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ON SOLVABLE GENERALIZED CALABI-YAU MANIFOLDS

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ABSTRACT. — We give an example of a compact 6-dimensional non-Kähler symplectic manifold (M, κ) that satisfies the Hard Lefschetz Condition. Moreover, it is showed that (M, κ) is a special generalized Calabi-Yau manifold.

RÉSUMÉ. — On donne un exemple d'une variété symplectique compacte (M, κ) de dimension 6 qui n'admet aucune structure Kählerienne, mais qui satisfait la condition de Lefschetz Forte et dont l'algèbre de DeRham est formelle ; de plus, on montre que (M, κ) peut être dotée d'une structure de Calabi-Yau généralisée spéciale.

1. Introduction

In the symplectic universe, a distinguished galaxy is represented by those objects satisfying the *Hard Lefschetz Condition* (HLC), i.e. those compact $2n$-dimensional symplectic manifolds (M, κ) for which the maps

$$[\kappa]^p : \mathrm{H}^{n-p}(M, \mathbb{R}) \to \mathrm{H}^{n+p}(M, \mathbb{R}), \quad 0 \leqslant p \leqslant n$$

are isomorphisms. A classical result states that, if (M, κ, J) is a compact Kähler manifold, then (M, κ) satisfies the HLC (see [7]), and $\wedge^*(M)$ is a formal DGA; moreover, HLC symplectic manifolds possess some of the cohomological properties of a Kähler manifold (e.g. the odd Betti numbers $b_{2p+1}(M)$ are even, $b_p(M) \leqslant b_{p+2}(M)$, $0 \leqslant p < n-1$). It is well known that, given a symplectic manifold (M, κ), it is possible to define a $\bigstar$ symplectic Hodge operator on $\wedge^p(M)$, for $0 \leqslant p \leqslant 2n$ and a codifferential operator

$d^\star$. Then $(\wedge^*, d^\star, d)$ is a differentiable Gerstenhaber-Batalin-Vilkovisky (dGBV) algebra, that is integrable (i.e. the $dd^\star$-lemma holds), if and only if (M, κ) satisfies the HLC (see [13], [10], [9], [6]), and in this case $(\wedge^*, d^\star, d)$ is formal in the sense of DSLA's (see e.g. [3]).

On the other hand, in recent years, a great deal of interest has been originated by generalizations of the notion of Calabi-Yau manifold (especially in connection with a more flexible theory of deformation of classical objects) and further special generalization can be investigated in real dimension 6 (see [2]).

In this paper we construct an example of a 6-dimensional compact symplectic manifold (M, κ), carrying a structure of special generalized Calabi-Yau manifold and satisfying HLC, whose De Rham algebra is formal (as DGA), but which does not admit any Kähler structure). Note that our manifold is solvable but not nilpotent and its cohomology is larger than the invariant one.

2. Special generalized Calabi-Yau manifolds

Let (M, κ) be a $2n$-dimensional symplectic manifold. An almost complex structure J on M is said to be κ-calibrated if, for any $x \in M$,

$$g_J[x](\cdot, \cdot) := \kappa[x](\cdot, J[x]\cdot)$$

is an almost Hermitian metric on M. Let (M, J, g) be an almost Kähler manifold and let ∇^{LC} be the Levi Civita connection of g. The *Chern connection* is the covariant exterior differential operator defined by

$$\nabla := \nabla^{LC} - \frac{1}{2} J \nabla^{LC} J.$$

∇ is characterized by the following three conditions:

$$\nabla J = 0, \quad \nabla g = 0, \quad T^\nabla = \frac{1}{4} N_J,$$

where T^∇ is the torsion of ∇ and N_J is the Nijenhuis tensor of J. Recall that, if J is an almost complex structure on M, then we have that

$$d : \Lambda_J^{p,q}(M) \to \Lambda_J^{p+2,q-1}(M) + \Lambda_J^{p+1,q}(M) + \Lambda_J^{p,q+1}(M) + \Lambda_J^{p-1,q+2}(M)$$

and so d splits accordingly as

$$d = A_J + \partial_J + \overline{\partial}_J + \overline{A}_J.$$

DEFINITION 2.1. — *A special generalized Calabi-Yau manifold is the datum of (M, κ, J, ϵ) where (M, κ) is a compact symplectic 6-dimensional manifold, J is a κ-calibrated almost complex structure on M and ϵ is a nowhere vanishing $(3,0)$-form on M satisfying*

    i) $\nabla \epsilon = 0$
    ii) $A_J(\bar{\epsilon}) + \overline{A}_J(\epsilon) = 0$.

Remark 2.2. — It can be proved (see [2]) that ϵ satisfies i) and ii) if and only if

$$(2.1) \qquad \begin{cases} \epsilon \wedge \bar{\epsilon} &= -ie^{\sigma} \frac{\kappa^3}{3!}, \quad \text{with } \sigma = \text{const.} \\ d\Re e\, \epsilon &= 0. \end{cases}$$

If J is a holomorphic structure, then $A_J = 0$, $\nabla = \nabla^{LC}$ and a special generalized Calabi-Yau manifold is a Calabi-Yau manifold of complex dimension 3.

