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THE HILBERT SCHEME OF SPACE CURVES OF SMALL DIAMETER

by Jan Oddvar KLEPPE

Abstract. — This paper studies space curves C of degree d and arithmetic genus g, with homogeneous ideal I and Rao module $M = \mathrm{H}^1_*(\tilde{I})$, whose main results deal with curves which satisfy ${}_0\mathrm{Ext}^2_R(M,M) = 0$ (e.g. of diameter, $\mathrm{diam} M \leqslant 2$). For such curves we find necessary and sufficient conditions for unobstructedness, and we compute the dimension of the Hilbert scheme, $\mathrm{H}(d,g)$, at (C) under the sufficient conditions. In the diameter one case, the necessary and sufficient conditions coincide, and the unobstructedness of C turns out to be equivalent to the vanishing of certain graded Betti numbers of the free minimal resolution of I. More generally by taking suitable deformations of C we show how to kill repeated direct free factors ("ghost-terms") in the minimal resolution of I, leading to a rather concrete description of the number of irreducible components of $\mathrm{H}(d,g)$ which contains an obstructed diameter one curve. We also show that every irreducible component of $\mathrm{H}(d,g)$ is reduced in the diameter one case.

Résumé. — Cet article concerne des courbes gauches C de degré d et de genre g, d'idéal homogène I et de module de Rao $M = \mathrm{H}^1_*(\tilde{I})$; les résultats principaux portent sur les courbes qui vérifient ${}_0\mathrm{Ext}^2_R(M,M) = 0$ (e.g. de diamètre, $\mathrm{diam} M \leqslant 2$). Pour de telles courbes nous trouvons des conditions nécessaires et suffisantes pour être non obstruées et nous calculons la dimension du schéma de Hilbert, $\mathrm{H}(d,g)$ en (C) sous des conditions suffisantes. Dans le cas du diamètre 1, les conditions nécessaires et suffisantes coïncident et la condition d'être non obstruée s'avère être équivalente à l'annulation de certains nombres de Betti gradués de la résolution libre minimale de I. Plus généralement, en prenant des déformations convenables de C, nous montrons comment éliminer les facteurs directs libres répétés ("termes-fantômes") dans la résolution minimale de I, conduisant à une description relativement concrète du nombre des composantes irréductibles de $\mathrm{H}(d,g)$ qui contiennent une courbe obstruée de diamètre 1. Nous prouvons aussi que chaque composante irréductible de $\mathrm{H}(d,g)$ est réduite au cas de diamètre 1.

Keywords: Hilbert scheme, space curve, Buchsbaum curve, unobstructedness, cup-product, graded Betti numbers, ghost term, linkage, normal module, postulation Hilbert scheme.
Math. classification: 14C05, 14H50, 14B10, 14B15, 13D10, 13D02, 13D07, 13C40.

1. Introduction and main results

The Hilbert scheme of space curves of degree d and arithmetic genus g, $\mathrm{H}(d,g)$, has received much attention over the last years after Grothendieck showed its existence [13]. At so-called special curves it has turned out that the structure of $\mathrm{H}(d,g)$ is difficult to describe in detail, and questions related to irreducibility and number of components, dimension and smoothness have been hard to solve. For particular classes of space curves, some results are known. In 1975 Ellingsrud [10] managed to prove that the open subset of $\mathrm{H}(d,g)$ of arithmetically Cohen-Macaulay curves (with a fixed resolution of the sheaf ideal $\mathcal{I}_C$) is smooth and irreducible, and he computed the dimension of the corresponding component. A generalization of this result in the direction of smoothness and dimension was already given in [18] (see Theorem 1.1(i) below) while the irreducibility was nicely generalized by Bolondi [3]. Later, Martin-Deschamps and Perrin gave a stratification $\mathrm{H}_{\gamma,\rho}$ of $\mathrm{H}(d,g)$ obtained by deforming space curves with constant cohomology [26]. Their results lead immediately to (iii) in the following

Theorem 1.1. — *Let C be a curve in $\mathbb{P}^3$ of degree d and arithmetic genus g, let $I = \mathrm{H}^0_*(\mathcal{I}_C) := \oplus \mathrm{H}^0(\mathcal{I}_C(v))$, $M = \mathrm{H}^1_*(\mathcal{I}_C)$ and $E = \mathrm{H}^1_*(\mathcal{O}_C)$ and suppose at least one of the following conditions:*

(i) ${}_v\mathrm{Hom}_R(I,M) = 0$ *for* $v = 0$ *and* $v = -4$,
(ii) ${}_v\mathrm{Hom}_R(M,E) = 0$ *for* $v = 0$ *and* $v = -4$, *or*
(iii) ${}_0\mathrm{Hom}_R(I,M) = 0, {}_0\mathrm{Hom}_R(M,E) = 0$ *and* ${}_0\mathrm{Ext}^2_R(M,M) = 0$.

Then $\mathrm{H}(d,g)$ is smooth at (C), i.e., C is unobstructed. Moreover if ${}_0\mathrm{Ext}^i_R(M,M) = 0$ for $i \geqslant 2$, then the dimension of the Hilbert scheme at (C) is

$$\dim_{(C)} \mathrm{H}(d,g) = 4d + {}_0\mathrm{hom}_R(I,E) + {}_{-4}\mathrm{hom}_R(I,M) + {}_{-4}\mathrm{hom}_R(M,E).$$

We may drop the condition ${}_0\mathrm{Ext}^i_R(M,M) = 0$ for $i \geqslant 2$ in Theorem 1.1 by slightly changing the dimension formulas (*cf.* Theorem 2.6 and Remark 2.7). Moreover we remark that once we have a minimal resolution of $\mathcal{I}_C$, we can easily compute ${}_0\mathrm{hom}_R(I,E)$ (as equal to $\delta^2(0)$ in Definition 2.1) while the dimensions of the other Hom-groups are at least easy to find provided C is Buchsbaum (Remark 2.7, (3.4) and (3.6)). Another result from Section 2 is that if a sufficiently general curve C of an irreducible component V of $\mathrm{H}_{\gamma,\rho}$ satisfies the vanishing of the two Hom-groups of Theorem 1.1 (iii), then V (up to possible closure in $\mathrm{H}(d,g)$) is an irreducible component of $\mathrm{H}(d,g)$ (Proposition 2.10).

A main goal of this paper is to see when the sufficient conditions of unobstructedness of Theorem 1.1 are also *necessary* conditions. Note that it has "classically" been quite hard to prove obstructedness because one essentially had to compute a neighborhood of (C) in $\mathrm{H}(d,g)$ to conclude ([34], [18], [7], [15]). Looking for another approach to prove obstructedness, we consider in Section 3 the *cup product* and its "images" in ${}_0\mathrm{Hom}_R(I,E)$, ${}_{-4}\mathrm{Hom}_R(I,M)^\vee$ and ${}_{-4}\mathrm{Hom}_R(M,E)^\vee$ via some natural maps, close to what Walter and Fløystad do in [40] and [11] (see also [29], [35]). These "images" correspond to *three* Yoneda pairings, one of which is the natural morphism

$$(1.1) \qquad {}_0\mathrm{Hom}_R(I,M) \times {}_0\mathrm{Hom}_R(M,E) \longrightarrow {}_0\mathrm{Hom}_R(I,E).$$

All three pairings are easy to handle because they are given by taking simple compositions of homomorphisms, *cf.* Proposition 3.6 and 3.8. If ${}_0\mathrm{Ext}^2_R(M,M) = 0$, it turns out that the non-vanishing of *one of* the three pairings is sufficient for obstructedness. In particular, for a Buchsbaum curve of diameter at most 2, we can, by using a natural decomposition of M, get the non-vanishing of (1.1) from the non-vanishing of some of the Hom-groups involved. More precisely we have (*cf.* Theorem 3.2 for a generalization to e.g. curves with ${}_0\mathrm{Ext}^2_R(M,M) = 0$ obtained by Liaison Addition)

THEOREM 1.2. — *Let C be a Buchsbaum curve in $\mathbb{P}^3$ of diameter at most 2 and let $M \cong M_{[c-1]} \oplus M_{[c]}$ be an R-module isomorphism where $M_{[t]}$, for $t = c-1$ and c, is the part of $M = \mathrm{H}^1_*(\mathcal{I}_C)$ supported in degree t. Then C is obstructed if one of the following conditions hold*

(a) ${}_0\mathrm{Hom}_R(I, M_{[t]}) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M_{[t]}, E) \neq 0$, *for* $t = c$ *or* $t = c-1$,
(b) ${}_{-4}\mathrm{Hom}_R(I, M_{[t]}) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M_{[t]}, E) \neq 0$, *for* $t = c$ *or* $t = c-1$,
(c) ${}_0\mathrm{Hom}_R(I, M_{[t]}) \neq 0$ *and* ${}_{-4}\mathrm{Hom}_R(M_{[t]}, E) \neq 0$, *for* $t = c$ *or* $t = c-1$.

Buchsbaum curves in $\mathbb{P}^3$ are rather well understood by studies of Migliore and others (*cf.* [30] for a survey of important results as well as for an introduction to Liaison Addition), and Theorem 1.2 takes some care of its obstructedness properties. Note also that since the main assumption ${}_0\mathrm{Ext}^2_R(M,M) = 0$ of Section 3 is liaison-invariant, there may be many more applications of Proposition 3.6 and 3.8.

Our results in Section 3 also allow an effective calculation of (at least the degree 2 terms of) the equations of the singularities of $\mathrm{H}(d,g)$ at some curves whose diameter is 2 or less (as illustrated in Example 3.12). To get equivalent conditions of unobstructedness and a complete picture of the equations of the singularities of $\mathrm{H}(d,g)$ more generally, we need a more general version of the cup product and we certainly need to include their higher Massey products (Laudal, [25] and [24]).

If we reformulate Theorem 1.1 by logical negation to necessary conditions of obstructedness (*cf.* Proposition 3.1) we get necessary conditions which are quite close (resp. equivalent) to the sufficient conditions of Theorem 1.1 in the diameter 2 case (resp. in the diameter 1 case). It is easy to substitute the non-vanishing of the Hom-groups of Theorem 1.2 by the non-triviality of certain graded Betti numbers in the minimal resolution,

$$0 \to \bigoplus_i R(-i)^{\beta_{3,i}} \to \bigoplus_i R(-i)^{\beta_{2,i}} \to \bigoplus_i R(-i)^{\beta_{1,i}} \to I \to 0,$$

of I (*cf.* Corollary 3.3). In the diameter one case, we get the following main result (*cf.* [26], pp. 185-193 for the case $M \cong k$).

Theorem 1.3. — *Let C be a curve in $\mathbb{P}^3$ whose Hartshorne-Rao module $M \neq 0$ is of diameter 1. Then C is obstructed if and only if*

$$\beta_{1,c} \cdot \beta_{2,c+4} \neq 0 \text{ or } \beta_{1,c+4} \cdot \beta_{2,c+4} \neq 0 \text{ or } \beta_{1,c} \cdot \beta_{2,c} \neq 0.$$

Moreover if C is unobstructed *and M is r-dimensional, then the dimension of the Hilbert scheme $\mathrm{H}(d,g)$ at (C) is*

$$\dim_{(C)} \mathrm{H}(d,g) = 4d + {}_0\hom_R(I,E) + r(\beta_{1,c+4} + \beta_{2,c}).$$

The Hilbert scheme of constant postulation (or the postulation Hilbert scheme), for which there are various notations, $\mathrm{GradAlg}(H)$, $\mathrm{Hilb}^H(\mathbb{P}^3)$ or just H_γ in the literature, has received much attention recently. We prove

Proposition 1.4. — *In addition to the general assumptions of Theorem 1.3, let $M_{-4} = 0$. Then*

$$\mathrm{H}_\gamma \text{ is singular at } (C) \text{ if and only if } \beta_{1,c+4} \cdot \beta_{2,c+4} \neq 0.$$

Moreover if H_γ is smooth at (C), then $\dim_{(C)} \mathrm{H}_\gamma = 4d + {}_0\hom_R(I,E) + r(\beta_{1,c+4} + \beta_{2,c} - \beta_{1,c})$.

In Section 4 we are concerned with curves which admit a generization (*i.e.*, a deformation to a "more general curve") or are generic in $\mathrm{H}_{\gamma,\rho}$, H_γ or $\mathrm{H}(d,g)$. Inspired by ideas of Martin-Deschamps and Perrin in [26] we prove some results, telling that we can kill certain repetitions in a minimal resolution ("ghost-terms") of the ideal $I(C)$, under deformation. Hence curves with such simplified resolutions exist. One result of particular interest is

Theorem 4.1 which considers the form of a minimal resolution of $I(C)$ given by a Theorem of Rao, *cf.* (3.1) and (3.2). We prove

THEOREM 1.5. — *If C is a generic curve of $\mathrm{H}_{\gamma,\rho}$ (or of H_{γ} or $\mathrm{H}(d,g)$), then C admits a minimal free resolution of the form*

$$0 \to L_4 \xrightarrow{\sigma\oplus 0} L_3 \oplus F_1 \to F_0 \to I(C) \to 0,$$

where $\sigma : L_4 \to L_3$ is given by the leftmost map in the minimal resolution of the Rao module M, cf. (3.1)*, and where F_1 and F_0 are without repetitions (i.e., without common direct free factors).*

Restricting to general Buchsbaum curves, we prove, under some conditions, that L_4 and F_1, and L_4 and $F_0(-4)$, have no common direct free factor (Proposition 4.2). We get

COROLLARY 1.6. — *Let C be a curve in $\mathbb{P}^3$ whose Rao module $M \neq 0$ is of diameter* 1 *and concentrated in degree c.*

(a) *If C is generic in $\mathrm{H}_{\gamma,\rho}$, then H_{γ} is smooth at (C). Moreover C is obstructed if and only if $\beta_{1,c} \cdot \beta_{2,c+4} \neq 0$. Furthermore if $\beta_{1,c} = 0$ and $\beta_{2,c+4} = 0$, then C is generic in $\mathrm{H}(d,g)$.*

(b) *If C is generic in H_{γ}, then C is unobstructed. Indeed both $\mathrm{H}(d,g)$ and H_{γ} are smooth at (C). In particular every irreducible component of $\mathrm{H}(d,g)$ whose generic curve C satisfies $\operatorname{diam} M = 1$ is reduced (i.e., generically smooth).*

Moreover we are able to make explicit various generizations of Buchsbaum curves of diameter at most two, allowing us in many cases to decide whenever an obstructed curve is contained in a *unique* component of $\mathrm{H}(d,g)$ or not (Proposition 4.6). Finally we show that any Buchsbaum curve whose Hartshorne-Rao module has diameter 2 or less, admits a generization in $\mathrm{H}(d,g)$ to an unobstructed curve, hence belongs to a *reduced* irreducible component of $\mathrm{H}(d,g)$. We believe that every irreducible component of $\mathrm{H}(d,g)$ whose generic curve C satisfies $\operatorname{diam} M \leqslant 2$ is *reduced*.

A first version of this paper (containing Theorem 2.6, Theorem 1.3, Theorem 1.5, Corollary 1.6, Proposition 4.6 and the "cup product part" of Proposition 3.6 and 3.8, see [19], available from my home-page) was written in the context of the group "Space Curves" of Europroj, and some main results were lectured at its workshop in May 1995, at the Emile Borel Center, Paris. Later we have been able to generalize several results (e.g. Theorem 1.2). The author thanks prof. O. A. Laudal at Oslo and prof. G. Bolondi at Bologna for interesting discussions on the subject.

