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RELATIVE AMPLENESS IN RIGID GEOMETRY

by Brian CONRAD (*)

ABSTRACT. — We develop a rigid-analytic theory of relative ampleness for line bundles and record some applications to faithfully flat descent for morphisms and proper geometric objects. The basic definition is fibral, but pointwise arguments from the algebraic and complex-analytic cases do not apply, so we use cohomological properties of formal schemes over completions of local rings on rigid spaces. An analytic notion of quasi-coherence is introduced so that we can recover a proper object from sections of an ample bundle via suitable Proj construction. The locus of relative ampleness in the base is studied, as is the behavior of relative ampleness with respect to analytification and arbitrary extension of the base field. In particular, we obtain a quick new proof of the relative GAGA theorem over affinoids.

RÉSUMÉ. — Nous développons une théorie analytique rigide de l'amplitude relative pour les fibrés en droites et notons quelques applications à la descente fidèlement plate de morphismes et d'objets géométriques propres. Nous utilisons la définition fibrée, mais nous ne pouvons pas appliquer les arguments ponctuels des cas algébrique et analytique complexe. Nous devons donc utiliser des propriétés cohomologiques des schémas formels sur les anneaux locaux complétés des espaces rigides. Nous introduisons une notion analytique de quasi-cohérence pour retrouver un objet à partir des sections d'un fibré ample par une construction Proj convenable. Nous étudions le lien d'amplitude relative et le comportement de l'amplitude relative par rapport à l'analytification et à l'extension du corps de base arbitraire. En particulier, nous obtenons une démonstration nouvelle rapide du théorème GAGA relatif sur les affinoïdes.

1. Introduction

1.1. Motivation

The aim of this paper is to develop a rigid-analytic theory of relative ampleness for line bundles, and to record some applications to rigid-analytic

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faithfully flat descent for morphisms and for proper geometric objects equipped with a relatively ample line bundle. (For coherent sheaves on rigid spaces, the theory of faithfully flat descent is established in [5] via Raynaud's theory of formal models [7].) Such a theory could have been worked out decades ago, as the tools we use have been known for a long time. Our primary motivation for working out the theory now is for applications in the context of p-adic modular forms on more general groups; as a first step in this direction, in [12] and [13] the theory in the present paper is used to develop a relative theory of arbitrary-level canonical subgroups in rather general rigid-analytic families of generalized elliptic curves and abelian varieties over p-adic fields. (The key point in this application is to have results not depending on the specification of discrete parameters such as the degree of a polarization.)

For a proper rigid space X over a non-archimedean field k, an invertible sheaf $\mathcal{L}$ on X is *ample* if some high tensor power $\mathcal{L}^{\otimes N}$ is the pullback of $\mathcal{O}_{\mathbf{P}^n_k}(1)$ under some closed immersion $j : X \hookrightarrow \mathbf{P}^n_k$ into a rigid-analytic projective space. The cohomological criterion for ampleness works, though the proof of the criterion in the algebraic and complex-analytic cases uses pointwise arguments and so some modifications are required in the rigid-analytic case. In [16, §4] a few aspects of a theory of ampleness are developed for quasi-compact separated rigid spaces X, taking the cohomological criterion as the starting point, but we develop what we need for k-proper X *ab ovo* because our intention is to develop a relative theory (for applications in [12] and [13]) and so we prefer to set up the absolute case in a way that best prepares us for relativization.

If $f : X \to S$ is a proper morphism between rigid spaces over k, and $\mathcal{L}$ is a line bundle on X, analogy with the case of proper schemes [15, IV$_3$, 9.6.4] motivates the following definition: $\mathcal{L}$ is *S-ample* (or *relatively ample* over S) if $\mathcal{L}_s$ is ample on the rigid space X_s over $k(s)$ for all $s \in S$. We adopt this as the initial definition because it is a property that can be checked in abstract situations. Does this definition satisfy all of the properties one desires? For example, is relative ampleness preserved by arbitrary extension on k (for quasi-separated S)? This would hold if the relationship between ampleness and projective embeddings relativizes, but even this relationship is not obvious:

1) Does there exist an admissible covering $\{S_\alpha\}$ of S such that each $X_\alpha = f^{-1}(S_\alpha)$ admits a closed S_α-immersion into some projective space $\mathbf{P}^{n_\alpha}_{S_\alpha}$ such that some positive tensor power $\mathcal{L}^{\otimes N_\alpha}|_{X_\alpha}$ is isomorphic to the pullback of $\mathcal{O}(1)$?

2) Is the graded $\mathcal{O}_S$-algebra $\mathcal{A} = \bigoplus_{n \geqslant 0} f_* \mathcal{L}^{\otimes n}$ locally finitely generated?

3) Can we recover X from $\mathcal{A}$ (as we can in the case of schemes, by using relative Proj)?

These properties of $\mathcal{A}$ are a trivial consequence of an affirmative answer to the question on relative projective embeddings (due to Köpf's relative GAGA theorems over affinoids [33, §5]), but in fact we have to argue in reverse: relative projective embeddings (given an S-ample $\mathcal{L}$) will be constructed by *using* the finiteness properties of $\mathcal{A}$ and the reconstruction of X from $\mathcal{A}$ (via an analytic Proj operation). Hence, we must study $\mathcal{A}$ in the absence of relative projective embeddings.

In the complex-analytic case, Grauert and Remmert [20] provided affirmative answers to the preceding questions in important special cases. Their method of proof cannot be used in the rigid-analytic case for two reasons: the pointwise nature of our fibral definition of S-ampleness provides no obvious link with admissibility properties for covers of S, and the arguments in [20] require the existence of a relative projective embedding (unrelated to $\mathcal{L}$). We therefore need a different approach. Grothendieck's existence theorem for proper formal schemes, applied over the adic noetherian rings $\widehat{\mathcal{O}}_{S,s}$ for $s \in S$, turns out to be the right tool for our needs. Our rigid-analytic arguments with formal schemes provide affirmative answers to all of the above questions concerning ampleness in the rigid-analytic case and they work *verbatim* in the complex-analytic case, giving new and shorter proofs of generalizations of the theorems of Grauert and Remmert. (The theory of formal schemes did not exist at the time that [20] was written.) See Theorem 3.2.4, Theorem 3.2.7, and Corollary 3.2.8 for these results.

There are more questions that naturally arise. If $f : X \to S$ is proper and $\mathcal{L}$ is an invertible sheaf on X, what can be said about properties of the locus $U_{\mathcal{L}}$ of $s \in S$ such that $\mathcal{L}_s$ is ample on X_s? For example, is $U_{\mathcal{L}}$ an admissible open (perhaps even Zariski-open)? If so, for quasi-separated S does the formation of $U_{\mathcal{L}}$ commute with any extension of the base field? As with the above questions, we give affirmative answers via the method of formal schemes and algebraization over the adic rings $\widehat{\mathcal{O}}_{S,s}$, except we prove that the ampleness locus $U_{\mathcal{L}}$ in S is merely a Zariski-open locus in a (canonical) Zariski-open subset of S rather than that it is a Zariski-open in S (Zariski-openness is not a transitive condition in analytic settings); see Theorem 3.2.9. This structure for $U_{\mathcal{L}}$ appears to be a new result even in the complex-analytic case.

1.2. Overview

Let us now briefly summarize the contents of this paper. In §2 we discuss quasi-coherent sheaves on rigid spaces and we construct rigid-analytic versions of the relative Spec and Proj operations on locally finitely generated quasi-coherent sheaves of algebras. (This is an extension of the techniques in [30, II].) These operations are used in §3 to develop the theory of relatively ample line bundles on rigid spaces that are proper over a base. As a simple application, in Example 3.2.6 we obtain quick proofs of Köpf's relative GAGA theorem over affinoids via the theory of relative ampleness and the GAGA theorems over a field. Applications to representability and faithfully flat descent are provided in §4, with close attention given to behavior with respect to analytification and change in the ground field. Further applications to a theory of analytification for locally separated algebraic spaces over non-archimedean fields will be given in [14].

1.3. Notation and terminology

A *non-archimedean field* is a field k equipped with a non-trivial non-archimedean absolute value with respect to which k is complete. An *analytic extension field* k'/k is an extension k' of k endowed with a structure of non-archimedean field such that its absolute value extends the one on k. When we work with rigid spaces, they will tacitly be assumed to be rigid spaces over a fixed non-archimedean ground field k unless we say otherwise.

Rigid-analytic projective n-space over k is denoted $\mathbf{P}^n_k$; the scheme-theoretic counterpart is denoted $\mathbf{P}^n_{\mathrm{Spec}(k)}$ when it arises. An analogous convention applies to affine n-space. An *algebraic k-scheme* is a scheme locally of finite type over k. (We do not require finite type, since when working with algebraic spaces over k it is more natural to use k-schemes that are locally of finite type.) We refer the reader to [11, §5] for a discussion of properties of the analytification functor $X \rightsquigarrow X^{\mathrm{an}}$ (resp. $\mathcal{F} \rightsquigarrow \mathcal{F}^{\mathrm{an}}$) from algebraic k-schemes (resp. $\mathcal{O}_X$-modules) to rigid spaces over k (resp. $\mathcal{O}_{X^{\mathrm{an}}}$-modules).

For a rigid space S, we use the phrase "locally on S" to mean "over the constituents in an admissible open covering of S". We refer the reader to [6, 9.3.6] for a discussion of change of the analytic base field, and we note that this construction requires the rigid space to be quasi-separated (*e.g.*, separated). Although non-separated algebraic k-schemes are quasi-separated in the sense of schemes, their analytifications are never quasi-separated in the sense of rigid geometry. Thus, in §A.2 we introduce the

property of *pseudo-separatedness* that is satisfied by analytifications of arbitrary algebraic k-schemes and we prove that this property is sufficient for the construction of reasonable change of base field functors. Thus, whenever discussing change of the base field for rigid spaces we assume the spaces are either quasi-separated or pseudo-separated. (In the context of Berkovich spaces, no such restrictions are required [3, §1.4].) A rigid space is pseudo-separated and quasi-separated if and only if it is separated.

In order to have proofs that translate *verbatim* into the complex-analytic case (up to replacing "admissible affinoid open" with "compact Stein set"), we now review some convenient terminology related to complex-analytic spaces. Let X be a complex-analytic space and let $K \subseteq X$ be a compact subset admitting a Hausdorff neighborhood. By [19, II, 3.3.1], if $\mathcal{G}$ is a sheaf of sets on a Hausdorff neighborhood of K in X then the natural map $\varinjlim \mathcal{G}(W) \to (\mathcal{G}|_K)(K)$ is bijective, where W runs over a base of opens containing K. Consequently, if $\mathcal{F}$ and $\mathcal{F}'$ are two coherent sheaves on an open neighborhood U of K in X then coherence of $\mathcal{H}om(\mathcal{F}, \mathcal{F}')$ over U implies that the natural map $\varinjlim \mathrm{Hom}_{\mathcal{O}_W}(\mathcal{F}|_W, \mathcal{F}'|_W) \to \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{F}|_K, \mathcal{F}'|_K)$ is bijective, where W ranges over open subsets of U containing K and $\mathcal{O}_K$ denotes $\mathcal{O}_X|_K$. An $\mathcal{O}_K$-module $\mathcal{G}$ is *coherent* if $\mathcal{G} \simeq \mathcal{F}|_K$ for a coherent sheaf $\mathcal{F}$ on an open neighborhood of K; as $\mathcal{F}$ is functorial and unique near K, a simple induction shows that if $\mathcal{G}$ is an $\mathcal{O}_K$-module and $\mathcal{G}|_{K_i}$ is coherent for compact subsets $K_1, \ldots, K_n \subseteq K$ that cover K then $\mathcal{G}$ is coherent. In the category of $\mathcal{O}_K$-modules, the full subcategory of coherent sheaves is stable under the formation of kernels, cokernels, images, and tensor products.

A *compact Stein set* in X is a compact subset K such that K admits a Hausdorff neighborhood and $\mathrm{H}^i(K, \mathcal{G}) = 0$ for all $i > 0$ and all coherent $\mathcal{G}$ on K. For such K, $\mathcal{G}(K)$ is a finite $\mathcal{O}_K(K)$-module for every coherent sheaf $\mathcal{G}$ on K. Any $x \in X$ has a base of neighborhoods that are compact Stein sets [21, Ch. III, 3.2], so for any compact subset $K \subseteq X$ admitting a Hausdorff neighborhood, the category of coherent $\mathcal{O}_K$-modules is stable under extensions in the category of $\mathcal{O}_K$-modules.

Contents

2. Spec and Proj

A basic tool in the rigid-analytic theory of ampleness is a rigid-analytic version of the Proj construction in the theory of schemes [15, II]. A minor nuisance in the global theory of Proj for schemes is that it does not have a convenient universal property, so the construction must be globally canonical. The construction of Proj for rigid spaces is similar to the construction used in the case of schemes, resting on representability of Spec functors associated to certain sheaves of algebras. Thus, as a prelude to Proj, in §2.1–§2.2 we develop quasi-coherence and Spec in the rigid-analytic setting. Beware that (as we explain in Remark 2.1.5) the rigid-analytic notion of quasi-coherence has some deficiencies, fortunately not relevant to our applications. Since most of the arguments prior to §2.3 are of a predictable nature (once the right definitions are set forth), the reader is advised to skip ahead to §2.3 and then read backwards into §2.1–§2.2 as the need arises.

The only serious issue in §2.1–§2.3 is to get analogues of all of the basic results as in the scheme-theoretic theories of relative Spec and Proj, even though rigid spaces do not arise from rings in quite the same way as do schemes in algebraic geometry. The same basic principle guides many of the proofs: we reduce relative problems to the fibral situation, in which case it is often possible to use analytification arguments to reduce ourselves to algebraic problems that have been solved for schemes. It is very important

in some of our later proofs with ample bundles on proper rigid spaces that Proj is linked to Spec much as in the case of schemes; using Köpf's work [33] as an alternative foundation to the theory of relative rigid geometry would lead to rather unpleasant proofs.

When contemplating functors on the rigid-analytic category and aiming to prove "algebraic representability" of such functors, one may be reminded of the work of Hakim [29] on "relative schemes". In [29, VIII] there is a general analytic-algebraic comparison discussion, and a comparison theorem concerning analytic Hilbert (and related) functors is proved. Our preliminary constructions with relative Spec and Proj are similar in spirit to those in [29], but otherwise Hakim's work has little overlap with what we do and seems to not be helpful for the problems that we need to solve.

2.1. Quasi-coherent sheaves

Let S be a rigid space. An $\mathcal{O}_S$-algebra $\mathcal{A}$ is *locally finitely generated* if there exists an admissible covering $\{U_i\}$ of S such that $\mathcal{A}|_{U_i}$ is is a quotient of a sheaf of polynomial algebras over $\mathcal{O}_{U_i}$ in finitely many variables. If these presentations can be chosen to have kernel ideals that are locally generated by finitely many sections, then $\mathcal{A}$ is *locally finitely presented*. (We shall see in Theorem 2.1.11 that in such cases any local presentation of $\mathcal{A}$ as a quotient of a polynomial algebra in finitely many variables has ideal of relations that is locally generated by finitely many sections.) The theory of locally finitely presented sheaves of algebras on analytic spaces is studied in [30, II] for complex-analytic spaces and for non-archimedean analytic spaces in a pre-Tate sense (*e.g.*, without non-rational points and without the Tate topology).

Locally finitely presented sheaves of algebras are the ones of most interest to us, but the finite presentation condition is sometimes inconvenient to check in abstract situations such as for sheaves $\mathcal{A} = \bigoplus_{n\geqslant 0} f_*\mathcal{L}^{\otimes n}$ as in §1.1. Hence, we need an alternative viewpoint on the theory of locally finitely presented sheaves of algebras so that the theory is applicable on rigid-analytic spaces and moreover is well-suited to our later work with analytic Proj. The starting point is:

DEFINITION 2.1.1. — *An $\mathcal{O}_S$-module $\mathcal{F}$ is* quasi-coherent *if, locally on S, $\mathcal{F}$ is a direct limit of coherent sheaves. That is, S has an admissible covering $\{S_i\}$ such that $\mathcal{F}|_{S_i}$ is a direct limit of a directed system $\{\mathcal{F}_{i,\alpha}\}$ of coherent $\mathcal{O}_{S_i}$-modules.*

In Corollary 2.1.12 we will record the equivalence of quasi-coherence and local finite presentation for locally finitely generated sheaves of algebras. We emphasize (as will be convenient) that in Definition 2.1.1 we do not require the auxiliary coherent sheaves $\mathcal{F}_{i,\alpha}$ to be subsheaves of $\mathcal{F}|_{S_i}$. Lemmas 2.1.8 and 2.1.9 will ensure that imposing this stronger property on the $\mathcal{F}_{i,\alpha}$'s does not affect the notion of quasi-coherence.

Example 2.1.2. — Let S_0 be an algebraic k-scheme, and let $S_0^{\mathrm{an}} \to S_0$ be the analytification morphism. For any quasi-coherent sheaf $\mathcal{F}_0$ on S_0, the $\mathcal{O}_{S_0^{\mathrm{an}}}$-module pullback $\mathcal{F}_0^{\mathrm{an}}$ is quasi-coherent on S_0^{an}.

Example 2.1.3. — If $\{\mathcal{A}^i\}_{i\in I}$ is an indexed set of coherent sheaves on S then $\bigoplus_{i\in I}\mathcal{A}^i$ is quasi-coherent.

Example 2.1.4. — Let M be a module over a k-affinoid algebra A, and let $\{M_i\}$ be its directed system of A-finite submodules. The associated coherent sheaves $\widetilde{M_i}$ on $X = \mathrm{Sp}(A)$ form a directed system and its direct limit depends functorially on M so we may denote it $\widetilde{M}$. For any affinoid open $U = \mathrm{Sp}(A')$ in X, the natural map $A' \otimes_A M \to \widetilde{M}(U)$ is an isomorphism. By Lemma 2.1.8, any quasi-coherent sheaf on an arbitrary rigid space X *locally* arises via this construction. As we explain in Remark 2.1.5, such locality is a non-trivial condition when X is affinoid.

For any map of rigid spaces $f : S' \to S$ over k and any $\mathcal{O}_S$-module $\mathcal{F}$ with pullback $\mathcal{F}' = f^*\mathcal{F}$, if $\mathcal{F}$ is quasi-coherent (resp. a locally finitely generated sheaf of algebras, resp. a locally finitely presented sheaf of algebras) on S then so is $\mathcal{F}'$ on S'.

Remark 2.1.5. — We will soon prove that the concept of quasi-coherence satisfies most of the reasonable properties one might expect, and in particular it will be adequate for our purposes. However, there are some defects: it is not true in general that a quasi-coherent sheaf on an affinoid space has vanishing higher cohomology (see Example 2.1.6, due to Gabber), and such a counterexample on an affinoid cannot be expressed as a direct limit of coherent sheaves (by Lemma 2.1.7 below). In particular, the construction in Example 2.1.4 does *not* generally give rise to all quasi-coherent sheaves on an affinoid space (thereby settling the "open question" in [17, Exercise 4.6.7] in the negative). Moreover, quasi-coherence is generally not preserved under direct limits, nor even under countably infinite direct sums (see Example 2.1.10). One may ask if there is a better definition of quasi-coherence than we have given, but we do not know any potential applications to motivate the search for a better definition.

Example 2.1.6. — We now give Gabber's elegant construction of a quasi-coherent sheaf $\mathcal{F}$ on an affinoid X such that $\mathrm{H}^1(X,\mathcal{F}) \neq 0$. Let X be the closed unit disc over k, and let $x', x'' \in X(k)$ be two distinct rational points. Let $U' = X - \{x'\}$ and let $U'' = X - \{x''\}$, and define $U = U' \cap U''$. Let

$$\mathcal{F}' = \bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{U'} e'_n, \quad \mathcal{F}'' = \bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{U''} e''_n$$

be two free sheaves with countably infinite rank on U' and U'' respectively. We shall glue these to define an $\mathcal{O}_X$-module $\mathcal{F}$ that has no nonzero global sections. Such an $\mathcal{F}$ is quasi-coherent because $\mathcal{F}|_{U'}$ and $\mathcal{F}|_{U''}$ are direct limits of coherent sheaves, and if t is the standard coordinate on X then the cohomology sequence associated to

$$0 \to \mathcal{F} \xrightarrow{t} \mathcal{F} \longrightarrow \mathcal{F}/t\mathcal{F} \to 0$$

provides an injection $\mathrm{H}^0(X,\mathcal{F}/t\mathcal{F}) \hookrightarrow \mathrm{H}^1(X,\mathcal{F})$. Hence, $\mathrm{H}^1(X,\mathcal{F}) \neq 0$ because $\mathcal{F}/t\mathcal{F}$ is a nonzero skyscraper sheaf supported at the origin.

To construct the gluing $\mathcal{F}$ with no nonzero global sections, we choose $h \in \mathcal{O}_X(U)$ with essential singularities at x' and x'' (many such h exist). We define $\mathcal{F}$ by identifying $\mathcal{F}'|_U$ and $\mathcal{F}''|_U$ with the free sheaf $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_U e_n$ via the conditions

$$e_{2m} = e'_{2m}|_U = e''_{2m}|_U + h e''_{2m+1}|_U,$$

$$e_{2m+1} = e''_{2m+1}|_U = e'_{2m+1}|_U + h e'_{2m+2}|_U$$

for $m \in \mathbf{Z}$. Let $f \in \mathcal{F}(X)$ be a global section, so on any open affinoid V in U we may write $f|_V = \sum f_{n,V} e_n|_V$ for sections $f_{n,V} \in \mathcal{O}_X(V)$ that vanish for all but finitely many n. Since U is connected and normal (even smooth), the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective because $\mathcal{F}|_U$ is a direct sum of copies of $\mathcal{O}_U$. Thus, $f = \sum f_n e_n$ for $f_n \in \mathcal{O}_X(U)$ that vanish for all but finitely many n (that is, f is a *finite* $\mathcal{O}_X(U)$-linear combination of e_n's even though U is not quasi-compact).

Using the definition of $\mathcal{F}$, the image of the injective restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ consists of those finite sums $f = \sum f_n e_n$ with $f_n \in \mathcal{O}_X(U)$ such that

- f_n is analytic at x' for even n and at x'' for odd n,
- $f_n + h f_{n-1}$ is analytic at x' for odd n and at x'' for even n.

Hence, if $f = \sum f_n e_n \in \mathcal{F}(U)$ comes from a nonzero element of $\mathcal{F}(X)$ then by choosing the maximal n_0 such that $f_{n_0} \neq 0$ we see that f_{n_0} and $h f_{n_0}$ are both analytic at a common point amongst x' and x'' (depending on the parity of n_0). The ratio $h = h f_{n_0}/f_{n_0}$ is therefore meromorphic at x' or x'',

a contradiction, so $\mathcal{F}(X) = 0$. (This construction makes sense in complex-analytic geometry as well, where it gives a "quasi-coherent" sheaf $\mathcal{F}$ on the open unit disc Δ in $\mathbf{C}$ such that $\mathcal{F}(\Delta) = 0$ and $\mathrm{H}^1(\Delta', \mathcal{F}) \neq 0$ for all open subdiscs Δ' centered at the origin with radius near 1.)

LEMMA 2.1.7. — *If $U \subseteq S$ is an admissible affinoid open and $\mathcal{F}$ is an $\mathcal{O}_S$-module such that $\mathcal{F}|_U$ is a direct limit of coherent $\mathcal{O}_U$-modules, then $\mathrm{H}^i(U, \mathcal{F}) = 0$ for all $i > 0$.*

In the complex-analytic case, this lemma holds for compact Stein sets U because direct limits commute with cohomology on any compact Hausdorff space [19, II, 4.12.1].

Proof. — We may rename U as S. Since higher derived-functor cohomology vanishes for coherent sheaves on affinoids, it suffices to prove that if S is a quasi-compact and quasi-separated rigid space then $\mathrm{H}^i(S, \cdot)$ commutes with the formation of direct limits for all $i \geqslant 0$. Consider the Čech to derived functor spectral sequence

$$\mathrm{E}_2^{p,q} = \check{\mathrm{H}}^p\big(S, \underline{\mathrm{H}}^q(\mathcal{F})\big) \Longrightarrow \mathrm{H}^{p+q}(S, \mathcal{F})$$

(where $\underline{\mathrm{H}}^q(\mathcal{F})$ is the presheaf $U \mapsto \mathrm{H}^q(U, \mathcal{F}|_U)$). Since $\mathrm{E}_2^{0,n} = 0$ for all $n > 0$, we can use induction and considerations with the cofinal system of finite covers by quasi-compact admissible opens to reduce to proving $\check{\mathrm{H}}^n(S, .)$ commutes with the formation of direct limits for all $n \geqslant 0$. This follows from the exactness of direct limits and the compatibility of direct limits and global sections on a quasi-compact and quasi-separated rigid space. □

LEMMA 2.1.8. — *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module on a rigid space S.*

1) The subsheaf of $\mathcal{F}$ generated by any two coherent subsheaves is again coherent, and there exists an admissible cover $\{S_j\}$ of S such that each $\mathcal{F}|_{S_j}$ is the direct limit of its directed system of coherent subsheaves.

2) If $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra and $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{A}$-modules that are quasi-coherent as $\mathcal{O}_S$-modules, then $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is a quasi-coherent $\mathcal{O}_S$-module.

3) If $\mathcal{F} = \varinjlim \mathcal{F}_i$ for coherent $\mathcal{F}_i$, then for any inclusion $V \subseteq U$ of open affinoids in S the natural map $\mathcal{O}_S(V) \otimes_{\mathcal{O}_S(U)} \mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism.

In the complex-analytic case, part 3) holds for compact Stein sets U and V, and parts 1) and 2) hold without change.

Proof. — Consider 1). Working locally, we may assume $\mathcal{F} = \varinjlim \mathcal{F}_i$ with each $\mathcal{F}_i$ coherent. If $\overline{\mathcal{F}}_i$ denotes the image of $\mathcal{F}_i$ in $\mathcal{F}$ then certainly $\varinjlim \overline{\mathcal{F}}_i = \mathcal{F}$. We claim that each $\overline{\mathcal{F}}_i$ is coherent. Fix i_0, and consider

the kernels

$$\mathcal{K}_i = \ker(\mathcal{F}_{i_0} \to \mathcal{F}_i)$$

for $i \geqslant i_0$. These form a directed system of coherent subsheaves of the coherent sheaf $\mathcal{F}_{i_0}$, so $\{\mathcal{K}_i\}$ stabilizes locally on S (see [22, Ch. 5, §6] for the analogous noetherian property in the complex-analytic case). If $\mathcal{K} \subseteq \mathcal{F}_{i_0}$ denotes the coherent direct limit of the $\mathcal{K}_i$'s, then we have $\overline{\mathcal{F}}_{i_0} \simeq \mathcal{F}_{i_0}/\mathcal{K}$ and hence $\overline{\mathcal{F}}_{i_0}$ is indeed coherent. It follows by similar methods that any map from a coherent sheaf to $\mathcal{F}$ has coherent image, so the subsheaf generated by any two coherent subsheaves of $\mathcal{F}$ is again coherent.

Now we turn to 2). Working locally and using 1), we may suppose

$$\mathcal{F} = \varinjlim \mathcal{F}_i, \quad \mathcal{G} = \varinjlim \mathcal{G}_j, \quad \mathcal{A} = \varinjlim \mathcal{A}_h$$

as $\mathcal{O}_S$-modules, where $\{\mathcal{F}_i\}$, $\{\mathcal{G}_j\}$, and $\{\mathcal{A}_h\}$ are directed systems of coherent subsheaves of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{A}$ respectively (note that $\mathcal{A}_h$ is not necessarily a subsheaf of $\mathcal{O}_S$-subalgebras in $\mathcal{A}$). The subsheaf

$$\mathcal{A}_h \cdot \mathcal{F}_i = \operatorname{image}(\mathcal{A}_h \otimes_{\mathcal{O}_S} \mathcal{F}_i \to \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{F} \to \mathcal{F})$$

inside of $\mathcal{F}$ is therefore coherent for all i and h, and likewise each $\mathcal{A}_h \cdot \mathcal{G}_j$ inside of $\mathcal{G}$ is coherent. Thus, the fake tensor product

$$(\mathcal{A}_h \cdot \mathcal{F}_i) \otimes_{\mathcal{A}_h} (\mathcal{A}_h \cdot \mathcal{G}_j) \stackrel{\text{def}}{=\!=} ((\mathcal{A}_h \cdot \mathcal{F}_i) \otimes_{\mathcal{O}_S} (\mathcal{A}_h \cdot \mathcal{G}_j))/(a_h f_i \otimes g_j - f_i \otimes a_h g_j)$$

is a coherent $\mathcal{O}_S$-module. These fake tensor products form a directed system of coherent sheaves whose limit is isomorphic to $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ as an $\mathcal{O}_S$-module.

Finally, for 3) we note that the natural maps $\varinjlim \mathcal{F}_i(U) \to \mathcal{F}(U)$ and $\varinjlim \mathcal{F}_i(V) \to \mathcal{F}(V)$ are isomorphisms because U and V are each quasi-compact and quasi-separated, and so we are reduced to checking that the natural map $\mathcal{O}_S(V) \otimes_{\mathcal{O}_S(U)} \mathcal{F}(U) \to \mathcal{F}(V)$ is an isomorphism for coherent $\mathcal{F}$. Since $\mathcal{F}|_U$ admits a presentation as the cokernel of a map $\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n}$, we are reduced to the trivial case $\mathcal{F} = \mathcal{O}_X$. □

LEMMA 2.1.9. — *As a full subcategory of the category of $\mathcal{O}_S$-modules on a rigid space S, the category of quasi-coherent sheaves is stable under the formation of kernels, cokernels, extensions, images, and tensor products. In particular, a direct summand of a quasi-coherent sheaf is quasi-coherent.*

Moreover, a quasi-coherent subsheaf of a coherent sheaf is coherent, and hence a quasi-coherent sheaf is coherent if and only if it is locally finitely generated as an $\mathcal{O}_S$-module.

As with the proof of Lemma 2.1.8, the proof of Lemma 2.1.9 works in the complex-analytic case if we replace affinoid opens with compact Stein sets in the proof.

Proof. — A direct limit of coherent subsheaves of a coherent sheaf is locally stationary, so all of the assertions follow immediately from Lemma 2.1.8 except for the case of extensions (the case of direct summands follows from the case of images). Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of $\mathcal{O}_S$-modules such that $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent. To prove that $\mathcal{F}$ is quasi-coherent, we may work locally on S. Thus, we may assume that S is affinoid and that $\mathcal{F}'$ and $\mathcal{F}''$ are each direct limits of their directed systems of coherent subsheaves. We write $\mathcal{F}'' = \varinjlim \mathcal{F}''_\alpha$ with coherent subsheaves $\mathcal{F}''_\alpha$, and pulling back by $\mathcal{F}''_\alpha \to \mathcal{F}''$ gives extensions

$$0 \to \mathcal{F}' \longrightarrow \mathcal{F}_\alpha \longrightarrow \mathcal{F}''_\alpha \to 0$$

with $\mathcal{F}$ equal to the direct limit of its subsheaves $\mathcal{F}_\alpha$. Hence, it suffices to prove that each $\mathcal{F}_\alpha$ is a direct limit of coherent sheaves on S; the point is to avoid further shrinking on S (that may depend on α).

More generally, consider a short exact sequence of $\mathcal{O}_S$-modules

$$0 \to \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}_0 \to 0 \tag{2.1.1}$$

with $\mathcal{G}_0$ coherent and $\mathcal{F}'$ equal to the direct limit of coherent subsheaves. Assume that S is affinoid. Under these assumptions, we wish to prove that $\mathcal{F}$ is a direct limit of coherent sheaves on S.

