the lines of the matrix $A_d(h,\Omega)$ (or equivalently, that of the corresponding polynomials) is zero. In the case, when $d < n$, these lines are nonzero-proportional to the coefficients strings of the polynomials $D\omega_i$, $d(i) = d$, thus, the latters are linearly dependent and Ω is h-degenerate. Let $d \geqslant n$. Let us show that there exists a nontrivial linear combination of $D\omega_i$, $d(i) = d$, that belongs to the gradient ideal. In this case the lines of $A_d(h,\Omega)$, which are linearly dependent, are nonzero-proportional to the coefficients strings of the polynomials $D\omega_i$, and $x^\ell y^{d-n-\ell}\,\partial h/\partial y$, $x^\ell y^{d-n-\ell}\,\partial h/\partial x$, $\ell = 0,\dots,d-n$, so, a nontrivial linear combination of these polynomials vanishes. The last $2(d-n+1)$ multiples of the partial derivatives are linearly independent, which follows from the statement that the partial derivatives of a generic homogeneous polynomial h are relatively prime. Therefore, a vanishing nontrivial linear combination of them and the polynomials $D\omega_i$ contains a nontrivial linear combination of $D\omega_i$. The latter linear combination is a one we are looking for. Proposition 2.4 is proved. □

Thus, the functions $P(h,\Omega)$ and $C(h,\Omega)$ have common zero set outside the discriminant hypersurface $S = \{\Sigma(h) = 0\}$: this set is the hypersurface of pairs (h,Ω) such that Ω is h-degenerate.

As it is shown below, equality (2.1) is implied by the following:

LEMMA 2.5. — *For any fixed generic homogeneous polynomial h of degree $n+1 \geqslant 2$ the functions $P(h,\Omega)$, $C(h,\Omega)$ have nonzero gradients in Ω on a Zariski open subset of the set of h-degenerate tuples Ω.*

The lemma is proved in 2.5. The function $C(h,\Omega)$ is (at most) double-valued (its different branches are obtained from each other by multiplication by ± 1). This follows from Remark 1.1. The previous Lemma implies that each one of the functions $C(h,\Omega)$, $P(h,\Omega)$ has simple zero at each irreducible component of their common zero hypersurface outside S. Hence, the ratio $C(h,\Omega)/P(h,\Omega)$ is a nowhere vanishing (at most) double-valued function holomorphic outside the hypersurface S. It has at most a polynomial growth, as (h,Ω) tends to S or to infinity by definition and a theorem of P. Deligne [2, thm., III.1.8]. Recall that the polynomial $\Sigma(h)$ is irreducible. Hence, the previous ratio is a power s of $\Sigma(h)$ up to multiplication by constant. This proves (2.1). To find s, we use the following

PROPOSITION 2.6. — *The function $C(h,\Omega)$ is homogeneous in h of the degree $-n^2$.*

Proof. — Let $b \in \mathbb{C} \setminus 0$. Let us compare $C(h,\Omega)$ and $C(bh,\Omega)$. By definition, for any $t \in \mathbb{C}$ the value at t of the function $\Delta(t)$ corresponding to a polynomial $H = h$ is equal to the value at bt of that corresponding to bh, i.e., $\Delta(t) = C(h,\Omega)t^{n^2} = C(bh,\Omega)(bt)^{n^2}$. Therefore, $C(bh,\Omega) = b^{-n^2}C(h,\Omega)$. This proves Proposition 2.6. □

By definition, a polynomial $P_d(h,\Omega)$ is independent on h for $d < n$ and is homogeneous in h of degree $2(d-n+1)$ for $d \geqslant n$. Therefore, the polynomial P is homogeneous in h of degree $n(n-1)$. Recall that $\deg \Sigma(h) = 2n$. Therefore, by Proposition 2.6, the power s in (2.1) is equal to $\frac{1}{2n}(-n^2 - n^2 + n) = \frac{1}{2} - n$. This proves (1.19) modulo the calculation of the constant C_n, which will be done in Section 3.

2.2. Marked basis of vanishing cycles

All the definitions and the statements of the present Subsection are contained in [1].

Firstly we recall the definition of a local vanishing cycle.

Lemma 2.7 (Morse lemma). — *A holomorphic function having a Morse critical point may be transformed to a sum of a nondegenerate quadratic form and a constant term by an analytic change of coordinates near this point.*

Corollary 2.8. — *Consider a holomorphic function in $\mathbb{C}^2$ having a Morse critical point with a critical value a. There exists a ball centered at the critical point whose intersection with each level curve corresponding to a value close to a of the function is diffeomorphic to an annulus.*

Definition 2.9. — *A generator of the first homology group of the latter intersection annulus (considered as a cycle in the homology of the global level curve) is called a local vanishing cycle corresponding to a.*

A local vanishing cycle is well defined up to change of orientation.

Definition 2.10. — *Let H be an ultra-Morse polynomial of degree $n+1 \geqslant 2$, a_j, $j = 1, \dots, n^2$, be its critical values, a be one of them. Let $t_0 \in B = \mathbb{C} \setminus \{a_1, \dots, a_{n^2}\}$, $\alpha : [0,1] \to \mathbb{C}$ be a path from t_0 to a such that*

$$\alpha[0,1) \subset B. \tag{2.2}$$

For any $s \in [0,1]$ close to 1 let $\delta(t), t = \alpha(s)$, be a local vanishing cycle on S_t corresponding to a. Consider the extension of δ along the path α up to a continuous family of cycles $\delta(s)$ in complex level curves $H = \alpha(s)$. The homology class $\delta = \delta(0) \in H_1(t_0)$ is called a cycle vanishing along α.

Definition 2.11. — *Let H, a_j, t_0 be as in the previous definition. Consider a set of paths α_j, $j = 1, \dots, \mu$, from t_0 to a_j that satisfy (2.2). Suppose these paths are neither pairwise nor self intersected. Then the set of cycles $\delta_j \in H_1(t_0)$ vanishing along $\alpha_j, j = 1. \dots, \mu$, is called a marked set of vanishing cycles on the level curve $H = t_0$.*

Lemma 2.12. — *Any marked set of vanishing cycles is a basis in the first integer homology group of the level curve.*

Lemma 2.13. — *The images of any vanishing cycle under monodromy operators along all the loops generate the previous homology group.*

2.3. A vanishing criterium for Abelian integral

Denote Ω^0 the space of polynomials in two complex variables (x, y). By Ω^1 denote the space of polynomial 1-forms. By $\Omega^1_n \subset \Omega^1$ denote the subspace of forms of degrees at most n.