1.1. Notations and terminology

A curve C in $\mathbb{P}^3$ is an *equidimensional, locally Cohen-Macaulay* subscheme of $\mathbb{P} := \mathbb{P}^3$ of dimension one with sheaf ideal $\mathcal{I}_C$ and normal sheaf $\mathcal{N}_C = \mathcal{H}om_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_C, \mathcal{O}_C)$. If $\mathcal{F}$ is a coherent $\mathcal{O}_\mathbb{P}$-Module, we let $\mathrm{H}^i(\mathcal{F}) = \mathrm{H}^i(\mathbb{P}, \mathcal{F})$, $\mathrm{H}^i_*(\mathcal{F}) = \oplus_v \mathrm{H}^i(\mathcal{F}(v))$ and $h^i(\mathcal{F}) = \dim \mathrm{H}^i(\mathcal{F})$, and we denote by $\chi(\mathcal{F}) = \Sigma(-1)^i h^i(\mathcal{F})$ the Euler-Poincaré characteristic. Moreover $M = M(C)$ is the Hartshorne-Rao module $\mathrm{H}^1_*(\mathcal{I}_C)$ or just the Rao module, $E = E(C)$ is the module $\mathrm{H}^1_*(\mathcal{O}_C)$ and $I = I(C)$ is the homogeneous ideal $\mathrm{H}^0_*(\mathcal{I}_C)$ of C. They are graded modules over the polynomial ring $R = k[X_0, X_1, X_2, X_3]$, where k is supposed to be algebraically closed of characteristic zero. The postulation γ (resp. deficiency ρ and specialization σ) of C is the function defined over the integers by $\gamma(v) = \gamma_C(v) = h^0(\mathcal{I}_C(v))$ (resp. $\rho(v) = \rho_C(v) = h^1(\mathcal{I}_C(v))$ and $\sigma(v) = \sigma_C(v) = h^1(\mathcal{O}_C(v))$). Let

$$\begin{aligned} s(C) &= \min\{n \mid h^0(\mathcal{I}_C(n)) \neq 0\}, \\ c(C) &= \max\{n \mid h^1(\mathcal{I}_C(n)) \neq 0\}, \\ e(C) &= \max\{n \mid h^1(\mathcal{O}_C(n)) \neq 0\}. \end{aligned}$$

Let $b(C) = \min\{n \mid h^1(\mathcal{I}_C(n)) \neq 0\}$ and let $\operatorname{diam} M(C) = c(C) - b(C) + 1$ be the diameter of $M(C)$ (or of C). If $c(C) < s(C)$ (resp. $e(C) < b(C)$), we say C has maximal rank (resp. maximal corank). A curve C such that $\mathfrak{m} \cdot M(C) = 0$, $\mathfrak{m} = (X_0, \dots, X_3)$, is a Buchsbaum curve. C is *unobstructed* if the Hilbert scheme of space curves of degree d and arithmetic genus g, $\mathrm{H}(d, g)$, is smooth at the corresponding point $(C) = (C \subseteq \mathbb{P}^3)$, otherwise C is *obstructed*. The open part of $\mathrm{H}(d, g)$ of *smooth connected* space curves is denoted by $\mathrm{H}(d, g)_S$, while $\mathrm{H}_{\gamma,\rho} = \mathrm{H}(d, g)_{\gamma,\rho}$ (resp. H_γ, resp. $\mathrm{H}_{\gamma,M}$) denotes the subscheme of $\mathrm{H}(d, g)$ of curves with constant cohomology, *i.e.*, γ_C and ρ_C do not vary with C, (resp. constant postulation γ, resp. constant postulation γ and constant Rao module M), *cf.* [26] for an introduction. The curve in a sufficiently small open irreducible subset of $\mathrm{H}(d, g)$ (small enough to satisfy all the openness properties which we want to pose) is called a *generic* curve of $\mathrm{H}(d, g)$, and accordingly, if we state that a generic curve has a certain property, then the curve belongs to an open irreducible subset of $\mathrm{H}(d, g)$ of curves having this property. A *generization* $C' \subseteq \mathbb{P}^3$ of $C \subseteq \mathbb{P}^3$ in $\mathrm{H}(d, g)$ is a generic curve of some irreducible subset of $\mathrm{H}(d, g)$ containing (C).

For any graded R-module N of finite type , we have the right derived functors $\mathrm{H}^i_\mathfrak{m}(N)$ and ${}_v\mathrm{Ext}^i_\mathfrak{m}(N, -)$ of $\Gamma_\mathfrak{m}(N) = \oplus_v \ker(N_v \to \Gamma(\mathbb{P}, \tilde{N}(v)))$ and $\Gamma_\mathfrak{m}(\mathrm{Hom}_R(N, -))_v$ respectively (*cf.* [14], exp. VI). We use small letters for the k-dimension and subscript v for the homogeneous part of degree v, e.g. ${}_v\mathrm{ext}^i_{\mathfrak{m}}(N_1,N_2) = \dim {}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_1,N_2)$.

2. Preliminaries. Sufficient conditions for unobstructedness.

In this section we recall the main Theorem on unobstructedness of space curves of this paper (Theorem 1.1 or Theorem 2.6). Theorem 2.6 is not entirely new. Indeed (i) and (i') were proved in [18] under the assumption "C generically a complete intersection" (combining [20], Rem. 3.7 and [22], (4.10.1) will lead to a proof), while the (iii) and (iii') part is a rather straightforward consequence of a theorem of Martin-Deschamps and Perrin which appeared in [26]. However, (ii) and (ii') seem new, even though at least (ii) is easily deduced from (i) by linkage. Indeed linkage preserves unobstructedness also in the non arithmetically Cohen-Macaulay (ACM) case provided we link carefully (Proposition 2.5). We will include a proof of Theorem 2.6, also because we need the arguments (e.g. the technical tools and the exact sequences which appear) later.

Let N, N_1 and N_2 be graded R-modules of finite type. Then recall that the right derived functors ${}_v\mathrm{Ext}^i_{\mathfrak{m}}(N,-)$ of ${}_v\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Hom}_R(N,-))$ are equipped with a spectral sequence ([14], exp. VI)

$$E_2^{p,q} = {}_v\mathrm{Ext}^p_R(N_1, \mathrm{H}^q_{\mathfrak{m}}(N_2)) \Rightarrow {}_v\mathrm{Ext}^{p+q}_{\mathfrak{m}}(N_1,N_2) \tag{2.1}$$

($\Rightarrow$ means "converging to") and a duality isomorphism ([23], Thm. 1.1);

$${}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_2,N_1) \cong {}_{-v-4}\mathrm{Ext}^{4-i}_R(N_1,N_2)^\vee \tag{2.2}$$

where $(-)^\vee = \mathrm{Hom}_k(-,k)$, which generalizes the Gorenstein duality

$${}_v\mathrm{H}^i_{\mathfrak{m}}(M) \simeq {}_{-v}\mathrm{Ext}^{4-i}_R(M,R(-4))^\vee.$$

These groups fit into a long exact sequence ([14], exp.VI)

$$\begin{aligned}\to {}_v\mathrm{Ext}^i_{\mathfrak{m}}(N_1,N_2) \to {}_v\mathrm{Ext}^i_R(N_1,N_2) \to \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\tilde N_1, \tilde N_2(v))\\ \to {}_v\mathrm{Ext}^{i+1}_{\mathfrak{m}}(N_1,N_2)\to\end{aligned} \tag{2.3}$$

which in particular relates the deformation theory of $(C \subseteq \mathbb{P}^3)$, described by $\mathrm{H}^{i-1}(\mathcal{N}_C) \cong \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\tilde I, \tilde I(v))$ for $i=1,2$ (*cf.* [18], Rem. 2.2.6 for a proof of this isomorphism), to the deformation theory of the homogeneous ideal $I = I(C)$, described by ${}_0\mathrm{Ext}^i_R(I,I)$, in the following exact sequence

$$\begin{aligned}{}_v\mathrm{Ext}^1_R(I,I) \hookrightarrow \mathrm{H}^0(\mathcal{N}_C(v)) \to {}_v\mathrm{Ext}^2_{\mathfrak{m}}(I,I) \xrightarrow{\alpha} {}_v\mathrm{Ext}^2_R(I,I)\\ \to \mathrm{H}^1(\mathcal{N}_C(v)) \to {}_v\mathrm{Ext}^3_{\mathfrak{m}}(I,I) \to 0.\end{aligned} \tag{2.4}$$

Let $M = \mathrm{H}^2_{\mathfrak{m}}(I)$. In this situation C. Walter proved that the map $\alpha : {}_v\mathrm{Ext}^2_{\mathfrak{m}}(I,I) \cong {}_v\mathrm{Hom}_R(I, \mathrm{H}^2_{\mathfrak{m}}(I)) \to {}_v\mathrm{Ext}^2_R(I,I)$ of (2.4) factorizes via ${}_v\mathrm{Ext}^2_R(M,M)$ in a natural way ([39], Thm. 2.3), the factorization is in fact given by a certain edge homomorphism of the spectral sequence (2.1) with $N_1 = M$, $N_2 = I$ and $p+q = 4$, *cf.* (2.15), (2.16) and (2.17) where the factorization of this map occurs. Fløystad furthered the study of α in [11]. Also in [29], (see [29] Sect. 0.e and Sect. 3), they need to understand α properly to make their calculations.

To compute the dimension of the components of $\mathrm{H}(d,g)$, we have found it convenient to introduce the following invariant, defined in terms of the graded Betti numbers of a minimal resolution of the homogeneous ideal I of C:

$$0 \to \bigoplus_i R(-i)^{\beta_{3,i}} \to \bigoplus_i R(-i)^{\beta_{2,i}} \to \bigoplus_i R(-i)^{\beta_{1,i}} \to I \to 0. \tag{2.5}$$

Definition 2.1. — *If C is a curve in $\mathbb{P}^3$, we let*

$$\delta^j(v) = \sum_i \beta_{1,i} \cdot h^j(\mathcal{I}_C(i+v)) - \sum_i \beta_{2,i} \cdot h^j(\mathcal{I}_C(i+v)) + \sum_i \beta_{3,i} \cdot h^j(\mathcal{I}_C(i+v)).$$

Lemma 2.2. — *Let C be any curve of degree d in $\mathbb{P}^3$. Then the following expressions are equal*

$$\begin{aligned} {}_0\mathrm{ext}^1_R(I,I) - {}_0\mathrm{ext}^2_R(I,I) = 1 - \delta^0(0) &= 4d + \delta^2(0) - \delta^1(0) \\ &= 1 + \delta^2(-4) - \delta^1(-4). \end{aligned}$$

Remark 2.3. — Those familiar with results and notations of [26] will recognize $1-\delta^0(0)$ as δ_γ and $\delta^1(-4)$ as $\epsilon_{\gamma,\delta}$ in their terminology. By Lemma 2.2 it follows that the dimension of the Hilbert scheme $\mathrm{H}_{\gamma,M}$ of constant postulation and Rao module, which they show is $\delta_\gamma + \epsilon_{\gamma,\delta} - {}_0\mathrm{hom}(M,M)$ (Thm. 3.8, page 171), is also equal to $1 + \delta^2(-4) - {}_0\mathrm{hom}(M,M)$.

Proof. — To see the equality to the left, we apply ${}_v\mathrm{Hom}_R(-,I)$ to the resolution (2.5). Since $\mathrm{Hom}_R(I,I) \cong R$ and since the alternating sum of the dimension of the terms in a complex equals the alternating sum of the dimension of its homology groups, we get

$$\dim R_v - {}_v\mathrm{ext}_R{}^1(I,I) + {}_v\mathrm{ext}_R{}^2(I,I) = \delta^0(v), v \in Z. \tag{2.6}$$

If $v = 0$ we get the equality of Lemma 2.2 to the left. The equality in the middle follows from [18], Lemma 2.2.11. We will, however, indicate how we can prove this and the right hand equality by using (2.2) and (2.3). Indeed by (2.2), ${}_v\mathrm{ext}^{4-i}_m(I,I) = {}_{-v-4}\mathrm{ext}^i_R(I,I)$. Hence

$${}_v\mathrm{ext}^2_m(I,I) - {}_v\mathrm{ext}^3_m(I,I) + \dim R_{-v-4} = \delta^0(-v-4), v \in Z \tag{2.7}$$

by (2.6). Combining (2.6) and (2.7) with the exact sequence (2.4), we get

$$\binom{v+3}{3} - \chi(\mathcal{N}_C(v)) = \delta^0(v) - \delta^0(-v-4),\ v \in Z \tag{2.8}$$

because $\dim R_v - \dim R_{-v-4} = \binom{v+3}{3}$. Therefore it suffices to prove

$$\delta^0(-v-4) = \delta^1(v) - \delta^2(v),\ v \geqslant -4. \tag{2.9}$$

Indeed using (2.8) and (2.9) for $v = 0$ we get the equality of Lemma 2.2 in the middle because $\chi(\mathcal{N}_C) = 4d$ holds for *any* curve (*cf.* Remark 2.4) while (2.9) for $v = -4$ takes care of the last equality appearing in Lemma 2.2.

To prove (2.9) we use the spectral sequence (2.1) together with (2.7). Letting $M = \mathrm{H}^2_{\mathfrak{m}}(I)$ and $E = \mathrm{H}^3_{\mathfrak{m}}(I)$ we get ${}_v\mathrm{Ext}^2_{\mathfrak{m}}(I,I) \cong {}_v\mathrm{Hom}_R(I,M)$ and ${}_v\mathrm{Ext}^2_R(I,E) \cong {}_v\mathrm{Ext}^5_{\mathfrak{m}}(I,I) = 0$ and an exact sequence

$$\begin{aligned} {}_v\mathrm{Ext}^1_R(I,M) \hookrightarrow {}_v\mathrm{Ext}^3_{\mathfrak{m}}(I,I) \to {}_v\mathrm{Hom}_R(I,E) \to {}_v\mathrm{Ext}^2_R(I,M) \\ \to {}_v\mathrm{Ext}^4_{\mathfrak{m}}(I,I) \twoheadrightarrow {}_v\mathrm{Ext}^1_R(I,E) \end{aligned} \tag{2.10}$$

where we have used that $v \geqslant -4$ implies ${}_v\mathrm{Hom}(I, \mathrm{H}^4_{\mathfrak{m}}(I)) = 0$ since $\mathrm{H}^4_{\mathfrak{m}}(I) = \mathrm{H}^4_{\mathfrak{m}}(R)$. As argued for (2.6), applying ${}_v\mathrm{Hom}(-,M)$ (resp. ${}_v\mathrm{Hom}(-,E)$) to the resolution (2.5), we get

$$\delta^1(v) = \sum_{i=0}^{2}(-1)^i\ {}_v\mathrm{ext}^i(I,M),\ \text{(resp. } \delta^2(v) = \sum_{i=0}^{2}(-1)^i\ {}_v\mathrm{ext}^i(I,E)). \tag{2.11}$$

So $\delta^1(v) - \delta^2(v)$ equals $\sum_{i=2}^4(-1)^i {}_v\mathrm{ext}^i_{\mathfrak{m}}(I,I)$ by (2.10), and since ${}_v\mathrm{Ext}^4_{\mathfrak{m}}(I,I) \cong {}_{-v-4}\mathrm{Hom}(I,I)^\vee \cong R^\vee_{-v-4}$ we get (2.9) from (2.7), and the proof of Lemma 2.2 is complete. ☐

Remark 2.4. — In [18], Lemma 2.2.11 we proved $\chi(\mathcal{N}_C(v)) = 2dv + 4d$ for *any* curve and any integer v by computing $\delta^0(v)$ for $v >> 0$. Indeed using the definition of $\delta^0(v)$, the sequence $0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_C \to 0$ and $\sum_j(-1)^j\sum_i i\cdot\beta_{j,i} = 0$, we get by applying Riemann-Roch to $\chi(\mathcal{O}_C(i+v))$,

$$\delta^0(v) = \sum_j\sum_i(-1)^j\beta_{j,i}\cdot\chi(\mathcal{O}_{\mathbb{P}}(i+v)) - (dv+1-g)\ ,\ v >> 0 \tag{2.12}$$

while duality on $\mathbb{P}$ and (2.5) show that the double sum of (2.12) equals $-\chi(\mathcal{I}_C(-v-4)) = \binom{v+3}{3} + \chi(\mathcal{O}_C(-v-4))$. We get $\chi(\mathcal{N}_C(v)) = 2dv + 4d$ by combining with (2.8).

Proposition 2.5. — *Let C and C' be curves in $\mathbb{P}^3$ which are linked (algebraically) by a complete intersection of two surfaces of degrees f and g. If*

$$\mathrm{H}^1(\mathcal{I}_C(v)) = 0 \text{ for } v = f, g, f-4 \text{ and } g-4,$$

then C is unobstructed if and only if C' is unobstructed.

One may find a proof in [22], Prop. 3.2. Proposition 2.5 allows us to complete the proof of the following main result on unobstructedness. It applies mostly to curves of small diameter, see also Miró-Roig's criterion for unobstructedness of Buchsbaum curves of diameter at most 2 ([33]).