Since S is affinoid, there is a short exact sequence

$$0 \to \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G}_0 \to 0 \tag{2.1.2}$$

where $\mathcal{E}$ is a locally free coherent sheaf. Applying $\mathrm{Hom}_{\mathcal{O}_S}(.\,, \mathcal{F}')$ gives an exact sequence

$$\mathrm{Hom}(\mathcal{K}, \mathcal{F}') \longrightarrow \mathrm{Ext}^1(\mathcal{G}_0, \mathcal{F}') \longrightarrow \mathrm{Ext}^1(\mathcal{E}, \mathcal{F}'),$$

with $\mathrm{Ext}^1(\mathcal{E}, \mathcal{F}') = \mathrm{H}^1(S, \mathcal{E}^\vee \otimes \mathcal{F}') = 0$ by Lemma 2.1.7 and the hypothesis on $\mathcal{F}'$. Hence, the extension structure (2.1.1) is the pushout of (2.1.2) by an $\mathcal{O}_S$-linear map $\mathcal{K} \to \mathcal{F}'$. Since we have assumed $\mathcal{F}'$ is a direct limit of coherent subsheaves, say $\mathcal{F}' = \varinjlim \mathcal{F}'_i$, the map $\mathcal{K} \to \mathcal{F}'$ uniquely factors through some map $\mathcal{K} \to \mathcal{F}'_{i_0}$ because S is quasi-compact. Thus, the pushouts of (2.1.2) along $\mathcal{K} \to \mathcal{F}'_i$ for $i \geqslant i_0$ provide coherent subsheaves of $\mathcal{F}$ whose direct limit is $\mathcal{F}$. □

Example 2.1.10. — If $\{\mathcal{F}_\alpha\}$ is a directed system of quasi-coherent sheaves, it is generally not true that $\varinjlim \mathcal{F}_\alpha$ is quasi-coherent. The essential problem is that there may not exist an admissible cover $\{S_i\}$ of S such that for each i the sheaf $\mathcal{F}_\alpha|_{S_i}$ on S_i is a direct limit of coherent sheaves for *all* α. To construct a counterexample, let S be the closed unit disc and let $\{x_n\}_{n \geqslant 1}$ be a sequence in $S(k) - \{0\}$ with $|x_n|$ strictly decreasing

to 0. Let $\mathcal{F}_n$ be the output of Gabber's construction in Example 2.1.6 for the pair of points $\{0, x_n\}$, so $\mathcal{F}_n$ cannot be expressed as a direct limit of coherent sheaves on any closed disc centered at 0 with radius $\geqslant |x_n|$. Let $\mathcal{F} = \bigoplus_{n\geqslant 1} \mathcal{F}_n$. If $\mathcal{F}$ is quasi-coherent then for some admissible open U containing the origin we can identify $\mathcal{F}|_U$ with a direct limit of coherent subsheaves $\mathcal{G}_\alpha$ on U. Thus, each subsheaf $\mathcal{F}_{n,\alpha} = \mathcal{F}_n|_U \cap \mathcal{G}_\alpha$ makes sense and is coherent by Lemma 2.1.9 ($\mathcal{F}_n|_U$ is quasi-coherent and $\mathcal{G}_\alpha$ is coherent). Since $\mathcal{F}_n|_U = \varinjlim \mathcal{F}_{n,\alpha}$, we conclude that each $\mathcal{F}_n|_U$ is a direct limit of coherent sheaves on U, and this is a contradiction for n so large that the disc centered at 0 with radius $|x_n|$ is contained in U.

THEOREM 2.1.11. — *Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras and suppose that $\mathcal{A}$ is an $\mathcal{O}_S$-algebra quotient of $\mathcal{O}_S[T_1, \ldots, T_n]$ on S, with kernel $\mathcal{I}$. The sheaf $\mathcal{A}$ is the direct limit of its coherent subsheaves, and for any open affinoid $U \subseteq S$ the sequence*

$$0 \to \mathcal{I}(U) \longrightarrow \mathcal{O}_S(U)[T_1, \ldots, T_n] \longrightarrow \mathcal{A}(U) \to 0$$

is short exact with the finitely generated ideal $\mathcal{I}(U)$ generating $\mathcal{I}|_U$ as an $\mathcal{O}_U[T_1, \ldots, T_n]$-module.

Proof. — Since $\mathcal{A}$ is quasi-coherent, the image $\mathcal{A}_d \subseteq \mathcal{A}$ of the coherent sheaf of polynomials of total degree $\leqslant d$ is coherent (because it is a quasi-coherent quotient of a coherent sheaf). Similarly, $\mathcal{I} = \varinjlim \mathcal{I}_d$ for the coherent intersection $\mathcal{I}_d$ of $\mathcal{I}$ with the sheaf of polynomials of total degree $\leqslant d$, so Lemma 2.1.7 implies $\mathrm{H}^1(U, \mathcal{I}) = 0$. This yields the asserted short exact sequence of U-sections. To show that $\mathcal{I}(U)$ generates $\mathcal{I}|_U$ as an $\mathcal{O}_U[T_1, \ldots, T_n]$-module it is equivalent to prove that $\mathcal{I}(U)$ generates $\mathcal{I}|_U$ as an $\mathcal{O}_U$-module, and this follows from Lemma 2.1.8.(3) because $\mathcal{I}$ is a direct limit of coherent sheaves. □

COROLLARY 2.1.12. — *Let $\mathcal{B}$ be a quasi-coherent and locally finitely generated sheaf of $\mathcal{O}_S$-algebras.*

- *If $\mathcal{N} \subseteq \mathcal{M}$ is an inclusion of $\mathcal{B}$-modules that are quasi-coherent over $\mathcal{O}_S$ with $\mathcal{M}$ a finite $\mathcal{B}$-module then $\mathcal{N}$ is a finite $\mathcal{B}$-module.*
- *Any finite $\mathcal{B}$-module that is quasi-coherent over $\mathcal{O}_S$ is finitely presented over $\mathcal{B}$.*
- *A locally finitely generated $\mathcal{O}_S$-algebra is locally finitely presented if and only if it is quasi-coherent.*

Proof. — By Lemma 2.1.9, it suffices to prove that quasi-coherent ideal sheaves in $\mathcal{B}$ are $\mathcal{B}$-finite. This follows from Theorem 2.1.11 by locally expressing the quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{B}$ as a quotient of a polynomial algebra in finitely many variables. □

Remark 2.1.13. — By [18, I.10], any point in a complex-analytic space S has a base of neighborhoods K that are compact Stein sets for which $\mathcal{O}_S(K)$ is noetherian. (This may be also deduced from the necessary and sufficient criterion in [41, Thm. 1] because any semi-analytic set in a real-analytic manifold is locally connected [4, Cor. 2.7].) Hence, Theorem 2.1.11 and Corollary 2.1.12 are valid in the complex-analytic case (with essentially the same proof) provided that we take U in the theorem to be one of these K's.

We now consider how quasi-coherence behaves with respect to extension of the base field. Let S be a rigid space over k that is either quasi-separated or pseudo-separated (see §A.2 for the definition of pseudo-separatedness), and let k'/k be an analytic extension field. Let S' be the rigid space $k'\widehat{\otimes}_k S$ over k'. Although there is generally no natural map of ringed topoi $S' \to S$ when $[k' : k]$ is infinite, by working over affinoids there is an evident exact "pullback" functor $\mathrm{Coh}(S) \to \mathrm{Coh}(S')$ on categories of coherent sheaves, and this functor is naturally compatible with tensor products, with base change $T \to S$ for quasi-separated or pseudo-separated T, and with further extension of the base field. We can naturally extend this to an exact functor (maintaining the same compatibility properties) $\mathrm{Qcoh}(S) \to \mathrm{Qcoh}(S')$ on categories of quasi-coherent sheaves, as follows.

First assume S is affinoid and $\mathcal{F}$ is a direct limit of coherent sheaves, so by Lemma 2.1.8.(1) $\mathcal{F}$ is the direct limit of its coherent subsheaves $\mathcal{F}_i$. Define

$$k'\widehat{\otimes}_k\mathcal{F} = \varinjlim k'\widehat{\otimes}_k\mathcal{F}_i,$$

where $k'\widehat{\otimes}_k\mathcal{F}_i$ is the coherent pullback of $\mathcal{F}_i$ to S'. This construction only depends on a cofinal system of $\mathcal{F}_i$'s, and so since the $k'\widehat{\otimes}_k\mathcal{F}_i$'s are a cofinal system of coherent subsheaves of $k'\widehat{\otimes}_k\mathcal{F}$ it follows that the definition of $k'\widehat{\otimes}_k\mathcal{F}$ in this special case satisfies all of the desired properties (including transitivity with respect to a further extension of the base field and compatibility with affinoid base change on S). By Lemma 2.1.8.(3) and the isomorphism (A.2.4), for any admissible open $U' \subseteq S'$ the restriction $(k'\widehat{\otimes}_k\mathcal{F})|_{U'}$ represents the functor $\mathcal{G}' \rightsquigarrow \mathrm{Hom}_{\mathcal{O}_S(S)}(\mathcal{F}(S), \mathcal{G}'(U'))$ on the category of $\mathcal{O}_{U'}$-modules. Thus, for any quasi-separated or pseudo-separated S it is straightforward to define the quasi-coherent sheaf $k'\widehat{\otimes}_k\mathcal{F}$

on S' by using Lemma 2.1.8.(3) to glue over open affinoids $k'\widehat{\otimes}_k V \subseteq S'$ for open affinoids $V \subseteq S$ on which $\mathcal{F}$ is a direct limit of coherent sheaves. The resulting functor $\mathcal{F} \rightsquigarrow k'\widehat{\otimes}_k\mathcal{F}$ is exact and is compatible with (i) tensor products, (ii) base change $T \to S$ for quasi-separated or pseudo-separated T, and (iii) further extension of the base field.

DEFINITION 2.1.14. — *If $\mathcal{F}$ is a quasi-coherent sheaf on a quasi-separated or pseudo-separated rigid space S over k, then for any analytic extension field k'/k the quasi-coherent sheaf $k'\widehat{\otimes}_k\mathcal{F}$ on $S' = k'\widehat{\otimes}_k S$ is the* pullback *of $\mathcal{F}$ to S'.*

Example 2.1.15. — Let $\mathcal{F}$ be a quasi-coherent sheaf on a quasi-separated rigid space S and let $U \subseteq S$ be an admissible affinoid open. Let $S' = k'\widehat{\otimes}_k S$ and $U' = k'\widehat{\otimes}_k U \subseteq S'$ be obtained by base change and let $\mathcal{F}' = k'\widehat{\otimes}_k\mathcal{F}$ on S' be the pullback of $\mathcal{F}$. The natural map $\mathcal{O}_{S'}(U') \otimes_{\mathcal{O}_S(U)} \mathcal{F}(U) \to \mathcal{F}'(U')$ is an isomorphism. Indeed, since $\mathcal{O}_S(U)$ is a k-affinoid algebra over which $\mathcal{O}_{S'}(U') = k'\widehat{\otimes}_k\mathcal{O}_S(U)$ is flat [11, 1.1.5 (1)], we may work locally on U to reduce to the case when $S = U$ is affinoid with $\mathcal{F} = \varinjlim \mathcal{F}_i$ for coherent sheaves $\mathcal{F}_i$. The problem thereby reduces to the case of coherent $\mathcal{F}$, and this case is trivial.

Example 2.1.16. — If $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras then the quasi-coherent $\mathcal{O}_{S'}$-module $k'\widehat{\otimes}_k\mathcal{A}$ has a natural structure of $\mathcal{O}_{S'}$-algebra, and it is locally finitely generated as an $\mathcal{O}_{S'}$-algebra if $\mathcal{A}$ is locally finitely generated as an $\mathcal{O}_S$-algebra. Indeed, by exactness and tensor-compatibility of $\mathcal{F} \rightsquigarrow k'\widehat{\otimes}_k\mathcal{F}$, it suffices to note that the natural map of $\mathcal{O}_{S'}$-algebras $\mathcal{O}_{S'}[T_1, \ldots, T_n] \to k'\widehat{\otimes}_k(\mathcal{O}_S[T_1, \ldots, T_n])$ is an isomorphism.

2.2. Relative Spec

We need relative Proj, so we first require a theory of relative Spec. The starting point is a generalization of an observation of Tate. In [15, II, Errata, 1.8.1], it is proved that for any commutative ring A and any locally ringed space X, the natural map

$$\mathrm{Hom}(X, \mathrm{Spec}\, A) \longrightarrow \mathrm{Hom}\big(A, \Gamma(X, \mathcal{O}_X)\big)$$

is a bijection. Since a rigid-analytic space is not a locally ringed space in the traditional sense (as it rests on a Grothendieck topology, albeit a mild one), we require an easy generalization of this result. Let us first introduce some convenient terminology.

Definition 2.2.1. — *A* Grothendieck-topologized space *is a site whose underlying category consists of a collection of subsets* (*called "open subsets"*) *in a set X such that*

- *the collection of open subsets is stable under finite intersections and contains both $\emptyset$ and X,*
- *$\{U\}$ is a covering of U for every open U,*
- *a covering $\{U_i\}_{i\in I}$ of any U satisfies $\bigcup U_i = U$ inside of X.*

We usually write X to denote this data, with the choice of Grothendieck topology understood from the context.

Definition 2.2.2. — *A* locally ringed Grothendieck-topologized space *is a Grothendieck-topologized space X endowed with a sheaf of rings $\mathcal{O}$ such that*

- *the stalks $\mathcal{O}_x = \varinjlim_{x\in U} \mathcal{O}(U)$ are local rings for all $x \in X$,*
- *for any U and $s \in \mathcal{O}(U)$, the set $U_s = \{x \in U \mid s_x \in \mathcal{O}_x^\times\}$ is open,*
- *for any U and subset $\Sigma \subseteq \mathcal{O}(U)$ the union $\cup_{s\in\Sigma} U_s$ is open and admits $\{U_s\}_{s\in\Sigma}$ as a covering.*

A *morphism* $(X, \mathcal{O}) \to (X', \mathcal{O}')$ between locally ringed Grothendieck-topologized spaces is a pair $(f, f^\sharp)$ where $f : X \to X'$ is a continuous map of sites (that is, formation of preimages carries opens to opens and coverings to coverings) and $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$ is a map of sheaves of rings such that $f_x^\sharp : \mathcal{O}'_{f(x)} \to \mathcal{O}_x$ is a map of local rings for all $x \in X$. (By the usual argument, f_* has a left-exact adjoint on the categories of sheaves of sets, so $(f, f^\sharp)$ gives rise to a map of ringed topoi.) The category of locally ringed Grothendieck-topologized spaces contains the category of locally ringed spaces as a full subcategory. If C is a ring and we consider the category of pairs $(X, \mathcal{O})$ for which $\mathcal{O}$ is a sheaf of C-algebras, then by only allowing morphisms such that $f^\sharp$ is a map of sheaves of C-algebras we get the category of *locally ringed Grothendieck-topologized spaces over C*; if $C = k$ is a non-archimedean field then this category contains the category of rigid spaces over k as a full subcategory.

Lemma 2.2.3. — *Let C be a commutative ring. For any C-algebra A and locally ringed Grothendieck-topologized space $(X, \mathcal{O}_X)$ over C, the natural map of sets* $\mathrm{Hom}_C(X, \mathrm{Spec}\, A) \to \mathrm{Hom}_{C\text{-alg}}(A, \Gamma(X, \mathcal{O}_X))$ *is a bijection, where the left side is the set of morphisms in the category of locally ringed Grothendieck-topologized spaces over C.*

Proof. — The proof for locally ringed spaces (and $C = \mathbf{Z}$) in [15, II, Errata, 1.8.1] carries over *verbatim*. □

Remark 2.2.4. — By the preceding lemma, scheme-theoretic fiber products of C-schemes serve as fiber products in the category of locally ringed Grothendieck-topologized spaces over C.

The next theorem summarizes most of the basic results concerning an analytic Spec functor on the category of locally finitely generated quasi-coherent sheaves of algebras (or equivalently, locally finitely presented sheaves of algebras) in the rigid-analytic case. Similar results are obtained in [30, II, §1] for complex-analytic spaces and a pre-Tate notion of analytic space over non-archimedean fields. Our argument is a variant that works for rigid spaces.

THEOREM 2.2.5. — *Let S be a rigid space, and let $\mathcal{A}$ be a sheaf of $\mathcal{O}_S$-algebras that is locally finitely generated and quasi-coherent.*

1) *The functor*

$$(f : X \to S) \rightsquigarrow \mathrm{Hom}_{\mathcal{O}_S\text{-alg}}(\mathcal{A}, f_*\mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(f^*\mathcal{A}, \mathcal{O}_X)$$

on rigid spaces over S is represented by an S-separated rigid space $f_{\mathcal{A}} : \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A} \to S$, and when S is affinoid with $\mathcal{A}$ a direct limit of coherent sheaves then the natural map

$$\mathrm{Hom}_{\mathcal{O}_S\text{-alg}}(\mathcal{A}, f_*\mathcal{O}_X) \longrightarrow \mathrm{Hom}_{\mathcal{O}_S(S)\text{-alg}}\big(\mathcal{A}(S), \mathcal{O}_X(X)\big) \tag{2.2.1}$$

is a bijection.

2) *The following properties hold:*

• *the formation of $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A} \to S$ is naturally compatible with base change on S, and with change of the base field for quasi-separated or pseudo-separated S,*

• *a surjective map $\mathcal{A} \to \mathcal{B}$ of quasi-coherent and locally finitely generated $\mathcal{O}_S$-algebras induces a closed immersion $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{B} \to \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$ defined by the coherent ideal in $\mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}}$ induced by the quasi-coherent ideal $\ker(\mathcal{A} \twoheadrightarrow \mathcal{B})$ in $\mathcal{A}$,*

• *for maps $\mathcal{C} \to \mathcal{A}$ and $\mathcal{C} \to \mathcal{B}$ between locally finitely generated and quasi-coherent $\mathcal{O}_S$-algebras, the canonical map*

$$\mathrm{Spec}^{\mathrm{an}}(\mathcal{A} \otimes_{\mathcal{C}} \mathcal{B}) \longrightarrow \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A} \times_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{C}} \mathrm{Spec}^{\mathrm{an}}\,\mathcal{B}$$

(which makes sense, by Lemma 2.1.8.(2)) is an isomorphism.

3) *If S_0 is an algebraic k-scheme and $\mathcal{A}_0$ is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras that is locally finitely generated, then the natural S_0^{an}-map*

$$\mathrm{Spec}^{\mathrm{an}}(\mathcal{A}_0^{\mathrm{an}}) \longrightarrow (\mathrm{Spec}\,\mathcal{A}_0)^{\mathrm{an}} \tag{2.2.2}$$

is an isomorphism. It respects base change on S_0 and change of the base field.

4) *Choose $x \in \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$ over $s \in S$. Let $\mathfrak{m}_x \subseteq \mathcal{A}_s$ be the maximal ideal induced by the composite map of stalks*

$$\mathcal{A}_s \longrightarrow (f_{\mathcal{A}*}\mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}})_s \longrightarrow \mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A},x}.$$

The natural map of maximal-adic completions $(\mathcal{A}_s)^\wedge_{\mathfrak{m}_x} \to \mathcal{O}^\wedge_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A},x}$ is an isomorphism.

Remark 2.2.6. — The map (2.2.2) is defined by the universal property of analytification and by considerations (locally over S_0) with the universal property of affine k-schemes in Lemma 2.2.3.

Proof. — In general, if $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$ exists for some pair $(\mathcal{A}, S)$ then for any admissible open $U \subseteq S$ the restriction of $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$ over U satisfies the universal property to be $\mathrm{Spec}^{\mathrm{an}}(\mathcal{A}|_U)$. Thus, it suffices to carry out the general existence proof locally on S. We may therefore assume that $\mathcal{A}$ is a quotient of some $\mathcal{O}_S[T_1, \ldots, T_n]$. Let $\mathcal{I}$ denote the kernel of the surjection

$$\mathcal{O}_S[T_1, \ldots, T_n] \twoheadrightarrow \mathcal{A}, \tag{2.2.3}$$

so $\mathcal{I}$ is quasi-coherent on S. By the universal property of $\mathbf{A}^n_k$ in the rigid-analytic category, it is clear that $\mathbf{A}^n_k \times S$ with its standard global coordinates is $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{O}_S[T_1, \ldots, T_n]$. Let $\pi : \mathbf{A}^n_k \times S \to S$ be the projection. For any $f : X \to S$, to give an $\mathcal{O}_S$-algebra map $\mathcal{A} \to f_*\mathcal{O}_X$ is the same as to give an S-map $X \to \mathbf{A}^n_k \times S$ such that the ideal sheaf $\mathcal{J} = \mathrm{image}(\pi^*\mathcal{I} \to \mathcal{O}_{\mathbf{A}^n \times S})$ pulls back to the ideal sheaf 0 on X. Since $\pi^*\mathcal{I}$ is quasi-coherent on $\mathbf{A}^n_k \times S$ (as $\mathcal{I}$ is quasi-coherent on S), $\mathcal{J}$ is a quasi-coherent subsheaf of $\mathcal{O}_{\mathbf{A}^n_k \times S}$. We conclude (Lemma 2.1.9) that $\mathcal{J}$ is a coherent ideal. The rigid-analytic zero locus of $\mathcal{J}$ on $\mathbf{A}^n_k \times S$ has the universal property to be $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$.

By construction, $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$ is S-separated. In the special case that $S = \mathrm{Sp}(C)$ is affinoid and $\mathcal{F}$ is a direct limit of coherent $\mathcal{O}_S$-modules, the natural map $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F}, f_*\mathcal{O}_X) \to \mathrm{Hom}_C(\mathcal{F}(S), \mathcal{O}_X(X))$ is a bijection for rigid spaces X over S (as one sees by passing to the limit on the trivial case of coherent $\mathcal{F}$). Hence, when the $\mathcal{O}_S$-algebra $\mathcal{A}$ is a direct limit of coherent sheaves, $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A} \to S$ represents the functor

$$X \rightsquigarrow \mathrm{Hom}_{C\text{-alg}}\big(\mathcal{A}(S), \mathcal{O}_X(X)\big)$$

on rigid spaces over S. This settles 1).

We now turn to 2) and 3). The compatibility of $\mathrm{Spec}^{\mathrm{an}}$ with both base change and fiber products is a trivial consequence of the universal property of $\mathrm{Spec}^{\mathrm{an}}$. Universal properties also provide the desired compatibility of

$\mathrm{Spec}^{\mathrm{an}}$ with respect to analytification and base change for algebraic k-schemes (the key point is Lemma 2.2.3), as well as the fact that a surjection $\mathcal{A} \twoheadrightarrow \mathcal{B}$ with quasi-coherent kernel $\mathcal{I}$ induces a closed immersion of $\mathrm{Spec}^{\mathrm{an}}$'s defined by the coherent ideal sheaf $\mathrm{im}(f_{\mathcal{A}}^{*}\mathcal{I} \to \mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}})$ in $\mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}}$.

Fix an analytic extension field k'/k. Let us construct a natural morphism $h_{\mathcal{A}}$ from $k'\widehat{\otimes}_k \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$ to $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}'$ over $S' = k'\widehat{\otimes}_k S$ for quasi-separated or pseudo-separated S, with $\mathcal{A}'$ denoting the quasi-coherent locally finitely generated $\mathcal{O}_{S'}$-algebra pullback $k'\widehat{\otimes}_k\mathcal{A}$ in the sense of Definition 2.1.14. First consider the case when $S = \mathrm{Sp}(C)$ is affinoid and $\mathcal{A}$ is a direct limit of coherent sheaves. The S-space $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$ represents the functor $X \rightsquigarrow \mathrm{Hom}_{C\text{-alg}}(\mathcal{A}(S), \mathcal{O}_X(X))$ on rigid spaces X over S. Example 2.1.15 ensures $\mathcal{A}'(S') = C' \otimes_C \mathcal{A}(S)$ for $C' = k'\widehat{\otimes}_k C = \mathcal{O}_{S'}(S')$, so $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}'$ represents the functor

$$X' \rightsquigarrow \mathrm{Hom}_{C'\text{-alg}}\big(\mathcal{A}'(S'), \mathcal{O}_{X'}(X')\big) = \mathrm{Hom}_{C\text{-alg}}\big(\mathcal{A}(S), \mathcal{O}_{X'}(X')\big) \tag{2.2.4}$$

on rigid spaces X' over S'. For $X' = k'\widehat{\otimes}_k \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$ there is an evident natural C-algebra map

$$\mathcal{A}(S) \longrightarrow \Gamma(\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}, \mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}}) \longrightarrow \Gamma(X', \mathcal{O}_{X'}),$$

so by (2.2.4) this is induced by a unique morphism $h_{\mathcal{A}} : X' \to \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}'$. By Lemma 2.1.8.(3), the formation of $h_{\mathcal{A}}$ is compatible with affinoid base change on S and further extension of the base field. In particular, the formation of $h_{\mathcal{A}}$ is compatible with replacing the affinoid S with any affinoid open $U \subseteq S$. The construction of $h_{\mathcal{A}}$ for affinoid S therefore uniquely extends to the case of any quasi-separated or pseudo-separated S in a manner that is compatible with any base change $T \to S$ for quasi-separated or pseudo-separated T. This extended construction of $h_{\mathcal{A}}$ respects any further extension of the base field.

To prove that $h_{\mathcal{A}}$ is an isomorphism, by working locally on S we can assume that $\mathcal{A}$ is a quotient of a polynomial algebra over $\mathcal{O}_S$. By exactness of the pullback functor $\mathrm{Qcoh}(S) \to \mathrm{Qcoh}(S')$ and the fact that $\mathrm{Spec}^{\mathrm{an}}$ converts surjections into closed immersions (cut out by the expected ideal sheaf), the isomorphism problem is reduced to the trivial case when $\mathcal{A}$ is a polynomial algebra over $\mathcal{O}_S$. Using $h_{\mathcal{A}}$, (2.2.2) is compatible with change of the base field.

Finally, we turn to 4). The ideal $\mathfrak{m}_x$ is maximal because the k-algebra $\mathcal{A}_s/\mathfrak{m}_x$ is a subalgebra of the k-finite residue field at x on $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$. It suffices to check that $(\mathcal{A}_s)^{\wedge}_{\mathfrak{m}_x} \to \mathcal{O}^{\wedge}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A},x}$ is an isomorphism for Artin local S because in the general case the inverse limits of these two completions on infinitesimal fibers over $s \in S$ are the two completions of interest in 4).

Thus, we now assume that S is Artin local. In this case we can use 3) to reduce the isomorphism problem to the obvious fact that if Z is an algebraic k-scheme and $\iota : Z^{\mathrm{an}} \to Z$ is its analytification then the induced maps $\mathcal{O}^{\wedge}_{Z,\iota(z)} \to \mathcal{O}^{\wedge}_{Z^{\mathrm{an}},z}$ are isomorphisms for all $z \in Z^{\mathrm{an}}$ (as these two complete local noetherian k-algebras pro-represent the same functor on the category of finite local k-algebras, and the map between them respects this functorial identification since analytification is fully faithful on finite local k-schemes). □

COROLLARY 2.2.7. — *Let $\mathcal{A}$ be a quasi-coherent and locally finitely generated $\mathcal{O}_S$-algebra. If $\mathcal{A}$ is $\mathcal{O}_S$-flat then* $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$ *is S-flat.*

Proof. — This follows from Theorem 2.2.5.(4). □

COROLLARY 2.2.8. — *Let $\mathcal{A}$ be a quasi-coherent and locally finitely generated $\mathcal{O}_S$-algebra. The structure map $f_{\mathcal{A}}$ is finite if and only if $\mathcal{A}$ is coherent, in which case $\mathcal{A} \to f_{\mathcal{A}*}\mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}}$ is an isomorphism. The functors*

$$(f : X \to S) \rightsquigarrow f_*\mathcal{O}_X, \quad \mathcal{A} \rightsquigarrow (\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A} \to S)$$

between the category of rigid spaces finite over S and the category of coherent sheaves of $\mathcal{O}_S$-algebras form an adjoint pair for which the adjunctions are isomorphisms, so these are quasi-inverse equivalences of categories.

The complex-analytic case is treated in [30, II, §3, §5], except that it is not proved in [30] that finiteness of $f_{\mathcal{A}}$ forces $\mathcal{A}$ to be coherent; our method yields this result in the complex-analytic case.

Proof. — First assume that $\mathcal{A}$ is coherent. We may work locally on S to show that $f_{\mathcal{A}}$ is finite and $\mathcal{A} \to f_{\mathcal{A}*}\mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}}$ is an isomorphism. Hence, we can assume $S = \mathrm{Sp}(C)$ is affinoid and $\mathcal{A}$ is associated to a finite C-algebra A. The bijection (2.2.1) and the universal property of $\mathrm{Sp}(A)$ in the rigid-analytic category imply that the finite map $h : X = \mathrm{Sp}(A) \to S$ (equipped with the canonical isomorphism $\mathcal{A} \to h_*\mathcal{O}_X$) satisfies the universal property to be $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}$.

Conversely, if $f_{\mathcal{A}}$ is finite then $f_{\mathcal{A}*}\mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}}$ is a coherent $\mathcal{O}_S$-module. Thus, to establish the coherence of the quasi-coherent $\mathcal{A}$ in such cases it suffices (by Lemma 2.1.9) to prove that in general the canonical map $\mathcal{A} \to f_{\mathcal{A}*}\mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}}$ is injective. The only proof of such injectivity that we could come up with rests on the theory of Proj and so it will be given later, with a proof that works in the complex-analytic case, in Theorem 2.3.12. (Corollary 2.2.8 will not be used until §3.)

If $f : X \to S$ is finite then the isomorphism property for the canonical S-map $X \to \mathrm{Spec}^{\mathrm{an}} f_*\mathcal{O}_X$ can be checked by working locally over affinoids in S. The asserted quasi-inverse equivalences of categories are thereby checked to be quasi-inverse to each other. □

Example 2.2.9. — Let $S = \mathrm{Sp}(C)$ be affinoid and let $\mathcal{A}$ be a quasi-coherent and locally finitely generated $\mathcal{O}_S$-algebra. Let $X = \mathrm{Spec}^{\mathrm{an}} \mathcal{A}$. Shrinking S if necessary, assume that $\mathcal{A}$ is a quotient of a polynomial algebra over $\mathcal{O}_S$ in finitely many variables. By Theorem 2.1.11 and Lemma 2.2.3, the C-algebra $\mathcal{A}(S)$ is finitely generated and X represents the functor

$$Z \rightsquigarrow \mathrm{Hom}_{C\text{-alg}}\left(\mathcal{A}(S), \mathcal{O}_Z(Z)\right) = \mathrm{Hom}_{\mathrm{Spec}\, C}\left(Z, \mathrm{Spec}\, \mathcal{A}(S)\right)$$

on the category of rigid spaces over S. This says that X is the relative analytification of the finite-type C-scheme $\mathrm{Spec}\, \mathcal{A}(S)$ in the sense of Köpf [33, Satz 1.2].

Example 2.2.10. — For any locally finitely generated quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}$ and $a \in \mathcal{A}(S)$, the sheaf

$$\mathcal{A}_a \stackrel{\mathrm{def}}{=\!=} \mathcal{A}[T]/(1 - aT)$$

is a locally finitely generated quasi-coherent $\mathcal{O}_S$-algebra. By functoriality of $\mathrm{Spec}^{\mathrm{an}}$ there is a canonical map

$$\xi_a : \mathrm{Spec}^{\mathrm{an}} \mathcal{A}_a \to \mathrm{Spec}^{\mathrm{an}} \mathcal{A}.$$

The canonical map $\mathcal{A} \to f_{\mathcal{A}*}\mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}}$ carries the global section a of $\mathcal{A}$ to a global section $\widetilde{a}$ of $\mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}}$. Universality forces ξ_a to be an isomorphism of $\mathrm{Spec}^{\mathrm{an}} \mathcal{A}_a$ onto the Zariski-open locus in $\mathrm{Spec}^{\mathrm{an}} \mathcal{A}$ where $\widetilde{a}$ is non-vanishing.