Let $H(x, y)$ be a complex polynomial. Define

$$K_H = \mathrm{d}\Omega^0 + \Omega^0 \mathrm{d}H = \left\{\mathrm{d}f + g\mathrm{d}H \mid f, g \in \Omega^0\right\}.$$

In the proof of Lemma 2.3 we use the following

Theorem 2.14 (see [5], [8], [9]). — *Let H be an arbitrary ultra-Morse polynomial of degree $n+1 \geqslant 2$. Let $\omega \in \Omega^1$ be a 1-form such that for any $t \in \mathbb{C}$ and any cycle $\gamma \in H_1(t)$*

$$\int_\gamma \omega = 0.$$

If $\deg \omega \leqslant n$, *then* $\mathrm{d}\omega = 0$.(1) *In general case, if there are no restriction on degree, then $\omega \in K_H$.*

Addendum to Theorem 2.14. — Theorem 2.14 holds if H is replaced by generic homogeneous polynomial.

Remark 2.15. — As it was shown in [5], the last statement of Theorem 2.14 (general case) holds for arbitrary complex polynomial in two variables with isolated critical points, provided that all the fibers $H^{-1}(t)$ are connected. A generalization of this fact to arbitrary dimension was proved in [3] (and also in [13] but under additional conditions on the polynomial).

Corollary 2.16. — *Theorem* 2.14 *holds true, if the assumption on the integral is replaced by*

$$\int_{\delta_t} \omega = 0,$$

where δ_t is a family of cycles vanishing to some critical value.

Proof. — The monodromy images of a vanishing cycle generate $H_1(t)$ (Lemma 2.13). This together with Theorem 2.14 implies the corollary. □

Proof. — of Theorem 2.14(2) For $t \in \mathbb{C}$ denote $S_t = \{H(x, y) = t\}$.

Let us firstly consider that $\omega \in \Omega^1_n$. Let us prove the first statement of Theorem 2.14. We give a sketch of the proof here: a more detailed proof may be found in [8], [9]. Consider a straight line L which is generic with

(1) This first statement of Theorem 2.14 was firstly proved by Yu. S. Ilyashenko [8], [9]. General Theorem 2.14 (including the second statement) was proved by L. Gavrilov [5].
(2) From unpublished paper by Yu. S. Ilyashenko

respect to H. This means that $H|_L$ is a polynomial of degree $n+1$, with exactly n critical points; denote them $q_1, \dots, q_n$. For any $s \in \mathbb{C}^2$ lying on a noncritical level curve of H, denote by $b_1(s), \dots, b_{n+1}(s)$ the intersection points of the level curve $S_{H(s)} : H = H(s)$ with L. Let $\gamma_j(s)$ be a real curve in $S_{H(s)}$ with the beginning at $b_j(s)$ and the endpoint s. Consider the function

$$Q(s) = \frac{1}{n+1} \sum_{1}^{n+1} \int_{\gamma_j(s)} \omega. \tag{2.3}$$

This function is well defined on any noncritical level curve S_t for $t \neq H(q_j), j = 1, \dots, n$. Indeed, it depends on s only, not on the choice of the curves $\gamma_j(s)$: replacing $\gamma_j(s)$ by another curve $\lambda_j(s) \subset S_t$ with the same endpoints adds to $Q(s)$ an integral of ω over the cycle $[\gamma_j(s) \circ \lambda_j^{-1}(s)] \in H_1(S_{H(s)})$; this integral is zero by assumption.

Formula (2.3) implies that

$$\mathrm{d}(Q \,|\, S_{H(s)}) = \omega \,|\, S_{H(s)} \tag{2.4}$$

When the above chosen s ranges over a small disc transversal to the level curves of H, the arcs $\gamma_j(s)$ may be chosen to depend analytically on s. Hence, Q is a holomorphic function in its domain. This function is bounded in any compact set and well defined outside a finite union of algebraic curves. By the theorem on removable singularities, Q may be holomorphically extended to all of $\mathbb{C}^2$.

The coefficients of the form ω are polynomials of degree less than $n+1$, and level curves of H near infinity ressemble those of the homogeneous polynomial h, which is the highest homogeneous part of H. Hence, the function Q grows no faster than a polynomial of degree no greater than $n+1$. By the Liouville theorem, Q itself is such a polynomial.

In the assumptions of Theorem 2.14 we constructed a polynomial Q of degree less than $n+2$ whose differential restricted to level curves of H coincides with ω, see (2.4).

Let $\omega = F\mathrm{d}x + G\mathrm{d}y$, $\mathrm{d}Q = Q_x\mathrm{d}x + Q_y\mathrm{d}y$. The difference between these forms vanishes on the Hamiltonian vector field $(H_y, -H_x)$. Hence

$$(Q_x - F)H_y - (Q_y - G)H_x = 0. \tag{2.5}$$

The polynomials H_y, H_x are relatively prime and their degrees equal n because H is ultra-Morse. The degrees of the polynomials in the parenthesis are less than $n+1$. Hence, $Q_x - F = cH_x$, $Q_y - G = cH_y$ for some $c \in \mathbb{C}$. Therefore, the form $\omega = \mathrm{d}Q - c\mathrm{d}H$ is exact. The first statement of Theorem 2.14 is proved.

Let us now consider that ω is a 1-form of arbitrary degree satisfying the conditions of Theorem 2.14. Let us prove the second statement of Theorem 2.14.

Formula (2.5) was obtained without any restriction to degrees of the coefficients of the form ω. Without these restrictions, it implies:

$$Q_x - F = gH_x, \quad Q_y - G = gH_y$$

for some polynomial g. Hence $\omega = \mathrm{d}Q - g\mathrm{d}H \in K_H$. Theorem 2.14 is proved $\square$

Proof of Addendum to Theorem 2.14 (Yu. S. Ilyashenko). — The previous proof works for any polynomial H with relatively prime first derivatives. In particular, it works for generic homogeneous polynomial h taken instead of H and proves the Addendum. $\square$

2.4. Nonvanishing and h-nondegeneracy. Proof of Lemma 2.3

We have already shown (Proposition 2.2) that $\Delta(t)$ is a polynomial. Let us prove (1.18). For $t = a_j(H)$, the j'th column of $\mathbb{I}(t;H)$ vanishes. Hence

$$\Delta\big(a_j(H)\big) = 0.$$

Therefore, Δ is divisible by $t - a_j(H)$ for any $j = 1, \dots, \mu = n^2$. Proposition 2.19 below shows that, in the assumptions of Lemma 2.3 $\deg \Delta = \mu$. (This together with the previous statement implies that $\Delta(t)$ is $\prod(t - a_j)$ up to constant factor.)