Theorem 2.6. — *If C is any curve in $\mathbb{P}^3$ of degree d and arithmetic genus g, satisfying (at least) one of the following conditions:*

(i) ${}_v\mathrm{Hom}_R(I, M) = 0$ *for* $v = 0$ *and* $v = -4$
(ii) ${}_v\mathrm{Hom}_R(M, E) = 0$ *for* $v = 0$ *and* $v = -4$
(iii) ${}_0\mathrm{Hom}_R(I, M) = 0, {}_0\mathrm{Hom}_R(M, E) = 0$ *and* ${}_0\mathrm{Ext}^2_R(M, M) = 0$,

then C is unobstructed. Moreover, in each case, the dimension of the Hilbert scheme $H(d, g)$ at $(C \subseteq \mathbb{P}^3)$ is given by

(i') $\dim_{(C)} H(d, g) = 4d + \delta^2(0) - \delta^1(0)$, *provided* (i) *holds* ,
(ii') $\dim_{(C)} H(d, g) = 4d + \delta^2(0) - \delta^1(0) + {}_{-4}\mathrm{hom}_R(I, M) + {}_0\mathrm{hom}_R(I, M) - {}_0\mathrm{ext}^2_R(M, M)$, *provided* (ii) *holds,*
(iii') $\dim_{(C)} H(d, g) = 4d + \delta^2(0) - \delta^1(0) + {}_{-4}\mathrm{hom}_R(I, M)$, *provided* (iii) *holds.*

Proof.

(i) Let $A = R/I$ and let Def_I (resp. Def_A) be the deformation functor of deforming the homogeneous ideal I as a graded R-module (resp. A as a graded quotient of R), defined on the category of local Artin k-algebras with residue field k. Let Hilb_C be the corresponding deformation functor of $C \subseteq \mathbb{P}^3$ (i.e the local Hilbert functor at C) defined on the same category. To see that C is unobstructed we just need, thanks to the duality (2.2), to interpret the exact sequence (2.4) in terms of deformation theory. Recalling that $\mathcal{O}_{C,x}$, $x \in C$ is unobstructed since $\mathcal{I}_{C,x}$ has projective dimension one (*cf.* [10]), we get that $\mathrm{H}^1(\mathcal{N}_C)$ contains all obstructions of deforming $C \subseteq \mathbb{P}^3$. By (2.1) and (2.2);

$$\text{(2.13)} \quad {}_0\mathrm{Ext}^2_{\mathfrak{m}}(I, I) \cong {}_0\mathrm{Hom}(I, M), \quad \text{and } {}_0\mathrm{Ext}^2_R(I, I) \cong {}_{-4}\mathrm{Ext}^2_{\mathfrak{m}}(I, I)^\vee \cong {}_{-4}\mathrm{Hom}(I, M)^\vee.$$

Using the vanishing of the first group of (2.13), we get $Def_I \cong \mathrm{Hilb}_C$ since (2.4) shows that their tangent spaces are isomorphic and since we have an injection of their obstruction spaces (similar to the proof of $Def_A \cong \mathrm{Hilb}_C$ in [20], Rem. 3.7, where the *former* functor must be isomorphic to the local Hilbert functor of constant postulation of C because it deforms the graded quotient A flatly, *i.e.,* has constant Hilbert function), *cf.* [18], Thm. 2.2.1 and [40], Thm. 2.3 where Walter manages to get rid of the "generically complete intersection" assumption of [18], § 2.2 by the argument in the line before (2.13) (see also [11], Prop. 3.13 or [26], VIII, for their tangent spaces). Now Def_I is smooth because ${}_0\mathrm{Ext}^2_R(I,I)$ vanishes by (2.13). This proves (i), and then (i') follows at once from Lemma 2.2.

(iii) One may deduce the unobstructedness of C from results in [26] by combining Thm. 1.5, page 135 with their tangent space descriptions, pp. 155-166. However, since we need the basic exact sequences below later (for which we have no complete reference), we give a new proof which also leads to another result (Proposition 2.10 (b)). Indeed for any curve we claim there is an exact sequence:

$$(2.14)\quad 0 \to T_{\gamma,\rho} \to {}_0\mathrm{Ext}^1_R(I,I) \xrightarrow{\beta} {}_0\mathrm{Hom}_R(M,E) \to {}_0\mathrm{Ext}^2_R(M,M) \\ \to {}_0\mathrm{Ext}^2_R(I,I) \to$$

where $T_{\gamma,\rho}$ is the tangent space of the Hilbert scheme of constant cohomology $\mathrm{H}_{\gamma,\rho}$ at (C). To prove it we use the spectral sequence (2.1) and the duality (2.2) *twice* (Walter's idea mainly, to see the factorization of α via ${}_0\mathrm{Ext}^2_R(M,M)$ in (2.4)), to get an isomorphism, resp. a surjection

$$(2.15)\quad {}_0\mathrm{Ext}^2_R(I,I) \cong {}_{-4}\mathrm{Ext}^2_{\mathfrak{m}}(I,I)^\vee \cong {}_{-4}\mathrm{Hom}(I,M)^\vee \cong {}_0\mathrm{Ext}^4_{\mathfrak{m}}(M,I)$$

$$(2.16)\quad \beta_1 : {}_0\mathrm{Ext}^1_R(I,I) \cong {}_{-4}\mathrm{Ext}^3_{\mathfrak{m}}(I,I)^\vee \twoheadrightarrow {}_{-4}\mathrm{Ext}^1_R(I,M)^\vee \cong {}_0\mathrm{Ext}^3_{\mathfrak{m}}(M,I)$$

Now replacing I by M as the first variable in (2.10) or using (2.1) directly, we get

$$(2.17)\quad 0 \to {}_0\mathrm{Ext}^1_R(M,M) \to {}_0\mathrm{Ext}^3_{\mathfrak{m}}(M,I) \xrightarrow{\beta_2} {}_0\mathrm{Hom}(M,E) \\ \to {}_0\mathrm{Ext}^2_R(M,M) \to {}_0\mathrm{Ext}^4_{\mathfrak{m}}(M,I) \to$$

which combined with (2.15) and (2.16) yield (2.14) because the composition β of β_1 (arising from duality used twice) and β_2 must be the natural one, *i.e.,* the one which sends an extension of ${}_0\mathrm{Ext}^1(I,I)$ (*i.e.,* a short exact sequence) onto the corresponding connecting homomorphism $M = \mathrm{H}^2_{\mathfrak{m}}(I) \to E = \mathrm{H}^3_{\mathfrak{m}}(I)$. And we get the claim by [26], Prop. 2.1, page 157, which implies $\ker\beta = T_{\gamma,\rho}$.

To see that C is unobstructed, we get by (2.14) and the vanishing of ${}_0\mathrm{Hom}_R(M,E)$ an isomorphism between the local Hilbert functor of constant cohomology at C and Def_I. The latter functor Def_I is isomorphic to $\mathrm{Hilb}_{(C)}$ because ${}_0\mathrm{Hom}_R(I,M) = 0$ (*cf.* the proof of (i)), while the former functor is smooth because ${}_0\mathrm{Ext}^2(M,M)$ contains in a natural way the obstructions of deforming a curve in $\mathrm{H}_{\gamma,\rho}$ (*cf.* [26], Thm. 1.5, page 135). This leads easily to the conclusion of (iii). Moreover note that we now get (iii') from Lemma 2.2 because $h^0(\mathcal{N}_C) = {}_0\mathrm{ext}^1_R(I,I)$ and ${}_0\mathrm{ext}^2_R(I,I) = {}_{-4}\mathrm{hom}_R(I,M)$.

(ii) The unobstructedness of C follows from Proposition 2.5. Indeed if we take a complete intersection $Y \supseteq C$ of two surfaces of degrees f and g such that the *conditions* of Proposition 2.5 hold (such Y exists), then the corresponding linked curve C' satisfies

(2.18)
$${}_v\mathrm{Hom}_R(I(C'),M(C')) \cong {}_v\mathrm{Hom}_R(M(C),E(C)) \text{ for } v=0 \text{ and } v=-4$$

because $M(C')$ (resp. $I(C')/I(Y)$) is the dual of $M(C)(f+g-4)$ (resp. $E(C)(f+g-4)$) and ${}_v\mathrm{Hom}_R(I(Y),M(C')) = 0$ for $v = 0,-4$ by assumption. Hence we conclude by Proposition 2.5 and Theorem 2.6 (i). It remains to prove the dimension formula in (ii'). For this we claim that the image of the map $\alpha : {}_0\mathrm{Ext}^2_{\mathfrak{m}}(I,I) \cong {}_0\mathrm{Hom}_R(I,M) \to {}_0\mathrm{Ext}^2_R(I,I)$ which appears in (2.4) for $v=0$, is isomorphic to ${}_0\mathrm{Ext}^2_R(M,M)$. Indeed α factorizes via ${}_0\mathrm{Ext}^2_R(M,M)$ in a natural way, and the factorization is given by a certain map of (2.14). Now ${}_v\mathrm{Hom}_R(M,E) = 0$ for $v=0$ and -4 implies that the maps ${}_v\mathrm{Ext}^2_R(M,M) \to {}_v\mathrm{Ext}^2_R(I,I)$ of (2.14) are injective for $v=0$ and $v=-4$. Dualizing one of them (the map for $v=-4$) we get a surjective composition;

$$(2.19)\quad {}_0\mathrm{Hom}_R(I,M) \cong {}_{-4}\mathrm{Ext}^2_R(I,I)^\vee \to {}_{-4}\mathrm{Ext}^2_R(M,M)^\vee \cong {}_0\mathrm{Ext}^2_R(M,M)$$

which composed with the other injective map above is precisely α. This proves the claim. Now by (2.4) and the proven claim;

$$h^0(\mathcal{N}_C) = {}_0\mathrm{ext}^1_R(I,I) + \dim\ker\alpha = {}_0\mathrm{ext}^1_R(I,I) + {}_0\mathrm{hom}_R(I,M) - {}_0\mathrm{ext}^2_R(M,M)$$

and we get the dimension formula by Lemma 2.2 and we are done. □

Remark 2.7. — Putting the arguments in the text at (2.10) and (2.11) together (and use that ${}_0\mathrm{Ext}^4_{\mathfrak{m}}(I,I)=0$), we get

$$\delta^2(0) = {}_0\mathrm{hom}_R(I,E). \tag{2.20}$$

Moreover if

$${}_0\mathrm{Ext}^i_R(M,M) = 0 \text{ for } 2 \leqslant i \leqslant 4. \tag{2.21}$$

then we may put the different expressions of $\dim_{(C)} \mathrm{H}(d,g)$ of Theorem 2.6 in one common formula;

$$\dim_{(C)} \mathrm{H}(d,g) = 4d + \delta^2(0) + {}_{-4}\mathrm{hom}_R(I,M) + {}_{-4}\mathrm{hom}_R(M,E). \tag{2.22}$$

Indeed

$${}_0\mathrm{Ext}^2_R(I,M) \cong {}_{-4}\mathrm{Ext}^2_{\mathfrak{m}}(M,I)^\vee \cong {}_{-4}\mathrm{Hom}(M,M)^\vee \cong {}_0\mathrm{Ext}^4_R(M,M) = 0,$$

and we have

$${}_0\mathrm{Ext}^1_R(I,M) \cong {}_{-4}\mathrm{Ext}^3_{\mathfrak{m}}(M,I)^\vee \cong {}_{-4}\mathrm{Hom}(M,E)^\vee$$

because ${}_{-4}\mathrm{Ext}^i_R(M,M) \cong {}_0\mathrm{Ext}^{4-i}_R(M,M)^\vee = 0$ for $i = 1,2$. Hence ${}_0\mathrm{ext}^3_{\mathfrak{m}}(I,I) = {}_0\mathrm{hom}_R(I,E) + {}_{-4}\mathrm{hom}(M,E)$ by (2.10). Using (2.13) and that $\alpha = 0$ in (2.4) for $v = 0$, we get

$$h^1(\mathcal{N}_C) = \delta^2(0) + {}_{-4}\mathrm{hom}_R(I,M) + {}_{-4}\mathrm{hom}(M,E), \tag{2.23}$$

and we conclude easily.

Using (2.23), we can generalize the vanishing result of $\mathrm{H}^1(\mathcal{N}_C)$ appearing in [22], Cor. 4.12, to

Corollary 2.8. — *Let C be any curve in $\mathbb{P}^3$, let $\operatorname{diam} M \leqslant 2$ and suppose $e(C) < s(C)$. If $\operatorname{diam} M \neq 0$, suppose also $e(C) \leqslant c + 1 - \operatorname{diam} M$ and $c(C) \leqslant s(C)$. Then*

$$\mathrm{H}^1(\mathcal{N}_C) = 0.$$

Proof. — Since $e(C) < s(C)$, we get $\delta^2(0) = 0$ by the definition of $\delta^2(0)$. Moreover suppose C is not ACM. Then $c(C) \leqslant s(C)$ and (2.5) imply ${}_0\mathrm{ext}^1_R(I,M) = 0$. Finally, since we have $\max\{i | \beta_{1,i} \neq 0\} \leqslant \max\{c(C) + 2, e(C)+3\}$ by Castelnuovo-Mumford regularity, we get ${}_{-4}\mathrm{hom}_R(I,M) = 0$ by (2.5) and we conclude by Remark 2.7. □

Hence curves of $\operatorname{diam} M \leqslant 2$ whose minimal resolution (2.5) is "close enough" to being linear satisfy $\mathrm{H}^1(\mathcal{N}_C) = 0$. Indeed $\mathrm{H}^1(\mathcal{N}_C) = 0$ for any curve of diameter one or two (resp. diameter zero) whose Betti numbers satisfy $\beta_{2,i} = 0$ for $i > \min\{c + 5 - \operatorname{diam} M, s + 3\}$, $\beta_{1,i} = 0$ for $i < c$ (resp. $\beta_{2,i} = 0$ for $i > s + 3$). Thus Corollary 2.8 generalizes [32], Prop. 6.1.

Remark 2.9. — (2.13), (2.14), (2.15), (2.16) and (2.17) are valid for *any* curve in $\mathbb{P}^3$. Moreover if $M_{-4}=0$, we get

$$_0\mathrm{Hom}(M,\mathrm{H}^4_{\mathfrak{m}}(I)) \cong {_0\mathrm{Ext}}^4_{\mathfrak{m}}(M,R)=0$$

since $\mathrm{H}^4_{\mathfrak{m}}(I)\cong \mathrm{H}^4_{\mathfrak{m}}(R)$ and one may see that the spectral sequence which converges to $_0\mathrm{Ext}^4_{\mathfrak{m}}(M,I)$ (*cf.* (2.15), (2.16) and (2.17)) consists of at most two non-vanishing terms. Hence we can continue the exact sequences (2.17) and (2.14) to the right with

$$_0\mathrm{Ext}^4_{\mathfrak{m}}(M,I)\cong {_0\mathrm{Ext}}^2(I,I)\to {_0\mathrm{Ext}}^1_R(M,E)\to {_0\mathrm{Ext}}^3_R(M,M).$$

The proof of Theorem 2.6 implies also the following result (see (i), mainly the argument from [20], Rem. 3.7, to get (a) and (iii), mainly (2.14) and the paragraph before (ii), to get (b)). Note that if C has seminatural cohomology (*i.e.*, maximal rank and maximal corank), then the assumptions of (a) and (b) obviously hold, and we get Prop. 3.2 of [27], ch. IV, which leads to [27], ch. V, Prop. 2.1 and to the unobstructedness of C in the case $\operatorname{diam} M\leqslant 2$ (the latter is also proved in [2]).

Proposition 2.10. — *Let C be any curve in $\mathbb{P}^3$ and let $M=\mathrm{H}^1_*(\mathcal{I}_C)$ and $E=\mathrm{H}^1_*(\mathcal{O}_C)$. Then*

(a) $_0\mathrm{Hom}_R(I,M)=0$ *implies* $\mathrm{H}_\gamma\cong H(d,g)$ *as schemes at* (C),
(b) $_0\mathrm{Hom}_R(M,E)=0$ *implies* $\mathrm{H}_{\gamma,\rho}\cong \mathrm{H}_\gamma$ *as schemes at* (C).