It is straightforward to check (via universal properties and working locally on S) that the formation of ξ_a is compatible with base change on S, with analytification of algebraic k-schemes, and (for quasi-separated or pseudo-separated S) with change of the base field. The open immersions ξ_a provide the gluing data to be used in the rigid-analytic relative theory of Proj.

Example 2.2.11. — We can enhance Example 2.2.9 to obtain a new quick construction of the relative analytification functor of Köpf in [33, §1]. More precisely, for any C-scheme $\mathcal{X}$ locally of finite type we seek to construct a map $\iota_{\mathcal{X}} : \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ that is a final object in the category of rigid spaces over $\mathrm{Sp}(C)$ equipped with a map to $\mathcal{X}$ as locally ringed Grothendieck-topologized spaces over C. (If C is k-finite, so $\mathcal{X}$ is an algebraic k-scheme, then $\mathrm{Sp}(C) = \mathrm{Spec}\, C$ and the usual analytification of $\mathcal{X}$

over k does the job; hence, in such cases there is no risk of confusion caused by the notation "$\mathcal{X}^{\mathrm{an}}$".)

It follows formally from Lemma 2.2.3 and universal properties that the formation of $\mathcal{X}^{\mathrm{an}}$ (when it exists) must be compatible with fiber products and with k-affinoid base change on C, and must carry open immersions to Zariski-open immersions and closed immersions to closed immersions. For example, the Zariski-open preimage in $\mathcal{X}^{\mathrm{an}}$ of a Zariski-open $\mathcal{U} \subseteq \mathcal{X}$ satisfies the universal property to be $\mathcal{U}^{\mathrm{an}}$, and the compatibility with fiber products follows from Remark 2.2.4. Existence of the relative analytification for affine $\mathcal{X}$ is settled by Example 2.2.9 and Theorem 2.2.5(4), and the automatic compatibility with open immersions and fiber products permits us to glue along Zariski-opens for the general case.

Since $\mathrm{Sp}(R) = \mathrm{Spec}(R)^{\mathrm{an}}$ for any C-algebra R that is k-finite, by considering morphisms from $\mathrm{Spec}(k') = \mathrm{Sp}(k')$ to $\mathcal{X}$ for varying finite extensions k'/k we see that $\iota_{\mathcal{X}} : \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ must be a bijection onto the set of points of $\mathcal{X}$ with residue field of finite degree over k. (Hence, if $f : \mathcal{X}' \to \mathcal{X}$ is a surjective map between locally finite type C-schemes then f^{an} must be surjective.) Similarly, by using R-points for varying k-finite C-algebras R we deduce from universal properties that for all $x \in \mathcal{X}^{\mathrm{an}}$ the natural map of complete local noetherian k-algebras $\mathcal{O}^{\wedge}_{\mathcal{X},\iota_{\mathcal{X}}(x)} \to \mathcal{O}^{\wedge}_{\mathcal{X}^{\mathrm{an}},x}$ is an isomorphism, so $\iota_{\mathcal{X}}$ is necessarily flat. In particular, pullback along $\iota_{\mathcal{X}}$ defines an exact analytification functor $\mathcal{F} \rightsquigarrow \mathcal{F}^{\mathrm{an}}$ from $\mathcal{O}_{\mathcal{X}}$-modules to $\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}$-modules carrying coherent sheaves to coherent sheaves.

We have seen that relative analytification carries Zariski-open/closed immersions to Zariski-open/closed immersions and carries fiber products to fiber products, so (as in the case $C = k$) it follows that the morphism $\mathcal{X}^{\mathrm{an}} \to \mathrm{Sp}(C)$ is pseudo-separated for any locally finite type C-scheme $\mathcal{X}$. The separatedness of $\mathrm{Sp}(C)$ therefore implies that $\mathcal{X}^{\mathrm{an}}$ is a pseudo-separated rigid space for any such $\mathcal{X}$, so it makes sense to address the compatibility of relative analytification with respect to any analytic extension field k'/k. Arguing as in the case $C = k$, we obtain a transitive natural $\mathrm{Sp}(k'\widehat{\otimes}_k C)$-isomorphism of k'-analytic rigid spaces

$$k'\widehat{\otimes}_k \mathcal{X} \simeq \left((k'\widehat{\otimes}_k C) \otimes_C \mathcal{X}\right)^{\mathrm{an}}.$$

This isomorphism is compatible with fiber products and k-affinoid base change on C, and for affine $\mathcal{X}$ it is compatible with the behavior of $\mathrm{Spec}^{\mathrm{an}}$ with respect to extension of the base field.

2.3. Relative Proj

Let S be a rigid space and let

$$\mathcal{A} = \bigoplus_{n \geqslant 0} \mathcal{A}^n \tag{2.3.1}$$

be a graded sheaf of $\mathcal{O}_S$-algebras such that

- each $\mathcal{A}^n$ is coherent as an $\mathcal{O}_S$-module (so $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra),
- $\mathcal{A}$ is locally finitely generated as an $\mathcal{O}_S$-algebra.

Locally on S, $\mathcal{A}$ is a quotient of a graded polynomial algebra in finitely many variables over $\mathcal{O}_S$, with each variable assigned a suitable non-negative degree. By Theorem 2.1.11, if $U \subseteq S$ is an admissible open such that $\mathcal{A}|_U$ is a graded quotient of a weighted polynomial algebra $\mathcal{O}_U[T_1, \ldots, T_m]$ then $\mathcal{A}(V)$ is a graded quotient of $\mathcal{O}(V)[T_1, \ldots, T_m]$ for any open affinoid $V \subseteq U$.

DEFINITION 2.3.1. — *Let $a \in \mathrm{H}^0(S, \mathcal{A}^n)$ be a global section with $n > 0$, so $\mathcal{A}_a$ is naturally a $\mathbf{Z}$-graded quasi-coherent sheaf of $\mathcal{O}_S$-algebras such that $1/a$ is a homogeneous global section with constant degree $-n$. The $\mathcal{O}_S$-algebra $\mathcal{A}_{(a)}$ is the degree-0 part of $\mathcal{A}_a$.*

LEMMA 2.3.2. — *The $\mathcal{O}_S$-algebra $\mathcal{A}_{(a)}$ is locally finitely generated and quasi-coherent. If $\mathcal{A}$ is a graded quotient of a polynomial algebra over $\mathcal{O}_S$ in finitely many homogeneous variables then $\mathcal{A}_{(a)}$ is a quotient of a polynomial algebra over $\mathcal{O}_S$ in finitely many variables.*

Proof. — Since $\mathcal{A}_a$ is quasi-coherent, by Lemma 2.1.9, its direct summand $\mathcal{A}_{(a)}$ as an $\mathcal{O}_S$-module is quasi-coherent. To check that $\mathcal{A}_{(a)}$ is locally finitely generated, we may work locally on S. Thus, we can assume that $\mathcal{A}$ is a quotient of a graded $\mathcal{O}_S$-algebra $\mathcal{O}_S[T_1, \ldots, T_r]$ with $\deg(T_j) = d_j \geqslant 0$. It follows that $\mathcal{A}_{(a)}$ is a quotient of the $\mathcal{O}_S$-algebra

$$\mathcal{O}_S\big[T^{d_1}T_1^n, \ldots, T^{d_r}T_r^n, T^e T_1^{e_1} \cdots T_r^{e_r} \mid ne = \sum e_j d_j,\ 0 \leqslant e_j < n\big]. \tag{2.3.2}$$

This sheaf has the form $\mathcal{O}_S \otimes_{\mathbf{Z}} A_{(a)}$ where $A = \mathbf{Z}[T_1, \ldots, T_r, a]$ is the graded noetherian polynomial ring over $\mathbf{Z}$ with $\deg(T_j) = d_j$ and $\deg(a) = n$. Since $A_{(a)}$ is of finite type over $\mathbf{Z}$ [15, II, 2.1.6 (iv), 2.2.5], we conclude that (2.3.2) is a quotient of a polynomial algebra over $\mathcal{O}_S$ in finitely many variables. □

In an evident manner, the formation of $\mathcal{A}_{(a)}$ is compatible with base change on S and (for quasi-separated or pseudo-separated S) with change of the base field. By Lemma 2.3.2, it makes sense to form the rigid space $\mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}_{(a)}$ over S, and the functor $(\mathcal{A}, a) \rightsquigarrow \mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}_{(a)}$ is compatible

with base change on S and (for quasi-separated or pseudo-separated S) with change of the base field.

For $f \in \mathrm{H}^0(S, \mathcal{A}^n)$ and $g \in \mathrm{H}^0(S, \mathcal{A}^m)$ with $n, m > 0$, there is an equality of $\mathcal{O}_S$-subalgebras of $\mathcal{A}_{fg}$:

$$(\mathcal{A}_{(f)})_{g^n/f^m} = \mathcal{A}_{(fg)} = (\mathcal{A}_{(g)})_{f^m/g^n}. \tag{2.3.3}$$

By Example 2.2.10, $\mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}_{(fg)}$ is naturally identified with a Zariski-open in each of $\mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}_{(f)}$ and $\mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}_{(g)}$. The data of these Zariski-opens for all f and g satisfy the triple overlap compatibilities as we vary f and g and shrink the base, so we can glue over S. To be precise:

Definition 2.3.3. — *Let $\mathcal{A}$ be as in* (2.3.1)*. For varying admissible opens $U, V \subseteq S$ and sections $f \in \mathrm{H}^0(U, \mathcal{A}^n)$ and $g \in \mathrm{H}^0(V, \mathcal{A}^m)$ with $n, m > 0$, we glue*

$$\mathrm{Spec}^{\mathrm{an}}((\mathcal{A}|_U)_{(f)})|_{U\cap V} \quad \textit{and} \quad \mathrm{Spec}^{\mathrm{an}}((\mathcal{A}|_V)_{(g)})|_{U\cap V}$$

via (2.3.3) *over $U \cap V$. The resulting rigid space over S is denoted* $\mathrm{Proj}^{\mathrm{an}}\, \mathcal{A}$.

Remark 2.3.4. — For any $f \in \mathcal{A}^n(S)$ with $n > 0$, the admissible open $\mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}_{(f)}$ in $\mathrm{Proj}^{\mathrm{an}}\, \mathcal{A}$ is Zariski-open. Indeed, for any admissible open $U \subseteq S$ and any section $g \in \mathcal{A}^m(U)$ with $m > 0$, the overlap of $\mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}_{(f)}$ and $\mathrm{Spec}^{\mathrm{an}}(\mathcal{A}|_U)_{(g)}$ in $\mathrm{Proj}^{\mathrm{an}}\, \mathcal{A}$ is the non-vanishing locus of f^m/g^n on $\mathrm{Spec}^{\mathrm{an}}(\mathcal{A}|_U)_{(g)}$.

We define $\mathcal{A}_{>N} = \bigoplus_{n>N} \mathcal{A}^n$ for $\mathcal{A}$ as in (2.3.1). This $\mathcal{A}$-ideal is quasi-coherent over $\mathcal{O}_S$. By Corollary 2.1.12, the $\mathcal{A}$-module $\mathcal{A}_{>N}$ is $\mathcal{A}$-finite. We let $\mathcal{A}_+$ denote $\mathcal{A}_{>0}$.

The definition of $\mathrm{Proj}^{\mathrm{an}}$ is too global. For example, there is so much data in the definition that it is not a tautology that the formation of $\mathrm{Proj}^{\mathrm{an}}$ respects base change (but see Theorem 2.3.6). To localize the situation, we will use the $\mathcal{A}$-finite $\mathcal{A}_+$. By the definition of $\mathrm{Proj}^{\mathrm{an}}$ there is a natural map

$$\mathrm{Proj}^{\mathrm{an}}(\mathcal{A}|_U) \longrightarrow (\mathrm{Proj}^{\mathrm{an}}\, \mathcal{A}) \times_S U \tag{2.3.4}$$

over admissible opens $U \subseteq S$; the existence of this map rests on the enormous amount of data that was glued in the definition of $\mathrm{Proj}^{\mathrm{an}}\, \mathcal{A}$. We claim that (2.3.4) is an isomorphism. By working locally on S, it is enough to consider admissible open $U \subseteq S$ over which the finite $\mathcal{A}|_U$-module $\mathcal{A}_+|_U$ is generated by finitely many homogeneous U-sections. To handle the isomorphism problem for (2.3.4) in this special case, it suffices to prove the very useful:

LEMMA 2.3.5. — *Suppose elements $f_j \in \mathrm{H}^0(S, \mathcal{A}^{i_j})$ for $1 \leqslant j \leqslant r$ and $i_j > 0$ generate the $\mathcal{A}$-finite quasi-coherent ideal sheaf $\mathcal{A}_+$ in $\mathcal{A}$. For any admissible open $U \subseteq S$ and $g \in \mathcal{A}^m(U)$ with $m > 0$, $\mathrm{Spec}^{\mathrm{an}}(\mathcal{A}|_U)_{(g)}$ is covered by its Zariski-open subsets $\mathrm{Spec}^{\mathrm{an}}(\mathcal{A}|_U)_{(gf_j)}$. In particular, the Zariski-opens $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(f_j)}$ in $\mathrm{Proj}^{\mathrm{an}}\,\mathcal{A}$ form a covering. The same holds with $\mathcal{A}_+$ replaced by $\mathcal{A}_{>N}$ for any $N \geqslant 0$.*

Proof. — The hypothesis on the f_j's is preserved by renaming U as S, and the statement to be proved is expressed in terms of $\mathrm{Spec}^{\mathrm{an}}$'s whose formation is local on S. Thus, we just have to check that if $g \in \mathcal{A}^m(S)$ with $m > 0$ then the $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(gf_j)}$'s are a set-theoretic cover of $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(g)}$. We may pass to fibers over points $\mathrm{Sp}(k')$ of S. Renaming k' as k, the compatibility with analytification in Theorem 2.2.5.(2) when $S = \mathrm{Sp}(k)$ reduces our problem to the known scheme version [15, II, 2.3.15] for Proj of a graded ring. □

THEOREM 2.3.6. — *The map $\mathrm{Proj}^{\mathrm{an}}\,\mathcal{A} \to S$ is separated. Moreover, the formation of $\mathrm{Proj}^{\mathrm{an}}\,\mathcal{A}$ is naturally compatible with base change on S, and with change of the base field when S is quasi-separated or pseudo-separated.*

Proof. — The isomorphism (2.3.4) and Lemma 2.3.5 imply that the construction of $\mathrm{Proj}^{\mathrm{an}}\,\mathcal{A}$ can be given locally over S with a finite amount of data that is insensitive to base change and change of the base field. Thus, by Remark 2.3.4 and Lemma A.2.4 the compatibilities with base change and (for quasi-separated or pseudo-separated S) change of the base field follow from the analogous compatibilities for the formation of $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(f)}$.

By Remark 2.3.4, for $f \in \mathcal{A}^n(S)$ and $g \in \mathcal{A}^m(S)$ with $n, m > 0$, the natural map

$$\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(fg)} \longrightarrow \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(f)} \times_S \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(g)}$$

fits into a cartesian square

$$\begin{array}{ccc} \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(fg)} & \longrightarrow & \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(f)} \times_S \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(g)} \\ \downarrow & & \downarrow \\ \mathbf{G}_m & \longrightarrow & \mathbf{A}^1_k \times \mathbf{A}^1_k \end{array}$$

with the right side given by $(g^n/f^m) \times (f^m/g^n)$, the left side given by $g^{n+m}/(fg)^m$, and the bottom side given by the closed immersion $t \mapsto (t, 1/t)$ onto the hyperbola $xy = 1$. The top side is therefore a closed immersion, so $\mathrm{Proj}^{\mathrm{an}}\,\mathcal{A}$ is separated over S. □

In Corollary 2.3.9 we will see that $\mathrm{Proj}^{\mathrm{an}}\,\mathcal{A}$ is S-proper.

Definition 2.3.7. — *Let $\mathcal{F}$ be a coherent sheaf on a rigid space S. Let $\mathrm{Sym}(\mathcal{F})$ be the locally finitely generated and quasi-coherent graded symmetric algebra of $\mathcal{F}$ over $\mathcal{O}_S$. The rigid space $\mathbf{P}^{\mathrm{an}}(\mathcal{F})$ is $\mathrm{Proj}^{\mathrm{an}}(\mathrm{Sym}(\mathcal{F}))$.*

Using Lemma 2.3.5, it follows exactly as in the case of schemes that $\mathbf{P}^{\mathrm{an}}(\mathcal{F})$ classifies isomorphism classes of pairs (f,θ) where $f : X \to S$ is a rigid space over S and $\theta : f^*\mathcal{F} \twoheadrightarrow \mathcal{L}$ is an $\mathcal{O}_X$-linear surjection onto an invertible sheaf. If $S = \mathrm{Sp}(A)$ is affinoid and $M = \mathrm{H}^0(S,\mathcal{F})$ then we write $\mathbf{P}^{\mathrm{an}}(M)$ rather than $\mathbf{P}^{\mathrm{an}}(\mathcal{F})$; in this case, an S-morphism $X \to \mathbf{P}^{\mathrm{an}}(M)$ is classified by an A-linear map $M \to \mathrm{H}^0(X,\mathcal{L})$ whose image generates all stalks of a line bundle $\mathcal{L}$. By Theorem 2.3.6, the formation of $\mathbf{P}^{\mathrm{an}}(\mathcal{F})$ commutes with base change on S and (for quasi-separated or pseudo-separated S) with change of the base field.

We now summarize some basic properties of $\mathrm{Proj}^{\mathrm{an}}$.

Theorem 2.3.8. — *Let S be a rigid space, and let $\mathcal{A} = \bigoplus_{n\geqslant 0}\mathcal{A}^n$ be a locally finitely generated and graded sheaf of $\mathcal{O}_S$-algebras such that $\mathcal{A}^n$ is coherent for all n.*

1) *If $\mathcal{A}$ is $\mathcal{O}_S$-flat then $\mathrm{Proj}^{\mathrm{an}}\,\mathcal{A}$ is S-flat.*

2) *Assume $(S,\mathcal{A})$ is the analytification of an analogous pair $(S_0,\mathcal{A}_0)$ with S_0 an algebraic k-scheme. For any $a \in \mathrm{H}^0(S_0,\mathcal{A}_0^n)$ with $n > 0$, the S-isomorphisms*

$$\mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(a^{\mathrm{an}})} \simeq \mathrm{Spec}^{\mathrm{an}}((\mathcal{A}_0)^{\mathrm{an}}_{(a)}) \simeq (\mathrm{Spec}((\mathcal{A}_0)_{(a)}))^{\mathrm{an}} \tag{2.3.5}$$

glue to define an S-isomorphism

$$\mathrm{Proj}^{\mathrm{an}}\,\mathcal{A} \simeq (\mathrm{Proj}\,\mathcal{A}_0)^{\mathrm{an}} \tag{2.3.6}$$

that is compatible with base change on S_0 and with change of the base field.

3) *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a graded map between locally finitely generated and graded $\mathcal{O}_S$-algebras with graded terms that are coherent and supported in nonnegative degrees. Assume that locally on S the map φ is surjective in all sufficiently large degrees. There exists a unique map*

$$\mathrm{Proj}^{\mathrm{an}}(\varphi) : \mathrm{Proj}^{\mathrm{an}}\,\mathcal{B} \to \mathrm{Proj}^{\mathrm{an}}\,\mathcal{A} \tag{2.3.7}$$

over S that is compatible with base change on S and with both (2.3.6) and the algebraic analogue of (2.3.7). The map (2.3.7) is a closed immersion, and for quasi-separated or pseudo-separated S it respects change of the base field.

Proof. — Corollary 2.2.7 gives (1) because for $a \in \mathcal{A}(U)$ the $\mathcal{O}_U$-algebra $(\mathcal{A}|_U)_{(a)}$ is an $\mathcal{O}_U$-module direct summand of $\mathcal{A}|_U$ and hence is $\mathcal{O}_U$-flat. Now consider (2). The diagram

$$\begin{array}{ccc} \mathrm{Spec}^{\mathrm{an}}\big(\mathcal{A}_{(a^{\mathrm{an}}a'^{\mathrm{an}})}\big) & \xrightarrow{\simeq} & \big(\mathrm{Spec}((\mathcal{A}_0)_{(aa')})\big)^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}^{\mathrm{an}}\big(\mathcal{A}_{(a^{\mathrm{an}})}\big) & \xrightarrow[\simeq]{} & \big(\mathrm{Spec}((\mathcal{A}_0)_{(a)})\big)^{\mathrm{an}} \end{array} \tag{2.3.8}$$

commutes and is cartesian with open immersions along the columns. By Lemma 2.3.5 the resulting map

$$\mathrm{Proj}^{\mathrm{an}}\,\mathcal{A} \longrightarrow (\mathrm{Proj}\,\mathcal{A}_0)^{\mathrm{an}} \tag{2.3.9}$$

over S_0^{an} is compatible with base change on S_0 and with change of the base field, and (via the cartesian property of (2.3.8)) it is an isomorphism.

To prove 3), note that uniqueness is clear by the Krull intersection theorem. For existence, first consider an element $b \in \mathrm{H}^0(S, \mathcal{B}^n)$ that lifts to an element $a \in \mathrm{H}^0(S, \mathcal{A}^n)$, with $n > 0$. We claim that the composite

$$\mathrm{Spec}^{\mathrm{an}}\,\mathcal{B}_{(b)} \hookrightarrow \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(a)} \hookrightarrow \mathrm{Proj}^{\mathrm{an}}\,\mathcal{A} \tag{2.3.10}$$

is independent of a, is compatible with base change on S, and (for quasi-separated or pseudo-separated S) is compatible with change in the base field. The independence of a holds for Artin local S due to 2) and the known algebraic analogue. The independence of a in general follows by passing to infinitesimal fibers. The same method establishes compatibility of (2.3.10) with any base change $S' \to S$, via passage to infinitesimal fibers over S and S'. Compatibility of (2.3.10) with change in the base field (for quasi-separated or pseudo-separated S) follows from Lemma 2.3.5 due to the compatibility of $\mathrm{Spec}^{\mathrm{an}}$ with change in the base field.

By working locally over admissible opens U in S such that $\varphi|_U$ is surjective in all degrees $\geqslant N_U$ for some $N_U > 0$ and such that a finite set of $\mathcal{B}|_U$-module generators of $\mathcal{B}_{>N_U}|_U$ lifts to $\mathcal{A}(U)$, we can glue to define a unique map $\mathrm{Proj}^{\mathrm{an}}(\varphi) : \mathrm{Proj}^{\mathrm{an}}\,\mathcal{B} \to \mathrm{Proj}^{\mathrm{an}}\,\mathcal{A}$ of the desired type in (3) except that we still need to show that $\mathrm{Proj}^{\mathrm{an}}(\varphi)$ is a closed immersion. For $a \in \mathrm{H}^0(S, \mathcal{A}^n)$ with image $b \in \mathrm{H}^0(S, \mathcal{B}^n)$, the map $\mathcal{A}_{(a)} \to \mathcal{B}_{(b)}$ is surjective. Thus, for such a and b the map $\mathrm{Spec}^{\mathrm{an}}\,\mathcal{B}_{(b)} \to \mathrm{Spec}^{\mathrm{an}}\,\mathcal{A}_{(a)}$ is a closed immersion by Theorem 2.2.5.(2). By Lemma 2.3.5 (and working locally on S), it follows that $\mathrm{Proj}^{\mathrm{an}}\,\mathcal{B} \to \mathrm{Proj}^{\mathrm{an}}\,\mathcal{A}$ is a closed immersion if

the commutative diagram (with open immersions along columns)

$$\begin{array}{ccc} \mathrm{Spec}^{\mathrm{an}}\mathcal{B}_{(b)} & \longrightarrow & \mathrm{Spec}^{\mathrm{an}}\mathcal{A}_{(a)} \\ \downarrow & & \downarrow \\ \mathrm{Proj}^{\mathrm{an}}\mathcal{B} & \longrightarrow & \mathrm{Proj}^{\mathrm{an}}\mathcal{A} \end{array} \tag{2.3.11}$$

is cartesian. The induced map from $\mathrm{Spec}^{\mathrm{an}}\mathcal{B}_{(b)}$ to the fiber product is an open immersion (as it is a $\mathrm{Proj}^{\mathrm{an}}\mathcal{B}$-map between admissible opens in $\mathrm{Proj}^{\mathrm{an}}\mathcal{B}$), so our problem is set-theoretic and hence it suffices to work on fibers over S. On such fibers, the settled part 2) reduces us to the known scheme analogue [15, II, 2.9.2 (i)]. □

By Theorem 2.3.8.(3), we obtain:

Corollary 2.3.9. — *Let S be a rigid space and let $\mathcal{A} = \bigoplus_{n\geqslant 0}\mathcal{A}^n$ be a locally finitely generated and graded sheaf of $\mathcal{O}_S$-algebras with coherent graded terms. Locally on S there exists a closed immersion of $\mathrm{Proj}^{\mathrm{an}}\mathcal{A}$ into a weighted projective space over the base. In particular, $\mathrm{Proj}^{\mathrm{an}}\mathcal{A} \to S$ is proper.*

Let $\mathcal{A}^{(d)}$ be the graded sheaf of algebras obtained from $\mathcal{A}$ by dropping all graded terms in degrees not divisible by some integer $d > 0$; clearly $\mathcal{A}^{(d)}$ has coherent graded terms, and it is locally finitely generated because $\mathbf{Z}[T_1, \ldots, T_r]^{(d)}$ is finitely generated over $\mathbf{Z}$ for any assigned values $\deg T_j = d_j \geqslant 0$ [15, II, 2.1.6 (iv)]. The natural isomorphisms $((\mathcal{A}|_U)^{(d)})_{(a)} \simeq (\mathcal{A}|_U)_{(a)}$ for sections $a \in \mathrm{H}^0(U, \mathcal{A}^{nd})$, $n > 0$, induce an S-map $\mathrm{Proj}^{\mathrm{an}}\mathcal{A}^{(d)} \to \mathrm{Proj}^{\mathrm{an}}\mathcal{A}$ whose formation is local on S.

Corollary 2.3.10. — *The canonical map $\mathrm{Proj}^{\mathrm{an}}\mathcal{A}^{(d)} \to \mathrm{Proj}^{\mathrm{an}}\mathcal{A}$ over S is an isomorphism, and its formation is compatible with base change on S and (for quasi-separated or pseudo-separated S) with change in the ground field.*

Proof. — The compatibility with base change and with change in the ground field follow from working locally on S and using Lemma 2.3.5 and the construction of the map. For Artin local S, (2.3.5) and (2.3.6) reduce the isomorphism problem to the known scheme case [15, II, 2.4.7 (i)]. For general S, since $\mathrm{Proj}^{\mathrm{an}}\mathcal{A}$ and $\mathrm{Proj}^{\mathrm{an}}\mathcal{A}^{(d)}$ are S-proper (Corollary 2.3.9), it remains to check that if $f : X \to Y$ is an S-map between rigid spaces proper over S then f is an isomorphism if it induces an isomorphism on infinitesimal fibers over S. This follows from Theorem A.2.6.(2). □

Example 2.3.11. — We now wish to give a Proj analogue of Example 2.2.9. Let $S = \mathrm{Sp}(C)$ be affinoid and assume that $\mathcal{A}$ as in Theorem 2.3.8 is a graded quotient of a weighted polynomial algebra over $\mathcal{O}_S$, so $\mathcal{A}(S)$ is a graded quotient of a weighted polynomial algebra over C in finitely many variables. For all $a \in \mathcal{A}(S)^n = \mathcal{A}^n(S)$, the natural maps

$$\mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}_{(a)} \longrightarrow \mathrm{Spec}\, \mathcal{A}_{(a)}(S) = \mathrm{Spec}\, \mathcal{A}(S)_{(a)}$$

glue to define a natural map of locally ringed Grothendieck-topologized spaces $\mathrm{Proj}^{\mathrm{an}}\, \mathcal{A} \to \mathrm{Proj}(\mathcal{A}(S))$ over $\mathrm{Spec}\, C$, and so consequently we obtain a natural S-map

$$\mathrm{Proj}^{\mathrm{an}}\, \mathcal{A} \longrightarrow \big(\mathrm{Proj}(\mathcal{A}(S))\big)^{\mathrm{an}} \tag{2.3.12}$$

to the relative analytification (over $\mathrm{Sp}\, C$) in the sense of Example 2.2.11. We claim that this is an isomorphism. Since $\mathcal{A}_{(a)}(S) = \mathcal{A}(S)_{(a)}$ for any homogeneous term $a \in \mathcal{A}_+(S) = \mathcal{A}(S)_+$, we may use Lemma 2.3.5 and the gluing constructions of each side of (2.3.12) to reduce the isomorphism problem to the analogue for Spec that was handled in Example 2.2.9.

The isomorphism (2.3.12) ensures that the relative analytification functor in Example 2.2.11 carries projective C-schemes to rigid spaces that are projective (and hence proper) over $\mathrm{Sp}(C)$. Since relative analytification carries closed immersions to closed immersions and surjections to surjections, by Chow's Lemma and Temkin's work on proper maps (see §A.1) it follows that relative analytification carries proper maps to proper maps.

THEOREM 2.3.12. — *Let S be a rigid space and let $\mathcal{A}$ be a quasi-coherent locally finitely generated $\mathcal{O}_S$-algebra. Let $f_{\mathcal{A}} : \mathrm{Spec}^{\mathrm{an}}\, \mathcal{A} \to S$ be the structure map. The canonical map $\mathcal{A} \to f_{\mathcal{A}*}\mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}}$ is injective. In particular,* $\mathrm{Spec}^{\mathrm{an}}$ *is a faithful functor.*

Remark 2.3.13. — For $s \in S$,

$$f_{\mathcal{A}}^{-1}(s) = \mathrm{Spec}^{\mathrm{an}}(\mathcal{A}_s/\mathfrak{m}_s\mathcal{A}_s) \simeq \mathrm{Spec}(\mathcal{A}_s/\mathfrak{m}_s\mathcal{A}_s)^{\mathrm{an}}$$

may be empty even if $\mathcal{A}_s \neq 0$. A simple example is to take S to be a connected smooth curve and $\mathcal{A} = \mathcal{O}_S[T]/(1 - fT)$ for a nonzero $f \in \mathcal{O}_S(S)$ with a zero at $s \in S$. The possibility of empty fibers at points where $\mathcal{A}$ has a nonzero stalk is the reason why the proof of Theorem 2.3.12 is not shorter or simpler.