The last degree equality is proved by comparison between H and its highest form h. The following proposition is the first step of this proof. It is stated in more general setting than needed for the proof of Lemma 2.3.

Proposition[3] 2.17. — *Let $\omega_1, \dots, \omega_\mu$ be a collection of polynomial* 1-*forms, $\mu = n^2$, d_i be the maximal degree of the polynomial coefficients of ω_i. Let β_i be the form obtained from ω_i by dropping all the terms of degree lower than d_i. Let H be an ultra-Morse polynomial,* $\deg H = n+1$, *and h its highest homogeneous part. Let $\gamma_j(t,H)$, $\gamma_j(t,h)$ be bases in $H_1(\{H=t\},\mathbb{Z})$ and $H_1(\{h=t\},\mathbb{Z})$ respectively,*

$$\mathbb{J} = \Big(\int_{\gamma_j(t;H)} \omega_i\Big), \quad \mathbb{K} = \Big(\int_{\gamma_j(t,h)} \beta_i\Big).$$

(3) From unpublished paper by Yu. S. Ilyashenko.

Suppose that the ratio

$$\sigma = \frac{\sum d_i + \mu}{n+1} \tag{2.6}$$

is integer. Then the determinant $\det \mathbb{J}$ *is a polynomial of degree no greater than* σ*. Moreover,*

$$\det \mathbb{K} = qt^{\sigma}, \quad q \in \mathbb{C}; \quad \deg(\det \mathbb{J} - \det \mathbb{K}) < \sigma.$$

Remark 2.18. — Proposition 2.17 holds true if the ultra-Morse polynomial H is replaced by its highest homogeneous part h in the definition of $\mathbb{J}$. This follows from the statement of the proposition in the case of homogeneous 1-forms ω_i.

Proof. — The determinants under consideration are polynomials (For the ultra-Morse polynomial H this follows from Proposition 2.2; for the homogeneous polynomial h this follows from the same Proposition applied to its ultra-Morse deformation). Let us first consider the polynomial $\det \mathbb{K}$. The ith string $\mathbb{K}^i$ of the matrix $\mathbb{K}$ is a homogeneous vector function of the form

$$\mathbb{K}^i = t^{\nu_i} q^i, \quad \nu_i = \frac{d_i + 1}{n+1}, \quad q^i \in \mathbb{C}^{\mu}.$$

Let Q be the matrix with the strings q^i. Then

$$\det \mathbb{K} = t^{\sigma} \det Q. \tag{2.7}$$

Simple rescaling arguments imply that the i'th string of $\mathbb{J}$ is $t^{\nu_i}(q^i + o(1))$, as $t \to \infty$. Hence,

$$\det \mathbb{J} = t^{\sigma}\big(\det Q + o(1)\big).$$

This implies the proposition. □

Proposition 2.17 reduces Lemma 2.3 to the homogeneous case, which is discussed below.

Recall that vanishing cycles of a generic homogeneous polynomial h are defined as the limits of vanishing cycles for an ultra-Morse perturbation H of h by lower terms, as these terms tend to zero.

PROPOSITION 2.19. — *Let* ω_i *be homogeneous polynomial* 1*-forms as in* (1.4)*,* h *be a generic homogeneous polynomial of degree* $n+1$*,* $t \in \mathbb{C} \setminus 0$*,* $\delta_j(t;h) \in H_1(t)$ *be a basis of cycles in its level curve* $h = t$*,* $j = 1, \dots, n^2$*. Let* $\mu = n^2$*,*

$$\mathbb{K} = \Big(\int_{\delta_j(t;h)} \omega_i\Big)$$

Then

$$\det \mathbb{K} = C(h, \Omega)\, t^{\mu}, \tag{2.8}$$

$C(h,\Omega)\neq 0$ if and only if Ω is h-nondegenerate

Proof. — Firstly we prove that the determinant (2.8) is a degree n^2 monomial. Then we prove the last nonvanishing criterium for $C(h,\Omega)$.

Proof of formula (2.8)[4]. — Let us calculate $\det\mathbb{K}$ using Proposition 2.17 applied to forms (1.4). For these forms the number σ from (2.6) is equal to μ. Indeed, $d_i=\ell(i)+m(i)+1$, $\nu_i=(d_i+1)/(n+1)$. Hence,

$$\sigma=\frac{1}{n+1}\sum_{\substack{0\leqslant k_1\leqslant n-1\\ 0\leqslant k_2\leqslant n-1}}(k_1+k_2+2)=\frac{n^2(n+1)}{n+1}=n^2=\mu.$$

This together with the statement of Proposition 2.17 on $\det\mathbb{K}$ implies (2.8). □

Proof of the statement that h-nondegeneracy implies $C(h,\Omega)\neq 0$[5]. — Let no linear combination of $D\omega_i$ belong to the gradient ideal of h. Let us show that $C(h,\Omega)\neq 0$. We prove this by contradiction. Suppose that $C(h,\Omega)=0$. Then $(\det\mathbb{K})(t)\equiv 0$. Hence, the determinant of the corresponding matrix $Q=\mathbb{K}(1)$ from (2.7) vanishes, so its strings are linearly dependent. The linear dependence for the strings of $\mathbb{K}(1)$ with coefficients σ_i

$$\sum\sigma_i q^i=0,$$

implies linear dependence for the strings of $\mathbb{K}(t)$ with the t-depending coefficients $\sigma_i t^{-\nu_i}$.