3. The solvmanifold M

In this section we will recall the construction of the solvmanifold M. Let $A \in \mathrm{SL}(2, \mathbb{Z})$ with two distinct real eigenvalues e^{λ} and $e^{-\lambda}$, where $\lambda > 0$. Let $P \in \mathrm{GL}(2, \mathbb{R})$ such that

$$PAP^{-1} = \Lambda = \begin{pmatrix} e^{-\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix}.$$

On $\mathbb{C}^2$, with coordinates (z, w), let $\sim$ be defined by

$$\begin{pmatrix} z' \\ w' \end{pmatrix} \sim \begin{pmatrix} z \\ w \end{pmatrix} \iff \begin{pmatrix} z' \\ w' \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix} + P \begin{pmatrix} m_1 + 2\pi i n_1 \\ m_2 + 2\pi i n_2 \end{pmatrix},$$

where $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. Then $\mathbb{C}^2 / \sim$ is a complex torus $\mathbb{T}^2$ and

$$\Lambda \left[\begin{pmatrix} z \\ w \end{pmatrix} \right] = \left[\Lambda \begin{pmatrix} z \\ w \end{pmatrix} \right]$$

is a well defined automorphism of $\mathbb{T}^2$. Indeed, if $\begin{pmatrix} z' \\ w' \end{pmatrix} \sim \begin{pmatrix} z \\ w \end{pmatrix}$, then

$$\Lambda \begin{pmatrix} z' \\ w' \end{pmatrix} = \Lambda \begin{pmatrix} z \\ w \end{pmatrix} + \Lambda P \begin{pmatrix} m_1 + 2\pi i n_1 \\ m_2 + 2\pi n_2 \end{pmatrix} =$$

$$= \Lambda \begin{pmatrix} z \\ w \end{pmatrix} + PA \begin{pmatrix} m_1 + 2\pi i n_1 \\ m_2 + 2\pi n_2 \end{pmatrix} = \Lambda \begin{pmatrix} z \\ w \end{pmatrix} + P \begin{pmatrix} m_1 + 2\pi i n_1 \\ m_2 + 2\pi n_2 \end{pmatrix},$$

so that $\Lambda \begin{pmatrix} z' \\ w' \end{pmatrix} \sim \Lambda \begin{pmatrix} z \\ w \end{pmatrix}$.

For example, take

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $\lambda = \log \frac{3+\sqrt{5}}{2}$ and we can choose

(3.1) $$P = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{\sqrt{5}-1}{2} \end{pmatrix}.$$

Set again

$$\lambda = \log \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \mu = \log \frac{\sqrt{5}-1}{2}$$

on $\mathbb{C}^3$, with coordinates (z_1, z_2, z_3), let us consider a multiplication $*$ defined by

$${}^t(z_1,\, z_2,\, z_3) * {}^t(w_1,\, w_2,\, w_3) = {}^t(z_1 + w_1,\, \mathrm{e}^{-w_1} z_2 + w_2,\, \mathrm{e}^{w_1} z_3 + w_3)$$

for any ${}^t(z_1,\, z_2,\, z_3), {}^t(w_1,\, w_2,\, w_3) \in \mathbb{C}^3$. Then $(\mathbb{C}^3, *)$ is a complex solvable Lie group. Let

$$
\begin{aligned}
T_1(z) &= {}^t(z_1 + \lambda,\, e^{-\lambda}z_2,\, e^{\lambda}z_3), \\
T_2(z) &= {}^t(z_1 + 2\pi i,\, z_2,\, z_3), \\
T_3(z) &= {}^t(z_1,\, z_2 + 1,\, z_3 - \mu), \\
T_4(z) &= {}^t(z_1,\, z_2 + \mu,\, z_3 + 1), \\
T_5(z) &= {}^t(z_1,\, z_2 + 2\pi i,\, z_3 - 2\pi i\mu,), \\
T_6(z) &= {}^t(z_1,\, z_2 + 2\pi i\mu,\, z_3 + 2\pi i)
\end{aligned}
$$

and let Γ be the subgroup generated by $T_1, \ldots, T_6$. Then Γ is a closed subgroup of the Lie group $(\mathbb{C}^3, *)$ and, consequently, the quotient $\mathbb{C}^3/\Gamma$ is a manifold. We have

(3.2) $$M := (\mathbb{C}^3, *)/\Gamma \simeq \frac{\mathbb{C} \times \mathbb{C}^2/P(\mathbb{Z}^2 + 2\pi i\mathbb{Z}^2)}{\Xi},$$

where

$$\Xi := < T_1, T_2 > .$$

Therefore, M is a non-nilpotent compact *complex solvmanifold* (see [11], [12]).

It is immediate to check that

(3.3) $$\varphi_1 := dz_1, \quad \varphi_2 := \mathrm{e}^{z_1} dz_2, \quad \varphi_3 := \mathrm{e}^{-z_1} dz_3$$

define holomorphic (with respect to the standard complex structure induced by $\mathbb{C}^3$) invariant 1-forms on M, such that

(3.4) $d\varphi_1 = 0, \quad d\varphi_2 = \varphi_1 \wedge \varphi_2, \quad d\varphi_3 = -\varphi_1 \wedge \varphi_3.$

By (3.2), the solvmanifold M is diffeomorphic to the product

(3.5) $$S^1 \times \frac{\mathbb{R} \times \mathbb{T}^2}{< T_1 >}.$$

4. Cohomology of M

We will compute the cohomology of the solvmanifold M.

THEOREM 4.1. — *We have* $b_1(M) = 2$, $b_2(M) = 5$ *and* $b_3(M) = 8$.

Proof. — We start by computing the second Betti number of M. By Hodge theorem, we determine the space of real harmonic two-forms on M. Let $\varphi \in \Lambda^2(M, \mathbb{R})$; then,

$$\varphi = \psi + \vartheta + \overline{\psi},$$

with $\psi \in \Lambda^{2,0}(M)$ and $\vartheta = \overline{\vartheta} \in \Lambda^{1,1}(M)$. Therefore, we can set

$$\psi = \psi_{12}\,\varphi_1 \wedge \varphi_2 + \psi_{13}\,\varphi_1 \wedge \varphi_3 + \psi_{23}\,\varphi_2 \wedge \varphi_3$$

$$\vartheta = \vartheta_{1\overline{1}}\,\varphi_1 \wedge \overline{\varphi}_1 + \vartheta_{1\overline{2}}\,\varphi_1 \wedge \overline{\varphi}_2 + \vartheta_{1\overline{3}}\,\varphi_1 \wedge \overline{\varphi}_3$$

$$-\overline{\vartheta}_{1\overline{2}}\,\varphi_2 \wedge \overline{\varphi}_1 + \vartheta_{2\overline{2}}\,\varphi_2 \wedge \overline{\varphi}_2 + \vartheta_{2\overline{3}}\,\varphi_2 \wedge \overline{\varphi}_3$$

$$-\overline{\vartheta}_{1\overline{3}}\,\varphi_3 \wedge \overline{\varphi}_1 - \overline{\vartheta}_{2\overline{3}}\,\varphi_3 \wedge \overline{\varphi}_2 + \vartheta_{3\overline{3}}\,\varphi_3 \wedge \overline{\varphi}_3,$$

where ψ_{ab}, $\vartheta_{a\overline{b}}$, $a, b = 1, 2, 3$ are C^∞ and Γ-invariant functions on $\mathbb{C}^3$ satisfying the following