Proof. — Since $\mathcal{A}$ is quasi-coherent, $\mathcal{A}(U) \to \prod_{s \in U} \mathcal{A}_s$ is injective for every admissible open U in S. Thus, it suffices to prove that $f_{\mathcal{A}}$ is injective on s-stalks for all $s \in S$. Our problem is therefore pointwise. Work locally on S so that $\mathcal{A}$ is a quotient of a sheaf of polynomial algebras

$\mathcal{O}_S[T_1,\ldots,T_n]$ modulo a quasi-coherent ideal $\mathcal{I}$. Choose $s \in S$. By passing to stalks at s, $\operatorname{Spec}\mathcal{A}_s$ is thereby identified with a closed subscheme of affine n-space over $\operatorname{Spec}\mathcal{O}_{S,s}$. Forming the projective closure in $\mathbf{P}^n_{\mathcal{O}_{S,s}}$ gives a graded quotient $\mathcal{A}'_s$ of the polynomial algebra $\mathcal{O}_{S,s}[Z_0,\ldots,Z_n]$ (with $\deg Z_j = 1$ for all j) such that defining $T_j = Z_j/Z_0$ identifies $(\mathcal{A}'_s)_{(Z_0)}$ and $\mathcal{A}_s$ as quotients of $\mathcal{O}_{S,s}[T_1,\ldots,T_n]$. Explicitly, $\mathcal{A}'_s$ is the quotient of $\mathcal{O}_{S,s}[Z_0,\ldots,Z_n]$ by the homogeneous ideal generated by the homogeneous polynomials g whose Z_0-dehomogenization in $\mathcal{O}_{S,s}[T_1,\ldots,T_n]$ vanishes in the quotient $\mathcal{A}_s$. In particular, Z_0 has image in $\mathcal{A}'_s$ that is not a zero-divisor.

By shrinking S around s and using the fact that a quasi-coherent ideal sheaf $\mathcal{J}$ in a locally finitely generated quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{B}$ is $\mathcal{B}$-finite (Corollary 2.1.12), there is a quasi-coherent graded quotient $\widetilde{\mathcal{A}}$ of $\mathcal{O}_S[Z_0,\ldots,Z_n]$ such that $\mathcal{A} = \widetilde{\mathcal{A}}_{(Z_0)}$ as quotients of $\mathcal{O}_S[T_1,\ldots,T_n]$ and $\mathcal{A}'_s$ is identified with the s-stalk of $\widetilde{\mathcal{A}}$. In particular, $\operatorname{Spec}^{\mathrm{an}}\mathcal{A} = \operatorname{Spec}^{\mathrm{an}}\widetilde{\mathcal{A}}_{(Z_0)}$ is identified with a Zariski-open U in the S-proper $\operatorname{Proj}^{\mathrm{an}}\widetilde{\mathcal{A}}$. The annihilator of Z_0 in $\widetilde{\mathcal{A}}$ is a quasi-coherent ideal sheaf $\mathcal{K}$ whose stalk at s is zero because Z_0 is not a zero-divisor on the s-stalk of $\widetilde{\mathcal{A}}_s = \mathcal{A}'_s$. Since $\mathcal{K}$ is $\widetilde{\mathcal{A}}$-finite we may therefore shrink S around s so that $\mathcal{K} = 0$, and hence Z_0 is *nowhere* a zero-divisor on $\widetilde{\mathcal{A}}$.

Let $h : P = \operatorname{Proj}^{\mathrm{an}}\widetilde{\mathcal{A}} \to S$ be the proper structure map, and let $H \subseteq P$ be the hypersurface $Z_0 = 0$ (via the canonical closed S-immersion of P into $\mathbf{P}^n_S$). Clearly H is a Cartier divisor in P, though this property may be lost upon base change to infinitesimal fibers over S. We write $\mathcal{O}_P(dH)$ to denote the dth (tensor) power of the inverse of the invertible ideal sheaf of H in $\mathcal{O}_P$ for $d \geqslant 0$. Let $U = P - H = \operatorname{Spec}^{\mathrm{an}}\mathcal{A}$ and let $j : U \hookrightarrow P$ be the inclusion, so $h \circ j = f_{\mathcal{A}}$ is the structure map for $U = \operatorname{Spec}^{\mathrm{an}}\mathcal{A}$. Let $\mathcal{A}_d \subseteq \mathcal{A}$ be the coherent image of the coherent sheaf of polynomials with total degree $\leqslant d$ in $\mathcal{O}_S[T_1,\ldots,T_n]$, so $\mathcal{A} = \varinjlim \mathcal{A}_d$. There is a natural map $\mathcal{A}_d \to h_*(\mathcal{O}_P(dH))$, and the map $\mathcal{A} \to f_{\mathcal{A}*}\mathcal{O}_{\operatorname{Spec}^{\mathrm{an}}\mathcal{A}}$ is identified with the direct limit of the composites

$$\mathcal{A}_d \longrightarrow h_*\big(\mathcal{O}_P(dH)\big) \longrightarrow h_*j_*j^*\mathcal{O}_P(dH) = h_*j_*\mathcal{O}_U = f_{\mathcal{A}*}\mathcal{O}_{\operatorname{Spec}^{\mathrm{an}}\mathcal{A}}.$$

The second step in this composite is an injection because $\mathcal{O}_P(dH) \to j_*\mathcal{O}_U$ is injective (as H is Cartier in P), so to complete the proof it suffices to prove that the map $\mathcal{A}_d \to h_*(\mathcal{O}_P(dH))$ is injective on s-stalks for all $d \geqslant 1$ and all $s \in S$. This is a map of coherent sheaves because h is proper, and so it suffices to check the injectivity condition on completed stalks at each point $s \in S$; we may also restrict attention to large d possibly depending on s.

The rigid-analytic theorem on formal functions functorially expresses the complete stalks in terms of the associated formal schemes over $\mathrm{Spf}(\mathcal{O}_{S,s}^{\wedge})$, and so by the algebraic theorem on formal functions it is equivalent to prove the corresponding injectivity claim for $P_s^{\mathrm{alg}} \stackrel{\mathrm{def}}{=\!=} \mathrm{Proj}(\mathcal{O}_{S,s}^{\wedge} \otimes_{\mathcal{O}_{S,s}} \widetilde{\mathcal{A}}_s)$ over $\mathrm{Spec}(\mathcal{O}_{S,s}^{\wedge})$. For large d (depending on s) the map from the dth graded piece of $\mathcal{O}_{S,s}^{\wedge} \otimes_{\mathcal{O}_{S,s}} \widetilde{\mathcal{A}}_s$ to the global sections of $\mathcal{O}_{P_s^{\mathrm{alg}}}(d)$ is an isomorphism. Thus, using Z_0 to identify $\mathcal{O}_{P_s^{\mathrm{alg}}}(1)$ with the inverse of the invertible ideal of the Cartier divisor $\{Z_0 = 0\} \cap P_s^{\mathrm{alg}}$ in $\mathbf{P}^n_{\mathrm{Spec}\,\mathcal{O}_{S,s}^{\wedge}}$ gives the desired injectivity result. □

3. Ample line bundles

We are now in position to develop the theory of ampleness in rigid geometry. We first discuss some aspects that are peculiar to the special case when the base is a field, and then we turn to the relative case. The case when the base is a field is essentially well-known, but we address it in some detail because there are intermediate results whose formulation in terms of $\mathrm{Spec}^{\mathrm{an}}$ and $\mathrm{Proj}^{\mathrm{an}}$ over $\mathrm{Sp}(k)$ will be convenient to use in proofs concerning the relative case over a general base.

3.1. The absolute case

We refer the reader to §A.1 for a discussion of rigid-analytic GAGA, including its relative version.

DEFINITION 3.1.1. — *Let $f : X \to \mathrm{Sp}(k)$ be proper. An invertible sheaf $\mathcal{L}$ on X is* ample *(or k-*ample*) if some large tensor power $\mathcal{L}^{\otimes N}$ is generated by global sections (i.e., $f^*f_*\mathcal{L}^{\otimes N} \to \mathcal{L}^{\otimes N}$ is surjective) and the resulting canonical map $X \to \mathbf{P}^{\mathrm{an}}(\mathrm{H}^0(X, \mathcal{L}^{\otimes N})) = \mathbf{P}^{\mathrm{an}}(f_*\mathcal{L}^{\otimes N})$ to a rigid-analytic projective space is a closed immersion.*

LEMMA 3.1.2. — *Let $f : X \to \mathrm{Sp}(k)$ be proper, and $\mathcal{L}$ an invertible sheaf on X. For an analytic extension field k'/k, $\mathcal{L}$ is k-ample on X if and only if $k'\widehat{\otimes}_k\mathcal{L}$ is k'-ample on $X' = k'\widehat{\otimes}_k X$.*

Proof. — The property of $\mathcal{L}^{\otimes N}$ being generated by its global sections is insensitive to extension of the ground field because the formation of both $f_*\mathcal{L}^{\otimes N}$ and the support of the cokernel of the map $f^*f_*\mathcal{L}^{\otimes N} \to \mathcal{L}^{\otimes N}$

commute with extension of the ground field (Theorem A.1.2 and Corollary A.2.7). Hence, it suffices to prove that a map $h : X \to P$ between proper rigid spaces over k is a closed immersion if and only if its extension of scalars $h' : X' \to P'$ over k' is a closed immersion. This follows from Theorem A.2.6.(3) (and also admits a short direct proof since proper quasi-finite maps are finite). □

Remark 3.1.3. — Upon using GAGA to algebraize an ample pair $(X, \mathcal{L})$, the algebraization of $\mathcal{L}$ is ample on the algebraization of X. It therefore follows from the algebraic theory that, in the absence of ampleness hypotheses on $\mathcal{L}$, if k_0 is a finite extension of k and X is a proper rigid space over k_0 then the line bundle $\mathcal{L}$ is k-ample on X if and only if $\mathcal{L}$ is k_0-ample on X.

A further application of GAGA shows that $\mathcal{L}$ is k-ample on X if and only if there exists a closed immersion $j : X \hookrightarrow \mathbf{P}^m_k$ and a positive integer N such that $\mathcal{L}^{\otimes N}$ is isomorphic to $j^*\mathcal{O}_{\mathbf{P}^m_k}(1)$. The rigid-analytic cohomological properties of $\mathcal{O}(1)$ on $\mathbf{P}^m_k$ are the key input in the proof of GAGA over k, and these properties imply that for an ample $\mathcal{L}$ on X and any coherent sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ has vanishing higher cohomology and is generated by global sections for all large n. See Theorem 3.1.5 for a converse.

The following lemma imposes very strong assumptions (the finiteness hypothesis on $\mathcal{A}$ and the existence of the m_s's), but we will prove in Theorem 3.2.7 below that these assumptions are satisfied if $\mathcal{L}_s = \mathcal{L}|_{X_s}$ is ample on X_s for every $s \in S$, in which case we will prove that the map in (3.1.2) below is an isomorphism.

LEMMA 3.1.4. — *Let $f : X \to S$ be proper and let $\mathcal{L}$ be an invertible sheaf on X. For each $s \in S$ assume that there exists an integer $m_s > 0$ such that*

$$f^* f_* \mathcal{L}^{\otimes m_s} \to \mathcal{L}^{\otimes m_s} \tag{3.1.1}$$

is surjective over $f^{-1}(W_s)$ for an admissible open $W_s \subseteq S$ containing s. Define the graded $\mathcal{O}_S$-algebra $\mathcal{A} = \bigoplus_{n \geqslant 0} f_* \mathcal{L}^{\otimes n}$, and assume that $\mathcal{A}$ is locally finitely generated as an $\mathcal{O}_S$-algebra.

There exists a canonical S-map

$$\phi_{\mathcal{A}} : X \longrightarrow \mathrm{Proj}^{\mathrm{an}} \mathcal{A} \tag{3.1.2}$$

that is compatible with flat base change on S, change of the base field (for quasi-separated or pseudo-separated S), and analytification (via (2.3.6) and the relative GAGA *isomorphism* (A.1.1) *for direct images).*

Proof. — Let $\mathcal{A}^r = f_*\mathcal{L}^{\otimes r}$ denote the rth graded piece of $\mathcal{A}$. For any $a \in \mathrm{H}^0(X, \mathcal{L}) = \mathcal{A}^1(X)$, the support of the coherent cokernel of the map $a : \mathcal{O}_X \to \mathcal{L}$ is an analytic set $Z_a \subseteq X$. For the Zariski-open complement $U_a = X - Z_a$ where a generates $\mathcal{L}$ we get a canonical S-map

$$\rho_a : U_a \longrightarrow \mathrm{Spec}^{\mathrm{an}}\, \mathcal{A}_{(a)} \hookrightarrow \mathrm{Proj}^{\mathrm{an}}\, \mathcal{A}$$

defined by the map $\mathcal{A}_{(a)} \to f_*\mathcal{O}_{U_a}$ sending $t/a^{\otimes r} \in \mathcal{A}_{(a)}(V) = \mathcal{A}(V)_{(a)}$ to the section $g \in (f_*\mathcal{O}_{U_a})(V) = \mathcal{O}_X(U_a \cap f^{-1}(V))$, where $t \in \mathcal{A}^r(V) = (f_*\mathcal{L}^{\otimes r})(V) = \mathcal{L}^{\otimes r}(f^{-1}(V))$ restricts to $g \cdot a^{\otimes r}$ on $U_a \cap f^{-1}(V)$ and the subset $V \subseteq S$ is admissible.

For any $a, a' \in \mathcal{L}(X)$, clearly ρ_a and $\rho_{a'}$ coincide on $U_a \cap U_{a'}$. Moreover, for any admissible open $V \subseteq S$ the map $\rho_a|_{U_a \cap f^{-1}(V)} = \rho_{a|f^{-1}(V)}$ is the canonical one associated to the morphism $f^{-1}(V) \to V$ and the section $a|_{f^{-1}(V)}$ of $\mathcal{L}|_{f^{-1}(V)}$. Thus, if $f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective on X then we can carry out the construction of ρ_a locally over S for local sections a of the coherent sheaf $f_*\mathcal{L}$; gluing the resulting maps ρ_a defines an S-map

$$\phi_{\mathcal{A}} : X \longrightarrow \mathrm{Proj}^{\mathrm{an}}\, \mathcal{A}. \tag{3.1.3}$$

For $m > 0$, the maps $\phi_{\mathcal{A}}$ and $\phi_{\mathcal{A}^{(m)}}$ are compatible via the isomorphism in Corollary 2.3.10, so the map (3.1.3) is unaffected by replacing $\mathcal{L}$ with any $\mathcal{L}^{\otimes m}$ for $m > 0$. Thus, we can carry out the construction of (3.1.3) if the sets

$$U'_m = \left\{ s \in S \,|\, f^*f_*\mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m} \text{ is surjective on stalks at all } x \in X_s \right\}$$

for $m \geqslant 1$ are admissible opens and form an admissible cover of S. We shall now show that these conditions on the U'_m's hold.

For a fixed m we have $U'_m = S - f(Z_m)$ where

$$Z_m = \left\{ x \in X \,|\, f^*f_*\mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m} \text{ is not surjective at } x \right\}.$$

Since Z_m is the support of the coherent sheaf $\mathrm{coker}(f^*f_*\mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m})$, it is Zariski-closed in X. By properness of f, the locus U'_m is therefore Zariski-open in S. Again using this properness, the existence of the m_s's as in (3.1.1) says that the U'_m's set-theoretically cover S, so $\{U'_m\}$ is an admissible covering because each U'_m is a Zariski-open in S. This completes the construction of (3.1.2) whenever m_s's exist as in (3.1.1). By construction, $\phi_{\mathcal{A}}$ is compatible with flat base change, change of the base field (for quasi-separated or pseudo-separated S), and analytification. □

Here is the cohomological criterion for ampleness over a field; see Corollary 3.2.5 for a relativization.

THEOREM 3.1.5. — *Let $f : X \to \mathrm{Sp}(k)$ be proper and let $\mathcal{L}$ be an invertible sheaf on X. If $\mathrm{H}^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$ for all coherent ideal sheaves $\mathcal{I}$ on X and all large n (depending on $\mathcal{I}$) then $\mathcal{L}$ is k-ample.*

Proof. — The proofs of the analogous theorem for schemes and complex-analytic spaces use pointwise arguments (such as ideal sheaves of points), and so admissibility problems arise when trying to use these proofs for rigid spaces. We shall adapt the principles of the algebraic and complex-analytic proofs via a suitable application of Lemma 3.1.4 and GAGA.

Pick $x \in X$, and let $\mathcal{I}_{\{x\}}$ be the coherent ideal associated to

$$i_x : \mathrm{Sp}\big(k(x)\big) \hookrightarrow X.$$

The short exact sequence

$$0 \to \mathcal{I}_{\{x\}} \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathcal{L}^{\otimes n} \longrightarrow i_{x*}(\mathcal{L}_x^{\otimes}/\mathfrak{m}_x\mathcal{L}_x^{\otimes n}) \to 0$$

induces a surjection $\mathrm{H}^0(X, \mathcal{L}^n) \twoheadrightarrow \mathcal{L}_x^{\otimes n}/\mathfrak{m}_x\mathcal{L}_x^{\otimes n}$ for large n depending on x. Thus, the map $f^*f_*\mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}$ between coherent sheaves is surjective on the stalks at x for large n depending on x. The Zariski-open locus $U_n = X - \mathrm{coker}(f^*f_*\mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n})$ therefore contains x when $n \geqslant n(x)$ for some $n(x) > 0$. Let $V_x = U_{n(x)} \cap \ldots \cap U_{2n(x)-1}$, so $f^*f_*\mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}$ is surjective on V_x for $n(x) \leqslant n < 2n(x)$.

For $n \geqslant 2n(x)$ we have $n = n(x)q + r$ with $q \geqslant 1$ and $n(x) \leqslant r < 2n(x)$, and the composite map of coherent sheaves

$$(f^*f_*\mathcal{L}^{\otimes n(x)})^{\otimes q} \otimes f^*f_*\mathcal{L}^{\otimes r} \longrightarrow f^*f_*\mathcal{L}^{\otimes n} \longrightarrow \mathcal{L}^{\otimes n}$$

is equal to

$$(f^*f_*\mathcal{L}^{\otimes n(x)})^{\otimes q} \otimes f^*f_*\mathcal{L}^{\otimes r} \longrightarrow (\mathcal{L}^{\otimes n(x)})^{\otimes q} \otimes \mathcal{L}^{\otimes r} = \mathcal{L}^{\otimes n}.$$

This second map is visibly surjective over V_x. Thus, $f^*f_*\mathcal{L}^{\otimes n}|_{V_x} \to \mathcal{L}^{\otimes n}|_{V_x}$ is surjective for *all* $n \geqslant n(x)$. The Zariski covering $\{V_x\}_{x \in X}$ of the quasi-compact X has a finite subcovering, so there exists $n_0 > 0$ such that $f^*f_*\mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}$ is surjective for $n \geqslant n_0$. That is, $\mathcal{L}^{\otimes n}$ is generated by $\mathrm{H}^0(X, \mathcal{L}^{\otimes n})$ for all $n \geqslant n_0$.

If $x \neq x'$ are distinct points in X, then using the short exact sequence

$$0 \to \mathcal{I}_{\{x,x'\}} \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathcal{I}_{\{x'\}} \otimes \mathcal{L}^{\otimes n} \longrightarrow i_{x*}(\mathcal{L}_x^{\otimes n}/\mathfrak{m}_x\mathcal{L}_x^{\otimes n}) \to 0$$

and the cohomological hypothesis on $\mathcal{L}$ shows that $\mathrm{H}^0(X, \mathcal{I}_{\{x'\}} \otimes \mathcal{L}^{\otimes n}) \to \mathcal{L}_x^{\otimes n}/\mathfrak{m}_x\mathcal{L}_x^{\otimes n}$ is surjective for large n (*a priori* depending on $\{x, x'\}$). In particular, there exists $n > 0$ and $s \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})$ that each depend on $\{x, x'\}$ such that

$$s_x \in \mathfrak{m}_x\mathcal{L}_x^{\otimes n}, \quad s_{x'} \notin \mathfrak{m}_{x'}\mathcal{L}_{x'}^{\otimes n}. \tag{3.1.4}$$

Consider the short exact sequence $0 \to \mathcal{K} \to f^*f_*\mathcal{L}^{\otimes n_0} \to \mathcal{L}^{\otimes n_0} \to 0$ with $\mathcal{K}$ defined to be the kernel term. Since the coherent sheaf $f_*\mathcal{L}^{\otimes n_0}$ on $\mathrm{Sp}(k)$ is globally free, its pullback by f is globally free of finite rank on X. Hence, $\mathcal{K}$ has a finite filtration whose successive quotients are isomorphic to coherent ideal sheaves. By the cohomological hypothesis on $\mathcal{L}$, we therefore have $\mathrm{H}^1(X, \mathcal{K} \otimes \mathcal{L}^{\otimes n}) = 0$ for large n. Using the projection formula, we get a commutative diagram

$$\begin{array}{ccc} f_*\mathcal{L}^{\otimes n_0} \otimes f_*\mathcal{L}^{\otimes n} & \longrightarrow & f_*(\mathcal{L}^{\otimes n_0} \otimes \mathcal{L}^{\otimes n}) \\ \simeq \downarrow & & \downarrow \simeq \\ f_*\big((f^*f_*\mathcal{L}^{\otimes n_0}) \otimes \mathcal{L}^{\otimes n}\big) & \longrightarrow & f_*\mathcal{L}^{\otimes n+n_0} \end{array}$$

on $\mathrm{Sp}(k)$. The bottom side is a surjection for large n since $\mathrm{H}^1(X, \mathcal{K} \otimes \mathcal{L}^{\otimes n}) = 0$ for large n. Thus, the top side is surjective for large n. It follows that the graded k-algebra $\bigoplus_{n \geqslant 0} \mathrm{H}^0(X, \mathcal{L}^{\otimes n})$ is *finitely generated*. We may therefore use Lemma 3.1.4 (with $m_s = n_0$ for the unique point s in $S = \mathrm{Sp}(k)$) to obtain a canonical morphism of rigid spaces

$$\iota : X \longrightarrow P = \mathrm{Proj}^{\mathrm{an}}\Big(\bigoplus_{n \geqslant 0} \mathrm{H}^0\big(X, \mathcal{L}^{\otimes n}\big)\Big).$$

By (3.1.4) and the definition of ι, if $x \neq x'$ are distinct points in X then $\iota(x) \neq \iota(x')$ because (by Remark 2.3.4) there is a Zariski-open in $\mathrm{Proj}^{\mathrm{an}}(\bigoplus_{n \geqslant 0} \mathrm{H}^0(X, \mathcal{L}^{\otimes n}))$ that contains $\iota(x)$ but does not contain $\iota(x')$. Thus, ι has finite fibers. The map $\iota : X \to P$ is proper, so X is finite over P. By Theorem 2.3.8 (2), the rigid space P is projective. Since finite maps of rigid spaces are classified by coherent sheaves of algebras as in Corollary 2.2.8, we may infer from Theorem 2.2.5(3) and GAGA on P that the P-finite rigid space X is the analytification of a scheme finite over a projective k-scheme. Hence, X is projective. By rigid GAGA on X, the cohomological hypothesis on $\mathcal{L}$ and the cohomological ampleness criterion in the algebraic case imply that the invertible sheaf $\mathcal{L}$ on X is ample in the sense of Definition 3.1.1. (By [15, III$_1$, 2.3.4.1] and GAGA, it now follows that ι is even an isomorphism.) □

By standard cohomological arguments, we deduce:

COROLLARY 3.1.6. — *Let X be a proper rigid space over k. An invertible sheaf $\mathcal{L}$ on X is ample on X if and only if $\mathcal{L}|_{X_{\mathrm{red}}}$ is ample on X_{red}.*

3.2. Relative ampleness

There are (at least) two ways to introduce the relative theory of ampleness, both of which will be proved to be equivalent. In the theory of schemes, relative ampleness is defined rather generally for any quasi-compact and separated morphism of schemes [15, II, 4.5.3, 4.6.1]. In the proper case, it admits a fibral characterization:

THEOREM 3.2.1. — *Let $f : X \to S$ be a proper and finitely presented morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. The set U of $s \in S$ such that $\mathcal{L}_s = \mathcal{L}|_{X_s}$ is ample on X_s is Zariski-open in S, and $\mathcal{L}|_{f^{-1}(U)}$ is U-ample. In particular, $\mathcal{L}$ is S-ample if and only if $\mathcal{L}_s$ is ample on X_s for all $s \in S$.*

Proof. — This is [15, IV$_3$, 9.6.4]. □

Theorem 3.2.1 motivates the following definition in the rigid-analytic case:

DEFINITION 3.2.2. — *Let $f : X \to S$ be a proper morphism of rigid spaces. An invertible sheaf $\mathcal{L}$ on X is* S-ample, *or* relatively ample over S, *if $\mathcal{L}_s$ is ample on X_s for every $s \in S$.*

It is obvious that S-ampleness is preserved by base change, but it is not obvious *a priori* (for quasi-separated or pseudo-separated S) whether or not relative ampleness is insenstive to change of the ground field; see Corollary 3.2.8 for the affirmative result. It is also not obvious if S-ample line bundles satisfy a relative version of the Cartan-Serre theorems A and B for $\mathcal{O}(1)$ (as was proved in the complex-analytic case by Grauert and Remmert [20] when projective embeddings are assumed to exist locally on the base).

A satisfactory relative theory of ampleness requires linking our fibral definition with properties of higher direct images and maps to relative projective spaces. It saves no effort to cheat by replacing Definition 3.2.2 with the property of being "relatively potentially very ample" (that is, locally over S, a high tensor power of $\mathcal{L}$ is a pullback of $\mathcal{O}(1)$ under a projective embedding over the base): at some point in the development of the theory (for applications in abstract situations) one has to confront the problem of proving the equivalence of "relatively potentially very ample" and fibral ampleness. Our strategy for proving such an equivalence is to prove that fibral ampleness implies that the hypotheses in Lemma 3.1.4 are satisfied and that (3.1.2) is an isomorphism in such cases. The development

of the relative theory cannot be substantially simplified by merely adopting a different initial definition.

The first step is the trivial observation that infinitesimal deformations of a projective algebraic rigid space remain projective algebraic if we can deform an ample line bundle:

LEMMA 3.2.3. — *Let A_0 be a finite local k-algebra, and let $X \to \mathrm{Sp}(A_0)$ be a proper morphism. If there exists a $\mathrm{Sp}(A_0)$-ample invertible sheaf $\mathcal{L}$ on X then $\mathcal{L}$ is k-ample on the k-proper X and hence X is projective algebraic over k.*

Proof. — Corollary 3.1.6 reduces us to the case when A_0 is replaced with its residue field k_0 of finite degree over k, and the equivalence of k_0-ampleness and k-ampleness was noted in Remark 3.1.3. □

Under the hypothesis in Lemma 3.2.3, $\operatorname{Spec} A_0$ is k-finite and

$$(\operatorname{Spec} A_0)^{\mathrm{an}} \simeq \operatorname{Sp} A_0.$$

Thus, it follows from GAGA that there exists a $\mathrm{Sp}(A_0)$-isomorphism $X \simeq Y^{\mathrm{an}}$ where Y is a projective scheme over $\operatorname{Spec}(A_0)$, and under this isomorphism we have $\mathcal{L} \simeq \mathcal{M}^{\mathrm{an}}$ for an invertible sheaf $\mathcal{M}$ on Y that must be A_0-ample (as this algebraic ampleness can be checked on the algebraic closed fiber, since A_0 is an Artin local ring and ampleness is insensitive to nilpotents on the algebraic side). In other words, the pair $(X, \mathcal{L})$ in Lemma 3.2.3 is "projective algebraic over $\operatorname{Spec}(A_0)$."

THEOREM 3.2.4. — *Let $f : X \to S$ be proper, $\mathcal{L}$ an S-ample invertible sheaf on X, and $\mathcal{F}$ a coherent sheaf on X. There exists an admissible open covering $\{U_\alpha\}$ of S and integers $n_\alpha > 0$ such that*

1) $\mathrm{R}^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})|_{U_\alpha} = 0$ for all $p > 0$, $n \geqslant n_\alpha$;

2) the natural map $f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})|_{f^{-1}(U_\alpha)} \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_{f^{-1}(U_\alpha)}$ is surjective for $n \geqslant n_\alpha$.

Proof. — Choose $s \in S$ and let $R_n = \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$. Let $X_n \to \mathrm{Sp}(R_n)$ be the nth infinitesimal neighborhood of the fiber X_s. By Corollary 3.1.6, the line bundle $\mathcal{L}_n = \mathcal{L}|_{X_n}$ is $\mathrm{Sp}(R_n)$-ample on X_n because $\mathcal{L}_s$ is ample on X_s. Define $\mathcal{F}_n = \mathcal{F}|_{X_n}$, so $\mathcal{F}_n$ is a coherent sheaf on X_n. We may apply Lemma 3.2.3 to the pair $(X_n, \mathcal{L}_n)$ over $\mathrm{Sp}(R_n)$. By rigid GAGA, the triple of rigid-analytic data $(X_n, \mathcal{L}_n, \mathcal{F}_n)$ is the analytification of a unique (up to unique isomorphism) triple of algebraic data $(Y_n, \mathcal{M}_n, \mathcal{G}_n)$ over $\operatorname{Spec}(R_n)$, where Y_n is a proper R_n-scheme, $\mathcal{M}_n$ is an R_n-ample invertible sheaf on Y_n, and $\mathcal{G}_n$ is a coherent sheaf on Y_n. By the uniqueness, this algebraic data

is compatible with change in n. By [15, I, 10.6, 10.11], this system of algebraic data defines a triple $(\mathfrak{Y}, \mathfrak{M}, \mathfrak{G})$ where $\widehat{f} : \mathfrak{Y} \to \mathrm{Spf}(\widehat{\mathcal{O}}_{S,s})$ is a proper formal scheme, $\mathfrak{M}$ is an *invertible* sheaf on $\mathfrak{Y}$, and $\mathfrak{G}$ is a coherent sheaf on $\mathfrak{Y}$. The $\mathrm{Spec}(R_n)$-fiber of $(\mathfrak{Y}, \mathfrak{M}, \mathfrak{G})$ is $(Y_n, \mathcal{M}_n, \mathcal{G}_n)$. We emphasize (for later purposes) that only the hypothesis of ampleness for $\mathcal{L}_s$ on X_s (and no hypotheses on higher infinitesimal fibers of f) has provided us with a proper formal scheme over $\mathrm{Spf}\,\widehat{\mathcal{O}}_{S,s}$ equipped with a line bundle $\mathfrak{M}$ that has ample reduction $\mathcal{M}_0$ modulo $\mathfrak{m}_s\widehat{\mathcal{O}}_{S,s}$. This will open the door to using the cohomological theory of formal schemes to solve our problems.

Step 1. — Since the reduction $\mathcal{M}_0$ of $\mathfrak{M}$ on Y_0 is ample, it follows from the Grothendieck algebraization theorem [15, III_1, 5.4.5] and Grothendieck's formal GAGA theorem [15, III_1, 5.1.6, 5.4.1] that the formal scheme data $(\mathfrak{Y}, \mathfrak{M}, \mathfrak{G})$ arises as the formal completion (along the closed fiber) of unique algebraic data $(Y, \mathcal{M}, \mathcal{G})$ where Y is a proper scheme over $\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s})$, $\mathcal{M}$ is an invertible sheaf on Y, and $\mathcal{G}$ is a coherent sheaf on Y. The reduction of this data modulo $\mathfrak{m}_s^{n+1}$ is the algebraized triple $(Y_n, \mathcal{M}_n, \mathcal{G}_n)$ as defined above via GAGA over R_n. In particular, since Y is $\widehat{\mathcal{O}}_{S,s}$-proper it follows from Theorem 3.2.1 that $\mathcal{M}$ is (relatively) ample on Y over $\mathrm{Spec}\,\widehat{\mathcal{O}}_{S,s}$ (as this is true for $\mathcal{M}_0$ on Y_0 over $\mathrm{Spec}\,k(s)$ because $\mathcal{L}_s$ is ample on X_s). By the *algebraic* cohomological theory of ample line bundles, for large n (depending on s) we have that $\mathrm{H}^p(Y, \mathcal{G} \otimes \mathcal{M}^{\otimes n})$ vanishes for all $p > 0$ and $\mathcal{G} \otimes \mathcal{M}^{\otimes n}$ is generated by global sections.