Finally, we shall in Section 4 see what happens to the unobstructedness of C when we impose on C different conditions of being "general enough". One result is already now clear, and it points out that the condition (iii) of Theorem 2.6 is the most important one for *generic* curves:

Proposition 2.11. — *Let C be a curve in $\mathbb{P}^3$, and suppose C is generic in the Hilbert scheme $H(d,g)$ and satisfies $_0\mathrm{Ext}^2_R(M,M)=0$. Then C is unobstructed if and only if*

$$_0\mathrm{Hom}_R(I,M)=0 \text{ and } {_0\mathrm{Hom}}_R(M,E)=0. \tag{2.24}$$

Proof. — One way is clear from Theorem 2.6. Now suppose C is unobstructed and generic with postulation γ and deficiency ρ. By generic flatness we see that $\mathrm{H}_{\gamma,\rho}\cong\mathrm{H}_\gamma\cong \mathrm{H}(d,g)$ near C from which we deduce an isomorphism of tangent spaces $\mathrm{T}_{\gamma,\rho}\cong {_0\mathrm{Ext}}^1_R(I,I)\cong \mathrm{H}^0(\mathcal{N}_C)$. We therefore conclude by the exact sequences (2.14) and (2.4), recalling that $\alpha : {_0\mathrm{Ext}}^2_{\mathfrak{m}}(I,I)\to {_0\mathrm{Ext}}^2_R(I,I)$, which appears in (2.4) for $v=0$ factorizes via $_0\mathrm{Ext}^2_R(M,M)$, *i.e.*, $\alpha=0$. □

Remark 2.12. — Combining (2.14) and (2.17) we get a *surjective* map $T_{\gamma,\rho} \to {}_0\mathrm{Ext}^1_R(M,M)$ whose kernel $T_{\gamma,M}$ is the tangent space of $\mathrm{H}_{\gamma,M}$ at (C). Now dualizing the exact sequence of (2.10) (for $v=-4$), one proves that the surjective map above fits into the exact sequence

(2.25)

$$k \to {}_0\mathrm{Hom}_R(M,M) \to {}_{-4}\mathrm{Hom}_R(I,E)^\vee \to T_{\gamma,\rho} \to {}_0\mathrm{Ext}^1_R(M,M) \to 0$$

and $k \to {}_0\mathrm{Hom}_R(M,M)$ is injective provided $M \neq 0$. We can use this surjectivity (and some considerations on the obstructions involved) to give a new proof of the smoothness of the morphism from $\mathrm{H}_{\gamma,\rho}$ to the "scheme" of Rao modules ([26], Thm. 1.5, page 135). Since ${}_{-4}\mathrm{hom}(I,E) = \delta^2(-4)$, *cf.* (2.11), the exact sequence above also leads to the dimension formula of $\mathrm{H}_{\gamma,M}$ we pointed out in Remark 2.3.

3. Sufficient conditions for obstructedness

In this section we will prove that the conditions (i), (ii), (iii) of Theorem 2.6 are both necessary and sufficient for unobstructedness provided M has diameter one. More generally we are, under the assumption ${}_0\mathrm{Ext}^2_R(M,M)=0$ (resp. $\operatorname{diam} M = 1$), able to make explicit conditions which imply (resp. are equivalent to) obstructedness. Indeed note that we can immediately reformulate the first part of Theorem 2.6 as

PROPOSITION 3.1. — *Let C be a curve in $\mathbb{P}^3$, and let ${}_0\mathrm{Ext}^2_R(M,M)=0$. If C is obstructed, then (at least) one of the following conditions hold*

(a) ${}_0\mathrm{Hom}_R(I,M) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M,E) \neq 0$,
(b) ${}_{-4}\mathrm{Hom}_R(I,M) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M,E) \neq 0$,
(c) ${}_0\mathrm{Hom}_R(I,M) \neq 0$ *and* ${}_{-4}\mathrm{Hom}_R(M,E) \neq 0$.

If C in addition is Buchsbaum, or more generally if the R-module M contains "a Buchsbaum component", by which we mean that M admits a decomposition $M = M' \oplus M_{[t]}$ as *R-modules* where the diameter of $M_{[t]}$ is 1 (*i.e.*, the surjection $M \to M_{[t]}$ splits as an R-linear map), then we have the following "converse" of Proposition 3.1.

THEOREM 3.2. — *Let C be a curve in $\mathbb{P}^3$, let $M = \mathrm{H}^1_*(\mathcal{I}_C)$ and $E = \mathrm{H}^1_*(\mathcal{O}_C)$ and suppose ${}_0\mathrm{Ext}^2_R(M,M)=0$. Moreover suppose there is an R-module isomorphism $M \cong M' \oplus M_{[t]}$ where the diameter of $M_{[t]}$ is 1 and $M_{[t]}$ supported in degree t. Then C is obstructed if at least one of the following conditions hold*

(a) ${}_0\mathrm{Hom}_R(I,M_{[t]}) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M_{[t]},E) \neq 0$, *or*

(b) ${}_{-4}\mathrm{Hom}_R(I, M_{[t]}) \neq 0$ *and* ${}_0\mathrm{Hom}_R(M_{[t]}, E) \neq 0$, *or*
(c) ${}_0\mathrm{Hom}_R(I, M_{[t]}) \neq 0$ *and* ${}_{-4}\mathrm{Hom}_R(M_{[t]}, E) \neq 0$.

Note that if we consider curves obtained by applying Liaison Addition to two curves where one of them is Buchsbaum of diameter 1, then we always have a decomposition of M as in Theorem 3.2 ([30], Thm. 3.2.3), see also [28] for some other cases. Moreover observe that if the module L_2 below has no generators in degree t and $t+4$, then the condition ${}_0\mathrm{Ext}^2_R(M, M) = 0$ holds if it holds for M', *i.e.*, ${}_0\mathrm{Ext}^2_R(M', M') = 0$ (Remark 3.5). We get Theorem 3.2 immediately from Proposition 3.6 and 3.8 which we prove shortly.

We can state Theorem 3.2 in terms of the non-triviality of certain graded Betti numbers of the homogeneous ideal $I = I(C)$. To see this, recall that once we have a minimal resolution of the Rao module M of free graded R-modules,

$$0 \to L_4 \xrightarrow{\sigma} L_3 \to L_2 \to L_1 \to L_0 \to M \to 0, \tag{3.1}$$

one may put the unique minimal resolution (2.5) of the homogeneous ideal I, $0 \to \oplus_i R(-i)^{\beta_{3,i}} \to \oplus_i R(-i)^{\beta_{2,i}} \to \oplus_i R(-i)^{\beta_{1,i}} \to I \to 0$, in the following form

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2 \to F_1 \to I \to 0, \tag{3.2}$$

i.e., where the composition of $L_4 \to L_3 \oplus F_2$ and the natural projection $L_3 \oplus F_2 \to F_2$ is zero ([37], Theorem 2.5). Note that any minimal resolution of I of the form (3.2) has well-defined modules F_2 and F_1. In particular $F_1 = \oplus_i R(-i)^{\beta_{1,i}}$. Moreover applying $\mathrm{Hom}(-, M)$ to (3.1) we get a minimal resolution of

$$\mathrm{Ext}^4_R(M, R) \cong \mathrm{Ext}^4_R(M' \oplus M_{[t]}, R) \cong \mathrm{Ext}^4_R(M', R) \oplus M_{[t]}(2t+4)$$

from which we see that L_4 contains $R(-t-4)^r$ as a direct summand where $r = \dim_k M_{[t]}$. Put

$$L_4 \cong L_4' \oplus R(-t-4)^r, F_2 \cong P_2 \oplus R(-t-4)^{b_1} \oplus R(-t)^{b_2}, F_1 \cong P_1 \oplus R(-t-4)^{a_1} \oplus R(-t)^{a_2} \tag{3.3}$$

where P_i, for $i = 1, 2$ are supposed to contain no direct factor of degree t and $t+4$. So a_1 and a_2 are exactly the first graded Betti number of I in the degree $t+4$ and t respectively, while b_1 and r (resp. b_2) are less than or equal to the corresponding Betti number of I in degree $t+4$ (resp. t) because L_4' and L_3 might contribute to the graded Betti numbers. If, however, M is of diameter 1 (and $M \cong M_{[t]}$), then $L_4' = 0$ and the generators of L_3 sit in degree $t+3$. In this case b_i and r are exactly equal to the corresponding graded Betti numbers in the minimal resolution (2.5). Now Theorem 3.2 translates to

Corollary 3.3. — *Let C be a curve in $\mathbb{P}^3$, let ${}_0\mathrm{Ext}^2_R(M,M)=0$ and suppose $M\cong M'\oplus M_{[t]}$ as R-modules where the diameter of $M_{[t]}$ is 1 and supported in degree t. Then C is obstructed if*

$$a_2\cdot b_1\neq 0 \text{ or } a_1\cdot b_1\neq 0 \text{ or } a_2\cdot b_2\neq 0.$$

This leads to one of the main Theorems of this paper, which solves the problem of characterizing obstructedness in the diameter 1 case (raised in [9]) completely.

Theorem 3.4. — *Let C be a curve in $\mathbb{P}^3$ whose Rao module $M\neq 0$ is of diameter 1 and concentrated in degree c, and let $\beta_{1,c+4}$ and $\beta_{1,c}$ (resp. $\beta_{2,c+4}$ and $\beta_{2,c}$) be the number of minimal generators (resp. minimal relations) of I of degree $c+4$ and c respectively. Then C is obstructed if and only if*

$$\beta_{1,c}\cdot\beta_{2,c+4}\neq 0 \text{ or } \beta_{1,c+4}\cdot\beta_{2,c+4}\neq 0 \text{ or } \beta_{1,c}\cdot\beta_{2,c}\neq 0.$$

Moreover if C is unobstructed *and M is r-dimensional (i.e., $r=\beta_{3,c+4}$), then the dimension of the Hilbert scheme* $\mathrm{H}(d,g)$ *at* (C) *is*

$$\dim_{(C)}\mathrm{H}(d,g)=4d+\delta^2(0)+r(\beta_{1,c+4}+\beta_{2,c}).$$

Proof of Corollary 3.3. — In the sequel we frequently use the triviality of the module structure of $M_{[t]}$ ($\mathfrak{m}\cdot M_{[t]}=0$). Now applying ${}_v\mathrm{Hom}_R(-,M_{[t]})$ to the minimal resolution (3.2) we have by (3.3),

$$(3.4)\qquad {}_0\mathrm{hom}_R(I,M_{[t]})=ra_2 \text{ and } {}_{-4}\mathrm{hom}_R(I,M_{[t]})=ra_1.$$

Moreover note that the assumption ${}_0\mathrm{Ext}^2_R(M,M)=0$ implies ${}_{-4}\mathrm{Ext}^2_R(M,M)=0$ by (2.19) and hence ${}_v\mathrm{Ext}^2_R(M_{[t]},M)=0$ for $v=0$ and -4 by the split R-linear map $M\to M_{[t]}$. By the duality (2.2) and the spectral sequence (2.1) (which converges to ${}_v\mathrm{Ext}^3_{\mathfrak{m}}(M_{[t]},I)$) we therefore get an exact sequence

$$(3.5)\quad 0\to {}_v\mathrm{Ext}^1_R(M_{[t]},M)\to {}_{-v-4}\mathrm{Ext}^1_R(I,M_{[t]})^\vee\to {}_v\mathrm{Hom}_R(M_{[t]},E)\to 0$$

for $v=0$ and -4. Since ${}_v\mathrm{Ext}^1_R(M_{[t]},M)^\vee\cong {}_{-v-4}\mathrm{Ext}^3_R(M,M_{[t]})$ by (2.2) and (2.1) and since we have ${}_{-v-4}\mathrm{Ext}^3_R(M,M_{[t]})\cong {}_{-v-4}\mathrm{Hom}_R(L_3,M_{[t]})$ by (3.1), we get ${}_v\mathrm{Ext}^1_R(M_{[t]},M)\cong {}_{-v-4}\mathrm{Hom}_R(L_3,M_{[t]})^\vee$. Interpreting ${}_{-v-4}\mathrm{Ext}^1_R(I,M_{[t]})$ similarly via the minimal resolution (3.2) of I, we get ${}_v\mathrm{Hom}_R(M_{[t]},E)\cong {}_{-v-4}\mathrm{Hom}_R(F_2,M_{[t]})^\vee$ for $v=0$ and -4 and hence

$$(3.6)\qquad {}_0\mathrm{hom}_R(M_{[t]},E)=rb_1 \quad\text{and}\quad {}_{-4}\mathrm{hom}_R(M_{[t]},E)=rb_2$$

by (3.3) and we conclude easily since $r\neq 0$. □

Remark 3.5. — For later use, note that

$$ {}_v\mathrm{Ext}^2_R(M_{[t]}, M))^\vee \cong {}_{-v-4}\mathrm{Ext}^2_R(M, M_{[t]}) \cong {}_{-v-4}\mathrm{Hom}_R(L_2, M_{[t]}). $$

Hence if we assume the latter group to vanish (instead of assuming ${}_0\mathrm{Ext}^2_R(M, M) = 0$), we get (3.5) and (3.6) for this v. In particular if ${}_v\mathrm{Ext}^2_R(M_{[t]}, M) = 0$ for $v = 0$ and -4, then (3.5) and (3.6) hold, as well as ${}_0\mathrm{Ext}^2_R(M, M) \cong {}_0\mathrm{Ext}^2_R(M', M')$ because

$$ {}_0\mathrm{Ext}^2_R(M', M_{[t]}) \cong {}_{-4}\mathrm{Ext}^2_R(M_{[t]}, M')^\vee = 0. $$

Proof of Theorem 3.4. — Combining Proposition 3.1 and Corollary 3.3 we immediately get the first part of the Theorem. Moreover since we by Remark 2.7 have

$$ \dim_{(C)} \mathrm{H}(d,g) = 4d + \delta^2(0) + {}_{-4}\mathrm{hom}_R(I, M) + {}_{-4}\mathrm{hom}_R(M, E), $$

we conclude by (3.4) and (3.6). □

To prove Theorem 3.2 the following key proposition is useful. As Fløystad points out in [11], if the image of the cup product $\langle\lambda,\lambda\rangle \in \mathrm{Ext}^2_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_C, \mathcal{I}_C)$, $\lambda \in \mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_C, \mathcal{I}_C)$, maps to a non-zero element $\bar{o} \in {}_0\mathrm{Hom}_R(I, E)$ via the right vertical map of (3.7) below, then C is obstructed. He makes several nice contributions to calculate $\bar{o}$, especially when M is a complete intersection (e.g. [11], Prop. 2.13 and §5), see also [29], §3 for further calculations and Laudal ([24], §2) for the theory of cup and Massey products. In general it is, however, quite difficult to prove that $\bar{o} \neq 0$, while the non-vanishing of the natural composition

$$ {}_0\mathrm{Hom}_R(I, M) \times {}_0\mathrm{Hom}_R(M, E) \longrightarrow {}_0\mathrm{Hom}_R(I, E) $$

is in some cases much easier to handle. This is the benefit of the following Proposition.

Proposition 3.6. — *Let C be a curve in $\mathbb{P}^3$, let $M = \mathrm{H}^1_*(\mathcal{I}_C)$ and $E = \mathrm{H}^1_*(\mathcal{O}_C)$ and suppose ${}_0\mathrm{Ext}^2_R(M, M) = 0$. If the obvious morphism*

$$ {}_0\mathrm{Hom}_R(I, M) \times {}_0\mathrm{Hom}_R(M, E) \longrightarrow {}_0\mathrm{Hom}_R(I, E) $$

(given by the composition) is non-zero, then C is obstructed. In particular if M admits a decomposition $M = M' \oplus M_{[t]}$ as R-modules where the diameter of $M_{[t]}$ is 1, then C is obstructed provided

$$ {}_0\mathrm{Hom}_R(I, M_{[t]}) \neq 0 \text{ and } {}_0\mathrm{Hom}_R(M_{[t]}, E) \neq 0. $$

Proof. — It is well known (*cf.* [25]) that if the Yoneda pairing (inducing the cup product)

$$ \langle -, - \rangle : \mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_C, \mathcal{I}_C) \times \mathrm{Ext}^1_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_C, \mathcal{I}_C) \to \mathrm{Ext}^2_{\mathcal{O}_\mathbb{P}}(\mathcal{I}_C, \mathcal{I}_C), $$

given by composition of resolving complexes, satisfies $\langle \lambda, \lambda \rangle \neq 0$ for some λ, then C is obstructed. If we let $p_1 : \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_C, \mathcal{I}_C) \to {}_0\mathrm{Hom}_R(I, M)$ and $p_2 : \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_C, \mathcal{I}_C) \to {}_0\mathrm{Hom}_R(M, E)$ be the maps induced by sending an extension onto the corresponding connecting homomorphisms, then $\langle -, - \rangle$ fits into a commutative diagram

$$\begin{array}{ccccc} \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_C, \mathcal{I}_C) & \times & \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_C, \mathcal{I}_C) & \longrightarrow & \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_C, \mathcal{I}_C) \\ \downarrow p_1 & & \downarrow p_2 & & \downarrow \\ {}_0\mathrm{Hom}_R(I, M) & \times & {}_0\mathrm{Hom}_R(M, E) & \longrightarrow & {}_0\mathrm{Hom}_R(I, E) \end{array} \tag{3.7}$$

where the lower horizontal map is given as in Proposition 3.6. By (2.4), ${}_0\mathrm{Ext}^1_R(I, I) = \ker p_1$, and p_1 is surjective because $\alpha = 0$ for $v = 0$. Moreover since the composition ${}_0\mathrm{Ext}^1_R(I, I) \hookrightarrow \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) \to {}_0\mathrm{Hom}_R(M, E)$ is surjective by the important sequence (2.14), there exists $(\lambda_1, \lambda_2) \in \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) \times {}_0\mathrm{Ext}^1_R(I, I)$ such that the composed map $p_2(\lambda_2)p_1(\lambda_1)$ is non-zero by assumption. Using $\lambda_2 \in {}_0\mathrm{Ext}^1_R(I, I) = \ker p_1$, we get

$$p_2(\lambda_1 + \lambda_2)p_1(\lambda_1 + \lambda_2) = p_2(\lambda_1)p_1(\lambda_1) + p_2(\lambda_2)p_1(\lambda_1)$$

i.e., either $\langle \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 \rangle$ or $\langle \lambda_1, \lambda_1 \rangle$ are non-zero, and C is obstructed.