Consider the following 1-form with algebraic coefficients:

$$\alpha=\sum h^{-\nu_i}\sigma_i\omega_i.$$

This form has zero integrals over all 1-cycles of the Riemann surfaces $h=t$, $t\in\mathbb{C}^*$, i.e., is exact on the nonsingular level curves of h. The form α has branching points at the lines $h=0$. A circuit around any one of these lines, which adds 2π to arg h, transforms α into

$$\Delta\alpha=\sum \mathrm{e}^{-2\pi i\nu_i}h^{-\nu_i}\sigma_i\omega_i.$$

For any $\nu\in\mathbb{Z}/(n+1)$ denote by χ_ν the character of the additive group $\mathbb{Z}_{n+1}$ determined by $\chi_\nu(1)=\mathrm{e}^{-2\pi i\nu}$. Then k circuits produce

$$\Delta^k\alpha=\sum\chi_{\nu_i}(k)h^{-\nu_i}\sigma_i\omega_i.$$

Let

$$A(\nu)=\bigl\{i\mid \nu_i=\nu,\,\sigma_i\neq 0\bigr\},\quad \alpha_\nu=\sum_{i\in A(\nu)}\sigma_i\omega_i.$$

[4] From unpublished paper by Yu. S. Ilyashenko.
[5] From unpublished paper by Yu. S. Ilyashenko

Note that ν_i range over the set $\{2/(n+1),\dots,(2n-1)/(n+1),2n/(n+1)\}$. Hence, for any ν only one of the two sets, $A(\nu+1)$ or $A(\nu-1)$, may be nonempty. In what follows, fix an arbitrary ν in such a way that $A(\nu)\neq\emptyset$, $A(\nu+1)=\emptyset$; no assumption about $A(\nu-1)$. The form

$$\beta_\nu=\frac{1}{n+1}\sum_{k=0}^{n}\chi_\nu^{-1}(k)\Delta^k\alpha$$

is exact when restricted to the nonsingular level curves of h and β polynomial. On the other hand, the sum of all the values of the character $\chi_\nu^{-1}\chi_{\nu_i}$ is zero provided that the character is not identically 1. Hence,

$$\beta_\nu=\sum_{i\in A(\nu)}h^{-\nu}\sigma_i\omega_i+\sum_{i\in A(\nu-1)}h^{-(\nu-1)}\sigma_i\omega_i.$$

The form

$$\beta=h^\nu\beta_\nu=\sum_{i\in A(\nu)}\sigma_i\omega_i+h\sum_{i\in A(\nu-1)}\sigma_i\omega_i$$

is exact on the level curves of h and is polynomial. By Addendum to Theorem 2.14, $\beta\in K_h$, that is

$$\beta=\mathrm{d}f+g\mathrm{d}h \tag{2.9}$$

for some polynomials f and g. Suppose that $A(\nu-1)\neq\emptyset$. Otherwise, the form $\sum_{i\in A(\nu)}\sigma_i\omega_i$ belongs to K_h, which is brought to contradiction even simpler, than it is done below for (2.9).

Taking the differential of both sides of (2.9) we get:

$$\Big(\sum_{i\in A(\nu)}\sigma_iD\omega_i)\mathrm{d}y\wedge\mathrm{d}x=-h\mathrm{d}\Big(\sum_{i\in A(\nu-1)}\sigma_i\omega_i\Big)-\mathrm{d}h\wedge\sum_{i\in A(\nu-1)}\sigma_i\omega_i+\mathrm{d}g\wedge\mathrm{d}h;$$

$\sigma_i\neq 0$ for $i\in A(\nu)$ by definition.

Together with the Euler identity $(n+1)h=zh_z+wh_w$, this implies that a nontrivial linear combination of the polynomials $D\omega_i$, namely,

$$\sum_{i\in A(\nu)}\sigma_iD\omega_i$$

belongs to the gradient ideal of h — a contradiction. □

Proof of the statement that h-degeneracy implies $C(h,\Omega)=0$. — Now let a nontrivial linear combination

$$\sum_{d(i)=d}c_iD\omega_i \tag{2.10}$$

vanish (in the case, when $d < n$) or belong to the gradient ideal of h (in the case, when $d \geqslant n$). Let us show that $C(h,\Omega) = 0$, or equivalently, determinant (2.8) vanishes.

In the first case, when $d < n$,

$$\sum c_i D\omega_i = D\omega' = 0; \quad \omega' = \sum c_i \omega_i, \tag{2.11}$$

so the form ω' is closed (thus, exact). Hence, its integral over any cycle in $H_1(t)$ vanishes. The string consisting of its integrals along basic cycles is a linear combination of strings of the Abelian integral matrix $\mathbb{K}$. Therefore, the determinant of the latter vanishes, and thus, $C(h,\Omega) = 0$.

Now let us consider the case, when $n \leqslant d \leqslant 2n-2$ and

$$\begin{cases} \text{for any } d' \leqslant n-1 \text{ the polynomials } (D\omega_j)_{d(j)=d'} \\ \text{are linearly independent.} \end{cases} \tag{2.12}$$

(If condition (2.12) is not satisfied, then the previous case takes place.) Let us show that $\det \mathbb{K} = 0$. To do this, let us introduce the following notations.

For $k \in \mathbb{N}$ denote $\widetilde{\Omega}^0_k$ ($\widetilde{\Omega}^1_k$) the space of homogeneous polynomials (respectively, 1-forms) of degree k in two complex variables. For $k < 0$ we put $\widetilde{\Omega}^0_k = 0$.

To show that $\det \mathbb{K} = 0$, we use the following properties of the operator D.

Remark 2.20. — For any $k \in \mathbb{N} \cup 0$ the operator D defined by (1.15) induces an isomorphism between the factor-space $\widetilde{\Omega}^1_k / \mathrm{d}\widetilde{\Omega}^0_{k+1}$ and the space $\widetilde{\Omega}^0_{k-1}$. This follows from the definition of D and the statement that each polynomial 2-form on $\mathbb{C}^2$ is exact.