It follows that for large n, $\mathfrak{G} \otimes \mathfrak{M}^{\otimes n}$ is generated by global sections and (by Grothendieck's scheme-theoretic theorem on formal functions)

$$\varprojlim_m \mathrm{H}^p(Y_m, \mathcal{G}_m \otimes \mathcal{M}_m^{\otimes n}) \simeq \mathrm{H}^p(Y, \mathcal{G} \otimes \mathcal{M}^{\otimes n}) = 0$$

for all $p > 0$. Rigid GAGA and Kiehl's rigid-analytic theorem on formal functions identify this inverse limit with

$$\varprojlim_m \mathrm{H}^p(X_m, \mathcal{F}_m \otimes \mathcal{L}_m^{\otimes n}) \simeq \widehat{\mathcal{O}}_{S,s} \otimes_{\mathcal{O}_{S,s}} \mathrm{R}^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})_s.$$

Thus, for some $n_s > 0$ depending on s we have $\mathrm{R}^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})_s = 0$ for all $p > 0$ and all $n \geqslant n_s$.

Step 2. — Now consider the maps

$$\theta_n : f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \longrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes n} \tag{3.2.1}$$

for $n \geqslant 0$. For fixed $n \geqslant 0$, $\mathrm{coker}\,\theta_n$ has analytic support Z_n in X. If Z_n is disjoint from a fiber X_s for some $s \in S$ then $S - f(Z_n)$ is a Zariski-open neighborhood of s over which θ_n is surjective.

We now fix $s \in S$ and shall show that for some $n'_s > 0$ and all $n \geqslant n'_s$, the map θ_n between coherent sheaves induces a surjection on stalks at each $x \in X_s$ (that is, $Z_n \cap X_s = \emptyset$ for all $n \geqslant n'_s$). To prove surjectivity on x-stalks it is equivalent to check surjectivity after applying the functor $\widehat{\mathcal{O}}_{X,x} \otimes_{\mathcal{O}_{X,x}} (\cdot)$ to such stalks. Thus, for all large n (depending on s), we want the natural map

$$\widehat{\mathcal{O}}_{X,x} \otimes_{\widehat{\mathcal{O}}_{S,s}} (\widehat{\mathcal{O}}_{S,s} \otimes_{\mathcal{O}_{S,s}} f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})_s) \to \widehat{\mathcal{O}}_{X,x} \otimes_{\mathcal{O}_{X,x}} (\mathcal{F} \otimes \mathcal{L}^{\otimes n})_x \tag{3.2.2}$$

to be surjective for all $x \in X_s$.

Recall the algebraic data $(Y, \mathcal{M}, \mathcal{G})$ over $\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s})$ and the formal scheme data $(\mathfrak{Y}, \mathfrak{M}, \mathfrak{G})$ over $\mathrm{Spf}(\widehat{\mathcal{O}}_{S,s})$ that we introduced above. By the theory of analytification, the underlying set of $X_s \simeq Y_0^{\mathrm{an}}$ is canonically identified with the set of closed points in the closed fiber Y_0 of Y. Moreover, the underlying topological spaces of Y_0 and $\mathfrak{Y}$ are canonically identified. Let $y \in Y_0$ be the closed point corresponding in this way to $x \in X_s$. Since the mth infinitesimal neighborhood of X_s in X is identified with the analytification Y_m^{an} of the mth infinitesimal neighborhood Y_m of Y_0 in Y in a manner that is compatible with change in m, it is easy to check that there are canonical isomorphisms of $\widehat{\mathcal{O}}_{S,s}$-algebras

$$\widehat{\mathcal{O}}_{\mathfrak{Y},y} \simeq \widehat{\mathcal{O}}_{Y,y} \simeq \widehat{\mathcal{O}}_{X,x}.$$

Using this isomorphism and both rigid and formal GAGA, it follows via the rigid-analytic and scheme-theoretic theorems on formal functions that (3.2.2) is identified with the natural map

$$\widehat{\mathcal{O}}_{\mathfrak{Y},y} \otimes_{\widehat{\mathcal{O}}_{S,s}} \mathrm{H}^0(\mathfrak{Y}, \mathfrak{G} \otimes \mathcal{M}^{\otimes n}) \longrightarrow \widehat{\mathcal{O}}_{\mathfrak{Y},y} \otimes_{\mathcal{O}_{\mathfrak{Y},y}} (\mathfrak{G} \otimes \mathfrak{M}^{\otimes n})_y. \tag{3.2.3}$$

In Step 1 we saw that $\mathfrak{G} \otimes \mathfrak{M}^{\otimes n}$ is generated by global sections for all large n (depending on s). The surjectivity of (3.2.3) is clear for such large n (*independent* of $y = x \in X_s$ for fixed s). Hence, for fixed $s \in S$ we have found n'_s such that (3.2.2) is surjective for all $x \in X_s$ when $n \geqslant n'_s$. That is, $Z_n \cap X_s = \emptyset$ for all $n \geqslant n'_s$, and so for each $n \geqslant n'_s$ there exists a Zariski-open neighborhood $S - f(Z_n)$ of s in S over which θ_n is surjective.

Step 3. — The preceding general analysis may be applied to $\mathcal{O}_X$ in the role of $\mathcal{F}$. Hence, we can find an admissible cover $\{U_\alpha\}$ of S so that (i) $f^* f_* \mathcal{L}^{\otimes m_\alpha} \to \mathcal{L}^{\otimes m_\alpha}$ is surjective over $f^{-1}(U_\alpha)$ for some $m_\alpha > 0$ and (ii) the coherent sheaf $f_* \mathcal{L}^{\otimes m_\alpha}|_{U_\alpha}$ is the quotient of a free $\mathcal{O}_{U_\alpha}$-module of finite rank. In particular, $\mathcal{L}^{\otimes m_\alpha}$ is a quotient of a free $\mathcal{O}_{f^{-1}(U_\alpha)}$-module $\mathcal{E}_\alpha$ of finite rank.

By using the "pointwise" conclusion of Step 2 for $\mathcal{F}$ and a fixed $s \in U_\alpha$ with n ranging through m_α consecutive integers at least as large as n'_s,

we arrive at a Zariski-open (and hence admissible) cover $\{U_{\alpha\beta}\}_{\beta\in B_\alpha}$ of U_α and integers $n_{\alpha\beta} > 0$ such that the restriction of θ_n over $f^{-1}(U_{\alpha\beta})$ is surjective for n satisfying $n_{\alpha\beta} \leqslant n < n_{\alpha\beta}+m_\alpha$. Since $f^*f_*\mathcal{L}^{\otimes m_\alpha} \to \mathcal{L}^{\otimes m_\alpha}$ is surjective over $f^{-1}(U_\alpha)$, we conclude that θ_n is surjective over $f^{-1}(U_{\alpha\beta})$ for *all* $n \geqslant n_{\alpha\beta}$. Renaming $\{U_{\alpha\beta}\}_{\alpha,\beta}$ as $\{U_\alpha\}$ and $n_{\alpha\beta}$ as n_α, we may suppose that θ_n is surjective over $f^{-1}(U_\alpha)$ for all $n \geqslant n_\alpha$ for a suitable $n_\alpha > 0$. We also may and do replace $\{U_\alpha\}$ with a refinement (still called $\{U_\alpha\}$) so that the restriction $f_\alpha : f^{-1}(U_\alpha) \to U_\alpha$ satisfies $\mathrm{R}^p f_{\alpha*}\mathcal{G} = 0$ for *all* coherent sheaves $\mathcal{G}$ on $f^{-1}(U_\alpha)$ and all $p \geqslant p_\alpha$ for some $p_\alpha > 0$. For example, by the theorem on formal functions, we can take each U_α to be sufficiently small so that there is a finite upper bound p_α on the number of affinoids that suffice to cover each fiber of f over U_α. (In the complex-analytic case we take $\{U_\alpha\}$ to be a collection of compact Stein sets whose interiors cover S, and since $f^{-1}(U_\alpha)$ is compact there is a finite upper bound p_α on the number of open Stein sets that cover each fiber of f over U_α.)

It remains to prove the vanishing of $\mathrm{R}^p f_*(\mathcal{F}\otimes\mathcal{L}^{\otimes n})|_{U_\alpha}$ for all $p > 0$ and *all* large n (depending on α and not on p). We now fix one U_α and use descending induction on $p > 0$: the inductive hypothesis is that for *all* coherent sheaves $\mathcal{G}$ on $f^{-1}(U_\alpha)$, the sheaf $\mathrm{R}^{p'} f_{\alpha*}(\mathcal{G}\otimes\mathcal{L}^{\otimes n})$ vanishes for all $p' \geqslant p$ and all sufficiently large n (that may depend on $\mathcal{G}$ and on p). This inductive hypothesis is satisfied for $p = p_\alpha$. We now assume that the inductive hypothesis is known for $p+1$ with some $p > 0$, and we seek to check it for p. It was arranged above (by pullback from U_α) that $\mathcal{L}^{\otimes m_\alpha}|_{f^{-1}(U_\alpha)}$ is the quotient of a free coherent sheaf $\mathcal{E}_\alpha$. Thus, for each coherent sheaf $\mathcal{G}$ on $f^{-1}(U_\alpha)$ we get a short exact sequence

$$0 \to \mathcal{K} \longrightarrow \mathcal{G}\otimes\mathcal{E}_\alpha \longrightarrow \mathcal{G}\otimes\mathcal{L}^{\otimes m_\alpha} \to 0$$

for some coherent sheaf $\mathcal{K}$ on $f^{-1}(U_\alpha)$. By the inductive hypothesis applied to $\mathcal{K}$, for some $n' \geqslant 0$ the sheaves $\mathrm{R}^{p'} f_{\alpha*}(\mathcal{K}\otimes\mathcal{L}^{\otimes n})$ vanish for all $p' \geqslant p+1$ and all $n \geqslant n'$. We conclude that for $n \geqslant n'$, $\mathrm{R}^p f_{\alpha*}(\mathcal{G}\otimes\mathcal{L}^{\otimes(n+m_\alpha)})$ is a quotient of a direct sum of finitely many copies of $\mathrm{R}^p f_{\alpha*}(\mathcal{G}\otimes\mathcal{L}^{\otimes n})$.

Choose $s \in U_\alpha$. Since $p > 0$, by Step 1 there exists a Zariski-open $V_s \subseteq U_\alpha$ containing s such that $\mathrm{R}^p f_{\alpha*}(\mathcal{G}\otimes\mathcal{L}^{\otimes n})|_{V_s} = 0$ if $n_{V_s} \leqslant n < n_{V_s} + m_\alpha$, with $n_{V_s} = \max(n', n_s) \geqslant n_s$. Combining this with the fact that $\mathrm{R}^p f_{\alpha*}(\mathcal{G}\otimes\mathcal{L}^{\otimes\nu})$ is a quotient of a direct sum of finitely many copies of $\mathrm{R}^p f_{\alpha*}(\mathcal{G}\otimes\mathcal{L}^{\otimes(\nu-m_\alpha)})$ for all $\nu \geqslant n_{V_s} + m_\alpha$ (since $n_{V_s} \geqslant n'$), it follows by induction on $\lfloor (n - n_{V_s})/m_\alpha \rfloor$ that $\mathrm{R}^p f_{\alpha*}(\mathcal{G}\otimes\mathcal{L}^{\otimes n})|_{V_s} = 0$ for *all* $n \geqslant n_{V_s}$. The Zariski-opens V_s ($s \in U_\alpha$) cover U_α set-theoretically, and so there exists a finite subcovering. The maximum of the n_{V_s}'s associated to the constituents of this finite subcovering is an $N > 0$ depending on $\mathcal{G}$

(and p and α) such that $\mathrm{R}^p f_{\alpha *}(\mathcal{G} \otimes \mathcal{L}^{\otimes n}) = 0$ on U_α for all $n \geqslant N$. This completes the inductive step, and so for each α there exists $N_\alpha > 0$ such that $\mathrm{R}^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})|_{U_\alpha} = 0$ for all $p > 0$ and all $n \geqslant N_\alpha$. □

COROLLARY 3.2.5. — *Let C be k-affinoid and let $f : X \to S = \mathrm{Sp}(C)$ be a proper map of rigid spaces. Let $\mathcal{L}$ be a line bundle $\mathcal{L}$ on X. If $\mathcal{L}$ is S-ample then for all coherent sheaves $\mathcal{F}$ on X and all sufficiently large n, $\mathcal{O}_X \otimes_C \mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is surjective and $\mathrm{H}^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $p > 0$. Conversely, if $\mathrm{H}^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$ for all coherent ideal sheaves $\mathcal{I}$ on X and all sufficiently large n (depending on $\mathcal{I}$) then $\mathcal{L}$ is S-ample.*

This corollary admits an evident generalization for any quasi-compact S, using higher direct images rather than cohomology modules. Note that $\mathrm{H}^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = \mathrm{H}^0(S, f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}))$ is C-finite.

Proof. — Since S is quasi-compact and $\mathrm{H}^p(X, \mathcal{G}) = \Gamma(S, \mathrm{R}^p f_*(\mathcal{G}))$ for all $p \geqslant 0$ and all coherent $\mathcal{G}$ on S (see Remark A.1.1), Theorem 3.2.4 provides the necessity of the asserted cohomological properties of S-ample line bundles on X. Now suppose that $\mathrm{H}^1(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) = 0$ for all coherent ideal sheaves $\mathcal{I}$ on X and all sufficiently large n (depending on $\mathcal{I}$). To prove that $\mathcal{L}$ is S-ample we pick $s \in S$ and we seek to show that $\mathcal{L}_s$ is ample on X_s. By Theorem 3.1.5, it suffices to prove that if $\mathcal{J}$ is a coherent ideal sheaf on X_s then $\mathrm{H}^1(X_s, \mathcal{J} \otimes \mathcal{L}_s^{\otimes n}) = 0$ for all large n. If $j : X_s \to X$ is the canonical closed immersion then $j_*\mathcal{J}$ is a coherent ideal on X and $\mathrm{H}^1(X_s, \mathcal{J} \otimes \mathcal{L}_s^{\otimes n}) \simeq \mathrm{H}^1(X, (j_*\mathcal{J}) \otimes \mathcal{L}^{\otimes n})$. Hence, we get the required vanishing for large n. □

Example 3.2.6. — We may now give a new quick proof of the relative version of GAGA over affinoids [33, §5–§6]. This consists of two parts: comparison of cohomology and an equivalence of categories of coherent sheaves. Let C be a k-affinoid algebra. For any locally finite type C-scheme $\mathcal{X}$ we write $\mathcal{F} \rightsquigarrow \mathcal{F}^{\mathrm{an}}$ to denote the exact pullback functor from $\mathcal{O}_{\mathcal{X}}$-modules to $\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}$-modules via the flat map $\iota_{\mathcal{X}} : \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ as in Example 2.2.11. As we saw in Example 2.3.11, for any proper map $f : \mathcal{X} \to \mathcal{Y}$ between locally finite type C-schemes, the relative analytification $f^{\mathrm{an}} : \mathcal{X}^{\mathrm{an}} \to \mathcal{Y}^{\mathrm{an}}$ over $\mathrm{Sp}(C)$ is proper. For any such f and any $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ there is a natural map of $\mathcal{O}_{\mathcal{Y}^{\mathrm{an}}}$-modules $\mathrm{R}^i f_*(\mathcal{F})^{\mathrm{an}} \to \mathrm{R}^i f_*^{\mathrm{an}}(\mathcal{F}^{\mathrm{an}})$ defined by δ-functoriality, and it makes sense to ask if this is an isomorphism for coherent $\mathcal{F}$. Since the source and target sheaves are coherent, it is equivalent to check the isomorphism property on completed stalks. Hence, the theorems on formal functions for schemes and rigid spaces (and the isomorphism $\mathcal{O}^{\wedge}_{\mathcal{Y}, \iota_{\mathcal{Y}}(y)} \simeq \mathcal{O}^{\wedge}_{\mathcal{Y}^{\mathrm{an}}, y}$ for

all $y \in \mathcal{Y}^{\mathrm{an}}$) reduce the problem to the special case when $\mathcal{Y} = \mathrm{Spec}(R)$ for a k-finite local C-algebra R. We may rename R as C, so $\mathcal{X}$ is k-proper and hence upon replacing the higher direct images with cohomology modules we can use GAGA over k to get the desired result.

In the case of C-proper $\mathcal{X}$ there remains the task of proving that $\mathcal{F} \rightsquigarrow \mathcal{F}^{\mathrm{an}}$ sets up an equivalence between the categories of coherent sheaves on $\mathcal{X}$ and on $\mathcal{X}^{\mathrm{an}}$. Since the k-affinoid C is a Jacobson ring whose maximal ideals have k-finite residue field, a coherent sheaf on the C-proper $\mathcal{X}$ vanishes if and only if it vanishes on fibers over closed points of $\mathrm{Spec}\, C$. It follows that a complex of coherent $\mathcal{O}_{\mathcal{X}}$-modules is exact if and only if its analytification is exact. Thus, exactly as in the complex-analytic case [26, XII, §4], Grothendieck's method (via his unscrewing lemma and Chow's lemma) reduces the equivalence problem for $\mathrm{Coh}(\mathcal{X}) \to \mathrm{Coh}(\mathcal{X}^{\mathrm{an}})$ in the general case to special case of C-projective $\mathcal{X}$. Theorem 3.2.4 over $S = \mathrm{Sp}(C)$ provides the required cohomological input to push through Serre's method in the projective case. (In particular, Serre's argument with Hom-sheaves and coherent ideal sheaves proves that $\mathcal{X} \rightsquigarrow \mathcal{X}^{\mathrm{an}}$ is fully faithful on proper C-schemes.)

An important property of $\mathrm{Proj}^{\mathrm{an}}$ is that, just as in algebraic geometry, it permits us to recover a proper object from the sections of powers of a given ample line bundle. More precisely, S-ampleness is equivalent to "relative very ampleness" locally on the base:

THEOREM 3.2.7. — *For an arbitrary proper map $f : X \to S$ and S-ample invertible sheaf $\mathcal{L}$ on X, the hypotheses on $\mathcal{L}$ and $\mathcal{A} = \bigoplus_{n \geqslant 0} f_*\mathcal{L}^{\otimes n}$ in Lemma* 3.1.4 *are satisfied and the map* (3.1.2) *is an isomorphism. Moreover, locally on S there is a large tensor power $\mathcal{L}^{\otimes N}$ generated by global sections such that the quasi-coherent graded $\mathcal{O}_S$-algebra $\bigoplus_{m \geqslant 0} f_*\mathcal{L}^{\otimes mN}$ is generated by its degree-*1 *term over its degree-*0 *term.*

In particular, there exists an admissible covering $\{S_i\}$ of S, positive integers n_i and N_i, and closed S_i-immersions $j_i : X_i \hookrightarrow \mathbf{P}^{n_i}_k \times S_i$ such that $j_i^*\mathcal{O}(1) \simeq \mathcal{L}|_{X_i}^{\otimes N_i}$, where $X_i = X|_{S_i}$.

Proof. — By Theorem 3.2.4 we deduce the existence of the m_s's as required in Lemma 3.1.4. By working locally on S, Theorem 3.2.4 reduces us to the case where there exists an $n_0 > 0$ such that $f^*f_*(\mathcal{L}^{\otimes n}) \to \mathcal{L}^{\otimes n}$ is surjective for all $n \geqslant n_0$. Before showing that $\mathcal{A}$ is locally finitely generated (so $\mathrm{Proj}^{\mathrm{an}}\, \mathcal{A}$ makes sense) and that the resulting map $X \to \mathrm{Proj}^{\mathrm{an}}\, \mathcal{A}$ in (3.1.2) is an isomorphism, we shall prove a weaker assertion: for fixed $s \in S$ we claim that there exists a Zariski-open neighborhood of s over

which the restriction of the natural map

$$\iota_n : X \longrightarrow P_n \stackrel{\text{def}}{=\!=} \mathbf{P}^{\text{an}}(f_*\mathcal{L}^{\otimes n}) \tag{3.2.4}$$

for $n \geqslant n_0$ is a closed immersion for all $n \geqslant n_s$ for some $n_s \geqslant n_0$ depending on s. In Steps 1 and 2 below we will prove this claim by using the method of formal schemes as in the proof of Theorem 3.2.4.

Step 1. — By Theorem A.2.6(2), to find a Zariski-open $U_n(s) \subseteq S$ around s such that the restriction of ι_n over $U_n(s)$ is a closed immersion (for some fixed $n \geqslant n_0$), it suffices to show that the induced rigid-analytic map

$$X_m \longrightarrow \mathbf{P}^{\text{an}}\big((f_*\mathcal{L}^{\otimes n})_s/\mathfrak{m}_s^{m+1}(f_*\mathcal{L}^{\otimes n})_s\big) \tag{3.2.5}$$

on the mth infinitesimal fiber is a closed immersion for all $m \geqslant 0$ (we view $(f_*\mathcal{L}^{\otimes n})_s/\mathfrak{m}_s^{m+1}(f_*\mathcal{L}^{\otimes n})_s$ as a coherent sheaf on $\text{Sp}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{m+1})$).

For $n \geqslant 0$, let $N_m^{(n)}$ denote the finite $\mathcal{O}_{S,s}/\mathfrak{m}_s^{m+1}$-module

$$(f_*\mathcal{L}^{\otimes n})_s/\mathfrak{m}_s^{m+1}(f_*\mathcal{L}^{\otimes n})_s.$$

By GAGA, if $n \geqslant n_0$ then we may identify the map (3.2.5) with the analytification of an "abstract" $\text{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{m+1})$-scheme morphism

$$Y_m \longrightarrow \mathbf{P}\big(N_m^{(n)}\big) \tag{3.2.6}$$

for a unique proper scheme Y_m over $\mathcal{O}_{S,s}/\mathfrak{m}_s^{m+1}$ as in the proof of Theorem 3.2.4. To find n_s for (3.2.4), it is necessary and sufficient to prove that the maps (3.2.6) are closed immersions for all $m \geqslant 0$ provided that n is sufficiently large (perhaps depending on s but not on m).

Let the $\widehat{\mathcal{O}}_{S,s}$-proper scheme Y and the ample line bundle $\mathcal{M}$ on Y be defined by algebraization of the formal-scheme data $(\mathfrak{Y}, \mathfrak{M})$ built from the algebraizations of the rigid-analytic infinitesimal fibers over the $\mathcal{O}_{S,s}/\mathfrak{m}_s^{m+1}$'s as in Step 1 of the proof of Theorem 3.2.4. The line bundle $\mathcal{M}^{\otimes n}$ is generated by global sections for all $n \geqslant n_0$, as this can be checked by using suitable completions as in (3.2.3) with $\mathfrak{G} = \mathcal{O}_{\mathfrak{Y}}$, so for $n \geqslant n_0$ we have a canonical map

$$Y \longrightarrow \mathbf{P}\big(\text{H}^0(Y, \mathcal{M}^{\otimes n})\big) \tag{3.2.7}$$

of proper schemes over $\text{Spec}(\widehat{\mathcal{O}}_{S,s})$.

Step 2. — We need to relate (3.2.7) with the "abstract" map (3.2.6). By the algebraic theory of ampleness, there exists $n_s \geqslant n_0$ such that (3.2.7) is a closed immersion for $n \geqslant n_s$. Thus, for all $m \geqslant 0$ and $n \geqslant n_s$, the map (3.2.7) induces compatible closed immersions of infinitesimal fibers

$$Y_m \longrightarrow \mathbf{P}(\text{H}^0(Y, \mathcal{M}^{\otimes n})/\mathfrak{m}_s^{m+1}\text{H}^0(Y, \mathcal{M}^{\otimes n})) \tag{3.2.8}$$

over $\mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{m+1})$. By the rigid-analytic and scheme-theoretic theorems on formal functions and GAGA, we have a natural isomorphism

$$\widehat{\mathcal{O}}_{S,s} \otimes_{\mathcal{O}_{S,s}} (f_*\mathcal{L}^{\otimes n})_s \simeq \varprojlim_r \mathrm{H}^0(X_r, \mathcal{L}_r^{\otimes n}) \simeq \varprojlim_r \mathrm{H}^0(Y_r, \mathcal{M}_r^{\otimes n}) = \mathrm{H}^0(Y, \mathcal{M}^{\otimes n}). \tag{3.2.9}$$

Thus, for all $n \geqslant 0$ there is a natural isomorphism of $\mathcal{O}_{S,s}/\mathfrak{m}_s^{m+1}$-modules (compatible with change in m)

$$N_m^{(n)} = \big(\widehat{\mathcal{O}}_{S,s} \otimes_{\mathcal{O}_{S,s}} (f_*\mathcal{L}^{\otimes n})_s\big)/\mathfrak{m}_s^{m+1}\big(\widehat{\mathcal{O}}_{S,s} \otimes_{\mathcal{O}_{S,s}} (f_*\mathcal{L}^{\otimes n})_s\big) \simeq \mathrm{H}^0(Y, \mathcal{M}^{\otimes n})/\mathfrak{m}_s^{m+1}\mathrm{H}^0(Y, \mathcal{M}^{\otimes n}). \tag{3.2.10}$$

Fix $n \geqslant n_s \geqslant n_0$. The map (3.2.5) is classified by the natural map

$$N_m^{(n)} = (f_*\mathcal{L}^{\otimes n})_s/\mathfrak{m}_s^{m+1}(f_*\mathcal{L}^{\otimes n})_s \to \mathrm{H}^0(X_m, \mathcal{L}_m^{\otimes n}), \tag{3.2.11}$$

and its "abstract" algebraization (3.2.6) is classified by the natural map

$$\mathrm{H}^0(Y, \mathcal{M}^{\otimes n})/\mathfrak{m}_s^{m+1}\mathrm{H}^0(Y, \mathcal{M}^{\otimes n}) \longrightarrow \mathrm{H}^0(Y_m, \mathcal{M}_m^{\otimes n}). \tag{3.2.12}$$

Consider the diagram

$$\begin{array}{ccc} \mathrm{H}^0(Y, \mathcal{M}^{\otimes n})/\mathfrak{m}_s^{m+1}\mathrm{H}^0(Y, \mathcal{M}^{\otimes n}) & \longrightarrow & \mathrm{H}^0(Y_m, \mathcal{M}_m^{\otimes n}) \\ \simeq \downarrow & & \downarrow \simeq \\ N_m^{(n)} & \longrightarrow & \mathrm{H}^0(X_m, \mathcal{L}_m^{\otimes n}) \end{array} \tag{3.2.13}$$

in which the left side is (3.2.10), the right side is the rigid-analytic GAGA comparison isomorphism, the top side is (3.2.12), and the bottom side is (3.2.11). By the definition of the left side of (3.2.13) in terms of inverse limits, (3.2.13) commutes. Thus, the left side of (3.2.13) identifies (3.2.8) and (3.2.6). Since (3.2.8) is a closed immersion for all $n \geqslant n_s$ and all $m \geqslant 0$, this implies that if $n \geqslant n_s$ then (3.2.6) is a closed immersion for all $m \geqslant 0$, and this is what we needed to prove.

Step 3. — In Steps 1 and 2, we have shown that for any $s \in S$ there exists $n_s \geqslant n_0$ such that for each $n \geqslant n_s$ the map (3.2.4) is a closed immersion over a Zariski-open neighborhood $U_n(s) \subseteq S$ around s that may depend on n. Since Zariski-open set-theoretic covers of S are admissible, by working locally on S we may now assume that for a suitably large N there is a closed S-immersion $j : X \hookrightarrow \mathbf{P}_k^m \times S$ such that $j^*\mathcal{O}(1) \simeq \mathcal{L}^{\otimes N}$.

Let $f : \mathbf{P}_k^m \times S \to S$ be the structure map. The local finite-generatedness of $\mathcal{A}$ reduces to the claim that for a coherent sheaf $\mathcal{F}$ on $\mathbf{P}_k^m \times S$, the graded sheaf of modules $\bigoplus_{n\geqslant 0} f_*\mathcal{F}(n)$ over $\bigoplus_{n\geqslant 0} f_*\mathcal{O}(n) = \mathcal{O}_S[Z_0, \ldots, Z_m]$ is locally generated by finitely many sections of bounded degree. Working

locally on S, by Theorem 3.2.4 we may suppose that there exists an $n_0 > 0$ such that $f^*(f_*\mathcal{F}(n)) \to \mathcal{F}(n)$ is surjective for all $n \geqslant n_0$ and $f_*\mathcal{F}(n_0)$ is the quotient of a locally free coherent sheaf $\mathcal{E}$. Similarly, we may suppose that for large r, say $r \geqslant r_0$, the maps

$$f_*f^*\big(f_*\mathcal{F}(n_0)\big)(r) \longrightarrow f_*\mathcal{F}(n_0+r)$$
$$\mathcal{E}\otimes f_*\mathcal{O}(r) \simeq f_*(f^*\mathcal{E})(r) \longrightarrow f_*f^*\big(f_*\mathcal{F}(n_0)\big)(r)$$

are surjective. The main point here is that (by Theorem 3.2.4) we may work locally on S and find r_0 so that $\mathrm{R}^1 f_*(\mathcal{K}(r)) = 0$ for all $r \geqslant r_0$ when $\mathcal{K}$ is the kernel of either of the two maps $f^*f_*\mathcal{F}(n_0) \twoheadrightarrow \mathcal{F}(n_0)$ or $\mathcal{E} \twoheadrightarrow f_*\mathcal{F}(n_0)$.

We conclude that for $r \geqslant r_0$ the natural composite map

$$\mathcal{E}\otimes f_*\mathcal{O}(r) \longrightarrow f_*\mathcal{F}(n_0)\otimes f_*\mathcal{O}(r) \longrightarrow f_*\mathcal{F}(n_0+r)$$

is surjective, so for all $r \geqslant r_0$ the map

$$f_*\mathcal{F}(n_0)\otimes f_*\mathcal{O}(r) \longrightarrow f_*\mathcal{F}(n_0+r)$$

is surjective. Thus, $\bigoplus_{n\geqslant 0} f_*\mathcal{F}(n)$ is generated over $\bigoplus_{n\geqslant 0} f_*\mathcal{O}(n)$ by its coherent subsheaf of terms in degrees $< n_0 + r_0$, so $\mathcal{A} = \bigoplus_{n\geqslant 0} f_*\mathcal{L}^{\otimes n}$ satisfies the hypotheses in Lemma 3.1.4 (taking $m_s = n_0$ and $W_s = S$ for all $s \in S$). We therefore have a canonical S-map

$$\iota : X \longrightarrow \mathrm{Proj}^{\mathrm{an}}\,\mathcal{A}.$$

It remains to show that ι is an isomorphism. By Theorem A.2.6, it suffices to prove that ι is an isomorphism on infinitesimal fibers over each $s \in S$. Fix a choice of s, and let $\iota_{s,m}$ be the map induced by ι on mth infinitesimal fibers over s. For Y and $\mathcal{M}$ as in Step 1 (with $\mathcal{M}$ ample on Y), we have a canonical map

$$j : Y \longrightarrow \mathrm{Proj}\,\Big(\bigoplus_{n\geqslant 0} \mathrm{H}^0(Y, \mathcal{M}^{\otimes n})\Big)$$

of proper schemes over $\mathrm{Spec}(\widehat{\mathcal{O}}_{S,s})$, and since (3.2.9) is an isomorphism for $n \geqslant 0$ we may use Theorem 2.3.8.(2) and Corollary 2.3.10 to identify j with the algebraization of the map of proper formal $\widehat{\mathcal{O}}_{S,s}$-schemes induced by algebraization of the $\iota_{s,m}$'s. Hence, $\iota_{s,m}$ is an isomorphism for all m if (and only if) j is an isomorphism, and the isomorphism property for j is a special case of [15, III_1, 2.3.4.1]. □

COROLLARY 3.2.8. — *Let $f : X \to S$ be a proper map of rigid spaces over k, and $\mathcal{L}$ an invertible sheaf on X. Assume S is quasi-separated or pseudo-separated, and let k'/k be an analytic extension. Let $f' : X' \to S'$ be the induced proper map of rigid spaces over k', with $\mathcal{L}'$ the pullback line bundle on X'.*

The line bundle $\mathcal{L}$ is S-ample if and only if $\mathcal{L}'$ is S'-ample.