Finally suppose the two last mentioned Hom-groups of Proposition 3.6 are non-vanishing. Then there exists a map $\psi \in {}_0\mathrm{Hom}_R(M_{[t]}, E)$ such that $\psi(m) \neq 0$ for some $m \in (M_{[t]})_t$. Since $M_{[t]}$ has diameter 1, we get ${}_0\mathrm{Hom}_R(I, M_{[t]}) \cong {}_0\mathrm{Hom}_R(R(-t)^{a_2}, M_{[t]}) \cong (M_{[t]})_t^{a_2}$ by (3.2) and (3.3), and we have $a_2 > 0$. Hence there is a map $\phi' \in {}_0\mathrm{Hom}_R(I, M_{[t]})$ such that $\phi'(1, 0, \ldots, 0) = m$ where $(1, 0, \ldots, 0)$ an a_2-tuple. Since ${}_0\mathrm{Hom}(I, M) \to {}_0\mathrm{Hom}(I, M_{[t]})$ is surjective by the existence of the R-split morphism $p : M \to M_{[t]}$ there is an element $\phi \in {}_0\mathrm{Hom}(I, M)$ which maps to ϕ'. Since the composition $\psi\phi' = \psi p \phi$ maps to a non-trivial element of ${}_0\mathrm{Hom}_R(I, E)$ by construction, we conclude by the first part of the proof. □

Remark 3.7. — Let C be a curve in $\mathbb{P}^3$ whose Rao module has *diameter* 1. From (2.4) and (2.14), *cf.* the proof above, we see at once that ${}_0\mathrm{Hom}_R(I, M) \neq 0$ and ${}_0\mathrm{Hom}_R(M, E) \neq 0$ if and only if we have the following strict inclusions of tangent spaces

$$T_{\gamma,\rho} \subsetneq {}_0\mathrm{Ext}^1_R(I, I) \subsetneq \mathrm{H}^0(\mathcal{N}_C) \tag{3.8}$$

where ${}_0\mathrm{Ext}^1_R(I, I)$ is the tangent space of the Hilbert scheme of constant postulation H_γ at (C). By Proposition 3.6, *C is obstructed if* (3.8) *holds*. If $M \cong k$, this conclusion follows also from [26], ch. X, Prop. 5.9, or from [29].

Along the same lines we are able to generalize a result of Walter [40]. If the diameter of M is 1 and if ${}_0\mathrm{Hom}_R(I, M) = 0$, then Walter proves Proposition 3.8 (a) below and he computes the completion of $\mathcal{O}_{\mathrm{H}(d,g),(C)}$

in detail. The first part of Proposition 3.6 and 3.8, however, requires only ${}_0\mathrm{Ext}^2_R(M,M)=0$. This vanishing condition, which one may show is invariant under linkage (by using (4.6)), holds for instance if the diameter of M is less or equal 2, or if M is generic of diameter 3 and the scheme of Rao modules is irreducible (*cf.* [27]).

Proposition 3.8. — *Let C be a curve in $\mathbb{P}^3$, let $M=\mathrm{H}^1_*(\mathcal{I}_C)$, $E=\mathrm{H}^1_*(\mathcal{O}_C)$ and let ${}_0\mathrm{Ext}^2_R(M,M)=0$.*

(a) *If the obvious morphism*

$$ {}_{-4}\mathrm{Hom}_R(I,M)\times {}_0\mathrm{Hom}_R(M,E)\longrightarrow {}_{-4}\mathrm{Hom}_R(I,E)$$

(given by the composition) is non-zero, then C is obstructed. In particular if M admits a decomposition $M=M'\oplus M_{[t]}$ as R-modules where the diameter of $M_{[t]}$ is 1, then C is obstructed provided

$$ {}_{-4}\mathrm{Hom}_R(I,M_{[t]})\neq 0 \text{ and } {}_0\mathrm{Hom}_R(M_{[t]},E)\neq 0.$$

(b) *If the morphism*

$$ {}_0\mathrm{Hom}_R(I,M)\times {}_{-4}\mathrm{Hom}_R(M,E)\longrightarrow {}_{-4}\mathrm{Hom}_R(I,E)$$

(given by the composition) is non-zero, then C is obstructed. In particular if M admits a decomposition $M=M'\oplus M_{[t]}$ as R-modules where the diameter of $M_{[t]}$ is 1, then C is obstructed provided

$$ {}_0\mathrm{Hom}_R(I,M_{[t]})\neq 0 \text{ and } {}_{-4}\mathrm{Hom}_R(M_{[t]},E)\neq 0.$$

Proof.

Step 1. In Step 1 we give a full proof of (a) under the extra temporary assumption $M_{-4}=0$. Denote by p_2' the restriction of p_2 (see (3.7)) to ${}_0\mathrm{Ext}^1_R(I,I)$ via the natural inclusion ${}_0\mathrm{Ext}^1_R(I,I)\hookrightarrow \mathrm{Ext}^1(\mathcal{I}_C,\mathcal{I}_C)$ and consider the commutative diagram

$$\begin{array}{lccccccc} & \langle -,-\rangle_0 : & {}_0\mathrm{Ext}^1_R(I,I) & \times & {}_0\mathrm{Ext}^1_R(I,I) & \longrightarrow & {}_0\mathrm{Ext}^2_R(I,I) \\ (3.9) & & \uparrow & & \downarrow p_2' & & \downarrow i \\ & & T_{\gamma,\rho} & \times & {}_0\mathrm{Hom}_R(M,E) & \longrightarrow & {}_0\mathrm{Ext}^1_R(M,E) \end{array}$$

where $\langle -,-\rangle_0$ is the Yoneda pairing. Indeed the restriction of ${}_0\mathrm{Ext}^1_R(I,I)$ to the subspace $T_{\gamma,\rho}$ in (3.9) makes the lower horizontal arrow well-defined in the commutative diagram above because of the natural map $T_{\gamma,\rho}\to {}_0\mathrm{Ext}^1_R(M,M)$ of Remark 2.12. Due to the exact sequence (2.14), continued as in Remark 2.9, the map p_2' is surjective and i is injective by the assumption ${}_0\mathrm{Ext}^2_R(M,M)=0$. Hence the pairing $\langle -,-\rangle_0$ factorizes via

$$\varphi' : T_{\gamma,\rho}\times {}_0\mathrm{Hom}_R(M,E)\longrightarrow {}_0\mathrm{Ext}^2_R(I,I) \tag{3.10}$$

and vanishes if we restrict φ' to ${}_{-4}\mathrm{Hom}_R(I,E)^\vee \times {}_0\mathrm{Hom}_R(M,E)$ via the map of Remark 2.12. (using the identity on ${}_0\mathrm{Hom}_R(M,E)$, because ${}_{-4}\mathrm{Hom}_R(I,E)^\vee$ maps to zero in ${}_0\mathrm{Ext}^1_R(M,M)$.

To prove (a) it suffices to prove $\langle\lambda,\lambda\rangle_0 \neq 0$ for some λ. We do this, we *claim that* there is another pairing $\varphi \neq 0$, which commutes with $\langle -,-\rangle_0$, and which essentially corresponds to the restriction of φ' above except for the exchange of variables, *i.e.*,

$$\varphi : {}_0\mathrm{Hom}_R(M,E) \times {}_{-4}\mathrm{Hom}_R(I,E)^\vee \longrightarrow {}_0\mathrm{Ext}^2_R(I,I) \tag{3.11}$$

(since $T_{\gamma,M} = \operatorname{coker}({}_0\mathrm{Hom}_R(M,M) \to {}_{-4}\mathrm{Hom}_R(I,E)^\vee)$ by Remark 2.12, we can continue the arguments below to see that the map φ of (3.11) extends to a somewhat more naturally defined pairing ${}_0\mathrm{Hom}_R(M,E) \times T_{\gamma,M} \to {}_0\mathrm{Ext}^2_R(I,I)$, but this observation does not really effect the proof). Now, to prove the claim there is, as in (3.7), a commutative diagram

$$\begin{array}{ccccc}
{}_{-4}\mathrm{Ext}^2_{\mathfrak{m}}(I,I) & \times & {}_0\mathrm{Ext}^1_R(I,I) & \longrightarrow & {}_{-4}\mathrm{Ext}^3_{\mathfrak{m}}(I,I) \\
\downarrow\cong & & \downarrow p_2' & & \downarrow \\
{}_{-4}\mathrm{Hom}_R(I,M) & \times & {}_0\mathrm{Hom}_R(M,E) & \longrightarrow & {}_{-4}\mathrm{Hom}_R(I,E)
\end{array}$$

where two of the vertical arrows are given by the spectral sequence (2.1) (*cf.* (2.10)) and where the lower pairing is the *non-vanishing* map of Proposition 3.8. Dualizing, we get the commutative diagram

$$\begin{array}{ccccc}
{}_0\mathrm{Ext}^1_R(I,I) & \times & {}_{-4}\mathrm{Ext}^3_{\mathfrak{m}}(I,I)^\vee & \longrightarrow & {}_{-4}\mathrm{Ext}^2_{\mathfrak{m}}(I,I)^\vee \\
\downarrow p_2' & & \uparrow & & \uparrow\cong \\
{}_0\mathrm{Hom}_R(M,E) & \times & {}_{-4}\mathrm{Hom}_R(I,E)^\vee & \longrightarrow & {}_{-4}\mathrm{Hom}_R(I,M)^\vee
\end{array}$$

where the *non-vanishing* lower arrow can be identified with the map φ of (3.11). Using the duality (2.2), we see that φ commutes with the Yoneda pairing $\langle -,-\rangle_0$, and the claim follows easily.

Now since $\varphi \neq 0$ and p_2' is surjective, there exists

$$(\lambda_2,\lambda_1) \in {}_0\mathrm{Hom}_R(M,E) \times {}_{-4}\mathrm{Hom}_R(I,E)^\vee$$

and $\lambda_2' \in {}_0\mathrm{Ext}^1_R(I,I)$ such that $p_2'(\lambda_2') = \lambda_2$ and such that $\langle\lambda_2',\lambda_1\rangle_0 = \varphi(\lambda_2,\lambda_1) \neq 0$. Note that $\langle\lambda_1,\lambda\rangle_0 = 0$ for any $\lambda \in {}_0\mathrm{Ext}^1_R(I,I)$ because $\langle\lambda_1,\lambda\rangle_0 = \varphi'(\lambda_1, p_2'(\lambda)) = 0$ by (3.10). It follows that

$$\langle\lambda_1+\lambda_2',\lambda_1+\lambda_2'\rangle_0 = \langle\lambda_2',\lambda_1\rangle_0 + \langle\lambda_2',\lambda_2'\rangle_0$$

i.e., either $\langle\lambda_1+\lambda_2',\lambda_1+\lambda_2'\rangle_0$ or $\langle\lambda_2',\lambda_2'\rangle_0$ are non-zero. Finally since the map α of (2.4) factors via ${}_0\mathrm{Ext}^2_R(M,M)$ for $v=0$, it follows that the map ${}_0\mathrm{Ext}^2_R(I,I) \to \mathrm{Ext}^2(\mathcal{I}_C,\mathcal{I}_C)$ is injective and maps obstructions to obstructions, *i.e.*, the Yoneda pairing $\langle -,-\rangle_0$ and the corresponding pairing $\langle -,-\rangle$ of (3.7) commute and vanish simultaneously. C is therefore obstructed.

Step 2. To prove (b) we use Step 1 and Proposition 2.5. Indeed let C be a curve as in (b) and let $Y \supseteq C$ be a complete intersection of two surfaces of degrees f and g such that the *conditions* of Proposition 2.5 hold and such that $\mathrm{H}^1(\mathcal{I}_C(f+g)) = 0$, $\mathrm{H}^1(\mathcal{O}_C(f-4)) = 0$ and $\mathrm{H}^1(\mathcal{O}_C(g-4)) = 0$ (such Y exists). Then we claim that the corresponding linked curve C' satisfies the conditions given in Step 1. Indeed slightly extending Remark 2.9, we have

$$\begin{aligned}
{}_0\mathrm{Hom}_R(I(C), M(C)) &\cong {}_0\mathrm{Hom}_R(M(C'), E(C'))\\
{}_{-4}\mathrm{Hom}_R(M(C), E(C)) &\cong {}_{-4}\mathrm{Hom}_R(I(C'), M(C'))\\
{}_{-4}\mathrm{Hom}_R(I(C), E(C)) &\cong {}_{-4}\mathrm{Hom}_R(I(C)/I(Y), E(C))\\
&\cong {}_{-4}\mathrm{Hom}_R(I(C')/I(Y), E(C'))
\end{aligned} \tag{3.12}$$

and we get the claim because

$${}_{-4}\mathrm{Hom}_R(I(C')/I(Y), E(C')) \to {}_{-4}\,\mathrm{Hom}_R(I(C'), E(C'))$$

is injective and $\mathrm{H}^1(\mathcal{I}_C(f+g)) \cong \mathrm{H}^1(\mathcal{I}_{C'}(-4))$. It follows that C' is obstructed by Step 1, and so is C by Proposition 2.5. Moreover if $M = M' \oplus M_{[t]}$ and the diameter of $M_{[t]}$ is 1, we conclude easily by arguing as in the very end of the proof of Proposition 3.6.

Step 3. Finally using the same idea as in Step 2, we prove that (b) and Proposition 2.5 imply (a). Indeed by Proposition 2.5 we can see that (a) and (b) are equivalent by making a suitable linkage, and the proof is complete. ☐

Focusing on the Hilbert scheme with constant postulation, H_γ, we have the following result, quite similar to Theorem 3.4.