(4.1) $\begin{cases} \vartheta_{1\overline{1}} = -\overline{\vartheta}_{1\overline{1}} \\ \vartheta_{2\overline{2}} = -\overline{\vartheta}_{2\overline{2}} \\ \vartheta_{3\overline{3}} = -\overline{\vartheta}_{3\overline{3}}. \end{cases}$

Then, the two-form φ is harmonic if and only if

(4.2) $\begin{cases} \partial\psi & = 0 \\ \partial\vartheta + \overline{\partial}\psi & = 0 \\ \partial^*\psi + \overline{\partial}^*\vartheta & = 0, \end{cases}$

where, as usual,

$$d = \partial + \overline{\partial}, \quad \partial^* = -*\partial*, \quad \overline{\partial}^* = -*\overline{\partial}*, \quad d^* = \partial^* + \overline{\partial}^*$$

and $* : \Lambda^{p,q}(M) \to \Lambda^{3-p,3-q}(M)$ is the Hodge operator is taken with respect to the Hermitian metric

$$g_0 = \sum_{h=1}^{3} \varphi_h \otimes \overline{\varphi}_h \,.$$

The first and the second equation of (4.2) are equivalent to say that $d\varphi = 0$, while the last equation means $d^*\varphi = 0$.

A straightforward computation, shows that (4.2) is equivalent to the following system of partial differential equations in the unknowns ψ_{hk}, $\vartheta_{h\overline{k}}$, $h, k = 1, 2, 3$

$$(4.3) \qquad e^{z_1} \frac{\partial \psi_{12}}{\partial z_3} - e^{-z_1} \frac{\partial \psi_{13}}{\partial z_2} + \frac{\partial \psi_{23}}{\partial z_1} = 0$$

$$(4.4) \qquad \frac{\partial \psi_{12}}{\partial \overline{z}_1} - e^{-z_1} \frac{\partial \vartheta_{1\overline{1}}}{\partial z_2} - \frac{\overline{\partial \vartheta_{1\overline{2}}}}{\partial \overline{z}_1} - \overline{\vartheta_{1\overline{2}}} = 0$$

$$(4.5) \qquad \frac{\partial \psi_{13}}{\partial \overline{z}_1} - e^{z_1} \frac{\partial \vartheta_{1\overline{1}}}{\partial z_3} - \frac{\overline{\partial \vartheta_{1\overline{3}}}}{\partial \overline{z}_1} + \overline{\vartheta_{1\overline{3}}} = 0$$

$$(4.6) \qquad \frac{\partial \psi_{23}}{\partial \overline{z}_1} + e^{z_1} \frac{\overline{\partial \vartheta_{1\overline{2}}}}{\partial \overline{z}_3} - e^{-z_1} \frac{\overline{\partial \vartheta_{1\overline{3}}}}{\partial \overline{z}_2} = 0$$

$$(4.7) \qquad e^{-\overline{z}_1} \frac{\partial \psi_{12}}{\partial \overline{z}_2} - e^{-z_1} \frac{\partial \vartheta_{1\overline{2}}}{\partial z_2} + \frac{\partial \vartheta_{2\overline{2}}}{\partial z_1} + \vartheta_{2\overline{2}} = 0$$

$$(4.8) \qquad e^{-\overline{z}_1} \frac{\partial \psi_{13}}{\partial \overline{z}_2} - e^{z_1} \frac{\partial \vartheta_{1\overline{2}}}{\partial z_3} - \frac{\overline{\partial \vartheta_{2\overline{3}}}}{\partial \overline{z}_1} + \overline{\vartheta_{2\overline{3}}} = 0$$

$$(4.9) \qquad e^{-\overline{z}_1} \frac{\partial \psi_{23}}{\partial \overline{z}_2} - e^{z_1} \frac{\partial \vartheta_{2\overline{2}}}{\partial z_3} - e^{-z_1} \frac{\overline{\partial \vartheta_{2\overline{3}}}}{\partial \overline{z}_2} = 0$$

$$(4.10) \qquad e^{\overline{z}_1} \frac{\partial \psi_{12}}{\partial \overline{z}_3} - e^{-z_1} \frac{\partial \vartheta_{1\overline{3}}}{\partial z_2} + \frac{\partial \vartheta_{2\overline{3}}}{\partial z_1} + \vartheta_{2\overline{3}} = 0$$

$$(4.11) \qquad e^{\overline{z}_1} \frac{\partial \psi_{13}}{\partial \overline{z}_3} - e^{z_1} \frac{\partial \vartheta_{1\overline{3}}}{\partial z_3} + \frac{\partial \vartheta_{3\overline{3}}}{\partial z_1} - \vartheta_{3\overline{3}} = 0$$

$$(4.12) \qquad e^{\overline{z}_1} \frac{\partial \psi_{23}}{\partial \overline{z}_3} - e^{z_1} \frac{\partial \vartheta_{2\overline{3}}}{\partial z_3} + e^{-z_1} \frac{\partial \vartheta_{3\overline{3}}}{\partial z_2} = 0$$

$$(4.13) \quad \frac{\overline{\partial \psi_{13}}}{\partial z_1} + e^{-z_1} \frac{\overline{\partial \psi_{23}}}{\partial z_2} + \overline{\psi_{13}} + \frac{\partial \vartheta_{1\overline{3}}}{\partial \overline{z}_1} + e^{-\overline{z}_1} \frac{\partial \vartheta_{2\overline{3}}}{\partial \overline{z}_2} - e^{\overline{z}_1} \frac{\partial \vartheta_{3\overline{3}}}{\partial \overline{z}_3} = 0$$