Proof. — Assume $\mathcal{L}$ is S-ample. To prove that $\mathcal{L}'$ is S'-ample we may work locally on S. Thus, by Theorem 3.2.7 we can assume that there is an isomorphism of line bundles $\mathcal{L}^{\otimes N} \simeq j^*\mathcal{O}(1)$ for some closed S-immersion $j : X \hookrightarrow \mathbf{P}^m_S$. Hence, ${\mathcal{L}'}^{\otimes N} \simeq {j'}^*\mathcal{O}(1)$ for the closed S'-immersion $j' : X' \hookrightarrow \mathbf{P}^m_{S'}$ induced by j. By the second part of Remark 3.1.3 on fibers over S', this implies that $\mathcal{L}'$ is S'-ample.

Now assume that $\mathcal{L}'$ is S'-ample. To prove that $\mathcal{L}$ is S-ample we can assume $S = \mathrm{Sp}(k_0)$ for a finite extension k_0/k. By hypothesis, $\mathcal{L}'$ is $k' \otimes_k k_0$-ample on $k'\widehat{\otimes}_k X = (k'\otimes_k k_0)\widehat{\otimes}_{k_0} X$. Without loss of generality (by Remark 3.1.3) we may replace k with k_0 and replace k' with a residue field of $k' \otimes_k k_0$ (see Corollary 3.1.6) so $S = \mathrm{Sp}(k)$ and $k'\widehat{\otimes}_k\mathcal{L}$ is k'-ample on $k'\widehat{\otimes}_k X$. By Lemma 3.1.2, $\mathcal{L}$ must be k-ample on X. □

We conclude our general discussion of relative ampleness with a result (inspired by Theorem 3.2.1) that concerns the locus of ample fibers with respect to a proper morphism.

THEOREM 3.2.9. — *Let $f : X \to S$ be proper, $\mathcal{L}$ an invertible sheaf on X. Define $U_{\mathcal{L}}$ to be the set of $s \in S$ such that $\mathcal{L}_s$ is ample on X_s. There exists a canonical Zariski-open set $W_{\mathcal{L}}$ in S containing $U_{\mathcal{L}}$ such that $U_{\mathcal{L}}$ is a Zariski-open locus in the rigid space $W_{\mathcal{L}}$ and the formation of $W_{\mathcal{L}}$ is compatible with base change on S and (for quasi-separated or pseudo-separated S) with change in the base field.*

The formation of $U_{\mathcal{L}}$ is compatible with base change on S and (for quasi-separated or pseudo-separated S) with change of the base field.

It may be that $U_{\mathcal{L}}$ is Zariski-open in S, but we do not see how to prove this (nor does it seem to be known in the complex-analytic case) because Zariski-openness is not a transitive condition in the analytic case.

Proof. — Once $U_{\mathcal{L}}$ is shown to be an admissible open, its formation obviously commutes with base change on S. Compatibility with change of the base field is not so trivial, and will follow from the description we shall give below for $U_{\mathcal{L}}$ (using $W_{\mathcal{L}}$).

Fix $s \in U_{\mathcal{L}}$. For all sufficiently large n (depending on s), $\mathcal{L}_s^{\otimes n}$ is generated by global sections. We claim that

$$(f_*\mathcal{L}^{\otimes n})_s \longrightarrow \mathrm{H}^0(X_s, \mathcal{L}_s^{\otimes n}) \tag{3.2.14}$$

is surjective for large n (depending on s). It suffices to check the surjectivity of (3.2.14) after making the faithfully flat base change $\mathcal{O}_{S,s} \to \widehat{\mathcal{O}}_{S,s}$. The formal-scheme method over $\widehat{\mathcal{O}}_{S,s}$ in the proof of Theorem 3.2.4 only requires

ampleness of $\mathcal{L}_s$, not S-ampleness of $\mathcal{L}$, and it identifies the $\widehat{\mathcal{O}}_{S,s}$-linear extension of (3.2.14) with the natural map

$$\mathrm{H}^0(Y, \mathcal{M}^{\otimes n}) \longrightarrow \mathrm{H}^0(Y_s, \mathcal{M}_s^{\otimes n}),$$

where $Y \to \mathrm{Spec}(\widehat{\mathcal{O}}_{S,s})$ is a proper morphism and $\mathcal{M}$ is an ample invertible sheaf on Y. By the algebraic theory of ampleness, we thereby deduce the desired surjectivity of (3.2.14) for $n \geqslant n_s$ where n_s depends on s. Since $\mathcal{L}_s$ is ample on X_s, we may and do also assume n_s is so large that $\mathcal{L}_s^{\otimes n}$ on X_s is generated by global sections for all $n \geqslant n_s$. By Nakayama's Lemma, the map of coherent sheaves

$$\varphi_n : f^* f_* \mathcal{L}^{\otimes n} \longrightarrow \mathcal{L}^{\otimes n}$$

is therefore surjective on stalks at all $x \in X_s$ for all $n \geqslant n_s$.

The Zariski-closed support $Z_n \subseteq X$ of the coherent $\mathrm{coker}(\varphi_n)$ is disjoint from X_s for $n \geqslant n_s$, and so for such n the Zariski-open $V_n = S - f(Z_n)$ contains s, with φ_n a surjection over $f^{-1}(V_n)$. Note that for any $n > 0$, the formation of the Zariski-open V_n is compatible with change of the base field, by Lemma A.2.4 and Corollary A.2.7; its compatibility with base change on S is clear. Moreover, by Lemma A.2.4, for any $n > 0$ the overlap

$$W_n = \bigcap_{i=n}^{2n-1} V_i = V_n \times_S \cdots \times_S V_{2n-1}$$

in S is a Zariski-open whose formation is compatible with change of the base field for quasi-separated or pseudo-separated S (and with base change on S for arbitrary S). Since any integer $m > 2n-1$ can be written in the form $m = nq + r$ with $q \geqslant 1$ and $n \leqslant r \leqslant 2n-1$, we see that $W_n \subseteq V_i$ for all $i \geqslant n$ (so $W_n = \cap_{i \geqslant n} V_i$). That is, φ_i is surjective over $f^{-1}(W_n)$ for all $i \geqslant n$. Since descending chains of analytic sets are locally stationary, $W_{\mathcal{L}} \stackrel{\mathrm{def}}{=\!=} \cup W_n$ is locally equal to $\cup_{n \leqslant N} W_n$ for all large N, and hence it is a Zariski-open in S that contains $U_{\mathcal{L}}$. The local finiteness of the union defining $W_{\mathcal{L}}$ ensures that the formation of $W_{\mathcal{L}}$ is compatible with arbitrary base change on S and with change of the base field for quasi-separated or pseudo-separated S (Lemma A.2.4), so we may replace S with $W_{\mathcal{L}}$ to get to the case when $\{W_n\}$ is a Zariski-open covering of S *provided* that we prove more: in this case we must prove that $U_{\mathcal{L}}$ is a Zariski-open in S and that its formation commutes with extension of the ground field when S is quasi-separated or pseudo-separated.

By working locally on S, we can reduce to the case where $f^* f_* \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}$ is surjective for all large n. Consider the resulting S-morphisms

$$\iota_n : X \longrightarrow \mathbf{P}^{\mathrm{an}}(f_* \mathcal{L}^{\otimes n})$$

for large n; the formation of the ι_n's is clearly compatible with change of the base field. By the formal scheme argument in the proof of Theorem 3.2.7, the ampleness of $\mathcal{L}_s$ for $s \in S$ implies that there is an integer $n'_s > 0$ such that the fiber map $(\iota_n)_s$ at s is a closed immersion for all $n \geqslant n'_s$. Define

$$U_n = \big\{ s \in S \,|\, (\iota_n)_s \text{ is a closed immersion} \big\}.$$

By Theorem A.2.6.(2), U_n is a Zariski-open set whose formation is compatible with change of the base field for quasi-separated or pseudo-separated S, and ι_n is a closed immersion over U_n. As with $W_{\mathcal{L}}$ above, the union $U = \bigcup U_n$ is Zariski-open. Also, the formation of U is compatible with change of the base field for quasi-separated or pseudo-separated S, and U contains $U_{\mathcal{L}}$.

We claim $U = U_{\mathcal{L}}$. Pick $s \in U_n$ for some fixed n. For the finite-dimensional $k(s)$-vector space

$$E_s = (f_*\mathcal{L}^{\otimes n})_s / \mathfrak{m}_s (f_*\mathcal{L}^{\otimes n})_s$$

we have a closed $\mathrm{Sp}(k(s))$-immersion

$$\iota : X_s \hookrightarrow \mathbf{P}^{\mathrm{an}}(E_s)$$

into a projective space over $k(s)$, with a natural map of invertible sheaves $\iota^*(\mathcal{O}_{\mathbf{P}^{\mathrm{an}}(E_s)}(1)) \to \mathcal{L}_s^{\otimes n}$ that is surjective and hence an isomorphism. This forces $\mathcal{L}_s$ to be ample, or in other words $s \in U_{\mathcal{L}}$. □

4. Applications and descent theory

The relative theories of $\mathrm{Proj}^{\mathrm{an}}$ and ampleness have several immediate applications in the direction of providing rigid-analytic versions of standard algebro-geometric existence theorems. In this section we will discuss some applications to representability and effective descent of geometric objects. In [14] some of these results will be applied to study the problem of analytification of locally separated algebraic spaces over non-archimedean fields.

4.1. Representability results

The simplest universal construction that we can carry out using $\mathrm{Proj}^{\mathrm{an}}$ is blowing up a rigid space along a coherent ideal sheaf. Let S be a rigid space and let $\mathcal{I}$ be a coherent ideal sheaf on S. The graded Rees algebra $\bigoplus_{n \geqslant 0} \mathcal{I}^n$ is locally finitely generated as an $\mathcal{O}_S$-algebra and has coherent graded terms. Thus, $\mathrm{Proj}^{\mathrm{an}}$ of this sheaf makes sense and is a rigid space proper over S:

DEFINITION 4.1.1. — *The* blow-up $\mathrm{Bl}_{\mathcal{I}}(S)$ *of* S *along* $\mathcal{I}$ *is the* S*-proper rigid space* $\mathrm{Proj}^{\mathrm{an}}(\bigoplus_{n\geqslant 0}\mathcal{I}^n)$.

By Lemma 2.3.5 the pullback ideal sheaf induced by $\mathcal{I}$ on $\mathrm{Bl}_{\mathcal{I}}(S)$ is invertible, and as in the case of schemes the map $\mathrm{Bl}_{\mathcal{I}}(S)\to S$ is the final object in the category of rigid spaces over S such that the coherent pullback ideal sheaf induced by $\mathcal{I}$ is invertible. More specifically, for an admissible open $U\subseteq S$ and any $a\in\mathcal{I}(U)$ viewed as a degree-1 section a_1 of $\bigoplus_{n\geqslant 0}\mathcal{I}^n$ over U, $\mathrm{Spec}^{\mathrm{an}}((\bigoplus_{n\geqslant 0}\mathcal{I}^n_{|U})_{(a_1)})\to S$ is the universal rigid space over U on which the pullback coherent ideal sheaf induced by $\mathcal{I}_{|U}$ is a globally free invertible sheaf admitting the pullback of a as a basis. This is proved exactly as in the algebraic case.

Universal properties provide canonical comparison morphisms that express the compatibility of rigid-analytic blow-up with respect to both flat base change on S and analytification of scheme-theoretic blow-up, and these comparison morphisms are isomorphisms. The behavior with respect to change of the base field also works out nicely, as follows. If Y is a quasi-separated or pseudo-separated rigid space over k, and k'/k is an analytic extension field, then for any coherent ideal $\mathcal{J}$ on Y the pullback $\mathcal{J}'=k'\widehat{\otimes}_k\mathcal{J}$ is a coherent ideal on $Y'=k'\widehat{\otimes}_k Y$, and $\mathcal{J}'$ is invertible if (and only if) $\mathcal{J}$ is invertible. Hence, for any k'/k and any quasi-separated or pseudo-separated S over k equipped with a coherent ideal $\mathcal{I}$, exactness properties of the functor $k'\widehat{\otimes}_k(\cdot)$ on coherent sheaves imply that the pullback of the invertible coherent $\mathcal{O}_{k'\widehat{\otimes}_k S}$-ideal $\mathcal{I}'=k'\widehat{\otimes}_k\mathcal{I}$ is an invertible coherent ideal on $k'\widehat{\otimes}_k\mathrm{Bl}_{\mathcal{I}}(S)$. We therefore obtain a canonical morphism over $S'=k'\widehat{\otimes}_k S$,

$$k'\widehat{\otimes}_k\mathrm{Bl}_{\mathcal{I}}(S)\longrightarrow\mathrm{Bl}_{\mathcal{I}'}(S'),$$

via the universal property of the target blow-up. This map is seen to be an isomorphism by direct inspection of the construction of blow-ups via Rees algebras.

Remark 4.1.2. — An entirely different approach to the construction of rigid-analytic blow-up and the verification of a few properties of such blowing up is given in [40] in the case of an algebraically closed base field.

A more interesting (but still easy) comparison with the algebro-geometric theory is the representability of Hilbert functors. Let $f:X\to S$ be a proper morphism of rigid spaces, and define the Hilbert functor

$$\mathbf{Hilb}_{X/S}$$

to classify closed immersions $Z\hookrightarrow X\times_S T$ such that Z is T-flat, for variable rigid spaces T over S. If $\mathcal{L}$ is an S-ample invertible sheaf on X, then the

Hilbert polynomials of the fibers of $Z \to T$ with respect to the T-ample $\mathcal{L}_T|_Z$ naturally break up $\mathbf{Hilb}_{X/S}$ into a disjoint union of Zariski-open and closed subfunctors

$$\mathbf{Hilb}^Q_{X/S} \quad \text{for } Q \in \mathbf{Q}[t]$$

(by Theorem A.1.6). Thus, representability of $\mathbf{Hilb}_{X/S}$ is equivalent to that of $\mathbf{Hilb}^Q_{X/S}$ for all $Q \in \mathbf{Q}[t]$ when there exists an S-ample $\mathcal{L}$ on X.

More generally, for a coherent sheaf $\mathcal{E}$ on X, we define the Quot-functor

$$\mathbf{Quot}_{\mathcal{E}/X/S}$$

to classify T-flat coherent quotients of $\mathcal{E}_T$ on X_T, for variable rigid spaces $T \to S$. If $\mathcal{L}$ is an S-ample invertible sheaf on X then $\mathbf{Quot}_{\mathcal{E}/X/S}$ naturally breaks up into a disjoint union of subfunctors

$$\mathbf{Quot}^Q_{\mathcal{E}/X/S}$$

that are both Zariski-open and Zariski-closed, where $\mathbf{Quot}^Q_{\mathcal{E}/X/S}$ classifies the flat coherent quotients whose fibral Hilbert polynomial with respect to $\mathcal{L}$ is $Q \in \mathbf{Q}[t]$ on all fibers.

The representability of $\mathbf{Quot}_{\mathcal{E}/X/S}$ in the complex-analytic category for S-projective X was proved by Grothendieck [23], based on the complex-analytic relative theorems A and B of Grauert-Remmert [20] that are analogues of Theorem 3.2.4 and Theorem 3.2.7. Since the complex-analytic versions of Theorems 3.2.4 and 3.2.7 mildly strengthen the Grauert-Remmert theorems by dropping the explicit assumption that there exist projective embeddings locally on the base (even though we have shown that this follows from the existence of fibrally ample line bundles), we get a mildly stronger form of Grothendieck's theorem in the complex-analytic case. In the rigid case we obtain:

THEOREM 4.1.3. — *Let $f : X \to S$ be proper and assume that there exists an S-ample invertible sheaf $\mathcal{L}$ on X. Fix $Q \in \mathbf{Q}[t]$ and a coherent sheaf $\mathcal{E}$ on X. The functor $\mathbf{Quot}^Q_{\mathcal{E}/X/S}$ is represented by a proper rigid space over S. Its formation is compatible with analytification of algebraic k-schemes, and (for quasi-separated or pseudo-separated S) its formation is compatible with change of the base field.*

In particular, $\mathbf{Hilb}_{X/S} = \mathbf{Quot}_{\mathcal{O}_X/X/S}$ exists as a separated rigid space over S and its formation is compatible with analytification and (for quasi-separated or pseudo-separated S) with change of the base field.

Proof. — By Theorem 3.2.4 and Theorem 3.2.7, Grothendieck's complex-analytic arguments work essentially *verbatim* to prove the rigid-analytic

version. The only extra ingredient that does not arise in the complex-analytic case is the compatibility with change in the base field, so we now address this aspect. Grothendieck's method uses the relative theorems A and B (that we have proved in the rigid-analytic case) to reduce the representability problems to the special case $\mathcal{E} = \mathcal{O}_X^{\oplus N}$, $X = \mathbf{CP}^n$, $S = \mathrm{Spec}(\mathbf{C})$ (as a complex-analytic space), and $\mathcal{L} = \mathcal{O}(1)$, and the rigid-analytic analogues of these reduction steps are compatible with change in the base field (by Lemma A.2.4 and Corollary A.2.7). In this special case Grothendieck's argument shows that the analytification of the separated algebraic universal object

$$\mathbf{Quot}^{Q}_{\mathcal{O}^{\oplus N}_{\mathbf{P}^n_{\mathrm{Spec}\,\mathbf{C}}}/\mathbf{P}^n_{\mathrm{Spec}\,\mathbf{C}}/\,\mathrm{Spec}\,\mathbf{C}}$$

is the complex-analytic universal object (for the same Q). His argument applies in the rigid-analytic case, and so the compatibility of the universal rigid-analytic objects with respect to change of the base field is a consequence of the general compatibility between rigid-analytification (of algebraic k-schemes) and extension of the analytic base field. □

Example 4.1.4. — If H_n denotes the Hilbert scheme for $\mathbf{P}^n_{\mathrm{Spec}(k)}$ and $Z_n \hookrightarrow H_n \times \mathbf{P}^n_{\mathrm{Spec}(k)}$ is the universal flat family over H_n, then the pair $(H_n^{\mathrm{an}}, Z_n^{\mathrm{an}})$ represents the Hilbert functor for $\mathbf{P}^n_k$ in the rigid-analytic category. This special case of Theorem 4.1.3 can also be deduced from Köpf's relative rigid-analytic GAGA (Example 3.2.6) but this alternative method of proof has no complex-analytic analogue (whereas the proof of Theorem 4.1.3 works simultaneously over both $\mathbf{C}$ and non-archimedean fields). The real content in Theorem 4.1.3 is its applicability to situations that are not tautologically projective over the base, but rather are equipped with a fibrally ample line bundle. That is, the usefulness of Theorem 4.1.3 in abstract situations rests on the analytic theory of relative ampleness in §3.

COROLLARY 4.1.5. — *Let S be a rigid space and let X and Y be two proper rigid spaces over S such that Y admits an S-ample line bundle and X is S-flat. The functor*

$$\mathbf{Hom}(X, Y) : T \rightsquigarrow \mathrm{Hom}_T(X_T, Y_T)$$

on rigid spaces over S is represented by an S-separated rigid space. The formation of this rigid space is compatible with analytification and (for quasi-separated or pseudo-separated S) with change of the base field.

Proof. — This goes exactly as in the case of schemes [24, §4 (c)], using Theorem 4.1.3. Grothendieck only gives a sketch of the proof, so we note that filling in the details requires using Theorem A.2.6.(1)–(2) (applied

to representing objects for suitable Hilbert functors), exactly as in the algebraic case. □

COROLLARY 4.1.6. — *Let S be a rigid-analytic space and let D be a diagram of finitely many proper flat rigid spaces X_i over S equipped with finitely many S-maps $f_{ijr} : X_i \to X_j$ ($r \in R_{ij}$). Assume that each X_i admits a closed S-immersion into a projective space over S. There exists a separated algebraic k-scheme $\mathcal{S}$, projective flat $\mathcal{S}$-schemes $\mathcal{X}_i$ equipped with finitely many $\mathcal{S}$-maps $\phi_{ijr} : \mathcal{X}_i \to \mathcal{X}_j$, a map $h : S \to \mathcal{S}^{\mathrm{an}}$, and S-isomorphisms $X_i \simeq \mathcal{X}_i^{\mathrm{an}} \times_{\mathcal{S}^{\mathrm{an}}} S$ that identify f_{ijr} with the pullback of ϕ_{ijr}^{an} for all i, j, r.*

Proof. — Let $X_i \hookrightarrow \mathbf{P}_S^{N_i}$ be a closed immersion over S. We may assume S is connected. By the constancy of Hilbert polynomials in proper flat rigid-analytic families over a connected base (Theorem A.1.6), all fibers of each X_i over S have a common Hilbert polynomial $P_i \in \mathbf{Q}[t]$ with respect to the embedding into $\mathbf{P}^N$. The universality in Theorem 4.1.3 and Corollary 4.1.5 therefore gives the desired result. □

The case of relative Picard functors will be addressed later in a special case in Theorem 4.3.3.

4.2. Faithfully flat descent

The condition of quasi-compactness is a very natural one in descent theory for schemes. However, when considering descent theory (or quotients by étale or flat equivalence relations) for rigid spaces it is inconvenient to impose quasi-compactness conditions on the structural morphisms. For example, one wants to consider the possibility of associating rigid spaces to algebraic spaces, say by forming an analytic quotient for an étale chart, yet (by Nagata's compactification theorem [36]) the analytification of a quasi-compact separated map f between algebraic k-schemes is quasi-compact in the sense of rigid geometry if and only if f is proper. (Recall from §1.3 that we take algebraic k-schemes to merely be locally of finite type over k, and not necessarily of finite type.) To avoid quasi-compactness conditions that are unduly restrictive in the setting of algebraic spaces, we shall use Berkovich spaces. The starting point is the following definition:

DEFINITION 4.2.1. — *Let $f : X \to Y$ be a flat map of rigid spaces. An* fpqc quasi-section *of f is a faithfully flat and quasi-compact map $U' \to Y$ equipped with a Y-map $s : U' \to X$. The map f* admits local fpqc quasi-sections *if there exists an admissible covering $\{Y_i\}$ of Y such that each*

restriction $f^{-1}(Y_i) \to Y_i$ has an fpqc *quasi-section. The notion of* étale quasi-section *is defined similarly, imposing the condition that $U' \to Y$ be an étale quasi-compact surjection; these are special kinds of* fpqc *quasi-sections.*

Since base change preserves the *fpqc* property for maps of rigid spaces (due to Raynaud's flat models theorem), base change preserves the property of admitting local *fpqc* quasi-sections. (The same goes in the étale case.) It is a tautology that any *fpqc* map of rigid spaces has an *fpqc* quasi-section, namely itself, and similarly for étale quasi-compact surjections. Our interest in local *fpqc* quasi-sections and local étale quasi-sections is that they exist in abundance for rigid-analytic maps that arise from algebraic geometry:

THEOREM 4.2.2. — *If $f : \mathcal{X} \to \mathcal{Y}$ is a faithfully flat map between algebraic k-schemes then $f^{\mathrm{an}} : \mathcal{X}^{\mathrm{an}} \to \mathcal{Y}^{\mathrm{an}}$ admits local fpqc quasi-sections. Likewise, if f is an étale surjection then f^{an} admits local étale quasi-sections.*

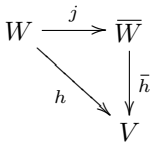
Proof. — We first treat the faithfully flat case. By [15, IV_4, 17.6.1], if we work Zariski-locally on $\mathcal{Y}$ we may suppose that $\mathcal{Y}$ is separated and quasi-compact (*e.g.*, affine) and that there exists a quasi-finite, flat, and separated surjection $\mathcal{Y}' \to \mathcal{Y}$ equipped with a $\mathcal{Y}$-map $\mathcal{X} \to \mathcal{Y}'$. Upon renaming $\mathcal{Y}'$ as $\mathcal{X}$, we may assume $\mathcal{X}$ is quasi-finite, flat, and separated over $\mathcal{Y}$. Our problem is therefore reduced to proving that if $\mathcal{U} \to \mathcal{S}$ is a quasi-finite, flat, and separated surjection of algebraic k-schemes and $\mathcal{S}$ is separated, then for any quasi-compact admissible open V in $\mathcal{S}^{\mathrm{an}}$ there exists a *quasi-compact* admissible open U in $\mathcal{U}^{\mathrm{an}}$ such that $U \to V$ is surjective; note that $U \to V$ is quasi-compact because V is separated and U is quasi-compact. (To solve the initial problem, we then take such V to range over the constituents of an admissible affinoid covering of $\mathcal{S}^{\mathrm{an}}$.)

Since $\mathcal{S}$ is separated, and hence $\mathcal{U}$ is separated, finite unions of quasi-compact admissible opens in $\mathcal{U}^{\mathrm{an}}$ are quasi-compact admissible opens. Thus, we can work locally on $\mathcal{S}$ and V, so we may assume $\mathcal{S}$ is quasi-compact and V is affinoid. By Zariski's Main theorem, there is a commutative diagram

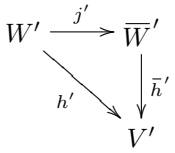
(4.2.1)
$$\begin{array}{ccc} \mathcal{U} & \hookrightarrow & \overline{\mathcal{U}} \\ & \searrow & \downarrow \\ & & \mathcal{S} \end{array}$$

where $\mathcal{U} \to \overline{\mathcal{U}}$ is an open immersion and $\overline{\mathcal{U}} \to \mathcal{S}$ is finite. Analytifying this and restricting over the open $V \subseteq \mathcal{S}^{\mathrm{an}}$ yields a commutative diagram of

rigid spaces

(4.2.2)


where $\overline{h}$ is finite, j is a Zariski-open immersion, and h is flat and surjective. Since V (and hence $\overline{W}$) is affinoid, we can use [3, §1.6] to realize this diagram of rigid spaces as being induced by an analogous diagram

(4.2.3)


in the category of Berkovich spaces over k, where V' is (strictly) k-affinoid, $\overline{h}'$ is finite, and j' is a *Zariski-open* immersion. Due to how we constructed (4.2.2) by localizing on the analytification of a similar diagram (4.2.1) of finite type k-schemes, it follows from [3, 3.2.10] that the map h' is *flat quasi-finite* in the sense of [3, §3.1].

By [3, 3.2.7], flat quasi-finite maps of Berkovich spaces are open. Since V' is compact and every point of W' admits a strictly k-affinoid neighborhood (as W' is Zariski-open in the strictly k-affinoid $\overline{W}'$), it therefore suffices to show that the open map h' is surjective. The map h' is induced by applying Berkovich-analytification and localization (on the base) to a surjective map $\mathcal{U} \to \mathcal{S}$ between algebraic k-schemes, and surjectivity in the category of schemes is preserved by change of the ground field. Thus, it follows that h' is surjective (*cf.* [3, 2.6.8]).

The étale case goes exactly the same way: replace "flat" with "étale" everywhere. □

THEOREM 4.2.3. — *Let $f : S' \to S$ be a faithfully flat of rigid spaces, and assume it admits local fpqc quasi-sections. The functor*

$$(X \to S) \rightsquigarrow (X' = X \times_S S', \varphi : X' \times_S S' \simeq S' \times_S X')$$

from rigid spaces over S to rigid spaces over S' equipped with descent data relative to f is a fully faithful functor.

Proof. — We are given rigid spaces X and Y over S and an S'-map $g' : X' \to Y'$ compatible with the descent data, and we wish to uniquely descend g' to an S-map $g : X \to Y$. Uniqueness is obvious, even without

requiring f to admit local *fpqc* quasi-sections, and so for the rest we may work locally on S. Thus, we can assume S is affinoid and that there exists an *fpqc* map $U' \to S$ equipped with an S-map $U' \to S'$. Let $\{S'_i\}$ be an admissible affinoid open covering of S', so there is a finite subset of indices I_0 such that the S'_i's for $i \in I_0$ pull back to an admissible covering of U'. Hence, $S'_0 = \coprod_{i \in I_0} S'_i$ is affinoid and *fpqc* over the affinoid S. Since faithfulness did not require quasi-compactness of f we can (without having required f to be quasi-separated) replace S' with S'_0 to reduce the problem of descent for morphisms to the case when S' is also affinoid (and in particular S' is *fpqc* over S).

Obviously we can assume X is affinoid, so X' is also affinoid. In order to get both X and Y to be affinoid, we need to be careful with respect to issues of admissibility. Let $\{U_i\}$ be an admissible affinoid open covering of Y, so the pullback covering $\{U'_i\}$ of Y' is admissible with the U'_i's quasi-compact. Consider the admissible open covering $\{{g'}^{-1}(U'_i)\}$ of X'. Note that each admissible open $W'_i = {g'}^{-1}(U'_i)$ in X' has identical preimages under the two projections from

$$X'' \stackrel{\text{def}}{=\!=} X' \times_X X' \simeq X \times_S (S' \times_S S')$$

down to X'. To conclude that each W'_i is the preimage of an admissible open W_i in X, we need a lemma.

LEMMA 4.2.4. — *Let $X' \to X$ be a faithfully flat map of rigid spaces, and assume it admits local fpqc quasi-sections. If W' is an admissible open in X' whose two pullbacks to $X' \times_X X'$ coincide, then W' is the preimage of an admissible open W in X.*

Proof. — Let W be the image of W' in X. It is necessary and sufficient to prove that W is an admissible open in X and $f^{-1}(W) = W'$. The hypothesis of equality $W' \times_X X' = X' \times_X W'$ as admissible opens in $X' \times_X X'$ is a set-theoretic condition, and by passing to fibers over points in X we see that an equivalent formulation of the hypothesis is that W' is a union of fibers of $X' \to X$. Hence, $W' = f^{-1}(W)$, and so our problem is precisely to prove the admissibility of W in X. By working locally on X, we may assume that there exists an *fpqc* map $U' \to X$ equipped with an X-map $U' \to X'$. If we let V' be the preimage of W' in U' then since $U' \to X$ is surjective and W' is a union of X-fibers in X' we see that (i) V' is a union of X-fibers in U', and (ii) the images of V' and W' in X coincide. We may therefore replace X' with U' and replace W' with V' to reduce to the case when $X' \to X$ is *fpqc*. We can assume X is affinoid, and by replacing X' with the disjoint union of the constituents of a finite

admissible affinoid covering we can also assume that X' is affinoid. Hence, we may assume that X and X' are quasi-compact and (quasi-)separated (so f is also quasi-compact and quasi-separated).

Let $\{W'_j\}$ be an admissible covering of W' by quasi-compact admissible opens that each lie over an affinoid open in X, so by [8, 5.11] $W_j = f(W'_j)$ is a quasi-compact admissible open in X and clearly W is the union of the W_j's. We will now check that $\{W_j\}$ is an admissible covering of W in X (whence W is admissible in X). Recalling the definition of admissibility, we have to show that for a morphism $h : \mathrm{Sp}(A) \to X$ whose image lands inside of W, the set-theoretic covering $\{h^{-1}(W_j)\}$ of $\mathrm{Sp}(A)$ admits a finite affinoid open refinement. Since h is quasi-compact (as X is quasi-separated and $\mathrm{Sp}(A)$ is quasi-compact), the $h^{-1}(W_j)$'s are quasi-compact admissible opens and we just have to show that finitely many of these cover $\mathrm{Sp}(A)$.