Proposition 3.9. — *Let C be a curve in $\mathbb{P}^3$ whose Rao module $M \neq 0$ is of diameter* 1 *and concentrated in degree c, and let $\beta_{1,c+4}$ and $\beta_{1,c}$ (resp. $\beta_{2,c+4}$ and $\beta_{2,c}$) be the number of minimal generators (resp. minimal relations) of degree $c+4$ and c respectively. Suppose also $M_{-4} = 0$. Then*

$$\mathrm{H}_\gamma \ \textit{is singular at } (C) \textit{ if and only if } \beta_{1,c+4} \cdot \beta_{2,c+4} \neq 0.$$

Moreover if H_γ *is smooth at (C) and M is r-dimensional (i.e., $r = \beta_{3,c+4}$), then*

$$\dim_{(C)} \mathrm{H}_\gamma = 4d + \delta^2(0) + r(\beta_{1,c+4} + \beta_{2,c} - \beta_{1,c}).$$

Proof. — Since the tangent space, resp. the obstructions, of H_γ at C is ${}_0\mathrm{Ext}^1_R(I,I)$, resp. sit in ${}_0\mathrm{Ext}^2_R(I,I)$, *cf.* the proof of (i) in Theorem 2.6, we have by Step 1 of the proof above that H_γ is not smooth at (C) provided $M_{-4} = 0$ and the conditions of Proposition 3.8 (a) hold. Hence if $\beta_{1,c+4} \cdot$

$\beta_{2,c+4} \neq 0$, it follows from (3.4) and (3.6) that H_γ is singular at (C) (only for this way here we need the assumption $M_{-4} = 0$). For the converse, suppose $\beta_{1,c+4} = 0$. Then ${}_0\mathrm{Ext}^2_R(I,I) \cong {}_{-4}\hom_R(I,M)^\vee = 0$. And if $\beta_{2,c+4} = 0$, we get by (3.6) and Proposition 2.10 an isomorphism between $\mathrm{H}_{\gamma,\rho}$ and H_γ at (C). The former scheme is smooth because ${}_0\mathrm{Ext}^2_R(M,M) = 0$, and we get the smoothness of the latter. Finally to see the dimension we use $\chi(\mathcal{N}_C) = 4d$ and (2.4) with $\alpha = 0$ for $v = 0$ to get

$${}_0\mathrm{ext}^1_R(I,I) = 4d + h^1(\mathcal{N}_C) - {}_0\hom_R(I,M),$$

and we conclude by (2.23), (3.4) and (3.6). $\square$

Remark 3.10. — Corollary 3.3 admits the following generalization. Instead of assuming the R-module isomorphism $M \cong M' \oplus M_{[t]}$, we suppose that M contains a minimal generator T of degree t and we replace $a_i \neq 0$ by the surjectivity of a certain non-trivial map as follows. Let $M \twoheadrightarrow M \otimes_R k \twoheadrightarrow k(-t)$ and $\eta(T) : {}_v\mathrm{Hom}_R(I,M) \to {}_v\mathrm{Hom}_R(I,k(-t))$ be maps induced by T. Note that $M \twoheadrightarrow M \otimes_R k$ is not necessarily a split R-homomorphism. So if F_{t-v} is a minimal generator of I of degree $t-v$ (inducing maps $R(-t+v) \hookrightarrow I$ and $\tau(F_{t-v}) : {}_v\mathrm{Hom}_R(I,k(-t)) \to {}_v\mathrm{Hom}_R(R(-t+v),k(-t)) \cong k$), we just suppose the surjectivity of the composition $\tau(F_{t-v})\eta(T)$ for $v = 0$ (resp. -4) instead of $a_2 \neq 0$ (resp. $a_1 \neq 0$), to get a generalization of Corollary 3.3. Hence if

$$\tau(F_t)\eta(T) \text{ is surjective for some minimal generator } F_t \text{ of } I,$$

$$\text{and } b_1 \neq 0 \text{ or } b_2 \neq 0\ ,$$

OR if

$$\tau(F_{t+4})\eta(T) \text{ is surjective for some minimal generator } F_{t+4} \text{ of } I,$$

$$\text{and } b_1 \neq 0,$$

then C is obstructed. There is no real change in the proof. Indeed looking to the very final part of Proposition 3.6 and to the proof of Corollary 3.3, noting that we don't need the surjectivity of

$${}_{-v-4}\mathrm{Ext}^1_R(I,k(-t))^\vee \to {}_v\mathrm{Hom}_R(k(-t),E)$$

in (3.5) (where we have replaced $M_{[t]}$ by $k(-t)$), we get the result. Finally note that it is easy to see that $\tau(F_{t-v})\eta(T)$ is surjective if the row in the matrix of relations (*i.e.*, the middle arrow) of (2.5) which corresponds to F_{t-v}, maps M_t to zero. If

$$\min\{i > t-v \mid \beta_{2,i} \neq 0\} > c-t,$$

then the entries of this row map M_t onto M_{c+j} for $j > 0$, i.e onto zero, and we have the mentioned surjectivity. This surjectivity holds in particular if $t = c$ (and L_4 contains generators of degree $c+4$, as always).

Remark 3.11. — We have by Proposition 3.6 and 3.8 the following three Yoneda pairings

$$
\begin{gathered}
{}_0\mathrm{Hom}_R(I,M) \times {}_0\mathrm{Hom}_R(M,E) \longrightarrow {}_0\mathrm{Hom}_R(I,E) \\
{}_0\mathrm{Hom}_R(I,M) \times {}_{-4}\mathrm{Hom}_R(I,E)^{\vee} \longrightarrow {}_{-4}\mathrm{Hom}_R(M,E)^{\vee} \\
{}_0\mathrm{Hom}_R(M,E) \times {}_{-4}\mathrm{Hom}_R(I,E)^{\vee} \longrightarrow {}_{-4}\mathrm{Hom}_R(I,M)^{\vee}.
\end{gathered}
$$

To illustrate, in a diagram, how the right hand sides contribute to $\mathrm{H}^1(\mathcal{N}_C)$, we suppose ${}_0\mathrm{Ext}^i_R(M,M) = 0$ for $i \geqslant 2$ to simplify. Then recall that ${}_0\mathrm{Ext}^2_R(I,M) = 0$ and ${}_0\mathrm{Ext}^1_R(I,M) \cong {}_{-4}\mathrm{Hom}(M,E)^{\vee}$ by Remark 2.7. Now (2.4) (resp. (2.10)) leads to the exactness of the horizontal (resp. vertical, with injective upper downarrow and surjective lower downarrow) sequence in the diagram

$$
\begin{array}{ccccccccc}
 & & & & & & {}_0\mathrm{Ext}^1_R(I,M) & \cong & {}_{-4}\mathrm{Hom}(M,E)^{\vee} \\
 & & & & & & \downarrow & & \\
0 & \to & {}_0\mathrm{Ext}^2_R(I,I) & \to & \mathrm{H}^1(\mathcal{N}_C) & \to & {}_0\mathrm{Ext}^3_{\mathfrak{m}}(I,I) & \to & 0 \\
 & & \downarrow\cong & & & & \downarrow & & \\
 & & {}_{-4}\mathrm{Hom}_R(I,M)^{\vee} & & & & {}_0\mathrm{Hom}_R(I,E) & &
\end{array}
.
$$

We will end this section by showing that there exists smooth connected space curves in any of the three cases (a), (b) and (c) of Theorem 3.2. The case (b) is treated in [40], where Walter manages to find obstructed curves of maximal rank (see also [4]). These curves make H_γ singular as well (Proposition 3.9). By linkage we can transfer the result in [40] to the case (c) and we get the existence of obstructed curves of maximal corank, whose local ring $\mathcal{O}_{\mathrm{H}(d,g),(C)}$ can be described exactly as in [40]. However, since we in the next section will see that a sufficiently general curve of $\mathrm{H}_{\gamma,\rho}$ does not verify neither (b) nor (c), the case (a) deserves special attention. We shall now see that there exist many smooth connected curves satisfying the conditions (a).

Example 3.12. — We claim that for any triple (r, a_2, b_1) of positive integers there exists a smooth connected curve C with minimal resolution as in (3.2) and (3.3) and $\operatorname{diam} M(C) = 1$, such that $s(C) = e(C) = c$, $h^0(\mathcal{I}_C(c)) = a_2$, $h^1(\mathcal{I}_C(c)) = r$, $h^1(\mathcal{O}_C(c)) = b_1$ and $a_1 = 0, b_2 = 0$. Hence

$$
{}_0\mathrm{hom}_R(I,M) = ra_2 \neq 0 \text{ and } {}_0\mathrm{hom}_R(M,E) = rb_1 \neq 0
$$

by (3.4) and (3.6). Since $a_2 = \beta_{1,c}$ and $b_1 = \beta_{2,c+4}$ the curves are obstructed by Theorem 3.4. To see their existence, put $a = a_2$ and $b = b_1$. If $a = 1$, we consider curves with Ω-resolution

$$0 \to \mathcal{O}_{\mathbb{P}}(-2)^{3r-1} \oplus \mathcal{O}_{\mathbb{P}}(-4)^{b} \to \mathcal{O}_{\mathbb{P}} \oplus \Omega^{r} \oplus \mathcal{O}_{\mathbb{P}}(-3)^{b-1} \to \mathcal{I}_C(c) \to 0.$$

By Chang's results ([5] or [40], Thm. 4.1) there exists smooth connected curves having Ω-resolution as above. Moreover $c = 1 + b + 2r$, the degree $d = \binom{c+4}{2} - 3r - 7$ and the genus $g = (c+1)d - \binom{c+4}{3} + 5$. If $a > 1$, curves with Ω-resolution

$$0 \to \mathcal{O}_{\mathbb{P}}(-1)^{a-2} \oplus \mathcal{O}_{\mathbb{P}}(-2)^{3r} \oplus \mathcal{O}_{\mathbb{P}}(-4)^{b} \to \mathcal{O}_{\mathbb{P}}^{a} \oplus \Omega^{r} \oplus \mathcal{O}_{\mathbb{P}}(-3)^{b-1} \\ \to \mathcal{I}_C(c) \to 0$$

exist, they are smooth and connected ([5] or [40], Thm. 4.1), $c = a + b + 2r + 1$, $d = \binom{c+4}{2} - 3a - 3r - 6$ and the genus $g = (c+1)d - \binom{c+4}{3} + 3a + 3$. We leave the verification of details to the reader, recalling only the exact sequences we frequently used in the verification;

$$(3.13) \quad 0 \to \Omega \to \mathcal{O}_{\mathbb{P}}(-1)^4 \to \mathcal{O}_{\mathbb{P}} \to 0 \\ \text{and } 0 \to \mathcal{O}_{\mathbb{P}}(-4) \to \mathcal{O}_{\mathbb{P}}(-3)^4 \to \mathcal{O}_{\mathbb{P}}(-2)^6 \to \Omega \to 0.$$

Putting the two sequences together, we get the Koszul resolution of the regular sequence $\{X_0, X_1, X_2, X_3\}$.

We will analyze these curves a little further, using Laudal's description of the completion of $O_{\mathrm{H}_{(d,g)},(C)}$ ([25], Thm. 4.2.4). This completion is $k[[\mathrm{H}^0(\mathcal{N}_C)^\vee]]/o(\mathrm{H}^1(N_C)^\vee)$, where o is a certain obstruction morphism (giving essentially the cup and Massey products). Now, consulting for instance the proof of Proposition 3.8, we see that the dual spaces of ${}_0\mathrm{Hom}_R(I, M)^\vee$ and ${}_0\mathrm{Hom}_R(M, E)^\vee$ inject into $\mathrm{H}^0(\mathcal{N}_C)^\vee$ and their intersection is empty. This implies

$$\mathrm{H}^0(N_C)^\vee \cong T_{\gamma,\rho}^\vee \oplus {}_0\mathrm{Hom}_R(I, M)^\vee \oplus {}_0\mathrm{Hom}_R(M, E)^\vee \text{ as } k\text{-vector spaces,}$$

and we can represent $k[[\mathrm{H}^0(\mathcal{N}_C)^\vee]]$ as $k[[Y_1, \dots, Y_m, Z_{11}, \dots, Z_{ar}, W_{11}, \dots, W_{rb}]]$, letting $Y_1, \dots, Y_m$, resp. $Z_{11}, \dots, Z_{ar}$, resp. $W_{11}, \dots, W_{rb}$ correspond to a basis of $T_{\gamma,\rho}^\vee$, resp. ${}_0\mathrm{Hom}_R(I, M)^\vee$, resp. ${}_0\mathrm{Hom}_R(M, E)^\vee$. Since $a_1 = 0, b_2 = 0$, we get by (3.4) and (3.6);

$${}_{-4}\mathrm{Hom}_R(I, M) = 0 \text{ and } {}_{-4}\mathrm{Hom}_R(M, E) = 0.$$

By Remark 2.7 and Definition 2.1, $h^1(\mathcal{N}_C) = \delta^2(0) = a_2 b_1$, and we can use Proposition 3.8 and its proof to conclude that, modulo $\mathfrak{m}_{\mathcal{O}}^3$ ($\mathfrak{m}_{\mathcal{O}}$ the maximal ideal of the completion of $\mathcal{O}_{\mathrm{H}(d,g),(C)}$), we have

$$\mathcal{O}_{H(d,g),(C)}/\mathfrak{m}_{\mathcal{O}}^3 = k[[Y_1,\dots,Y_l,Z_{11},\dots,Z_{ar},W_{11},\dots,W_{rb}]]/\mathfrak{a} \tag{3.14}$$

where the ideal $\mathfrak{a}$ is generated by the components of the matrix given by the product

$$\begin{bmatrix} Z_{11} & \dots & Z_{1r} \\ Z_{21} & \dots & Z_{2r} \\ \vdots & & \vdots \\ Z_{a1} & \dots & Z_{ar} \end{bmatrix} \begin{bmatrix} W_{11} & \dots & W_{1b} \\ W_{21} & \dots & W_{2b} \\ \vdots & & \vdots \\ W_{r1} & \dots & W_{rb} \end{bmatrix}. \tag{3.15}$$

Note that (3.15) corresponds precisely to the composition given by the pairing of Proposition 3.6! As in [40], proof of Thm. 0.5, we believe that the Massey products corresponding to (3.15) vanish, *i.e.,* the right-hand side of (3.14) is exactly the completion of $\mathcal{O}_{\mathrm{H}(d,g),(C)}$.

The simplest case is $(r,a_2,b_1) = (1,1,1)$, which yields curves C with $s(C) = 4, d = 18$ and $g = 39$ (Sernesi's example [38] or [8]), while the case $(r,a_2,b_1) = (2,1,1)$ yields curves C with $s(C) = 6, d = 32$ and $g = 109$. More generally, the curves of the case $(r,1,1)$ satisfy $h^1(\mathcal{N}_C) = a_2b_1 = 1$, *i.e.,* the ideal $\mathfrak{a}$ of (3.14) is generated by the single element

$$\sum_{i=1}^{r} Z_{1i} \cdot W_{i\,1}. \tag{3.16}$$

For Sernesi's example ($r = 1$), we recognize the known fact that this curve sits in the intersection of two irreducible components of $\mathrm{H}(d,g)$, while for $r > 1$, the irreducibility of (3.16) can be used to see that C belongs to a unique irreducible component of $\mathrm{H}(d,g)$. Other examples of singularities of $\mathrm{H}(d,g)$ which belong to a unique irreducible component are known ([21], Rem. 3b) and [16], Thm. 3.10). In the next section we prove the irreducibility/reducibility by studying in detail the possible generizations of a Buchsbaum curve.

4. The minimal resolution of a general space curve

In this section we study generizations of space curves C and how suitable generizations will simplify the minimal resolution of $I(C)$. By a *generization* we mean a deformation to a "more general curve", *cf.* Subsection 1.1. The general philosophy is that a sufficiently general curve of any irreducible component of $\mathrm{H}(d,g)$ should have as few repeated direct factors "as possible" in consecutive terms of the minimal resolution. We prove below a general result in this direction (Theorem 4.1) and a more restricted one (Proposition 4.2) for curves with special Rao modules, using some nice ideas from [26] where they make explicit some cancellations in the minimal resolution under flat deformation, in a special case ($M \cong k$) which has the potential of being generalized. More recently several papers have appeared using "consecutive cancellations" to relate graded Betti numbers with the same Hilbert function (see [36], [31] and its references). Recalling the notations (3.1) and (3.2) from Rao's theorem ([37], Thm. 2.5), we show

Theorem 4.1. — *Let C be a curve in $\mathbb{P}^3$ with postulation γ and Rao module $M = M(C)$ and suppose the homogeneous ideal $I(C)$ has a minimal free resolution of graded R-modules;*

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2 \to F_1 \to I(C) \to 0. \tag{4.1}$$

If there exists a direct free factor F satisfying $F_2 \cong F_2' \oplus F$ and $F_1 \cong F_1' \oplus F$, then there is a generization $C' \subseteq \mathbb{P}^3$ of $C \subseteq \mathbb{P}^3$ in the Hilbert scheme $\mathrm{H}(d,g)$ (in fact in $\mathrm{H}_{\gamma,M}$, i.e., with constant postulation and Rao module) whose homogeneous ideal $I(C')$ has a minimal free resolution of the following form

$$0 \to L_4 \xrightarrow{\sigma \oplus 0} L_3 \oplus F_2' \to F_1' \to I(C') \to 0.$$

Now suppose $M = M(C)$ admits an R-*module* decomposition $M = M' \oplus M_{[t]}$ where the diameter of $M_{[t]}$ is 1 (e.g. C is Buchsbaum). Let $0 \to L_4' \xrightarrow{\sigma'} L_3' \to L_2' \to L_1' \to L_0' \to M' \to 0$ be the minimal resolution of M' and let

$$0 \to R(-t-4)^r \xrightarrow{\sigma_{[t]}} R(-t-3)^{4r} \to \cdots \to R(-t)^r \to M_{[t]} \to 0$$

be the corresponding resolution of $M_{[t]}$ (which is "r times" the Koszul resolution of the R-module $k \cong R/(X_0, X_1, X_2, X_3)$.) By the Horseshoe lemma the minimal resolution of M is the direct sum of these two resolutions. Looking to (3.3), we get $a_1 \cdot b_1 = 0$ and $a_2 \cdot b_2 = 0$ for a general curve C of $\mathrm{H}(d,g)$ by Theorem 4.1. Hence the corresponding singularities of $\mathrm{H}(d,g)$ given by Corollary 3.3 can not occur for a general C, neither can the remaining class of singularities due to