$$(4.14) \quad \frac{\overline{\partial \psi_{12}}}{\partial z_1} - e^{z_1} \frac{\overline{\partial \psi_{23}}}{\partial z_3} - \overline{\psi_{12}} + \frac{\partial \vartheta_{1\overline{2}}}{\partial \overline{z}_1} - e^{-\overline{z}_1} \frac{\partial \vartheta_{2\overline{2}}}{\partial \overline{z}_2} - e^{\overline{z}_1} \frac{\overline{\partial \vartheta_{2\overline{3}}}}{\partial \overline{z}_3} = 0$$

$$(4.15) \quad e^{-z_1}\frac{\partial\overline{\psi_{12}}}{\partial z_2} + e^{z_1}\frac{\partial\overline{\psi_{13}}}{\partial z_3} + \frac{\partial\overline{\vartheta_{1\bar{1}}}}{\partial\overline{z}_1} + e^{-\overline{z}_1}\frac{\partial\overline{\vartheta_{1\bar{2}}}}{\partial\overline{z}_2} + e^{\overline{z}_1}\frac{\partial\overline{\vartheta_{1\bar{3}}}}{\partial\overline{z}_3} = 0\,.$$

Since Γ acts as a group of translations on the variable $x_4 = \Im m\, z_1$, the functions ψ_{hk}, $\vartheta_{h\bar{k}}$ $h,\, k = 1,\, 2,\, 3$ are periodic with respect to x_4. Therefore, we can take the Fourier expansion of these functions. Let us denote by $z = (z_1, z_2, z_3)$. We set:

$$\psi_{hk}(z,\overline{z}) = \sum_m \Psi_{hk\,m}(x_1, z_2, z_3, \overline{z}_2, \overline{z}_3)\,\exp(m\frac{z_1 - \overline{z}_1}{2})$$

$$\vartheta_{h\bar{k}}(z,\overline{z}) = \sum_m \Theta_{h\bar{k}\,m}(x_1, z_2, z_3, \overline{z}_2, \overline{z}_3)\,\exp(m\frac{z_1 - \overline{z}_1}{2})\,,$$

for $h,\, k = 1,\, 2,\, 3,\, m \in \mathbb{Z}$.

Now we put the previous expressions for the unknowns ψ_{hk}, $\vartheta_{h\bar{k}}$ into equations (4.3)...(4.15). By a direct computation, we get the following:

$$(4.16) \qquad e^{z_1}\frac{\partial\Psi_{12\,m}}{\partial z_3} - e^{-z_1}\frac{\partial\Psi_{13\,m}}{\partial z_2} + \frac{\partial\Psi_{23\,m}}{\partial z_1} + \frac{m}{2}\Psi_{23\,m} \quad = \quad 0$$

$$(4.17) \qquad -e^{-z_1}\frac{\partial\Theta_{1\bar{1}\,m}}{\partial z_2} + \frac{\partial\Psi_{12\,m}}{\partial\overline{z}_1} - \frac{m}{2}\Psi_{12\,m} \quad -$$

$$\overline{(\frac{\partial\Theta_{1\bar{2}\,-m}}{\partial\overline{z}_1} + (1 - \frac{m}{2})\Theta_{1\bar{2}\,-m})} \quad = \quad 0$$

$$(4.18) \qquad -e^{z_1}\frac{\partial\Theta_{1\bar{1}\,m}}{\partial z_3} + \frac{\partial\Psi_{13\,m}}{\partial\overline{z}_1} - \frac{m}{2}\Psi_{13\,m} \quad -$$

$$\overline{(\frac{\partial\Theta_{1\bar{3}\,-m}}{\partial\overline{z}_1} - (1 + \frac{m}{2})\Theta_{1\bar{3}\,-m})} \quad = \quad 0$$

$$(4.19) \qquad e^{z_1}\overline{\frac{\partial\Theta_{1\bar{2}\,-m}}{\partial\overline{z}_3}} - e^{-z_1}\overline{\frac{\partial\Theta_{1\bar{3}\,-m}}{\partial\overline{z}_2}} + \frac{\partial\Psi_{23\,m}}{\partial\overline{z}_1} - \frac{m}{2}\Psi_{23\,m} \quad = \quad 0$$

$$(4.20) \quad e^{-\overline{z}_1}\frac{\partial\Psi_{12\,m}}{\partial\overline{z}_2} - e^{-z_1}\frac{\partial\Theta_{1\bar{2}\,m}}{\partial z_2} + \frac{\partial\Theta_{2\bar{2}\,m}}{\partial z_1} + (1 + \frac{m}{2})\Theta_{2\bar{2}\,m} \quad = \quad 0$$

$$(4.21) \qquad\qquad\qquad e^{-\overline{z}_1}\frac{\partial\Psi_{13\,m}}{\partial\overline{z}_2} - e^{z_1}\frac{\partial\Theta_{1\bar{2}\,m}}{\partial z_3} \quad -$$

$$\overline{(\frac{\partial\Theta_{2\bar{3}\,-m}}{\partial\overline{z}_1} - (1 + \frac{m}{2})\Theta_{2\bar{3}\,-m})} \quad = \quad 0$$

$$(4.22) \qquad\qquad e^{-\overline{z}_1}\frac{\partial\Psi_{23\,m}}{\partial\overline{z}_2} - e^{z_1}\frac{\partial\Theta_{2\bar{2}\,m}}{\partial z_3} - e^{-z_1}\overline{\frac{\partial\Theta_{2\bar{3}\,-m}}{\partial\overline{z}_2}} \quad = \quad 0$$

$$(4.23) \quad e^{\overline{z}_1}\frac{\partial \Psi_{12\,m}}{\partial \overline{z}_3} - e^{-z_1}\frac{\partial \Theta_{1\overline{3}\,m}}{\partial z_2} + \frac{\partial \Theta_{2\overline{3}\,m}}{\partial z_1} + (\frac{m}{2}+1)\Theta_{2\overline{3}\,m} \quad = \quad 0$$