Corollary 2.21. — *The dimension of the factor-space $\widetilde{\Omega}^1_k / \mathrm{d}\widetilde{\Omega}^0_{k+1}$ is equal to k.*

Proposition 2.22. — *Let h be a generic homogeneous polynomial of degree $n+1 \geqslant 2$, D be the operator defined by* (1.15). *Let $n \leqslant d \leqslant 2n-1$, ∇^d be the intersection of the space $\widetilde{\Omega}^0_d$ with the gradient ideal of h. The dimension of the linear space ∇^d is equal to $2(d-n+1)$. The operator D induces an isomorphism $D : h\widetilde{\Omega}^1_{d-n} \to \nabla^d$.*

Proof. — The image $D(h\widetilde{\Omega}^1_k)$ is contained in ∇^d by definition and Euler identity. The dimensions of the both spaces $h\widetilde{\Omega}^1_{d-n}$ and ∇^d are equal to $2(d-n+1)$. For the former space this statement follows from definition. The latter space has the basis of the $2(d-n+1)$ polynomials $x^\ell y^{d-n-\ell}\, \partial h/\partial y$, $x^\ell y^{d-n-\ell}\, \partial h/\partial x$, $0 \leqslant \ell \leqslant d-n$, cf. the proof of

Proposition 2.4. Now for the proof of Proposition 2.22 it suffices to show that the restriction of D to $h\widetilde{\Omega}^1_{d-n}$ has zero kernel. We prove this statement by contradiction. Suppose the contrary, i.e., there exists a nonzero $\omega \in \widetilde{\Omega}^1_{d-n}$ such that $\mathrm{d}(h\omega) = 0$. Then $h\omega$ is a closed polynomal homogeneous form, and hence, $h\omega = \mathrm{d}Q$, Q is a homogeneous polynomial of degree $d+2 < 2n+2 = 2\deg h$. It follows from definition and genericity of h that Q vanishes on the $n+1$ zero lines of h and has order 2 zero on each of them. Therefore, $\deg Q \geqslant 2\deg h$ — a contradiction. Proposition 2.22 is proved. □

For a 1-form $\widetilde{\omega}$ by $I_{\widetilde{\omega}}$ denote the string of the integrals of $\widetilde{\omega}$ over the basic cycles in $H_1(t)$.

Let c_i be the coefficients of linear combination (2.10), ω' be the corresponding 1-form defined in (2.11) (recall that $n \leqslant d \leqslant 2n-2$). To show that $\det \mathbb{K} = 0$, we construct a vanishing t-depending nontrivial linear combination of the strings of $\mathbb{K}$. To do this, we prove that there exists an $\omega'' \in \widetilde{\Omega}^1_{d-n}$ such that

$$\omega = \omega' + h\omega'' \in \mathrm{d}\Omega^0. \tag{2.13}$$

Then the string I_ω of the integrals of the form ω vanishes. On the other hand, I_ω is a nontrivial t-depending linear combination of strings of the matrix $\mathbb{K}$. Indeed, by definition,

$$I_\omega = I_{\omega'} + I_{h\omega''} = I_{\omega'} + tI_{\omega''}.$$

The string $I_{\omega'}$ is the linear combination of strings of $\mathbb{K}$ with the coefficients c_i from (2.10); this combination is nontrivial by assumption. Now for the proof of the previous statement on I_ω it suffices to show that the string $I_{\omega''}$ is a linear combination of strings of $\mathbb{K}$ with constant coefficients. The $d-n$ forms $(\omega_j)_{d(j)=d-n-1}$ form a basis in the factor-space of Ω^1_{d-n} modulo closed forms (by Corollary 2.21 and assumption (2.12)). Therefore, ω'' is equal (modulo closed forms) to a linear combination

$$\omega'' = \sum_{d(j)=d-n-1} c'_j \omega_j.$$

Hence, $I_{\omega''}$ is the linear combination of strings of $\mathbb{K}$ with the coefficients c'_j. Therefore, (2.13) implies that $\det \mathbb{K} = 0$.

Inclusion (2.13) is equivalent to the equation

$$-D\omega' = D(h\omega'').$$

By assumption, the polynomial $-D\omega'$ (which is the linear combination (2.10) taken with the sign '$-$') belongs to the gradient ideal. This together with Proposition 2.22 implies existence of a solution $\omega'' \in \Omega^1_{d-n}$ to the last equation. This solution is a form ω'' we are looking for. The proof of Lemma 2.3 is completed. $\square$

2.5. Simplicity of zeros of $P(h,\Omega)$ and $C(h,\Omega)$. Proof of Lemma 2.5

We have already shown (Lemma 2.3) that the zero set of $C(h,\Omega)$ consists of those pairs (h,Ω) such that Ω is h-degenerate. Now let us prove that for any fixed generic h the gradient in Ω of the function $C(h,\Omega)$ does not vanish at the points (h,Ω) of its zero set satisfying the following genericity conditions:

$$\text{(2.14)} \qquad \begin{cases} \text{there exists a unique } d \geqslant n \text{ such that a nontrivial linear} \\ \text{combination of } D\omega_i,\ d(i)=d, \text{ belongs to the gradient ideal} \\ \text{of } h; \text{ this linear combination is unique up to multiplication} \\ \text{by constant,} \end{cases}$$

i.e., the rank (modulo the gradient ideal) of the system of the $2n-d-1$ polynomials $D\omega_i$ is equal to $2n-d-2$.

For any fixed generic h the set of the form tuples Ω satisfying (2.14) is Zariski open and dense in the zero set of $C(h,\Omega)$. This follows from the statement that for each d, $n \leqslant d \leqslant 2n-1$, the intersection ∇^d of the gradient ideal of h with the space $\widetilde{\Omega}^0_d$ of homogeneous polynomials of degree d has codimension $2n-d-1$ in $\widetilde{\Omega}^0_d$, which is equal to the number of the forms ω_i, $d(i)=d$: $\dim \nabla^d = 2(d-n+1)$ by the previous Proposition. Fix a point (h,Ω) satisfying (2.14). Let us show that the gradient of C in the variables Ω at (h,Ω) does not vanish. There is an index ℓ such that the $2n-d-2$ polynomials $D\omega_i$, $i \neq l$, are linearly independent modulo ∇^d (the last condition of (2.14)). Let us fix such an ℓ. Then the gradient of the function C along the space of forms ω_ℓ (with fixed ω_i corresponding to $i \neq \ell$) is nonzero. Indeed, let q_ℓ be a homogeneous 1-form of the degree $d+1$. The derivative of the function C in ω_ℓ in the direction q_ℓ is equal to its value

$$C(h, \omega_1, \dots, \omega_{\ell-1}, q_\ell, \omega_{\ell+1}, \dots, \omega_{n^2})$$

at h, the forms ω_j with $j \neq \ell$ and q_ℓ. This value is nonzero for a typical q_ℓ, namely, when the $2n-d-2$ polynomials $D\omega_i$, $i \neq \ell$, and Dq_ℓ are linearly independent modulo ∇^d (recall that the latter has codimension $2n-d-1$ in $\widetilde{\Omega}^0_d$). This proves the statement of Lemma 2.5 for $C(h,\Omega)$. The proof of the analogous statement for $P(h,\Omega)$ repeats the previous one with obvious changes. Lemma 2.5 is proved. The proof of Theorem 1.17 is completed.