Consider the cartesian diagram

$$\begin{array}{ccc} Z' & \xrightarrow{h'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Sp}(A) & \xrightarrow[h]{} & X \end{array}$$

where we note that f' is quasi-compact because f is quasi-compact. Due to the surjectivity of the columns, we just have to check that finitely many of the admissible opens ${f'}^{-1}(h^{-1}(W_j)) = {h'}^{-1}(f^{-1}(W_j))$ cover Z'. Since $f^{-1}(W_j) = f^{-1}(f(W'_j))$ contains W'_j, it is enough to check that finitely many of the ${h'}^{-1}(W'_j)$'s cover Z'. The morphism h' factors through the subset $f^{-1}(W)$ that is equal to the *admissible open* W', so since Z' is quasi-compact and $\{W'_j\}$ is an admissible covering of W' we are done. □

Continuing with the proof of Theorem 4.2.3, by Lemma 4.2.4 we get admissible opens W_i in X that form a set-theoretic cover of X such that their preimages W'_i form an admissible cover of X'. Recall that X' is affinoid, so finitely many $W'_{i_1}, \ldots, W'_{i_n}$ suffice to give an admissible cover of X'. Thus, $\{W_{i_1}, \ldots, W_{i_n}\}$ is a set-theoretic cover of the affinoid X. If this covering of X were *admissible*, then it would be permissible to replace X with W_{i_j} and Y with U_{i_j} so as to reduce to the case where Y is affinoid but perhaps X is not affinoid. Working locally on X would then bring us to the case where X and Y (and S and S') are simultaneously affinoid.

In order to complete this reduction to the affinoid case, we must show that if $X' \to X$ is a faithfully flat map of affinoids and $\{W_i\}$ is a collection of admissible opens in X whose preimage $\{W'_i\}$ is an admissible covering of X', then $\{W_i\}$ is admissible, or equivalently, has a finite affinoid open

refinement. If $\{V_{ij}\}_{j\in J_i}$ is an admissible affinoid covering of W_i then the preimage $\{V'_{ij}\}_{j\in J_i}$ is an admissible covering of W'_i, so since $\{W'_i\}$ is assumed to be an admissible covering of X' it follows that the entire collection $\{V'_{ij}\}_{(i,j)}$ is an *admissible* covering of X'. Since $\{W_i\}$ is admissible if its refinement $\{V_{ij}\}_{(i,j)}$ is, we may therefore reduce to the case where the W_i's are all affinoid. Since X' is quasi-compact and $\{W'_i\}$ is an admissible cover, there is a finite subcover $\{W'_{i_1},\ldots,W'_{i_n}\}$. Thus, the W_{i_j}'s for $1\leqslant j\leqslant n$ form a finite affinoid open refinement of $\{W_i\}$ that set-theoretically covers the *affinoid* X. Such $\{W_{i_j}\}$ is automatically admissible. This concludes the reduction of descent for morphisms to the special case when X, Y, S, and S' are all affinoid.

Writing our situation in terms of rings, we have to check that if $B\to B'$ is a faithfully flat map of k-affinoids, then B is the equalizer of the two maps $B'\rightrightarrows B'\widehat{\otimes}_B B'$. Since the specification of an element of B is (thanks to the Maximum Modulus Principle) functorially the same as a morphism $\mathrm{Sp}(B)\to\mathbf{A}^1_k$, it suffices to show that the functor represented by $\mathbf{A}^1_k$ satisfies the sheaf axioms for *fpqc* coverings. The advantage of this formulation is that it allows us to work locally on both $\mathrm{Sp}(B)$ and $\mathrm{Sp}(B')$. Hence, by choosing a faithfully flat formal model for $\mathrm{Sp}(B')\to\mathrm{Sp}(B)$ (in the sense of Raynaud) and working over suitable formal open affines, we may reduce to showing that for a faithfully flat map $\mathcal{B}\to\mathcal{B}'$ of topologically finitely presented R-algebras (with R denoting the valuation ring of k), the evident equalizer sequence of R-modules

$$0\to\mathcal{B}\longrightarrow\mathcal{B}'\longrightarrow\mathcal{B}'\widehat{\otimes}_{\mathcal{B}}\mathcal{B}' \tag{4.2.4}$$

is exact. Modulo any ideal of definition of the valuation ring of k, (4.2.4) is an equalizer sequence for a faithfully flat extension of rings, and so is exact by ordinary *fpqc* descent theory. By left-exactness of inverse limits, we conclude that (4.2.4) is exact, as desired. □

COROLLARY 4.2.5. — *For any rigid space S and morphism $X\to S$, the functor* $\mathrm{Hom}_S(\cdot,X)$ *on the category of rigid spaces over S satisfies the sheaf axioms relative to faithfully flat maps that admit local fpqc quasi-sections.*

Proof. — The functor $\mathrm{Hom}_S(\cdot,X)$ is obviously a separated presheaf with respect to such coverings. To verify the gluing axioms, consider a faithfully flat S-map $Y'\to Y$ admitting local *fpqc* quasi-sections and an S-map $Y'\to X$ whose two pullbacks to $Y'\times_S Y'$ coincide. We view the S-map $Y'\to X$ as a Y'-morphism $Y'\to X\times_S Y'$. Using descent for morphisms (Theorem 4.2.3) relative to the covering $Y'\to Y$ we can descend this to a Y-morphism $Y\to X\times_S Y$, and composing this morphism with the first

projection gives the desired (unique) S-map $Y \to X$ whose composite with the S-map $Y' \to Y$ is the initially given S-map $Y' \to X$. □

Descent theory also works for admissible covers:

COROLLARY 4.2.6. — *Let $X' \to X$ be an faithfully flat map of rigid spaces, and assume it admits local fpqc quasi-sections. Let $\{W_i'\}$ be an admissible covering of X' such that $W_i' \times_X X' = X' \times_X W_i'$ in $X' \times_X X'$ for all i. Each W_i' is the preimage of a unique admissible open W_i in X, and $\{W_i\}$ is an admissible covering of X.*

Proof. — Lemma 4.2.4 provides us with an admissible open W_i in X whose preimage is W_i', and W_i is visibly the image of W_i' in X; uniqueness of W_i descending W_i' is obvious since $X' \to X$ is surjective. We have to prove that $\{W_i\}$ is an admissible cover of X.

Let W be the abstract gluing of the W_i's along the $W_i \cap W_j$'s, so the W_i's form an admissible covering of W and there is a canonical (bijective) morphism $f : W \to X$ that we want to be an isomorphism. All we know is that over each $W_i \subseteq X$ this map restricts to an isomorphism, but we cannot conclude anything from this precisely because we do not yet know if $\{W_i\}$ is an admissible covering of X.

Let $W' = W \times_X X'$. Since W_i in W maps isomorphically to W_i in X, the preimage of W_i in W' is canonically isomorphic (via the second projection $W' \to X'$) to the preimage of W_i under $X' \to X$. We assumed that $\{W_i'\}$ is an admissible covering of X', so we conclude that $f' : W' \to X'$ is an isomorphism locally on the target and hence is an isomorphism. Let $g' : X' \to W'$ be the inverse of f'. We shall now apply the sheaf property of representable functors in Corollary 4.2.5 (in the category of rigid spaces over $S = \mathrm{Sp}(k)$) in order to uniquely fill in a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & W' \\ \downarrow & & \downarrow \\ X & \xrightarrow[g]{?} & W \end{array} \tag{4.2.5}$$

and this g will be an inverse to f.

Since the functor represented by W satisfies the sheaf axiom relative to $X' \to X$, we just have to check that the two composite maps

$$X' \times_X X' \rightrightarrows X' \xrightarrow{g'} W' \to W \tag{4.2.6}$$

coincide. We have $W' \times_W W' \simeq W \times_X (X' \times_X X')$, and the projection $W'' \to X''$ is an *isomorphism* because it is the composite $W' \times_W W' \simeq$

$W \times_X (X' \times_X X') \simeq W' \times_X X' \simeq X' \times_X X'$ whose final step is an isomorphism because $W' \to X'$ is an isomorphism. Thus, to check that the two composites in (4.2.6) coincide, it is enough to check after composition with the isomorphism $W' \times_W W' \simeq X' \times_X X'$ on the source, but the resulting two composites $W' \times_W W' \rightrightarrows W$ are both visibly equal to the canonical structure map. We therefore obtain the unique morphism g making (4.2.5) commute.

By using the existence of the commutative squares

$$\begin{array}{ccc} W' & \xrightarrow{g'} & X' \\ \downarrow & & \downarrow \\ W & \xrightarrow[g]{} & X \end{array} \qquad\qquad \begin{array}{ccc} X' & \xrightarrow{f'} & W' \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & W \end{array}$$

with inverses in the top row and common faithfully flat morphisms in the vertical direction, it follows that the bottom rows must be inverses. Thus, f is an isomorphism, so $\{W_i\}$ is an admissible covering of X. □

THEOREM 4.2.7. — *Let $f : X \to Y$ be a map of rigid spaces, and let $Y' \to Y$ be a faithfully flat map that admits local fpqc quasi-sections. Let* **P** *denote any of the properties: quasi-compact, quasi-separated, quasi-compact with finite fibers, separated, proper, monomorphism, surjective, open immersion, isomorphism, closed immersion, finite, flat, smooth, fibers with dimension $\leqslant n$, all non-empty fibers with pure dimension n, étale, fpqc, faithfully flat with local fpqc quasi-sections, and étale with local étale quasi-sections. The map f satisfies property* **P** *if and only if the base change $f' : X' = X \times_Y Y' \to Y'$ satisfies property* **P**.

Proof. — See Theorem A.1.5 (with $\mathcal{F} = \mathcal{O}_X$) for the equivalence of flatness for f and f'. It is obvious in all other cases that if f satisfies property **P** then so does f'. If f' is étale then it is clear that f must be flat with étale fibers and so f is étale. For the converse implication for other **P**, we may work locally on Y so that Y is affinoid and there exists an *fpqc* map $U' \to Y$ equipped with a Y-map $U' \to Y'$. If $\{Y'_i\}$ is an admissible covering of Y' by affinoids then some finite subset $\{Y'_i\}_{i \in I_0}$ pulls back to an admissible covering of U' and so we may replace Y' with $\coprod_{i \in I_0} Y'_i$ so that Y' is also affinoid. In particular, $Y' \to Y$ is an *fpqc* morphism (between affinoids).

Suppose that f' is flat (resp. étale) with local *fpqc* quasi-sections (resp. with local étale quasi-sections). We wish to show that the flat (resp. étale) map f has local *fpqc* quasi-sections (resp. has local étale quasi-sections).

Since Y' is affinoid, by replacing Y' with the disjoint union of the constituents of a suitable finite affinoid covering we may suppose that there exists an *fpqc* (resp. étale) map $U' \to Y'$ and a Y'-map $U' \to X'$. If $\{X_j\}$ is an admissible affinoid covering of X then its pullback $\{X'_j\}$ is an admissible affinoid covering of X' (as Y and Y' are affinoid) and so some finite subset $\{X'_j\}_{j\in J_0}$ pulls back to an admissible covering of U'. Thus, the affinoid $V' = \coprod_{j\in J_0} X'_j$ has structure map to Y' that is *fpqc* (resp. étale, quasi-compact, and surjective). Since $X \to Y$ is flat (resp. étale), it follows that the affinoid $V = \coprod_{j\in J_0} X_j$ equipped with its evident Y-map to X is an *fpqc* quasi-section (resp. étale quasi-section) for $X \to Y$. This completes the proof that the property of being faithfully flat with local *fpqc* quasi-sections (resp. étale with local étale quasi-sections) descends through the faithfully flat base change $Y' \to Y$.

We now explain how the descent of the "open immersion" property is reduced to the descent of the "isomorphism" property. If f' is an open immersion, then its admissible open image W' in Y' has both pullbacks to $Y'' = Y' \times_Y Y'$ equal to the image of the open immersion f''. Hence, by Lemma 4.2.4 we conclude that W' is the preimage of an admissible open W in Y that must coincide with the image of f. Thus, f uniquely factors through a map $X \to W$ that becomes an isomorphism after making the *fpqc* base change $W' \to W$. This reduces the "open immersion" case to the "isomorphism" case.

Once we take care of descending the properties of quasi-compactness and properness, all of the other properties follow immediately (by arguing as in the case of schemes; for example, quasi-separatedness means quasi-compactness of the diagonal morphism). Thus, we now only consider the problem of proving that f is quasi-compact (resp. proper) when f' is quasi-compact (resp. proper). We may assume Y and Y' are affinoid.

Assuming that f' is quasi-compact, so X' is quasi-compact, we wish to conclude that X is quasi-compact. For this, it suffices to prove quite generally that any rigid space X admitting an *fpqc* map from a quasi-compact rigid space X' must be quasi-compact. By choosing an admissible affinoid covering of X and passing to a finite refinement whose (quasi-compact) preimages are an admissible covering of the quasi-compact X', we can find finitely many admissible affinoid opens $U_1, \ldots, U_n$ in X whose preimages in X' form an admissible covering of X. By Corollary 4.2.6, the U_i's form an admissible covering of X, so indeed X is quasi-compact. This settles descent of quasi-compactness.

Now we assume f' is proper and we wish to deduce that f is proper. By descent of quasi-compactness applied to both f' and its diagonal $\Delta_{f'}$, we at least know that f is quasi-compact and quasi-separated. Since Y is affinoid, X and Y must both be quasi-compact and quasi-separated. Thus, f admits a formal model $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}_1$. By Raynaud's flat models theorem [8, 5.10 (c)], there is also a formal model $\mathfrak{Y}' \to \mathfrak{Y}_2$ of the flat map of affinoids $Y' \to Y$ such that $\mathfrak{Y}' \to \mathfrak{Y}_2$ is a flat morphism of formal schemes. By using a base change to a formal model of Y that dominates the two formal models $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ of Y, we may suppose that the same formal model $\mathfrak{Y}$ of Y is being used for the formal models of both f and $Y' \to Y$. Using the theory of rig-points as developed in [7], especially [7, 3.5], the surjectivity of $Y' \to Y$ implies that the flat formal morphism $\mathfrak{Y}' \to \mathfrak{Y}$ is surjective and hence is faithfully flat (and quasi-compact).

Since f' is proper, the formal model $\mathfrak{f}'$ must be proper. The *fpqc* property of $\mathfrak{Y}' \to \mathfrak{Y}$ allows us to use scheme-theoretic *fpqc*-descent modulo an ideal of definition of the valuation ring of k to infer that the formal model $\mathfrak{f}$ must be proper. Hence, $f = \mathfrak{f}^{\mathrm{rig}}$ is proper (see §A.1). □

THEOREM 4.2.8 (Bosch-Görtz). — *Let $f : S' \to S$ be a faithfully flat map of rigid spaces, and assume f admits local fpqc quasi-sections. The functor $\mathcal{F} \rightsquigarrow (\mathcal{F}', \varphi)$ from coherent sheaves $\mathcal{F}$ on S to coherent sheaves $\mathcal{F}'$ on S' equipped with descent data $\varphi : p_1^*\mathcal{F}' \to p_2^*\mathcal{F}'$ on $S' \times_S S'$ relative to f is an equivalence of categories.*

Proof. — The hard part is the case when f is *fpqc*, and this was settled by Bosch and Görtz [5, Thm. 3.1]. The general case may be deduced from this by a standard argument, as we now explain. It is obvious that $\mathcal{F} \rightsquigarrow (\mathcal{F}', \varphi)$ is a faithful functor. To prove full faithfulness and effectivity of descent, we may work locally on S. Thus, the hypothesis concerning local *fpqc* quasi-sections allows us to assume that there exists an *fpqc* map $U' \to S$ and an S-map $U' \to S'$. We may assume S is affinoid, so U' is quasi-compact. Let $\{S'_i\}$ be an admissible affinoid covering of S', and let I_0 be a finite set of indices such that the S'_i's for $i \in I_0$ pull back to an admissible cover of U'. For each finite set of indices I containing I_0, the affinoid $S'_I = \coprod_{i \in I} S'_i$ is *fpqc* over the affinoid S. The admissibility of the covering $\{S'_i\}$ and the full faithfulness with respect to each covering $S'_I \to S$ gives full faithfulness in general.

It remains to prove effectivity of descent. Let $(\mathcal{F}', \varphi)$ be a coherent sheaf on S' equipped with descent data with respect to $S' \to S$. Making a base change by $U' \to S$ yields a descent problem with respect to $S' \times_S U' \to U'$. This latter map has a section, so by a standard argument (as in [9, 6.1/3])

descent with respect to this covering is effective. This provides a coherent sheaf $\mathcal{F}'_{U'}$ on U' descending the pullback of $\mathcal{F}'$ to $S' \times_S U'$, and full faithfulness as proved above provides canonical descent data on $\mathcal{F}'_{U'}$ with respect to the *fpqc* map $U' \to S$. By the settled *fpqc* case, we may descend $\mathcal{F}'_{U'}$ to a coherent sheaf $\mathcal{F}$ on S, and this is easily seen to provide the solution to the initial descent problem with respect to $S' \to S$. □

THEOREM 4.2.9. — *Let $f : S' \to S$ be a faithfully flat of rigid spaces, and assume it admits local fpqc quasi-sections. If X' is an S'-proper rigid space equipped with descent data relative to f and if there exists an S'-ample line bundle $\mathcal{L}'$ on X' compatible with the descent data, then the descent of X' to a rigid space X over S is effective* (*and the descent of $\mathcal{L}'$ to a line bundle on the S-proper X is S-ample*).

Proof. — Theorem 3.2.7 ensures that (3.1.2) makes sense and is an isomorphism in the relatively ample case. Hence, any proper map $f' : X' \to S'$ equipped with an S'-ample $\mathcal{L}'$ is encoded in terms of the coherent sheaves $f'_*({\mathcal{L}'}^{\otimes n})$ for $n \geqslant 0$ via the $\mathrm{Proj}^{\mathrm{an}}$ construction. This description of X' is functorial with respect to isomorphisms in $(X', \mathcal{L}')$ and it respects flat base change on S'. The descent data on $(X', \mathcal{L}')$ induces descent data on the coherent sheaves $f'_*({\mathcal{L}'}^{\otimes n})$, and by descending $f'_*({\mathcal{L}'}^{\otimes n})$ to a coherent sheaf $\mathcal{A}_n$ on S (Theorem 4.2.8) we immediately see that $\mathcal{A} \overset{\mathrm{def}}{=\!=} \bigoplus_{n \geqslant 0} \mathcal{A}_n$ is canonically a graded $\mathcal{O}_S$-algebra that descends the graded $\mathcal{O}_{S'}$-algebra $\mathcal{A}' \overset{\mathrm{def}}{=\!=} \bigoplus_{n \geqslant 0} f'_*({\mathcal{L}'}^{\otimes n})$.

By Theorem 3.2.7, $\mathcal{A}'$ is locally finitely generated over $\mathcal{O}_{S'}$. We claim that $\mathcal{A}$ is locally finitely generated over $\mathcal{O}_S$. Since $S' \to S$ is faithfully flat and admits local *fpqc* quasi-sections, there is an admissible covering of S by quasi-compact opens U_i for which there exists an *fpqc* map $U'_i \to U_i$ and a U_i-map $h_i : U'_i \to S'$ for each i. Each U'_i is quasi-compact, so the locally finitely generated graded $\mathcal{O}_{U'_i}$-algebra $h_i^* \mathcal{A}'$ is a generated by terms in degrees $\leqslant N_i$ for some large N_i. By faithful flatness of $U'_i \to U_i$, the graded $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$ with coherent graded terms must be generated by its coherent subsheaf of terms in degrees $\leqslant N_i$, so $\mathcal{A}$ is locally finitely generated. Hence, the rigid space $X = \mathrm{Proj}^{\mathrm{an}}(\mathcal{A})$ over S makes sense and provides the required S-proper descent of X' with respect to $S' \to S$. By applying Theorem 4.2.8 to the map $X' \to X$, $\mathcal{L}'$ descends to a coherent sheaf $\mathcal{L}$ on X that is necessarily a line bundle, and $\mathcal{L}$ is clearly S-ample. □

4.3. Picard groups

Let $f : \mathcal{X} \to \mathcal{S}$ be a proper flat map of finite presentation between two schemes. Assume $\mathcal{O}_{\mathcal{S}} = f_*\mathcal{O}_{\mathcal{X}}$ and that the same holds after any base change on $\mathcal{S}$; this condition is satisfied if f has geometrically reduced and geometrically connected fibers. Assume that there is given a section $e \in \mathcal{X}(\mathcal{S})$, so pairs $(\mathcal{L}, i)$ consisting of a line bundle on $\mathcal{X}$ and a trivialization $i : \mathcal{O}_{\mathcal{S}} \simeq e^*\mathcal{L}$ admit no non-trivial automorphisms. Let $\mathbf{Pic}_{\mathcal{X}/\mathcal{S},e}$ be the functor whose value on any $\mathcal{S}$-scheme $\mathcal{S}'$ is the group of isomorphism classes of pairs $(\mathcal{L}, i)$ of $e_{\mathcal{S}'}$-rigidified line bundles on $\mathcal{X} \times_{\mathcal{S}} \mathcal{S}'$. This group functor is a sheaf for the *fpqc* topology on the category of $\mathcal{S}$-schemes.

THEOREM 4.3.1 (Artin). — *With notation and hypotheses as above, $\mathbf{Pic}_{\mathcal{X}/\mathcal{S},e}$ is a locally separated algebraic space whose structural morphism to $\mathcal{S}$ is locally of finite presentation. It contains a unique open $\mathcal{S}$-subgroup $\mathbf{Pic}^{\tau}_{\mathcal{X}/\mathcal{S},e}$ of finite presentation whose fiber over each $s \in \mathcal{S}$ is the $k(s)$-subgroup $\mathbf{Pic}^{\tau}_{\mathcal{X}_s/k(s),e}$ classifying line bundles representing torsion elements of the Néron-Severi group of the geometric fiber at s.*

Proof. — The existence result is [1, 5.3, 7.3], and the structure of $\mathbf{Pic}^{\tau}_{\mathcal{X}/\mathcal{S},e}$ is [27, XIII, 4.7]. The local separatedness comes out of the method of proof that $\mathbf{Pic}_{\mathcal{X}/\mathcal{S},e}$ is an algebraic space, but it can also be proved abstractly by using the valuative criterion for a map of finite type between noetherian schemes to be an immersion [38, 2.13, p. 100]. (The passage to the noetherian case is standard, or alternatively [38, 2.13, p. 100] can be proved without noetherian hypotheses.) □

We wish to use Theorem 4.3.1 to prove an analogous representability result for rigid spaces. However, since we have not presented a theory of analytification for locally separated algebraic spaces we will only treat a special case that is sufficient for work with abelian schemes in [12]: we shall impose the condition of geometrically integral fibers. The elimination of this restriction will be addressed in [14]. Since we work with geometrically integral fibers, we only require Grothendieck's earlier theorem [25, Thm. 3.1] that in the setting of Theorem 4.3.1 if the proper flat map $\mathcal{X} \to \mathcal{S}$ is projective locally over $\mathcal{S}$ and if the geometric fibers are integral then $\mathbf{Pic}_{\mathcal{X}/\mathcal{S},e}$ is represented by an $\mathcal{S}$-scheme $\mathrm{Pic}_{\mathcal{X}/\mathcal{S},e}$ that is moreover separated over $\mathcal{S}$.

The rigid-analytic input for what follows is a proper flat map $f : X \to S$ between rigid spaces, and we assume that $\mathrm{H}^0(X_s, \mathcal{O}_{X_s}) = k(s)$ for all $s \in S$ (a condition that is satisfied if each fiber X_s is geometrically reduced and geometrically connected in the sense of [11]). The rigid-analytic theorem on cohomology and base change (whose proof goes as in the algebraic case,

with the help of [31]) implies that the natural map $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism and that this property persists after any base change on S and (for quasi-separated or pseudo-separated S) any extension on the base field; we say $\mathcal{O}_S = f_*\mathcal{O}_X$ *universally*. Assume that there is given a section $e \in X(S)$. Since e-rigidified line bundles $(\mathcal{L}, i)$ on X admit no non-trivial automorphisms, the functor $\mathbf{Pic}_{X/S,e}$ classifying $e_{S'}$-rigidified line bundles on $X \times_S S'$ for variable rigid spaces S' over S is a sheaf for the topology generated by faithfully flat maps that admit local *fpqc* quasi-sections. We will be interested in the case when f has geometrically integral fibers.

LEMMA 4.3.2. — *Let $F : \mathcal{X} \to \mathcal{S}$ be a proper flat map of algebraic k-schemes and assume $\mathcal{O}_{\mathcal{S}} = F_*\mathcal{O}_{\mathcal{X}}$ universally. Let $e \in \mathcal{X}(\mathcal{S})$ be a section. Assume moreover that F is projective locally over $\mathcal{S}$ and that its geometric fibers are integral. The analytification of the separated $\mathcal{S}$-scheme* $\mathrm{Pic}_{\mathcal{X}/\mathcal{S},e}$ *represents* $\mathbf{Pic}_{\mathcal{X}^{\mathrm{an}}/\mathcal{S}^{\mathrm{an}},e^{\mathrm{an}}}$.

Proof. — Theorem 4.3.1 provides a locally separated algebraic space group $\mathcal{P}$ over $\mathcal{S}$ equipped with a universal e-rigidified line bundle $(\mathcal{L}, i)$ on $\mathcal{X} \times_{\mathcal{S}} \mathcal{P}$. By results of Grothendieck (mentioned above), $\mathcal{P}$ is a separated $\mathcal{S}$-scheme under our assumptions on F. Thus, we may analytify to get a separated $\mathcal{S}^{\mathrm{an}}$-group $\mathcal{P}^{\mathrm{an}}$ that is equipped with a canonical e^{an}-rigidified line bundle $(\mathcal{L}^{\mathrm{an}}, i^{\mathrm{an}})$ on $\mathcal{X}^{\mathrm{an}} \times_{\mathcal{S}^{\mathrm{an}}} \mathcal{P}^{\mathrm{an}}$. We must prove that this analytified structure is the universal e^{an}-rigidified line bundle on base changes of $\mathcal{X}^{\mathrm{an}} \to \mathcal{S}^{\mathrm{an}}$ in the category of rigid spaces. That is, for any map of rigid spaces $T \to \mathcal{S}^{\mathrm{an}}$ and any e_T^{an}-rigidified line bundle $(\mathcal{M}, \iota)$ on $\mathcal{X}^{\mathrm{an}} \times_{\mathcal{S}^{\mathrm{an}}} T$, we must show that there exists a unique $\mathcal{S}^{\mathrm{an}}$-map $T \to \mathcal{P}^{\mathrm{an}}$ with respect to which the pullback of $(\mathcal{L}^{\mathrm{an}}, i^{\mathrm{an}})$ is (uniquely) isomorphic to $(\mathcal{M}, \iota)$.

By working Zariski-locally on $\mathcal{S}$ and locally on T we may assume $\mathcal{S} = \mathrm{Spec}(\mathcal{A})$ is affine and $T = \mathrm{Sp}(A)$ is affinoid. The composite map $T \to \mathcal{S}$ uniquely factors through the natural k-map $\mathrm{Sp}(A) \to \mathrm{Spec}(A)$, and by universal properties of affinoids, affine schemes, and analytification, the natural map $\mathcal{X}^{\mathrm{an}} \times_{\mathcal{S}^{\mathrm{an}}} T \to \mathcal{X} \times_{\mathcal{S}} \mathrm{Spec}(A)$ is the relative analytification in the sense of Example 2.2.11.

It now suffices to prove that if A is any k-affinoid algebra and $\mathcal{Z} \to \mathrm{Spec}(A)$ is any proper A-scheme with associated relative analytification $Z \to \mathrm{Sp}(A)$, then (i) pullback along the canonical map $Z \to \mathcal{Z}$ of locally ringed Grothendieck-topologized spaces sets up an equivalence $\mathrm{Coh}(\mathcal{Z}) \simeq \mathrm{Coh}(Z)$ (denoted $\mathcal{F} \rightsquigarrow \mathcal{F}^{\mathrm{an}}$) between categories of coherent sheaves, and (ii) $\mathcal{F}$ is a line bundle if and only if $\mathcal{F}^{\mathrm{an}}$ is a line bundle. The equivalence between categories of coherent sheaves was explained in Example 3.2.6, and the line bundle aspect is immediate via properness over the base and

calculations on infinitesimal fibers over $\mathrm{Sp}(A)$ and over the Zariski-dense subset $\mathrm{MaxSpec}(A)$ in the Jacobson scheme $\mathrm{Spec}\, A$. □

THEOREM 4.3.3. — *With notation and hypotheses as above, assume X admits an S-ample line bundle. The functor $\mathbf{Pic}_{X/S,e}$ is represented by a rigid space $\mathrm{Pic}_{X/S,e}$ over S whose structural map to S is separated, and there is a unique Zariski-open S-subgroup $\mathrm{Pic}^{\tau}_{X/S,e}$ whose fiber over each $s \in S$ classifies line bundles numerically equivalent to zero. The formation of this representing object respects analytic extension on k when S is quasi-separated or pseudo-separated, and it also respects analytification.*

Proof. — We may now work locally on S, so we may use the S-ample line bundle and Theorem 3.2.7 to reduce to the case when X is closed in some $\mathbf{P}^N_S$. By Corollary 4.1.6 we may assume that $X \to S$ is the pullback of the analytification of a projective flat map $\mathcal{X} \to \mathcal{S}$ between algebraic k-schemes, and that e is the pullback of the analytification of a section in $\mathcal{X}(\mathcal{S})$. By [15, 12.2.4 (viii)], the locus $\mathcal{U}$ of $s \in \mathcal{S}$ such that $\mathcal{X}_s$ is geometrically integral is Zariski-open. The map $S \to \mathcal{S}^{\mathrm{an}}$ factors through the Zariski-open $\mathcal{U}^{\mathrm{an}} \subseteq \mathcal{S}^{\mathrm{an}}$, so we may rename $\mathcal{U}$ as $\mathcal{S}$ and hence the hypotheses of Grothendieck's version of Theorem 4.3.1 (with projectivity and fibral geometric integrality hypotheses) are satisfied by $\mathcal{X} \to \mathcal{S}$. Lemma 4.3.2 therefore gives the desired existence result. In the case of quasi-separated or pseudo-separated S, the behavior with respect to extension on k follows from the general compatibility of analytification and change of the analytic base field. □

Appendix A. Some background results

A.1. Review of properness, flatness, and quasi-finiteness

We use Kiehl's definition of properness [6, 9.6.2/2], in terms of which Kiehl proved the basic cohomological results for coherent sheaves on rigid spaces. In particular, Kiehl proved that coherence is preserved under higher direct images of proper maps [31, 2.6] and he proved the rigid-analytic theorem on formal functions [31, 3.7].

Remark A.1.1. — Let $f : X \to Y$ be a proper map and let $\mathcal{F}$ be coherent on X, so $\mathrm{R}^i f_* \mathcal{F}$ is coherent on Y. By a Leray spectral sequence argument, it follows from the Tate/Kiehl acyclicity theorem that for any open affinoid U in Y, the natural map $\mathrm{H}^i(f^{-1}(U), \mathcal{F}) \to (\mathrm{R}^i f_* \mathcal{F})(U)$ is an isomorphism.