$$(4.24) \quad e^{\overline{z}_1}\frac{\partial \Psi_{13\,m}}{\partial \overline{z}_3} - e^{z_1}\frac{\partial \Theta_{1\overline{3}\,m}}{\partial z_3} + \frac{\partial \Theta_{3\overline{3}\,m}}{\partial z_1} + (\frac{m}{2}-1)\Theta_{3\overline{3}\,m} \quad = \quad 0$$

$$(4.25) \quad e^{\overline{z}_1}\frac{\partial \Psi_{23\,m}}{\partial \overline{z}_3} - e^{z_1}\frac{\partial \Theta_{2\overline{3}\,m}}{\partial z_3} + e^{-z_1}\frac{\partial \Theta_{3\overline{3}\,m}}{\partial z_2} \quad = \quad 0$$

$$(4.26) \quad \frac{\partial \overline{\Psi_{13-m}}}{\partial z_1} + (\frac{m}{2}+1)\overline{\Psi_{13-m}} + e^{-z_1}\frac{\partial \overline{\Psi_{23-m}}}{\partial z_2} + \frac{\partial \Theta_{1\overline{3}m}}{\partial \overline{z}_1} \quad +$$

$$-\frac{m}{2}\,\Theta_{1\overline{3}m} + e^{-\overline{z}_1}\frac{\partial \Theta_{2\overline{3}m}}{\partial \overline{z}_2} - e^{\overline{z}_1}\frac{\partial \Theta_{3\overline{3}m}}{\partial \overline{z}_3} \quad = \quad 0$$

$$(4.27) \quad \frac{\partial \overline{\Psi_{12-m}}}{\partial z_1} + (\frac{m}{2}+1)\overline{\Psi_{12-m}} - e^{-z_1}\frac{\partial \overline{\Psi_{23-m}}}{\partial z_3} + \frac{\partial \Theta_{1\overline{2}m}}{\partial \overline{z}_1} \quad +$$

$$-\frac{m}{2}\,\Theta_{1\overline{2}m} - e^{-\overline{z}_1}\frac{\partial \Theta_{2\overline{2}-m}}{\partial \overline{z}_2} - e^{\overline{z}_1}\frac{\partial \Theta_{2\overline{3}-m}}{\partial \overline{z}_3} \quad = \quad 0$$

$$(4.28) \quad e^{-z_1}\frac{\partial \overline{\Psi_{12-m}}}{\partial z_2} + e^{z_1}\frac{\partial \overline{\Psi_{13-m}}}{\partial z_3} + \frac{\partial \Theta_{1\overline{1}-m}}{\partial \overline{z}_1} - \frac{m}{2}\,\overline{\Theta_{1\overline{1}-m}} \quad +$$

$$e^{-\overline{z}_1}\frac{\partial \overline{\Theta_{1\overline{2}-m}}}{\partial \overline{z}_2} + e^{\overline{z}_1}\frac{\partial \overline{\Theta_{1\overline{3}-m}}}{\partial \overline{z}_3} \quad = \quad 0$$

Notice that equations (4.16)...(4.28) are identities in m. By equations (4.22), (4.25), (4.26) and (4.27) we get that

$$\frac{\partial \Theta_{2\overline{3}\,m}}{\partial z_2} = \frac{\partial \Theta_{2\overline{3}\,m}}{\partial z_3} = \frac{\partial \Theta_{2\overline{3}\,m}}{\partial \overline{z}_2} = \frac{\partial \Theta_{2\overline{3}\,m}}{\partial \overline{z}_3} = 0 \quad \forall m \in \mathbb{Z}.$$

Consequently,

$$\Theta_{2\overline{3}\,m} = \Theta_{2\overline{3}\,m}(x_1).$$

Since $\Theta_{2\overline{3}\,m}$ is Γ-invariant, it follows that it is periodic. Therefore by (4.23) we obtain

$$\vartheta_{2\overline{3}}(z,\overline{z}) = (u+iv)e^{\overline{z}_1 - z_1} \quad u,\, v \in \mathbb{R}.$$

By proceeding in a similar way, by taking into account (4.1) and the crucial fact that the functions ψ_{hk}, $\vartheta_{h\overline{k}}$, $h,k = 1,2,3$ are Γ-invariant, we can solve the initial system of partial differential equations (4.3)...(4.15). A direct

computation shows that

$$
\begin{aligned}
\psi_{12}(z,\overline{z}) &= 0 \\
\psi_{13}(z,\overline{z}) &= 0 \\
\psi_{23}(z,\overline{z}) &= x+iy \quad x,\,y \in \mathbb{R} \\
\vartheta_{1\overline{1}}(z,\overline{z}) &= is \quad s \in \mathbb{R} \\
\vartheta_{1\overline{2}}(z,\overline{z}) &= 0 \\
\vartheta_{1\overline{3}}(z,\overline{z}) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_{2\overline{2}}(z,\overline{z}) &= 0 \\
\vartheta_{2\overline{3}}(z,\overline{z}) &= (u+iv)e^{\overline{z}_1 - z_1} \quad u,\,v \in \mathbb{R} \\
\vartheta_{3\overline{3}}(z,\overline{z}) &= 0\,.
\end{aligned}
$$

This proves that $b_2(M) = 5$. Needless to say that we could perform same computations for

$$
N := \frac{\mathbb{R} \times \mathbb{T}^2}{<T_1>}\,,
$$

sparing one dimension, but losing the complete use of complex variable.

In order to compute the first Betti number of M, we may perform the same method as before or refer to Theorem 4 of [11] for a more general result. We get $b_1(M) = 2$.