3. The constant C_n

Formula (1.13) for the constant C_n given by Theorem 1.10 is proved in 3.1–3.4.

3.1. The plan of the proof of the formula for the constant C_n

Everywhere below we suppose that $H(x,y) = h(x,y) = x^{n+1} + y^{n+1}$ and ω_i are the forms from (1.3). To find the constant C_n, we calculate the corresponding value $C(h)$ from (1.5) explicitly for this concrete $h(x,y)$ and then find C_n from formula (1.9), which expresses the value $C(h)$ via C_n (a more explicit version (3.7) of this formula will be proved in 3.4).

Let us sketch the calculations.

Recall the notation: $(\ell(j), m(j))$, $j = 1,\dots,n^2$, is the lexicographic sequence of the n^2 pairs (ℓ, m), $0 \leqslant \ell, m \leqslant n-1$. For any $j = 1,\dots,n^2$ put

$$I_j = \int_0^1 x^{\ell(j)} \left(1 - x^{n+1}\right)^{\frac{m(j)+1}{n+1}} \mathrm{d}x, \tag{3.1}$$

$$IP = \prod_{j=1}^{n^2} I_j. \tag{3.2}$$

$$\sigma = \prod_{1 \leqslant \ell < k \leqslant n+1} (\varepsilon^k - \varepsilon^\ell)^2, \quad \varepsilon = \mathrm{e}^{\frac{2\pi i}{n+1}}. \tag{3.3}$$

In 3.2 we express $C(h)$ via σ and IP: we show that

$$C(h) = \sigma^n IP. \tag{3.4}$$

In 3.3 we calculate IP: we show that

$$IP = \frac{(2\pi)^{\frac{1}{2}n(n+1)}(n+1)^{-\frac{1}{2}(n^2+4n+3)}((n+1)!)^n}{\prod_{m=1}^{n-1}(m+n+1)!}. \tag{3.5}$$

Remark 3.1. — The integrals I_j are expressed via appropriate values of B(or Γ)-function. It appears that due to the fact that the product IP contains *all* the integrals I_j, one can kill all the Γ-function values in the expression for IP by using Gauss-Legendre formula (3.19) for product of Γ-function values over appropriate arithmetic progression segment.

In 3.4 we calculate σ: we show that

$$\sigma = (-1)^{\frac{1}{2}n(n-1)}(n+1)^{n+1}. \tag{3.6}$$

Then we prove using (1.9) that

$$C(h) = C_n(-1)^{\frac{1}{4}n(n+1)(1-2n)}(n+1)^{\frac{1}{2}(1-3n)}. \tag{3.7}$$

Now formula (1.13) for C_n follows from the two latter formulas and (3.4), (3.5).

3.2. Calculation of $C(h)$. Proof of (3.4)

By definition, $C(h)$ is equal to the value of the determinant $\Delta(t)$ at $t = 1$. Let us calculate this value.

Let $F = \{H(x, y) = 1\}$. The fiber F admits the action of the group $\mathbb{Z} = \{(\ell, m) \mid \ell, m = 0, \dots, n\}$ by multiplication by ε^ℓ and ε^m of the coordinates x and y respectively.

Let $(\ell(j), m(j))$, $j = 1, \dots, n^2$, be the lexicographic sequence of the pairs (ℓ, m), $0 \leqslant \ell, m \leqslant n-1$. We calculate the value $\Delta(1)$ for appropriate basis $\delta_1, \dots, \delta_{n^2}$ in $H_1(F, \mathbb{Z})$ (defined below) such that each δ_j with $j > 1$ is obtained from δ_1 by the action of the element $(\ell(j), m(j)) \in \mathbb{Z}$. (This basis is completely defined by choice of δ_1. The basic cycles δ_j are *not necessarily vanishing.*)

To define δ_1, let us consider the fiber F as a covering over the x-axis having the branching points with the x-coordinates ε^j, $j = 0, \dots, n$. It is the Riemann surface defined by the equation $y = (1 - x^{n+1})^{1/(n+1)}$.

Consider the radial segments $[0, 1]$ and $[0, \varepsilon]$ of the branching points 1 and ε respectively in the x-axis; the former being oriented from 0 to 1, and the latter being oriented from ε to 0. Their union is an oriented piecewise-linear curve (denote it by ϕ). Let ϕ_0 and ϕ_1 be its liftings to the covering F such that ϕ_0 contains the point $(0, 1)$ and ϕ_1 is obtained from ϕ_0 by multiplication of the coordinate y by ε. The curves ϕ_i, $i = 0, 1$, are oriented from their common origin $(\varepsilon, 0)$ to their common end $(1, 0)$.

Definition 3.2. — *Let F, ϕ, ϕ_0, ϕ_1 be the same, as above. Define $\delta_1 \in H_1(F, \mathbb{Z})$ to be the homology class represented by the union of the oriented curve ϕ_0 and the curve ϕ_1 taken with the inverse orientation.*

Proposition 3.3. — *Let F, δ_1, $\ell(i)$, $m(i)$ be as above, $\delta_j \in H_1(F, \mathbb{Z})$, $j = 2, \dots, n^2$, be the homology classes obtained from δ_1 by the actions of the elements $(\ell(j), m(j)) \in \mathbb{Z}$. The classes δ_j, $j = 1, \dots, n^2$, generate the homology group $H_1(F, \mathbb{Z})$.*

Remark 3.4. — The complete $\mathbb{Z}_{n+1} \oplus \mathbb{Z}_{n+1}$-orbit of the cycle δ_1 has $(n+1)^2$ elements. The discussion below shows that they are linearly depending and the n^2 cycles obtained from δ_1 by the actions of the n^2 elements (ℓ, m), $0 \leqslant \ell, m \leqslant n-1$, form a basis in $H_1(F, \mathbb{Z})$.