Proposition 4.2. — *Let C be a curve in $\mathbb{P}^3$ and let $M(C) \cong M' \oplus M_{[t]}$ as R-modules where $M_{[t]}$ is r-dimensional of diameter 1 and supported in degree t. Moreover suppose the homogeneous ideal $I(C)$ has a minimal resolution of the following form;*

$$0 \to L_4' \oplus R(-t-4)^r \xrightarrow{\sigma' \oplus \sigma_{[t]} \oplus 0} L_3' \oplus R(-t-3)^{4r} \oplus F_2 \to F_1 \to I(C) \to 0, \tag{4.2}$$

where $F_2 \cong P'_2 \oplus R(-t-4)^{b_1}$ *and* $F_1 \cong P'_1 \oplus R(-t)^{a_2}$ *and where* P'_2 *(resp.* P'_1*) is without direct free factors generated in degree* $t+4$ *(resp.* t*).*

(a) *Let* $r \cdot b_1 \neq 0$ *and let* m_1 *be a number satisfying* $0 \leqslant m_1 \leqslant \min\{r, b_1\}$*. Then there is a generization* $C' \subseteq \mathbb{P}^3$ *of* $C \subseteq \mathbb{P}^3$ *in* $\mathrm{H}(d,g)$ *(in fact in* H_γ*, i.e., with constant postulation* γ*) such that* $I(C')$ *has a minimal free resolution of the following form;*

$$0 \to L'_4 \oplus R(-t-4)^{r-m_1} \to L'_3 \oplus R(-t-3)^{4r} \oplus P'_2 \oplus R(-t-4)^{b_1-m_1} \\ \to F_1 \to I(C') \to 0,$$

and such that $M(C') \cong M' \oplus M(C')_{[t]}$ *as* R*-modules for some* $r - m_1$ *dimensional module* $M(C')_{[t]}$ *supported in degree* t*. Moreover if* L'_2 *does not contain a direct free factor generated in degree* $t+4$*, then*

$$_0\mathrm{hom}_R(M(C')_{[t]}, E(C')) = (r - m_1)(b_1 - m_1).$$

(b) *Suppose* L'_2 *is without direct free factors generated in degree* t*. If* $r \cdot a_2 \neq 0$ *and if* m_2 *is a number satisfying* $0 \leqslant m_2 \leqslant \min\{r, a_2\}$*, then there is a generization* $C' \subseteq \mathbb{P}^3$ *of* $C \subseteq \mathbb{P}^3$ *in* $\mathrm{H}(d,g)$ *(with constant specialization) such that* $I(C')$ *has a minimal free resolution of the following form;*

$$0 \to L'_4 \oplus R(-t-4)^{r-m_2} \to L'_3 \oplus G_2 \to G_1 \oplus R(-t)^{a_2-m_2} \to I(C') \to 0$$

for some R*-free modules* G_2 *and* G_1 *where* G_1 *is without direct free factors generated in degree* t*. Moreover* $M(C') \cong M' \oplus M(C')_{[t]}$ *as* R*-modules for some* $r - m_2$ *dimensional module* $M(C')_{[t]}$ *supported in degree* t*, and we have*

$$_0\mathrm{hom}_R(I(C'), M(C')_{[t]}) = (r - m_2)(a_2 - m_2).$$

Once we have proved a key lemma, the proof of Theorem 4.1 is straightforward while the proof of Proposition 4.2 is a little bit more technical. Note that the assumptions on L'_2 in Proposition 4.2 (a) and (b) show that if ${}_0\mathrm{Ext}^2_R(M', M') = 0$ then ${}_0\mathrm{Ext}^2_R(M, M) = 0$ (Remark 3.5), indicating that our results of this section combine nicely with Theorem 3.2. We delay the proof of these results until the end of this section.

Now combining these two results with Theorem 3.4 in the diameter one case, we get

Corollary 4.3. — *Let* C *be a curve in* $\mathbb{P}^3$ *whose Rao module* $M \neq 0$ *is of diameter* 1 *and concentrated in degree* c*, and let* $\beta_{1,c+4}$ *and* $\beta_{1,c}$ *(resp.* $\beta_{2,c+4}$ *and* $\beta_{2,c}$*) be the number of minimal generators (resp. minimal relations) of degree* $c+4$ *and* c *respectively.*

(a) *If C is generic in $\mathrm{H}_{\gamma,\rho}$, then H_γ is smooth at (C). Moreover C is obstructed if and only if $\beta_{1,c}\cdot\beta_{2,c+4}\neq 0$. Furthermore if $\beta_{1,c}=0$ and $\beta_{2,c+4}=0$, then C is generic in $\mathrm{H}(d,g)$.*

(b) *If C is generic in H_γ, then C is unobstructed. Indeed both $\mathrm{H}(d,g)$ and H_γ are smooth at (C). In particular every irreducible component of $\mathrm{H}(d,g)$ whose generic curve C satisfies $\operatorname{diam} M(C)\leqslant 1$ is reduced (i.e., generically smooth).*

Proof.

(a) C is generic in $\mathrm{H}(d,g)$ by Proposition 2.10 because ${}_0\mathrm{Hom}_R(M,E)={}_0\mathrm{Hom}_R(I,M)=0$ by (3.4) and (3.6). The other statements follow directly from Theorem 3.4, Theorem 4.1 and Proposition 3.9.

(b) If C is generic in H_γ, then we immediately have $\beta_{1,c}\cdot\beta_{2,c}=0$ and $r\cdot\beta_{2,c+4}=0$ by Theorem 4.1 and Proposition 4.2. Since $r>0$ we see by Theorem 3.4 that $\mathrm{H}(d,g)$ (and of course H_γ by (a)) is smooth at (C). Finally if C is a generic curve of some irreducible component of $\mathrm{H}(d,g)$ satisfying $\operatorname{diam} M(C)\leqslant 1$ and γ is the postulation of C, then C is generic in H_γ and we conclude easily. □

Corollary 4.3 (a) generalizes [4] Prop. 1.1 which tells that a curve C of maximal rank or maximal corank of $\operatorname{diam} M(C)=1$, which is generic in $\mathrm{H}_{\gamma,\rho}$, is unobstructed.

Even though we can extend the next corollary to Buchsbaum curves satisfying ${}_0\mathrm{Ext}^2_R(M,M)=0$ (*i.e.*, ${}_0\mathrm{Hom}_R(L_2,M)=0$), we have chosen to formulate it for the somewhat more natural set of Buchsbaum curves C of $\operatorname{diam} M(C)\leqslant 2$. Note that Buchsbaum curves of maximal rank satisfy $\operatorname{diam} M(C)\leqslant 2$ ([30], Cor. 3.1.4, [9], Cor. 2.8), and that Corollary 4.3 and 4.4 (and [33]) give answers to the problems on unobstructedness of Buchsbaum curves raised by Ellia and Fiorentini in [9].

Corollary 4.4. — *Let C be a Buchsbaum curve of $\operatorname{diam} M(C)\leqslant 2$. Then there exists a generization C' of C in $H(d,g)$ such that C' is Buchsbaum (or ACM with $L_4=0$) and such that the modules of the three sets*

$$\{F_2,F_1\},\{L_4,F_2\}\text{ and }\{L_4,F_1(-4)\}$$

in its minimal resolution, $0\to L_4\xrightarrow{\sigma\oplus 0} L_3\oplus F_2\to F_1\to I(C')\to 0$, are without common direct free factors. Hence ${}_0\mathrm{Hom}_R(I(C'),M(C'))={}_0\mathrm{Hom}_R(M(C'),E(C'))=0$ and $\mathrm{H}(d,g)$ is smooth at (C').

Proof. — Firstly note that since the module structure of M of any Buchsbaum curve is trivial, we get from the resolution (4.1) that ${}_0\mathrm{Hom}_R(I,M)\cong{}_0\mathrm{Hom}_R(F_1,M)$. Since $M\cong\ker \mathrm{H}^3_*(\tilde\sigma\oplus 0)$, it follows that the latter group vanishes if and only if L_4 and $F_1(-4)$ are without common direct free factors. Moreover by arguing as in the proof of Corollary 3.3 we get

$$ {}_{-4}\mathrm{Hom}_R(F_2, M)^\vee \cong {}_0\mathrm{Hom}_R(M, E) $$

which vanishes if and only if L_4 and F_2 are without common direct free factors.

Now, by Theorem 4.1, $\{F_2, F_1\}$ have no common direct free factors, and writing $M(C) \cong M_{[c]} \oplus M_{[c-1]}$ as R-modules, we can successively apply Proposition 4.2 to $M_{[c]}$ and $M_{[c-1]}$. Indeed the (a) part of Proposition 4.2 with $M_{[t]} = M_{[c]}$ and $m_1 = \min\{r, b_1\}$ shows that $\{L_4, F_2\}$ for some generization of C are without common direct free factors of degree $c+4$. Then we proceed by (b) to see that $\{L_4, F_1(-4)\}$ for some further generization of C are without common direct free factors of degree $c+4$. Similarly we use Proposition 4.2 with $M_{[t]} = M_{[c-1]}$ to see that there remains, up to a suitable generization C', also no common direct free factor of degree $c+3$ in $\{L_4, F_2\}$ and $\{L_4, F_1(-4)\}$. Hence we have ${}_0\mathrm{Hom}_R(I(C'), M(C')) = {}_0\mathrm{Hom}_R(M(C'), E(C')) = 0$ by the first part of the proof and we conclude by Proposition 2.10. □

We should have liked to generalize Corollary 4.4 to the arbitrary case of diameter 2 by dropping the Buchsbaum assumption. In particular if we could prove a result analogous to Corollary 4.4 for curves whose Rao module M is the generic module of diameter two (*cf.* [27] for existence and minimal resolution), we would be able to answer affirmatively the following question (which we believe is true).

Question. — Is any irreducible component of $\mathrm{H}(d, g)$ whose Rao module of its generic curve is concentrated in at most two consecutive degrees, generically smooth?

In our corollaries we have used Theorem 4.1 and Proposition 4.2 to consider generic curves, or to get the existence of a certain generization, with nice obstruction properties. We may, however, also use our results to study many different generizations of a given curve C, see the works of Amasaki, Ellia and Fiorentini and others ([1], [38], [8], [22]) for similar approaches. Hence we may see when C sits in the intersection of different integral components of $\mathrm{H}(d, g)$. There may be quite a lot of such irreducible components of $\mathrm{H}(d, g)$ [12]. We will soon look closely to the possible generizations of a curve of diameter one in the case $\beta_{1,c} \cdot \beta_{2,c+4} \neq 0$. To get a flavour of the other possibilities, we consider the following example of a non-generic curve of $\mathrm{H}_{\gamma,M}$.

Example 4.5. — In [4] and [40] one proves the existence of an obstructed curve of $\mathrm{H}(33,117)_S$ of maximal rank with one-dimensional Rao module. Since the degrees of the minimal generators of $I(C)$ are given in [4] and $M = H^1(\mathcal{I}_C(5))$, we easily find the minimal resolution to be

$$0 \to R(-9) \to R(-10)^2 \oplus R(-9) \oplus R(-8)^4 \\ \to R(-9) \oplus R(-8) \oplus R(-7)^5 \to I(C) \to 0.$$

It follows from Theorem 3.4 of this paper that C is obstructed. By Proposition 4.2 (resp. Theorem 4.1) there exists a generization C_1 (resp. C_2) of C, obtained by removing the direct factor $R(-9)$ from L_4 and F_2 (resp. from F_2 and F_1). The curve C_1 is ACM, hence unobstructed, and belongs to a unique irreducible component V of $\mathrm{H}(33,117)_S$. Moreover the curve C_2 is unobstructed by Theorem 3.4. Now looking only to the semicontinuity of $h^1(\mathcal{I}_C(5))$ and $h^1(\mathcal{O}_C(5))$, there is a priori a possibility that C_2 may belong to V. By Corollary 4.3 (a) or by Proposition 2.10, however, C_2 is generic in $\mathrm{H}(33,117)_S$ since we may suppose C_2 is generic in $\mathrm{H}(33,117)_\gamma$. Hence the irreducible component W of $\mathrm{H}(33,117)_S$ to which C_2 belongs, satisfies $W \neq V$!! Since C is contained in the intersection of the components, we get the main example of [4] from our results.

As an illustration of the main results of this section, we restrict to curves which are generic in $\mathrm{H}_{\gamma,M}$, or more generally to curves which satisfy $a_1 \cdot b_1 = 0$ and $a_2 \cdot b_2 = 0$ (letting $a_1 = \beta_{1,c+4}$, $a_2 = \beta_{1,c}$, $b_1 = \beta_{2,c+4}$ and $b_2 = \beta_{2,c}$). Thus we consider the case

$$a_1 = 0, b_2 = 0 \text{ and } (a_2 \neq 0 \text{ or } b_1 \neq 0) \tag{4.3}$$

where proper generizations as in Proposition 4.2 occur, to give a rather complete picture of the existing generizations in $\mathrm{H}(d,g)$ (caused by simplifications of the minimal resolution). Let $n(C) = (r, a_1, a_2, b_1, b_2)$ be an associated 5-tuple. Only for curves satisfying $a_1 = 0$ and $b_2 = 0$ we allow the writing $n(C) = (r, a_2, b_1)$ as a triple. Thanks to [3] we remark that any curve D satisfying $n(D) = n(C)$ and $\gamma_D(v) = \gamma_C(v)$ for $v \neq c$, belongs to the same *irreducible* family $\mathrm{H}_{\gamma,M}$ as C, *i.e.*, a further generization of C and D in $\mathrm{H}_{\gamma,M}$ lead to the "same" generic curve. Now given a curve C with $n(C) = (r, a_2, b_1)$, we have by Proposition 4.2:

(4.4) For any pair (i,j) of non-negative integers such that $r - i - j \geqslant 0$, $a_2 - i \geqslant 0$ and $b_1 - j \geqslant 0$, there exists a generization C_{ij} of C in $\mathrm{H}(d,g)$ such that $n(C_{ij}) = (r-i-j, a_2 - i, b_1 - j)$.

Note that if we link C to C_l as in Proposition 2.5, we get, by combining (2.18), (3.4) and (3.6) that the 5-tuple $n(C_l) = (r(C_l), a_1(C_l), a_2(C_l), b_1(C_l), b_2(C_l))$ is equal to (r, b_2, b_1, a_2, a_1) where $n(C) = (r, a_1, a_2, b_1, b_2)$. In particular if C satisfies (4.3), then the linked curve C_l also does.

As an example, let $n(C) = (4, 3, 2)$ (such curves exist by Example 3.12). By (4.4) we have 10 different generizations C_{ij} among which two curves correspond to the triples $n(C_{22}) = (0, 1, 0)$ and $n(C_{31}) = (0, 0, 1)$, *i.e.*, they correspond to two *unobstructed* ACM curves with different postulation. Hence they belong to two different irreducible components of $\mathrm{H}(d, g)$ having (C) in their intersection. Pushing this argument further, we get at least

Proposition 4.6. — *Let C be a curve in $\mathbb{P}^3$ whose Rao module $M \neq 0$ is r-dimensional and concentrated in degree c, let $a_1 = \beta_{1,c+4}$ and $a_2 = \beta_{1,c}$ (resp. $b_1 = \beta_{2,c+4}$ and $b_2 = \beta_{2,c}$) be the number of minimal generators (resp. minimal relations) of degree $c + 4$ and c respectively, and suppose*

$$a_1 = 0, b_2 = 0 \text{ and } a_2 \cdot b_1 \neq 0.$$

(a) *If $r < a_2+b_1$, then C sits in the intersection of at least two irreducible components of $\mathrm{H}(d, g)$. Moreover, the generic curve of any component containing C is arithmetically Cohen-Macaulay, and the number $n(\mathrm{comp}, C)$ of irreducible components containing C satisfies*

$$\min\{a_2, r\} + \min\{b_1, r\} - r + 1 \leqslant n(\mathrm{comp}, C) \leqslant r + 1.$$

In the case $s(C) = e(C) = c$, we have equality to the left.