Since M is compact and parallelizable, we obtain that

$$
b_3(M) = 2(b_2(M) - b_1(M) + 1) = 8\,.
$$

$\square$

We can give also an alternative proof, using spectral sequences, to compute the Betti numbers of

$$
N = \frac{\mathbb{R} \times \mathbb{T}^2}{<T_1>}
$$

and then use Kunneth formula for

$$
M = S^1 \times N\,.
$$

N can be described as the $\mathbb{T}^2$-bundle over S^1

$$
[0\,,\lambda] \times \mathbb{T}^2
$$

where $\{0\} \times \mathbb{T}^2$ and $\{\lambda\} \times \mathbb{T}^2$ are identified via Λ;
in other words, let $\mathfrak{U} := \{U_1,\,U_2,\,U_3\}$ be the covering of S^1 given by:

$$
U_1 := \left\{ e^{i\theta}\ \Big|\ -\frac{\pi}{6} - \epsilon \leqslant \theta \leqslant \frac{\pi}{2} + \epsilon \right\}
$$

$$U_2 := \left\{ e^{i\theta} \mid \frac{\pi}{2} - \epsilon \leqslant \theta \leqslant \frac{7\pi}{6} + \epsilon \right\}$$

$$U_3 := \left\{ e^{i\theta} \mid \frac{7\pi}{6} - \epsilon \leqslant \theta \leqslant -\frac{\pi}{6} + \epsilon \right\}$$

then N is the fibre bundle defined by the transition functions

$$\psi_{12} : U_1 \cap U_2 \longrightarrow Aut(\mathbb{T}^2) , \ \psi_{12} \equiv \sqrt[3]{\Lambda}$$

$$\psi_{23} : U_2 \cap U_3 \longrightarrow Aut(\mathbb{T}^2) , \ \psi_{23} \equiv \sqrt[3]{\Lambda}$$

$$\psi_{31} : U_3 \cap U_1 \longrightarrow Aut(\mathbb{T}^2) , \ \psi_{31} \equiv \sqrt[3]{\Lambda}$$

we shall compute $b_1(N)$ and $b_2(N)$ by means of Leray's spectral sequence associated to the double complex

$$K^{p,q} := C^p(\mathfrak{U}, \wedge^q(\mathfrak{U}))$$

clearly the spectral sequence degenerate at E_2 and

$$E_\infty^{p,q} = E_2^{p,q} = H_\delta(\mathfrak{U}, \mathfrak{H}^q(\mathbb{T}^2)) , \ p = 0, 1$$

(where $\mathfrak{H}^q(\mathbb{T}^2)$ is the locally constant sheaf $H_{DR}^q(\mathbb{T}^2)$), all other elements being zero.

Note that:

if

$$a = (a_1, a_2, a_3) \in C^0(\mathfrak{U}, \mathfrak{H}^q(\mathbb{T}^2))$$

then

$$\delta a = (a_1 - (\sqrt[3]{\Lambda})^*(a_2), a_2 - (\sqrt[3]{\Lambda})^*(a_3), a_3 - (\sqrt[3]{\Lambda})^*(a_1))$$

now:

- $H^1(N) = E_2^{0,1} \oplus E_2^{1,0}$
- $H^2(N) = E_2^{0,2} \oplus E_2^{1,1}$

and

(1) $E_2^{0,1} = H_\delta^0(\mathfrak{U}, \mathfrak{H}^1(\mathbb{T}^2)) = \{\Lambda - \text{invariant elements of } H_{DR}^1(\mathbb{T}^2)\}$
    $= \{0\}$

(2) $E_2^{1,0} = H_\delta^1(\mathfrak{U}, \mathbb{R}) \simeq \mathbb{R}$

(3) $E_2^{0,2} = H_\delta^0(\mathfrak{U}, \mathfrak{H}^2(\mathbb{T}^2)) = \{\Lambda - \text{invariant elements of } H_{DR}^2(\mathbb{T}^2)\}$
    $\simeq \mathbb{R}^4$

(4) $E_2^{1,1} = H_\delta^1(\mathfrak{U}, \mathfrak{H}^1(\mathbb{T}^2)) = \{0\}$; in fact, given

$$b = (b_{12}, b_{23}, b_{31}) \in C^1(\mathfrak{U}, \mathfrak{H}^1(\mathbb{T}^2))$$

we have:

$$\delta(a_1, a_2, a_3) = b$$

with

(a) $a_2 = (I - \Lambda^*)^{-1}((\sqrt[3]{\lambda})^* b_{31} + b_{23} + ((\sqrt[3]{\Lambda})^2)^* b_{12})$

(b) $a_1 = b_{12} + (\sqrt[3]{\Lambda})^* a_2$

(c) $a_3 = ((\sqrt[3]{\Lambda})^{-1})^* (a_2 - b_{23})$

Consequently:

$$b_1(N) = 1 \ , \ b_2(N) = 4$$

Let α_h, α_{3+h} denote the real and imaginary part of φ_h, $h = 1, 2, 3$ (see Section 3). Then Theorem 4.1 implies the following

COROLLARY 4.2. — *We have*

$$\mathrm{H}^1(M, \mathbb{R}) \ = \langle\, [\alpha_1], [\alpha_4]\,\rangle$$

$$\mathrm{H}^2(M, \mathbb{R}) \ = \langle\, [\alpha_1 \wedge \alpha_4], [\alpha_2 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5], [\alpha_2 \wedge \alpha_3 - \alpha_5 \wedge \alpha_6],$$

$$[\cos(2x_4)(\alpha_2 \wedge \alpha_3 + \alpha_5 \wedge \alpha_6) - \sin(2x_4)(\alpha_2 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5)],$$

$$[\sin(2x_4)(\alpha_2 \wedge \alpha_3 + \alpha_5 \wedge \alpha_6) + \cos(2x_4)(\alpha_2 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5)]\,\rangle.$$

Proof. — A straightforward computation shows that the above forms are harmonic with respect to the Hermitian metric

$$g_0 = \sum_{h=1}^{3} \varphi_h \otimes \overline{\varphi}_h$$

on $M = \mathbb{C}^3 / \Gamma$, where φ_h are the 1-forms on M defined by (3.3). □