There are some subtle aspects of the rigid-analytic theory of properness that have been resolved by recent work of Temkin (using Berkovich spaces). Let us review the consequences of Temkin's work for our purposes. For a quasi-compact map $f : X \to Y$ between locally finite type schemes over a non-archimedean field k, elementary methods ensure that properness of f^{an} implies properness of f [11, 5.2.1(2)]. Likewise, in Raynaud's theory of formal models (see [7], [8], and [28, Appendix]), for a quasi-compact map $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ between flat and locally topologically finitely presented formal schemes over the valuation ring R of k, it is elementary to prove that properness of $\mathfrak{f}^{\mathrm{rig}}$ implies properness of $\mathfrak{f}$ [35, 2.5, 2.6]. In practice one needs the converses to these two assertions: by [2, 3.4.7] and [42, Cor. 4.4, Cor. 4.5], we have

- A quasi-compact map $f : X \to Y$ between algebraic k-schemes is proper if and only if f^{an} is proper.
- If $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is a quasi-compact map between locally topologically finitely presented and flat formal schemes over the valuation ring R of k, then properness of $\mathfrak{f}$ is equivalent to properness of $\mathfrak{f}^{\mathrm{rig}}$.
- Properness is preserved under composition in the rigid-analytic category.
- If Y is a rigid space and $h : X' \to X$ is a Y-morphism between rigid spaces that are separated over Y, and if X' is proper over Y (so h is proper [6, 9.6.2/4]), then the Zariski-closed set $h(X')$ in X is proper over Y with respect to the rigid-analytic structure defined by any coherent ideal sheaf on X whose zero locus is $h(X')$.

Let us now turn to "algebraic" cohomological properties of proper maps. The rigid-analytic analogue of Serre's GAGA theorem in the projective case over a field is proved in [33, pp. 43–53], and Kiehl's rigid-analytic theorem on formal functions allows one to relativize this to higher direct images with respect to projective morphisms. Using Chow's Lemma and Grothendieck's method in [26, XII], this generalizes to the case of an arbitrary proper map $f : X \to Y$ between algebraic k-schemes: there are δ-functorial GAGA isomorphisms

$$\mathrm{R}^i f_*(\mathcal{F})^{\mathrm{an}} \simeq \mathrm{R}^i f_*^{\mathrm{an}}(\mathcal{F}^{\mathrm{an}}) \tag{A.1.1}$$

for any coherent sheaf $\mathcal{F}$ on X. We shall require the following cohomological comparison isomorphism:

THEOREM A.1.2. — *Let $f : X \to \mathrm{Sp}(A)$ be a proper map to a k-affinoid, and let $\mathcal{F}$ be a coherent sheaf on X. If k'/k is an analytic extension field and $f' : X' \to \mathrm{Sp}(A')$ and $\mathcal{F}'$ are induced by f and $\mathcal{F}$, then the natural*

δ-functorial map $k'\widehat{\otimes}_k \mathrm{H}^i(X,\mathcal{F}) \simeq A' \otimes_A \mathrm{H}^i(X,\mathcal{F}) \to \mathrm{H}^i(X',\mathcal{F}')$ between finite A'-modules is an isomorphism.

The natural map in this theorem is defined using Čech theory since the functor $\mathcal{F} \rightsquigarrow \mathcal{F}'$ is only defined for quasi-coherent $\mathcal{O}_X$-modules and not for general $\mathcal{O}_X$-modules. The relativization of the theorem to the case of coherent higher direct images (via Definition 2.1.14) is an immediate consequence of Remark A.1.1.

Proof. — The first isomorphism follows from [11, 1.1.5(1)] since $\mathrm{H}^i(X,\mathcal{F})$ is a finite A-module. The rest of the theorem is therefore an immediate consequence of two ingredients: the computation of the cohomology in terms of a Čech complex of k-Banach spaces of countable type (these are spaces to which the Banach open mapping theorem applies), and Kiehl's result [31, 2.5*ff.*] that the differentials in this complex have *closed* image since f is proper. □

The following theorem is the rigid-analytic analogue of a classical result in the complex-analytic case [22, 3.1.2]; see [30, I, §1, Thm. 1] for a discussion in terms of a cruder notion of rigid-analytic space (avoiding the possibility of non-rational points). We give a proof, due to lack of a reference.

THEOREM A.1.3 (Structure theorem for locally quasi-finite maps). — *Let $f : X \to Y$ be a separated morphism of rigid spaces, and let $\Sigma \subseteq f^{-1}(y)$ be a finite set of isolated points for some $y \in Y$. There exists an admissible open U around y and an admissible open $V \subseteq f^{-1}(U)$ around Σ such that the restriction $f_V : V \to U$ of f is a finite map. Moreover, the natural map of $\mathcal{O}_{Y,y}$-algebras*

$$(f_{V*}\mathcal{O}_V)_y \longrightarrow \prod_{x\in\Sigma} \mathcal{O}_{X,x} \tag{A.1.2}$$

is an isomorphism. In particular, $\mathcal{O}_{X,x}$ is a finite $\mathcal{O}_{Y,y}$-module for all $x \in f^{-1}(y)$.

Proof. — Once we find U and V such that $f_V : V \to U$ is finite, the fact that (A.1.2) is an isomorphism (and hence $\mathcal{O}_{X,x}$ is $\mathcal{O}_{Y,y}$-finite) follows from [34, Satz 2.2]. The key problem is the existence of V. In principle, the proof in the complex-analytic case is what one uses. However, the possibility of non-rational points leads to some complications.

As a first step, we may assume Y is affinoid, so X is separated. This ensures that overlaps of affinoids in X are affinoid. If $x_0 \neq x_1$ are distinct points in X, then $X - \{x_i\}$ is Zariski-open (hence admissible), so there

exist open affinoids $U_i \subseteq X - \{x_{1-i}\}$ around x_i. The overlap $U_0 \cap U_1$ is an open affinoid in U_i not containing x_i. By the Grauert-Gerritzen description of affinoid subdomains of an affinoid in terms of rational subdomains and Weierstrass subdomains [6, 7.5.3/3], there exists an open affinoid $V_i \subseteq U_i$ around x_i that is disjoint from $U_0 \cap U_1$. Thus, any two distinct points in X admit disjoint admissible open neighborhoods, so we may find pairwise disjoint admissible open affinoids around each of the finitely many points in Σ. This permits us to reduce to the case where Σ consists of a single point x (the case $\Sigma = \emptyset$ is trivial: take $V = \emptyset$). We may then work locally around both y and x, so we can also assume X and Y are affinoid and $f^{-1}(y) = \Sigma = \{x\}$.

When x is k-rational, we may choose closed immersions $Y \hookrightarrow \mathbf{B}^m$ and $X \hookrightarrow \mathbf{B}^n$ such that $y = f(x) \mapsto 0$ and $x \mapsto 0$ respectively (with $\mathbf{B}^r$ denoting the closed unit polydisc in r-space). We can then argue (with the help of the Weierstrass preparation theorem) in a manner quite similar to the complex-analytic proof [22, Ch. 3, §1]. It remains to reduce to the case when the finitely many points in $f^{-1}(y)$ are k-rational. The key geometric input we need is the following immediate consequence of a lemma of Kisin [32, 2.3]:

LEMMA A.1.4. — *For any finite map $g : T' \to T$ between affinoids and any analytic set $Z \subseteq T$ (e.g., a point), any admissible open neighborhood of $g^{-1}(Z)$ in the affinoid T' contains $g^{-1}(U)$ for some admissible open U around Z*

Continuing with the proof of Theorem A.1.3, by choosing a sufficiently large finite extension k' of k and letting $f' : X' \to Y'$ denote the base change of f, we may assume that the finitely many points in Y' over y are k'-rational and that their finite f'-fibers consist of k'-rational points. Let $\pi_X : X' \to X$ and $\pi_Y : Y' \to Y$ denote the two finite projections. By the special case already treated, there exists an admissible open affinoid $U' \subseteq Y'$ around $\pi^{-1}(y)$ and an admissible open $V' \subseteq f'^{-1}(U')$ around $\pi_X^{-1}(x) = f'^{-1}(\pi_Y^{-1}(y))$ such that $f'_{V'} : V' \to U'$ is finite. Since π_Y is a finite map between affinoids, by Lemma A.1.4 we may shrink U' around $\pi_Y^{-1}(y)$ so that U' is the base change of some admissible open affinoid U in Y around y.

Using Lemma A.1.4 again, there exists an admissible open $W \subseteq X$ around $\{x\} = f^{-1}(y)$ such that $W' \subseteq V'$, and W must contain $f^{-1}(U_1)$ for some admissible open affinoid $U_1 \subseteq U$ around y. Renaming U_1 as Y and $f^{-1}(U_1)$ as X, we reduce to the case where f is a map between affinoids and f' is finite. It is clear that f is finite. □

For a morphism of rigid spaces $f : X \to Y$, an $\mathcal{O}_X$-module $\mathcal{F}$ is *Y-flat* if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module for all $x \in X$. This is only a reasonable definition when $\mathcal{F}$ is quasi-coherent (see Definition 2.1.1), and this is the only case that we shall need (*e.g.*, see Corollary 2.2.7). If this condition holds with $\mathcal{F} = \mathcal{O}_X$, then f is a *flat* map.

THEOREM A.1.5. — *Let $f : X \to Y$ be a morphism of rigid spaces and let $\mathcal{F}$ be a coherent sheaf on X. Let $Y' \to Y$ be a map of rigid spaces, and let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $X' = X \times_Y Y'$. If $\mathcal{F}$ is Y-flat then $\mathcal{F}'$ is Y'-flat, and the converse holds if $Y' \to Y$ is faithfully flat and admits local fpqc quasi-sections in the sense of Definition* 4.2.1.

Proof. — Choose $x' \in X'$ over points $x \in X$ and $y' \in Y'$ with common image $y \in Y$. (Note that for any point $x \in X$ mapping to a point $y \in Y$ we can find points $x' \in X'$ and $y' \in Y'$ over x and y respectively when $Y' \to Y$ is surjective.) Let $\mathcal{F}_n$ and $\mathcal{F}'_n$ denote the restrictions to $\mathcal{F}$ and $\mathcal{F}'$ to the nth infinitesimal neighborhoods X'_n and X_n of the fibers of f and f' over y and y' respectively. By the local flatness criterion [37, 22.3], Y'-flatness of $\mathcal{F}'$ at x' is equivalent to $\mathcal{O}_{Y',y'}/\mathfrak{m}_{y'}^{n+1}$-flatness of $\mathcal{F}'_{n,x'}$ for all $n \geqslant 0$, and similarly the Y-flatness of $\mathcal{F}$ at x is equivalent to $\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n+1}$-flatness of $\mathcal{F}_{n,x}$ for all $n \geqslant 0$. Since $\mathcal{O}_{Y',y'}/\mathfrak{m}_{y'}^{n+1}$ and $\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n+1}$ are k-finite, we have $\mathcal{O}_{X'_n,x'} \simeq \mathcal{O}_{X_n,x} \otimes_{\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n+1}} \mathcal{O}_{Y',y'}/\mathfrak{m}_{y'}^{n+1}$ with an ordinary tensor product. Hence, we have

$$\mathcal{F}'_{n,x'} = \mathcal{F}_{n,x} \otimes_{\mathcal{O}_{X_n,x}} \mathcal{O}_{X'_n,x'} \simeq \mathcal{F}_{n,x} \otimes_{\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n+1}} \mathcal{O}_{Y',y'}/\mathfrak{m}_{y'}^{n+1},$$

so Y-flatness of $\mathcal{F}$ implies Y'-flatness of $\mathcal{F}'$.

Conversely, assume that $\mathcal{F}'$ is Y'-flat and that f is faithfully flat with local *fpqc* quasi-sections. We want to prove that $\mathcal{F}_x$ is $\mathcal{O}_{Y,y}$-flat for all $y \in Y$ and $x \in X_y$. We can work locally on Y, and so we can assume that there exists a Y-map $U' \to Y'$ with U' faithfully flat and quasi-compact over Y. We can assume Y is affinoid, so U' is quasi-compact. Hence, there exists a finite set of admissible open affinoids Y'_i in Y' whose pullback to U' is an admissible covering. The affinoid $\coprod Y'_i$ is *fpqc* over Y, so we may replace Y' with $\coprod Y'_i$ to reduce to the case when Y' is *fpqc* over Y. The case when Y' is Y-finite is trivial, as then X' is X-finite and we can use ordinary tensor products rather than completed tensor products when working on X'. Hence, it suffices to prove that if $Y' \to Y$ is *fpqc* then there exists a finite flat map $Z \to U$ over an admissible open $U \subseteq Y$ around y such that $Y'(Z)$ is non-empty (as then pullback from Y' to Z along a Y-map $Z \to Y'$ carries $\mathcal{F}'$ to a Z-flat coherent sheaf on $X' \times_{Y'} Z = X_U \times_U Z$,

thereby reducing us to the trivial case of descent of flatness through a finite flat covering).

Our problem now has nothing to do with $\mathcal{F}$, but is rather the statement that $Y' \to Y$ admits a section after base change to a finite flat neighborhood of $y \in Y$. By the structure theorem for rigid-analytic locally quasi-finite morphisms (Theorem A.1.3), the Y-flatness of Y' allows us to argue exactly as in the case of schemes [15, IV_4, 17.6.1] to reduce the problem to that of non-emptiness of the Cohen-Macaulay locus in the non-empty fiber Y'_y. That is, it suffices to verify the non-emptiness of the CM-locus on a non-empty rigid space Y' over a non-archimedean field k. We may suppose Y' is affinoid, and since a finite flat algebra over a CM ring is CM it is enough to show that a finite Noether normalization map from Y' onto a unit polydisc is flat over a non-empty admissible open locus in the polydisc. Such flatness follows from the elementary "generic flatness" of finite surjective morphisms to a smooth base. □

With notation as in Theorem A.1.5, if X and Y are quasi-separated then it is easy to check (using Raynaud's theory of formal models, especially [8, 4.1], and the faithful flatness of $k'\widehat{\otimes}_k C$ over C for any k-affinoid C [11, 1.1.5 (1)]) that for any analytic extension field k'/k and any coherent sheaf $\mathcal{F}$ on X, the sheaf $\mathcal{F}$ is Y-flat if and only if the induced coherent sheaf $\mathcal{F}'$ on $X' = k'\widehat{\otimes}_k X$ is flat over $Y' = k'\widehat{\otimes}_k Y$. It therefore makes sense to consider the interaction of fibral Euler characteristics and extension on k for flat families of coherent sheaves. This is addressed in the second part of:

THEOREM A.1.6. — *Let $f : X \to Y$ be proper and let $\mathcal{F}$ be a coherent Y-flat sheaf on X. The Euler characteristic*

$$y \longmapsto \chi_y(\mathcal{F}_y) = \sum (-1)^i \dim_{k(y)} \mathrm{H}^i(X_y, \mathcal{F}_y)$$

is locally constant for the Zariski topology on Y.

If Y is quasi-separated or pseudo-separated, then for $\chi \in \mathbf{Z}$ the formation of the Zariski-open locus $U_\chi = \{y \in Y \,|\, \chi_y(\mathcal{F}_y) = \chi\}$ commutes with any extension on k.

See Definition A.2.1 for the notion of pseudo-separatedness.

Proof. — The algebraic methods in [39, §5] give the local constancy for the Zariski topology. Thus, we may assume that for some $\chi \in \mathbf{Z}$ and all points y we have $\chi_y(\mathcal{F}_y) = \chi$ with Y a quasi-separated or pseudo-separated space, and we must prove that this property is preserved after arbitrary extension on k. We can assume that Y is connected, and the case of finite extension on k is trivial. By [11, 3.2.3], for a suitable finite extension k'/k the rigid space $k' \otimes_k Y$ is a union of finitely many connected components

that are geometrically connected over k' (that is, each connected component remains connected after arbitrary analytic extension on k'). Thus, we may suppose that Y is geometrically connected over k. In view of the disjoint Zariski-open stratification that has already been proved in the general case, when the base is geometrically connected the preservation of the property of having constant fibral Euler characteristics is obviously preserved by extension of the base field. □

A.2. Pseudo-separatedness and change of base field

If $f : \mathcal{X} \to \mathcal{Y}$ is a non-separated map between algebraic k-schemes, the relative analytic diagonal map $\Delta_{f^{\mathrm{an}}} = \Delta_f^{\mathrm{an}}$ is never quasi-compact (even if f is quasi-compact) and so f^{an} is never quasi-separated when f is not separated. Since extension of the base field makes sense without restrictions in algebraic geometry, we want to be able to say $k'\widehat{\otimes}_k\mathcal{S}^{\mathrm{an}} \simeq (k' \otimes_k \mathcal{S})^{\mathrm{an}}$ for any algebraic k-scheme $\mathcal{S}$. However, if $\mathcal{S}$ is not separated then we cannot make sense of such an isomorphism until we provide an intrinsic method to define $k'\widehat{\otimes}_k S$ for a class of non-separated rigid spaces S that includes those arising as analytifications of algebraic k-schemes. Hence, we seek a property of morphisms of rigid spaces that is weaker than separatedness and is satisfied by all maps of the form f^{an} with f a map between algebraic k-schemes.

Definition A.2.1. — *A map of rigid spaces $f : X \to S$ is* pseudo-separated *(or X is S-*pseudo-separated*) if the diagonal $\Delta_f : X \to X \times_S X$ factors as $X \xrightarrow{i} Z \xrightarrow{j} X \times_S X$ with i a Zariski-open immersion and j a closed immersion. A rigid space X is* pseudo-separated *if $X \to \mathrm{Sp}(k)$ is pseudo-separated.*

Example A.2.2. — If $f : \mathcal{X} \to \mathcal{S}$ is a map between algebraic k-schemes then Δ_f is a quasi-compact immersion of schemes. Hence, it makese sense to form a scheme-theoretic closure for Δ_f, so Δ_f factors as a Zariski-open immersion followed by a closed immersion. Since $\Delta_{f^{\mathrm{an}}} = \Delta_f^{\mathrm{an}}$, it follows that f^{an} is pseudo-separated.

Example A.2.3. — A map of rigid spaces $f : X \to S$ is quasi-separated and pseudo-separated if and only if it is separated.

It is trivial to check that pseudo-separatedness is preserved by base change (this is why we omit a "denseness" condition on i in the definition), but the non-uniqueness of the factorization in the definition implies

that pseudo-separatedness does not descend through *fpqc* base change. The non-uniqueness of the factorization also implies that pseudo-separatedness is generally not local on the base. Also, the failure of Zariski-openness to be transitive in the analytic setting implies that pseudo-separatedness is generally not preserved under composition. However, it is easy to check that if X and Y are rigid spaces over a rigid space S and there is given an S-map $f : X \to Y$ then f is pseudo-separated when X and Y are S-pseudo-separated, and conversely if Y is S-separated and f is pseudo-separated then X is S-pseudo-separated. In particular, if X and Y are pseudo-separated rigid spaces then any map $X \to Y$ is pseudo-separated.

Recall the following elementary fact (see [11, 3.1.1] for a proof):

LEMMA A.2.4. — *Let $\iota : U \to X$ be a Zariski-open immersion into a quasi-separated rigid space X, with Zariski-closed complement $j : Z \hookrightarrow X$. For any analytic extension field k'/k, the induced maps $\iota' : U' \to X'$ and $j' : Z' \to X'$ are respectively open and closed immersions, with $\iota'(U')$ a Zariski-open and $j'(Z')$ its Zariski-closed complement in X'.*

The reason that we are interested in pseudo-separatedness is that it provides an alternative to quasi-separatedness that is sufficient for the formation of change of base field functors. To explain this, let X be a pseudo-separated rigid space over k and let k'/k be an analytic extension field. For any two affinoid opens U and V in X, the overlap $W = U \cap V$ is an admissible open in each of U and V, and moreover each inclusion $W \to U$ and $W \to V$ factors as a Zariski-open immersion followed by a closed immersion. Since U, V, and W are all separated, it makes sense to form the rigid spaces $k'\widehat{\otimes}_k U$, $k'\widehat{\otimes}_k V$, and $k'\widehat{\otimes}_k W$, and so we can consider the maps

$$k'\widehat{\otimes}_k W \longrightarrow k'\widehat{\otimes}_k U, \quad k'\widehat{\otimes}_k W \longrightarrow k'\widehat{\otimes}_k V. \tag{A.2.1}$$

By Lemma A.2.4, each of these maps is a Zariski-open immersion followed by a closed immersion. We claim that each is an admissible open immersion. To see this, first note that for each affinoid open $W_i \subseteq W$, the affinoid open $k'\widehat{\otimes}_k W_i$ in $k'\widehat{\otimes}_k W$ is also an affinoid open in each of $k'\widehat{\otimes}_k U$ and $k'\widehat{\otimes}_k V$. It then follows easily from the Zariski open/closed factorization that the collection of affinoids $k'\widehat{\otimes}_k W_i$ in each of $k'\widehat{\otimes}_k U$ and in $k'\widehat{\otimes}_k V$ is an admissible collection, so indeed the maps in (A.2.1) are admissible open immersions.

It is now a simple exercise to check that by gluing $k'\widehat{\otimes}_k U$ and $k'\widehat{\otimes}_k V$ along $k'\widehat{\otimes}_k (U \cap V)$ for all pairs of affinoid opens $U, V \subseteq X$, the triple overlap compatibility conditions are satisfied and hence we get a well-defined rigid space $k'\widehat{\otimes}_k X$ with the affinoids $k'\widehat{\otimes}_k U$ as an admissible open covering.

(If X is separated then this construction agrees with the one in the quasi-separated case; also see Example A.2.3.) Exactly as in the quasi-separated case, $X \rightsquigarrow k'\widehat{\otimes}_k X$ is naturally a functor from pseudo-separated rigid spaces over k to rigid spaces over k', and it carries fiber products to fiber products and Zariski open/closed immersion to Zariski open/closed immersions. In particular, $k'\widehat{\otimes}_k X$ is pseudo-separated over k'. Transitivity with respect to further extension of the base field and the exact "pullback" functor $\mathrm{Coh}(X) \to \mathrm{Coh}(k'\widehat{\otimes}_k X)$ on categories of coherent sheaves are defined as in the quasi-separated case. Finally, Lemma A.2.4 carries over (with the same proof) in the pseudo-separated case, and all appearances of "quasi-separated" in [11] may be replaced with "pseudo-separated" (see the end of [11, §3.1]).

Remark A.2.5. — The failure of pseudo-separatedness to be preserved under composition can be rectified by generalizing the definition of pseudo-separatedness to require Δ_f to merely factor as a finite composite of maps that are Zariski-open or Zariski-closed immersions. It is straightforward to check that what we have said above remains valid under this more general definition, and the reader will easily check that this generalization suffices for all applications of pseudo-separatedness in this paper; however, it is not clear if this proposed generalization of pseudo-separatedness of $\mathcal{X}^{\mathrm{an}}$ for an analytifiable algebraic space $\mathcal{X}$ (in the sense of [14]) forces $\mathcal{X}$ to be locally separated. For this reason, we have decided to adopt the more restrictive definition of pseudo-separatedness that is given above.

THEOREM A.2.6. — *Let $f : X \to Y$ be an S-map between proper rigid spaces over S. Let U be the locus of $s \in S$ such that f induces an isomorphism (resp. closed immersion) on the infinitesimal fibers (resp. on the fiber) over s.*

1) *If X is S-flat and f_s is an isomorphism for some $s \in S$ then f induces an isomorphism on infinitesimal fibers over s.*

2) *The locus U is Zariski-open in S, and $f|_U : X_U \to Y_U$ is an isomorphism (resp. closed immersion).*

3) *If S is quasi-separated or pseudo-separated then the formation of U is compatible with change of the base field.*

Proof. — First we prove 1), so assume that X is S-flat and pick $s \in S$ such that f_s is an isomorphism. For all $x \in X_s$ we have $f^{-1}(f(x)) = \{x\}$, so f is locally quasi-finite at x. By Theorem A.1.3, the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is finite. This is an isomorphism modulo $\mathfrak{m}_s$, and hence it is surjective (by Nakayama's Lemma). The kernel ideal $K \subseteq \mathcal{O}_{Y,f(x)}$ is finitely generated

and $K/\mathfrak{m}_s K = 0$ since $\mathcal{O}_{Y,f(x)}$ is noetherian and $\mathcal{O}_{X,x}$ is $\mathcal{O}_{S,s}$-flat. Thus, $K = 0$ and hence f is an isomorphism near any $x \in X_s$. More specifically, since $f^{-1}(f(x)) = \{x\}$ for all $x \in X_s$, we conclude that on infinitesimal fibers over s the maps induced by f are proper, bijective (hence finite), and local isomorphisms, so these induced maps must be isomorphisms.

Now we turn to 2). If f_s is a closed immersion then the maps induced by f on infinitesimal fibers over s are proper and quasi-finite, hence finite, and so are closed immersions since they are closed immersions modulo $\mathfrak{m}_s$. To prove that f is a closed immersion over a Zariski-open neighborhood of s when f restricts to a closed immersion on all infinitesimal fibers over s, we first note quite generally that the proper map f is a closed immersion if and only if the diagonal map $\Delta_f : X \to X \times_Y X$ is an isomorphism. Indeed, Δ_f is an isomorphism if and only if f is a monomorphism, and proper monomorphisms are closed immersions (as proper monics are visibly quasi-finite, hence finite, and a finite monomorphism to $\mathrm{Sp}(k')$ is readily seen to be either empty or $\mathrm{Sp}(k')$ for finite extensions k'/k). Thus, since $X \times_Y X$ is S-proper we can replace f with Δ_f if necessary so as to reduce the proof of 2) to the case of isomorphisms.

Fix $s \in U$. The map f is quasi-finite at all $x \in X_s$ and hence $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is finite for such x. Since this is an isomorphism modulo all powers of $\mathfrak{m}_s$, and hence modulo all powers of $\mathfrak{m}_{f(x)}$, this finite map becomes an isomorphism after faithfully flat base change to $\widehat{\mathcal{O}}_{Y,f(x)}$ and hence it is an isomorphism. That is, f is a local isomorphism near all points x in the fiber X_s. Since f is proper, it follows from the theorem on formal functions that for all $x \in X_s$ the natural map

$$f_*(\mathcal{O}_X)_{f(x)} \longrightarrow \mathcal{O}_{X,x} \tag{A.2.2}$$

of (finite) $\mathcal{O}_{Y,f(x)}$-modules is an isomorphism after applying the functor $\widehat{\mathcal{O}}_{Y,f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} (\cdot)$ and so it is an isomorphism. We shall now use sheaves of differentials to exploit the isomorphism (A.2.2).

We refer to [10, §1–§2] for a discussion of the basic properties of the coherent sheaves of relative differentials in the rigid case. The coherent sheaf $\Omega^1_{X/Y}$ is supported on a Zariski-closed set $Z \subseteq X$, so the Zariski-closed image of Z in S is disjoint from U because f is an isomorphism on infinitesimal fibers over all $s \in U$. Consider the coherent sheaf $f_*\mathcal{O}_X$ on Y. Let W be the Zariski-open locus in Y where $f_*\mathcal{O}_X$ is Y-flat. For any $s \in U$ and $y \in Y_s$ the fiber $f^{-1}(y)$ consists of a single point x, and $y \in W$ since (A.2.2) is an isomorphism and the natural map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism. Thus, the Zariski-closed image of $Y - W$ in S is disjoint from U. Letting $V \subseteq S$ be the Zariski-open complement of the union of

images of $Y - W$ and Z in S, we conclude that for a suitable Zariski-open V in S containing U, the coherent sheaf $f_*\mathcal{O}_X|_{Y_V}$ is Y_V-flat and $\Omega^1_{X/Y}|_{Y_V} = 0$.

The vanishing of $\Omega^1_{X/Y}|_{Y_V}$ forces $f|_V : X_V \to Y_V$ to be a *finite* map. Indeed, since f is proper we just have to check quasi-finiteness of $f|_V$. On fibers over Y_V we are reduced to the assertion that if T is a quasi-compact rigid space over a non-archimedean field k and $\Omega^1_{T/k} = 0$, then T consists of finitely many points. For any $t \in T$, the $\mathfrak{m}_t$-adic completion of $\Omega^1_{T/k,t}$ is the module $\widehat{\Omega}^1_{\widehat{\mathcal{O}}_{T,t}/k}$ of continuous k-linear Kähler differentials, and so if $\Omega^1_{T/k,t} = 0$ then $\widehat{\mathcal{O}}_{T,t}$ has vanishing Zariski cotangent space. Thus, the vanishing of $\Omega^1_{T/k}$ forces T to be 0-dimensional. Being quasi-compact and 0-dimensional, T has only finitely many points. Back in our original situation, we conclude that $f|_V$ must be finite, and hence flat (as $f_*\mathcal{O}_X|_{Y_V}$ is Y_V-flat).

A finite flat map has a degree that is locally constant in the Zariski topology on a rigid space. We will show that $f|_V$ has constant degree equal to 1 over a suitable Zariski-open of S contained in V, so f restricts to an isomorphism over this Zariski-open. Let $\pi_Y : Y \to S$ be the projection. If W' denotes the Zariski-open locus in Y where $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism, then the intersection

$$\widetilde{U} = V \cap \big(S - \pi_Y(Y - W')\big) = V \times_S \big(S - \pi_Y(Y - W')\big) \subseteq S \tag{A.2.3}$$

is a Zariski-open in S that contains U, and over this open the restriction of f is finite flat of degree 1. Thus, $\widetilde{U} = U$ and $f|_U$ is an isomorphism. This proves 2).

Now we assume S is quasi-separated or pseudo-separated, and we shall prove 3). We want to prove that the formation of U is compatible with change of the analytic base field. Since $\widetilde{U} = U$, it suffices to check that all of the ingredients entering into the construction (A.2.3) of $\widetilde{U}$ are compatible with change of the base field.

The theorem on formal functions ensures that the image of an analytic set Z under a proper map f coincides with the support of the coherent $f_*(\mathcal{O}_Z)$. The formation of direct images of coherent sheaves under a proper map naturally respects change of the base field, by Theorem A.1.2, so to show that the formation of $f(Z)$ respects change in the base field it suffices to study supports of coherent sheaves.

Recall from [11, 1.1.5(1)] that $A \to A' = k'\widehat{\otimes}_k A$ is faithfully flat for any k-affinoid A, and there is a canonical isomorphism

$$M' \underset{=}{\text{def}} A' \otimes_A M \simeq k'\widehat{\otimes}_k M \tag{A.2.4}$$

for finite A-modules M. Since the support of a coherent sheaf is the zero locus of its coherent annihilator ideal sheaf, it follows from (A.2.4) and the faithful flatness of $A \to A'$ that change of the base field commutes with formation of the support of a coherent sheaf. By faithful flatness considerations we likewise verify the compatibility of change of the base field and the formation of the kernel/cokernel of a map between coherent sheaves. Since Lemma A.2.4 ensures that Zariski-open/closed immersions behave well with respect to change of the base field, we conclude that the formation of $S - \pi_Y(Y - W')$ in (A.2.3) is compatible with extension on k.

To analyze V in (A.2.3), it remains to check that if $\mathcal{F}$ is a coherent sheaf on a quasi-separated or pseudo-separated rigid space Y then the formation of the Zariski-open locus of Y-flatness (or equivalently, of local freeness) for $\mathcal{F}$ is compatible with change of the base field. Working on the constituents of an admissible affinoid cover of Y, our problem is reduced to the claim that if A is k-affinoid, M is a finite A-module, $A' = k'\widehat{\otimes}_k A$, and $M' = A' \otimes_A M \simeq k'\widehat{\otimes}_k M$, then under the map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ the preimage of the Zariski-open locus of A-flatness for M is the Zariski-open locus of A'-flatness for M'. This claim follows from the faithful flatness of A' over A. □

A useful consequence of Lemma A.2.4 and the proof of Theorem A.2.6.(3) is:

COROLLARY A.2.7. — *For a coherent sheaf $\mathcal{F}$ on a quasi-separated or pseudo-separated rigid space S, the Zariski-open locus where $\mathcal{F}$ is locally free of a fixed rank $n \geqslant 0$ is compatible with change of the base field. In particular, the Zariski-closed support of $\mathcal{F}$ is compatible with change of the base field.*

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