Proof. — Let $\Gamma = \bigcup_{j=0}^{n} [0, \varepsilon^j]$ be the union of the radial segments of the branching points of the fiber F in the x-axis. Let $\widetilde{\Gamma} \subset F$ be the preimage of the set Γ under the projection of F to the x-axis. The set $\widetilde{\Gamma}$ is a deformation retract of the fiber F (hence, the inclusion $\widetilde{\Gamma} \to F$ is a homotopy equivalence). This follows from the statement that Γ is a deformation retract of the x-axis that contains all the branching points and from the covering homotopy theorem. The group $H_1(\widetilde{\Gamma}, \mathbb{Z})$ is generated by δ_j by construction. Hence, this remains valid for the whole fiber F. This proves Proposition 3.3. □

Let us calculate the value $\Delta = \Delta(1)$ in the basis δ_j from Proposition 3.3. To do this, we use the following

Remark 3.5. — Let ω_j be the forms (1.3), δ_j be as in Proposition 3.3, $I_{j,r} = I_{j,r}(1)$ be the corresponding integrals from (1.1), $(\ell(j), m(j))$ be the lexicographic integer pair sequence, $0 \leqslant \ell, m \leqslant n-1$. For any $j, r = 1, \ldots, n^2$

$$I_{j,r} = \varepsilon^{k(j,r)} I_{j,1}, \quad k(j,r) = \ell(r)\big(\ell(j)+1\big) + m(r)\big(m(j)+1\big). \tag{3.8}$$

Formula (3.8) implies the following

Corollary 3.6. — *Let ω_j be the forms* (1.3)*, F, δ_j be as in Proposition* 3.3*, $\mathbb{I} = (I_{j,r}) = (I_{j,r}(1))$ be the corresponding matrix of the integrals* (1.1)*, Δ be its determinant. Put*

$$\Pi = \prod_{j=1}^{n^2} I_{j,1}.$$

Let $(\ell(j), m(j))$ be the above lexicographic sequence, $k(j,r)$ be the same, as in (3.8)*, $G = (g_{jr})$ be the $n^2 \times n^2$-matrix with the elements*

$$g_{jr} = \varepsilon^{k(j,r)}. \tag{3.9}$$

Then

$$\Delta = \Pi \det G. \tag{3.10}$$

Thus, to find Δ, which is equal to $C(h)$, it suffices to calculate the expressions Π and $\det G$ from (3.10). Firstly we calculate $\det G$ explicitly. We show that

$$\det G = (n+1)^{-2n} \sigma^n. \tag{3.11}$$

Then we express Π via IP. We show that

$$\Pi = (n+1)^{2n} IP. \tag{3.12}$$

This will prove (3.4).

Proof of formula (3.11) for $\det G$. — In the proof of (3.11) we use the formula

$$\prod_{\ell=1}^{n} (1-\varepsilon^\ell) = n+1, \tag{3.13}$$

which follows from the fact that its left-hand side is equal to the value at $x=1$ of the polynomial $(x^{n+1}-1)/(x-1) = \sum_{j=0}^{n} x^j$. The matrix G is the tensor square of the matrix

$$Q = (q_{jr}) = \begin{pmatrix} 1 & \varepsilon & \cdots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \cdots & \varepsilon^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \varepsilon^n & \cdots & \varepsilon^{n(n-1)} \end{pmatrix}.$$

This follows from definition. Therefore, $\det G$ is the $2n$-th power of $\det Q$, which is van der Monde:

$$\det G = (\det Q)^{2n} = \Big(\prod_{1\leqslant \ell < k \leqslant n} (\varepsilon^k - \varepsilon^\ell) \Big)^{2n} = \Big(\prod_{1\leqslant \ell < k \leqslant n} (\varepsilon^k - \varepsilon^\ell)^2 \Big)^{n}. \tag{3.14}$$

The product in the right-hand side of (3.14) is equal to $(n+1)^{-2}\sigma$: the defining expression (3.3) for σ is obtained by multiplying the previous product by

$$\prod_{1\leqslant \ell \leqslant n} (1-\varepsilon^\ell)^2 = (n+1)^2$$

(see (3.13)). This together with (3.14) proves (3.11). □

Proof of formula (3.12) for Π. — Let us express $I_{j,1}$ via the integral I_j from (3.1). We show that

$$I_{j,1} = (1-\varepsilon^{m(j)+1})(1-\varepsilon^{\ell(j)+1}) I_j. \tag{3.15}$$

This together with the definition of Π (in Corollary 3.6) and (3.13) will imply (3.12).

Let ϕ_0, ϕ_1 be the oriented curves from Definition 3.2. Then

$$\begin{aligned} I_{j,1} &= \int_{\delta_1} x^{\ell(j)} y^{m(j)+1} \mathrm{d}x \\ &= \int_{\phi_0} x^{\ell(j)} y^{m(j)+1} \mathrm{d}x - \int_{\phi_1} x^{\ell(j)} y^{m(j)+1} \mathrm{d}x. \end{aligned} \tag{3.16}$$

The second integral in the right-hand side of (3.16) is equal to the first one times $\varepsilon^{m(j)+1}$ (by definition). Analogously, the first integral in its turn is the integral along the segment $[0,1]$ (which is equal to I_j) minus the one along the segment $[0,\varepsilon]$ oriented from 0 to ε. The integral along the last segment is equal to $\varepsilon^{\ell(j)+1}I_j$. This together with the two previous statements implies (3.15). Formula (3.12) is proved. The proof of formula (3.4) is completed. $\square$

3.3. Calculation of IP. Proof of (3.5)

To calculate $IP = \prod_{j=1}^{n^2} I_j$, we firstly express it via appropriate values of B- and Γ-functions. Recall their definitions:

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x, \quad \Gamma(a) = \int_0^{+\infty} x^{a-1}\mathrm{e}^{-x}\mathrm{d}x.$$

The variable change $u = x^{n+1}$ transforms integral (3.1) to

$$\frac{1}{n+1}\int_0^1 u^{\frac{\ell(j)+1}{n+1}-1}(1-u)^{\frac{m(j)+1}{n+1}}\mathrm{d}u = \frac{1}{n+1}B\Big(\frac{\ell(j)+1}{n+1}, \frac{m(j)+1}{n+1}+1\Big).$$

Therefore,

$$IP = (n+1)^{-n^2}\prod_{0\leqslant \ell,m\leqslant n-1} B\Big(\frac{\ell+1}{n+1}, \frac{m+1}{n+1}+1\Big). \tag{3.17}$$

To calculate the product in the right-hand side of (3.17), we use the following expression of B-function via Γ-function:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\cdot$$