Remark 4.3. — We stress that $\mathrm{H}^1_{\mathrm{inv}}(M, \mathbb{R}) = \mathrm{H}^1(M, \mathbb{R})$, but $\mathrm{H}^2_{\mathrm{inv}}(M, \mathbb{R})$ $\subsetneq \mathrm{H}^2(M, \mathbb{R})$. Indeed, by Corollary 4.2 we have

$$\mathrm{H}^1_{\mathrm{inv}}(M, \mathbb{R}) = \langle\, [\alpha_1], [\alpha_4]\,\rangle = \mathrm{H}^1(M, \mathbb{R})$$

and

$$\mathrm{H}^2_{\mathrm{inv}}(M, \mathbb{R}) = \langle\, [\alpha_1 \wedge \alpha_4], [\alpha_2 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5], [\alpha_2 \wedge \alpha_3 - \alpha_5 \wedge \alpha_6]\,\rangle \subsetneq \mathrm{H}^2(M, \mathbb{R}).$$

This is in contrast with the nilpotent case.

5. Main Theorem

We are ready to prove the following

THEOREM 5.1. — *Let $M = \mathbb{C}^3 / \Gamma$. Then*

   i) *there exists a symplectic structure κ on M such that (M, κ) satisfies the Hard Lefschetz Condition;*

  ii) *the De Rham complex of M is formal;*

 iii) *there exists on M a structure of special generalized Calabi-Yau manifold;*

iv) *M has no Kähler structures.*

Proof. — i) We start showing that M satisfies the Hard Lefschetz Condition. Let φ_1, φ_2, φ_3 be the differential defined by (3.3). As before, denote by α_h, α_{3+h} the real and imaginary part of φ_h, $h = 1, 2, 3$, respectively. By (3.4), we easily get that

(5.1)
$$\begin{cases} d\alpha_1 = d\alpha_4 = 0 \\ d\alpha_2 = \alpha_1 \wedge \alpha_2 - \alpha_4 \wedge \alpha_5 \\ d\alpha_3 = -\alpha_1 \wedge \alpha_3 + \alpha_4 \wedge \alpha_6 \\ d\alpha_5 = \alpha_1 \wedge \alpha_5 - \alpha_2 \wedge \alpha_4 \\ d\alpha_6 = -\alpha_1 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \, . \end{cases}$$

Let us define

(5.2)
$$\kappa := \alpha_1 \wedge \alpha_4 + \alpha_3 \wedge \alpha_5 + \alpha_6 \wedge \alpha_2$$

Then κ is closed and non degenerate and so it defines a symplectic structure on M. Let $\{\xi_1, \ldots, \xi_6\}$ be the dual basis of $\{\alpha_1, \ldots, \alpha_6\}$. Let us define a κ-calibrated almost complex structure J on M, by

(5.3)
$$J\xi_1 = \xi_4 \qquad J\xi_3 = \xi_5 \qquad J\xi_6 = \xi_2$$
$$J\xi_4 = -\xi_1 \qquad J\xi_5 = -\xi_3 \qquad J\xi_2 = -\xi_6$$

Let $L : \wedge^p(M) \to \wedge^{p+2}(M)$

$$L(\alpha) = \kappa \wedge \alpha$$

and

$$d^c = J^{-1}dJ \, .$$

Then we have the following identity

(5.4)
$$[\Delta, L] = -[d, d^c]$$

(see e.g. [1]), where Δ is the Hodge Laplacian with respect to the Riemannian metric

$$g_J[x](\cdot, \cdot) = \kappa[x](\cdot, J\cdot) \, .$$

Now, observe that

$$J\alpha_1 = -\alpha_4 \, , \quad J\alpha_4 = \alpha_1 \, , \quad J(\alpha_2 \wedge \alpha_3 - \alpha_5 \wedge \alpha_6) = -\alpha_2 \wedge \alpha_3 + \alpha_5 \wedge \alpha_6$$

and that

$$\alpha_1 \wedge \alpha_4 \, , \quad \alpha_2 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \, ,$$

$$(\cos(2x_4)(\alpha_2 \wedge \alpha_3 + \alpha_5 \wedge \alpha_6) - \sin(2x_4)(\alpha_2 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5)) \, ,$$

$$(\sin(2x_4)(\alpha_2 \wedge \alpha_3 + \alpha_5 \wedge \alpha_6) + \cos(2x_4)(\alpha_2 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5)) \, ,$$

are J-invariant. Hence, by (5.4), we get that $[\Delta, L] = 0$ on the above forms. Therefore, given $a \in \mathrm{H}^{3-p}(M, \mathbb{R})$, $p = 1, 2$, let $h(a)$ be its harmonic representative. Then $0 \neq L^p(h(a))$ is harmonic. Therefore

$$L^p : \mathrm{H}^{3-p}(M, \mathbb{R}) \to \mathrm{H}^{3+p}(M, \mathbb{R})$$

is injective and, by Poincaré duality, bijective. Hence, (M, κ) satisfies the Hard Lefschetz Condition.

ii) Summarizing, we have that the whole cohomology of M expressed through its harmonic representative is given by:

$$\mathrm{H}^1(M, \mathbb{R}) = \langle\, [\alpha_1],\, [\alpha_4]\,\rangle$$

$$\begin{aligned}
\mathrm{H}^2(M, \mathbb{R}) =\ & \langle\, [\alpha_1 \wedge \alpha_4], [\alpha_2 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5], [\alpha_2 \wedge \alpha_3 - \alpha_5 \wedge \alpha_6], \\
& [\cos(2x_4)(\alpha_2 \wedge \alpha_3 + \alpha_5 \wedge \alpha_6) - \sin(2x_4)(\alpha_2 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5)], \\
& [\sin(2x_4)(\alpha_2 \wedge \alpha_3 + \alpha_5 \wedge \alpha_6) + \cos(2x_4)(\alpha_2 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5)]\,\rangle \\
=\ & \langle\, [\alpha_1 \wedge \alpha_4],\, [\alpha],\, [\beta],\, [\gamma],\, [\delta]\,\rangle
\end{aligned}$$

$$\begin{aligned}
\mathrm{H}^3(M, \mathbb{R}) =\ & \langle\, [\alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_5 \wedge \alpha_6], \\
& [\alpha_1 \wedge \alpha_2 \wedge \alpha_6 - \alpha_1 \wedge \alpha_3 \wedge \alpha_5], \\
& [\alpha_2 \wedge \alpha_3 \wedge \alpha_4 - \alpha_4 \wedge \alpha_5 \wedge \alpha_6], \\
& [\alpha_3 \wedge \alpha_4 \wedge \alpha_5 - \alpha_2 \wedge \alpha_4 \wedge \alpha_6], \\
& [\cos(2x_4)(\alpha_2 \wedge \alpha_3 \wedge \alpha_4 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6) + \\
& + \sin(2x_4)(\alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_5)]\,, \\
& [\sin(2x_4)(\alpha_2 \wedge \alpha_3 \wedge \alpha_4 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6) + \\
& - \cos(2x_4)(\alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_4 \wedge \alpha_5)]\,, \\
& [-\cos(2x_4)(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_5 \wedge \alpha_6) + \\
& + \sin(2x_4)(\alpha_1 \wedge \alpha_2 \wedge \alpha_6 + \alpha_1 \wedge \alpha_3 \wedge \alpha_5)]\,, \\
& [-\sin(2x_4)(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_5 \wedge \alpha_6) + \\
& - \cos(2x_4)(\alpha_1 \wedge \alpha_2 \wedge \alpha_6 + \alpha_1 \wedge \alpha_3 \wedge \alpha_5)]\,,
\end{aligned}$$

$$\mathrm{H}^4(M,\mathbb{R}) = \langle\, [\alpha_2 \wedge \alpha_3 \wedge \alpha_5 \wedge \alpha_6]\,,$$
$$[\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5]\,,$$
$$[\alpha_1 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 - \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4]\,,$$
$$[\cos(2x_4)(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6)+$$
$$+\sin(2x_4)(\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5)]\,,$$
$$[\sin(2x_4)(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6)+$$
$$-\cos(2x_4)(\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5)]\,\rangle$$
$$\mathrm{H}^5(M,\mathbb{R}) = \langle\, [\alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6], [\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_5 \wedge \alpha_6]\rangle$$

It is easy to check that we are in a very special situation, i.e. for M the product of harmonic forms is harmonic and, as it is very easy to see, this is enough to establish that $(\wedge^*(M),\, d)$ is formal; in fact, we have the following, immediate to prove

LEMMA 5.2. — *Let (N, g) be a compact Riemannian manifold such that the space of harmonic forms of N is closed under multiplication; then the map*

$$h : (H^*_{DR}(N),\, 0) \longrightarrow (\wedge^*(N),\, d)$$

assigning to every De Rham class its harmonic representative is a DGA quasi-isomorphism and consequently N is formal.

moreover, giving a closer look, if we set:

$$V_1 := \langle \alpha_1,\, \alpha_4 \rangle \qquad\qquad V_2 = \langle \alpha,\, \beta,\, \gamma,\, \delta \rangle$$

then we see that

$$\mathfrak{M} = \wedge V_1 \otimes \wedge V_2$$

is the minimal model of the De Rham algebra of M. Hence ii) is proved.

iii) Next, we show that (M, κ), where κ is the symplectic form defined by (5.2), has a structure of special generalized Calabi-Yau manifold. Set

$$\tilde{\epsilon} := (\alpha_1 + i\alpha_4) \wedge (\alpha_6 + i\alpha_2) \wedge (\alpha_3 + i\alpha_5)\,.$$

Then, by definition, ϵ is a $(3,0)$-form with respect to the κ-calibrated almost complex structure defined by (5.3). By a direct computation we get

$$\Re e\, \tilde{\epsilon} = -\alpha_1 \wedge \alpha_3 \wedge \alpha_6 - \alpha_1 \wedge \alpha_2 \wedge \alpha_5 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6 - \alpha_2 \wedge \alpha_3 \wedge \alpha_4\,.$$

By setting

$$\epsilon := \frac{\sqrt{3}}{2}\tilde{\epsilon}\,,$$

we have

$$\epsilon \wedge \overline{\epsilon} = -i\kappa^3$$

and

$$d\Re e\,\epsilon = 0\,.$$

Therefore (M, κ, J, ϵ) is a special generalized Calabi-Yau manifold and iii) is proved.

iv) Finally, we show that M has no Kähler structures. First of all, we observe that the standard holomorphic structure on $M = \mathbb{C}^3/\Gamma$, induced by $\mathbb{C}^3$, is not Kähler, since φ_2, φ_3 are holomorphic non closed 1-forms on M. In order to prove that M has no Kähler structures, we recall the Main Theorem of [8].

A compact solvmanifold admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus. In particular, a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus.

For the nilpotent case see [4].

Assume now that $M = \mathbb{C}^3/\Gamma$ has a Kähler structure. Then M is covered by a torus and M is the total space of a holomorphic torus bundle $\{M, \pi, B, F\}$ over a complex torus B of complex dimension 1. In view of the Théorème Principal II, p. 192 of [5], we have that, if M is Kähler, then

$$b_1(M) = b_1(F) + b_1(B)\,.$$

This is absurd, since $b_1(M) = 2$, $b_1(F) = 4$ and $b_1(B) = 2$. The theorem is proved. $\qquad\square$

Remark 5.3. — Notice that the solvmanifold M is a *balanced manifold*. Indeed,

$$g_0 = \sum_{h=1}^{3} \varphi_h \otimes \overline{\varphi}_h\,.$$

defines an Hermitian metric on M, whose Kähler form is

$$\kappa_0 = \frac{i}{2} \sum_{h=1}^{3} \varphi_h \wedge \overline{\varphi}_h$$

and, by a direct computation, we have that

$$d(\kappa_0 \wedge \kappa_0) = 0\,.$$

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