Therefore, by (3.17),

$$IP = (n+1)^{-n^2}\frac{\Big(\prod_{\ell=0}^{n-1}\Gamma(\frac{\ell+1}{n+1})\Big)^n\Big(\prod_{m=0}^{n-1}\Gamma(\frac{m+1}{n+1}+1)\Big)^n}{\prod_{\ell,m=0}^{n-1}\Gamma(\frac{\ell+m+2}{n+1}+1)}\cdot \tag{3.18}$$

To calculate the products in (3.18), we use the following identities for Γ-function [3]:

$$\Gamma(n) = (n-1)! \quad \text{for any } n\in\mathbb{N},$$

$$\prod_{\ell=0}^{n}\Gamma\Big(z+\frac{\ell}{n+1}\Big) = (2\pi)^{\frac{1}{2}n}(n+1)^{\frac{1}{2}-(n+1)z}\Gamma\big((n+1)z\big) \tag{3.19}$$

(Gauss-Legendre formula). One gets

$$\prod_{\ell=0}^{n-1}\Gamma\Big(\frac{\ell+1}{n+1}\Big)=(2\pi)^{\frac12 n}(n+1)^{-\frac12}, \tag{3.20}$$

$$\prod_{m=0}^{n-1}\Gamma\Big(\frac{m+1}{n+1}+1\Big)=(2\pi)^{\frac12 n}(n+1)^{\frac12-(n+2)}(n+1)! \tag{3.21}$$

by applying (3.19) to $z=1/(n+1)$ and $z=(n+2)/(n+1)$ respectively and subsequent substitutions $\Gamma(1)=1$, $\Gamma(n+2)=(n+1)!$. Let us calculate the double product in ℓ,m in (3.18). For any fixed $m=0,\dots,n-1$

$$\begin{aligned}&\prod_{\ell=0}^{n-1}\Gamma\Big(\frac{\ell+m+2}{n+1}+1\Big)\\&\qquad=\Big(\Gamma\Big(\frac{m+1}{n+1}+1\Big)\Big)^{-1}(2\pi)^{\frac12 n}(n+1)^{\frac12-(m+n+2)}(m+n+1\Big)!\end{aligned}$$

by (3.19) applied to $z=(m+1)/(n+1)+1$. Therefore,

$$\begin{aligned}&\prod_{\ell,m=0}^{n-1}\Gamma\Big(\frac{\ell+m+2}{n+1}+1\Big)\\&\qquad=\Big(\prod_{m=0}^{n-1}\Gamma\Big(\frac{m+1}{n+1}+1\Big)\Big)^{-1}(2\pi)^{\frac12 n^2}(n+1)^{\frac12 n-\sum_{m=0}^{n-1}(m+n+2)}\\&\qquad\qquad\times\prod_{m=0}^{n-1}(m+n+1)!\end{aligned}$$

Substituting formula (3.21) for the first product in the right-hand side of the last formula and summarizing the power of $n+1$ yields

$$\begin{aligned}&\prod_{\ell,m=0}^{n-1}\Gamma\Big(\frac{\ell+m+2}{n+1}+1\Big)\\&\qquad=(2\pi)^{\frac12(n^2-n)}(n+1)^{-\frac32(n^2-1)}\prod_{m=1}^{n-1}(m+n+1)!.\end{aligned} \tag{3.22}$$

Substituting (3.20)–(3.22) to (3.18) yields (3.5).

3.4. The constants σ and $C(h)$. Proof of (3.6) and (3.7)

Proof of (3.6). — By definition, see (3.3), one has

$$\begin{aligned}\sigma &= (-1)^{\frac{1}{2}n(n+1)} \prod_{1\leqslant \ell<k\leqslant n+1} \big((\varepsilon^k-\varepsilon^\ell)(\varepsilon^\ell-\varepsilon^k)\big)\\ &= (-1)^{\frac{1}{2}n(n+1)} \prod_{1\leqslant k\leqslant n+1} \Big(\prod_{\substack{1\leqslant \ell\leqslant n+1\\ \ell\neq k}} (\varepsilon^k-\varepsilon^\ell)\Big)\\ &= (-1)^{\frac{1}{2}n(n+1)} \prod_{1\leqslant k\leqslant n+1} \Big(\prod_{\ell=1}^{n} \varepsilon^k(1-\varepsilon^\ell)\Big).\end{aligned}$$

The second (inner) product in the right-hand side of the last formula is equal to $(n+1)\varepsilon^{nk}$ by (3.13), so,

$$\sigma = (-1)^{\frac{1}{2}n(n+1)}\varepsilon^{\frac{1}{2}n(n+1)(n+2)}(n+1)^{n+1} = (-1)^{\frac{1}{2}n(n-1)}(n+1)^{n+1}.$$

□

Proof of (3.7). — Let us calculate the corresponding values $\Sigma(h)$ and $P(h)$ from formula (1.9) for $C(h)$. By definition, in our case each matrix $A_d(h)$ is diagonal of the size $2(d-n+1)$ with the diagonal elements equal to $n+1$, so, its determinant is equal to $(n+1)^{2(d-n+1)}$. Therefore,

$$P(h) = \prod_{d=n}^{2n-2} \det A_d(h) = (n+1)^{n(n-1)}.$$

Let us calculate $\Sigma(h)$. By (1.7), (3.3) and (3.6),

$$\Sigma(h) = \Big(\prod_{1\leqslant \ell<k\leqslant n+1} \mathrm{e}^{\frac{\pi i}{n+1}}(\varepsilon^k-\varepsilon^\ell)\Big)^2 = (-1)^n\sigma = (-1)^{\frac{1}{2}n(n+1)}(n+1)^{n+1}.$$

The two previous formulas together with (1.9) imply (3.7). The proof of (1.13) is completed. □

4. Acknowledgements

I am grateful to Yu. S. Ilyashenko for statement of the problem, his attention and helpful discussions. I wish to thank him and the referee for valuable remarks on the text. Research supported by part by Russian Foundation for Basic Research (RFBR) grant 02-02-00482.

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Manuscrit reçu le 19 juillet 2005,
accepté le 21 novembre 2005.

Alexey A. GLUTSYUK
Laboratoire J.-V. Poncelet
(UMI 2615 du CNRS et
Université Indépendante de Moscou)
Permanent address:
CNRS UMR 5669
Unité de Mathématiques Pures et Appliquées
École Normale Supérieure de Lyon
46 allée d'Italie
69364 Lyon Cedex 07 (France)