(b) *If $r \geqslant a_2 + b_1$ and $s(C) = e(C) = c$, then C is an obstructed curve which belongs to a unique irreducible component of $\mathrm{H}(d, g)$.*

Proof. — We firstly prove (b). Let C' be any generization of C in $\mathrm{H}(d, g)$ and let $n(C') = (r', a'_1, a'_2, b'_1, b'_2)$ be the associated 5-tuple where $r' = 0$ corresponds to the ACM case of C'. Since $s(C) = c$ and since the number $s(C)$ increases under generization by the semicontinuity of $h^0(\mathcal{I}_C(v))$, we get $s(C') \geqslant c$ as well as $h^0(\mathcal{I}_{C'}(c)) = a'_2$ and $b'_2 = 0$. Similarly $e(C) = c$ implies $h^1(\mathcal{O}_{C'}(c)) = b'_1$ and $a'_1 = 0$. Applying these considerations to $C' = C$, we get $\chi(\mathcal{I}_C(c)) \leqslant 0$ by the assumption $r \geqslant a_2 + b_1$.

Now let C' be the generic curve of an irreducible component containing C. By Proposition 4.2 we get $r'a'_2 = 0$ and $r'b'_1 = 0$ which combined with $\chi(\mathcal{I}_{C'}(c)) = \chi(\mathcal{I}_C(c)) \leqslant 0$ yields $a'_2 = 0$ and $b'_1 = 0$. Hence $n(C') = (r - a_2 - b_1, 0, 0, 0, 0)$ for any generic curve of $\mathrm{H}(d, g)$. Since $\gamma_{C'}(v) = \gamma_C(v)$ for $v \neq c$ by semicontinuity and the vanishing of $\mathrm{H}^1(\mathcal{I}_C(v))$, any such C' belongs to the same irreducible component of $\mathrm{H}(d, g)$ by the irreducibility of $\mathrm{H}_{\gamma_{C'}, M(C')}$. Moreover C is obstructed by Theorem 3.4, and (b) is proved.

(a) Suppose $r < a_2 + b_1$. To get the lower bound of $n(\mathrm{comp}, C)$ (which in fact is $\geqslant 2$), we use (4.4) to produce several generic curves of $\mathrm{H}(d,g)$ which are generizations of C. Indeed let $m(a) = \min\{a_2, r\}$ and $m(b) = \min\{b_1, r\}$. By (4.4) there exist generizations C_0, C_1,…,$C_{m(a)+m(b)-r}$ such that $n(C_0) = (0, a_2 - m(a), b_1 + m(a) - r)$, $n(C_1) = (0, a_2 - m(a) + 1, b_1 + m(a) - r - 1)$,…, $n(C_{m(a)+m(b)-r}) = (0, a_2 + m(b) - r, b_1 - m(b))$. Since the curves C_i are ACM and have different postulations, they belong to $m(a)+m(b)-r+1$ different components, and we get the minimum number of irreducible components as stated in the proposition.

To see that the generic curve C' of *any* component containing C is ACM, we recall that $r'a'_2 = 0$ and $r'b'_1 = 0$ by Proposition 4.2 with notations as in the first part of the proof. Suppose $r' \neq 0$. Then $a'_2 = 0$ and $b'_1 = 0$. To get a contradiction, we remark that $\gamma_{C'}(v) = \gamma_C(v)$ for $v < c$, from which we get $h^0(\mathcal{I}_{C'}(c)) + b'_2 = h^0(\mathcal{I}_C(c)) - a_2$ since a_2 (resp. b'_2) is the only possibly non-vanishing graded Betti number of $I(C)$ (resp. $I(C')$) in degree c. Hence $h^0(\mathcal{I}_{C'}(c)) \leqslant h^0(\mathcal{I}_C(c)) - a_2$ and similarly we have the "dual" result $h^1(\mathcal{O}_{C'}(c)) \leqslant h^1(\mathcal{O}_C(c)) - b_1$. Adding the inequalities, we get

$$\chi(\mathcal{I}_{C'}(c)) + h^1(\mathcal{I}_{C'}(c)) \leqslant \chi(\mathcal{I}_C(c)) + h^1(\mathcal{I}_C(c)) - a_2 - b_1 < \chi(\mathcal{I}_C(c)),$$

i.e., a contradiction because $\chi(\mathcal{I}_{C'}(c)) = \chi(\mathcal{I}_C(c))$. Now using the fact that the generic curve C' of any irreducible component containing C is ACM and that $\mathrm{H}_{\gamma_{C'}, M(C')}$ is irreducible, we prove easily that $n(\mathrm{comp}, C) \leqslant r+1$ because there are at most $r+1$ different postulations $\gamma_{C'}$. Indeed since $M(C') = 0$, $\gamma_{C'}(v) = \gamma_C(v)$ for $v \neq c$ and

$$\gamma_{C'}(c) + \sigma_{C'}(c) = \chi(\mathcal{I}_{C'}(c)) = \chi(\mathcal{I}_C(c)) = \gamma_C(c) + \sigma_C(c) - r$$

where $\sigma_C(v) = h^1(\mathcal{O}_C(v))$, we see that the different choices of $\gamma_{C'}$ can happen in degree $v = c$ only, and that they are given by $\gamma_{C'}(c) = \gamma_C(c) - i$ where i is chosen among $\{0, 1, 2, \dots, r\}$.

Suppose $s(C) = e(C) = c$. Since in this case $\gamma_C(c) = a_2$ and $\sigma_C(c) = b_1$ by arguments as in the first part of the proof, we can easily limit the (at most) $r+1$ different choices of the postulation $\gamma_{C'}(c) = \gamma_C(c) - i$ above by choosing

$$m(a) \leqslant i \leqslant r - m(b)$$

i.e., $n(\mathrm{comp}, C)$ equals precisely $m(a)+m(b)-r+1$, and we are done. □

Example 4.7. — Now we reconsider some particular cases of Example 3.12, even though Proposition 4.6 is well adapted to treat the whole example in detail. Recall that for any triple (r, a_2, b_1) of natural numbers, there exists a smooth connected curve C with $n(C) = (r, a_2, b_1)$ and $s(C) = e(C) = c(C)$ by Example 3.12. In particular

(a) For every integer $r > 0$ there exists a smooth connected curve C, with triple $n(C) = (r, r, r)$, of degree d and genus g as in Example 3.12, which is contained in $r + 1$ irreducible components of $\mathrm{H}(d, g)_S$. Moreover the generic curves of all the components containing C are ACM.

(b) For every $r > 0$ there exists an obstructed, smooth connected curve with triple $(r, a_2, b_1) = (2t, t, t)$ or $(2t + 1, t, t)$, of degree d and genus g as given by Example 3.12, which belongs to a unique irreducible component of $\mathrm{H}(d, g)_S$ by Proposition 4.6. In particular the obstructed curve C with $(r, a_2, b_1) = (2, 1, 1)$ belongs to a *unique* irreducible component of $\mathrm{H}(32, 109)_S$, confirming what we saw in Example 3.12.

To prove Theorem 4.1 and Proposition 4.2 we need a lemma for deforming a module N, which basically is known (and related to [26], Prop. 2.1, p. 140). For our purpose it suffices to see that if we can lift a (three term) resolution with augmentation N to a *complex*, then the complex defines a flat deformation of N. In the case $N = I(C)$ where C has e.g. codimension 2 in $\mathbb{P}^3$, we also know that a deformation of an ideal $I(C)$ is again an ideal, *i.e.*,

Lemma 4.8. — *Let C be a curve in $\mathbb{P}^3$ whose homogeneous ideal $I(C)$ has a minimal resolution of the following form*

$$(L^\bullet) \quad 0 \to \bigoplus_i R(-i)^{\beta_{3,i}} \xrightarrow{\varphi} \bigoplus_i R(-i)^{\beta_{2,i}} \xrightarrow{\psi} \bigoplus_i R(-i)^{\beta_{1,i}} \to I(C) \to 0.$$

Let A be a finitely generated k-algebra, B the localization of A in a maximal ideal $\wp$, and suppose there exists a complex

$$(L_B^\bullet) \quad \bigoplus_i R_B(-i)^{\beta_{3,i}} \xrightarrow{\varphi_B} \bigoplus_i R_B(-i)^{\beta_{2,i}} \xrightarrow{\psi_B} \bigoplus_i R_B(-i)^{\beta_{1,i}},$$

$R_B = R \otimes_k B$, such that $L_B^\bullet \otimes_B (B/\wp) \cong L^\bullet$. Then $(L_B^\bullet)$ is acyclic, φ_B is injective and the cokernel of ψ_B is a flat deformation of $I(C)$ as an ideal (so $\operatorname{coker}(\psi_B) \subseteq R_B$ defines a flat deformation of $C \subseteq \mathbb{P}^3$ with constant postulation). Moreover for some $a \in A - \wp$, we can extend this conclusion to A_a via $\operatorname{Spec}(B) \hookrightarrow \operatorname{Spec}(A_a)$, i.e., there exists a flat family of curves $C_{\operatorname{Spec}(A_a)} \subseteq \mathbb{P}^3 \times \operatorname{Spec}(A_a)$ whose homogeneous ideal $I(C_{A_a})$ has a resolution (not necessarily minimal) of the form

$$\begin{aligned}(L_{A_a}^\bullet) \quad 0 \to \bigoplus R_{A_a}(-i)^{\beta_{3,i}} \to \bigoplus R_{A_a}(-i)^{\beta_{2,i}} \\ \to \bigoplus R_{A_a}(-i)^{\beta_{1,i}} \to I(C_{A_a}) \to 0.\end{aligned}$$

Proof (sketch). — If $E = \operatorname{coker}\varphi$ and $E_B = \operatorname{coker}\varphi_B$, then one proves easily that $E_B \otimes_B (B/\wp) = E$, $\operatorname{Tor}_1(E_B, B/\wp) = 0$ and that φ_B is injective. By the local criterion of flatness, E_B is a flat deformation of E. Letting $Q_B = \operatorname{coker}(E_B \to \oplus_i R_B(-i)^{\beta_{1,i}})$, we can argue as we did for E_B to see that Q_B is a flat deformation of $I(C)$ and that $L_B^\bullet$ augmented by Q_B is exact.

To prove that Q_B is an ideal in R_B, we can use the isomorphisms $\mathrm{H}^{i-1}(\mathcal{N}_C) \cong \mathrm{Ext}^i_{\mathcal{O}_\mathbb{P}}(\tilde{I}, \tilde{I})$ for $i = 1, 2$, interpreted via deformation theory and repeatedly applied to $B_{i+1} \to B_i$ for $i \geqslant 1$ ($B_i = B/\wp^i$), to see that a deformation of the $\mathcal{O}_\mathbb{P}$-Module $\tilde{I}$ (such as $\tilde{Q_B}$) corresponds to a deformation of the curve C in the usual way, *i.e.*, via the cokernel of $\tilde{i}$: $\tilde{Q_B} \to \tilde{R_B}$. We get in particular a morphism $\mathrm{H}^0_*(\tilde{i})$: $Q_B \to R_B$ which proves what we want (one may give a direct proof using Hilbert-Burch theorem (*cf.* [26], page 37-38)).

Finally we easily extend the morphism i and any morphism of the resolution $L_B^\bullet$ to be defined over $A_{a'}$, for some $a' \in A - \wp$ (such that $L^\bullet_{A_{a'}}$ is a complex). By shrinking $\operatorname{Spec} A_{a'}$ to $\operatorname{Spec} A_a$, $a \in A - \wp$, we get the exactness of the complex and the flatness of $I(C_{A_a})$ because these properties are open. □

Proof of Theorem 4.1. — Suppose that F has rank s and consider the s by s submatrix $M(\psi)$ of ψ in

$$0 \to L_4 \xrightarrow{\sigma\oplus 0\oplus 0} L_3 \oplus F_2' \oplus F \xrightarrow{\psi} F_1' \oplus F \to I(C) \to 0$$

which corresponds to $F \to F$. As in the "Lemma de générisation simplifiantes" ([26], page 189), we can change the $0's$ on the diagonal of $M(\psi)$ to some $\lambda_1, \dots, \lambda_s$ where the $\lambda_i's$ are indeterminates of degree zero. Keeping $\sigma \oplus 0 \oplus 0$ unchanged, we still have a complex which by Lemma 4.8 implies the existence a flat family of curves over $\operatorname{Spec}(A_a)$, $A = k[\lambda_1, \dots, \lambda_s]$, for some $a \in A - (\lambda_1, \dots, \lambda_s)$. Let $\lambda := \Pi_{i=1}^s \lambda_i$ be the product. Since any curve C' of the family given by $\operatorname{Spec}(A_{\lambda a})$ has a resolution where F is redundant (F, and only F, is missing in its *minimal* resolution), and since we may still interpret the Rao module $M(C')$ as $\ker \mathrm{H}^3_*(\tilde{\sigma} \oplus 0 \oplus 0)$ with $\sigma \oplus 0 \oplus 0$ as above (so the whole family given by $\operatorname{Spec}(A_a)$ has constant Rao modules), we conclude easily. □

Remark 4.9. — Slightly extending the proof and using Bolondi's result on the irreducibility of $\mathrm{H}_{\gamma,M}$ ([3]), one may prove that set U of points (C) of the scheme $\mathrm{H}_{\gamma,M}$ whose modules F_2 and F_1 of the minimal resolution (4.1) of $I(C)$ are without common direct free factors, form an *open* (and nonempty if a curve with minimal resolution (4.1) exists) irreducible subset of $H_{\gamma,M}$.

Proof of Proposition 4.2.

(a) Since we have the assumption that $M \cong M' \oplus M_{[t]}$ as *R-modules*, the minimal resolution (3.1) of M is given as the direct sum of the resolution of

M' and the one of $M_{[t]}$ which is "r-times" the Koszul resolution associated with the regular sequence $\{X_0, X_1, X_2, X_3\}$. The matrix associated to $\sigma_{[t]}$ (resp. $\sigma = \sigma' \oplus \sigma_{[t]}$) will have the form

$$\begin{bmatrix} \underline{X} & 0 & \dots & 0 \\ 0 & \underline{X} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \underline{X} \end{bmatrix} \text{(resp. } \begin{bmatrix} \sigma' & 0 \\ 0 & \sigma_{[t]} \end{bmatrix}) \tag{4.5}$$

where $\underline{X}$ is $(X_0, X_1, X_2, X_3)^T$ and each "row" in the left matrix is a $4 \times r$ matrix, etc. Let $\eta_j \colon R(-t-4) \to L'_4 \oplus R(-t-4)^r$ be the map into the j-th direct factor of $R(-t-4)^r$, $1 \leqslant j \leqslant r$, and let

$$\pi_i : L'_3 \oplus R(-t-3)^{4r} \oplus P'_2 \oplus R(-t-4)^{b_1} \to R(-t-4)$$

be the projection onto the i-th factor of $R(-t-4)^{b_1}$, $1 \leqslant i \leqslant b_1$. Similar to what was observed by Martin Deschamps and Perrin in the case $M \cong k$ ([26], page 189) we can change the 0 component in the matrix of $\sigma \oplus 0$ which corresponds to $\pi_i \eta_j \colon R(-t-4) \to R(-t-4)$, to some indeterminate of degree zero. To get a complex we need to change four columns of the matrix A associated to $L_3 \oplus F_2 \to F_1$ as follows. Let $r_1 := \operatorname{rank} F_1$ and look to the column (a_k), $1 \leqslant k \leqslant r_1$, of A which corresponds to the map $R(-t-4) \to F_1$ from the i-th factor of $R(-t-4)^{b_1}$. Put $a_k = \sum_{l=0}^{3} \gamma^i_{k,l} X_l$ for every $1 \leqslant k \leqslant r_1$. Since the resolution is minimal, such $\gamma^i_{k,l}$ exist, but they are not necessarily unique. Since the column of the matrix of $\sigma \oplus 0$ which corresponds to η_j consists of only 0's and $\underline{X}$ (*cf.* (4.5)) there are precisely four columns $[H^j_{k,0}, H^j_{k,1}, H^j_{k,2}, H^j_{k,3}]$, $1 \leqslant k \leqslant r_1$, of A satisfying $\sum_{l=0}^{3} H^j_{k,l} X_l = 0$ for every k which may contribute to the composition $(\sigma \oplus 0)\eta_j$. Now if we change the trivial map $\pi_1 \eta_1$ to the multiplication by an indeterminate λ_1 and simultaneously change the four columns $[H^1_{k,0}, H^1_{k,1}, H^1_{k,2}, H^1_{k,3}]$ of A to $[H^1_{k,0} - \gamma^1_{k,0}\lambda_1, H^1_{k,1} - \gamma^1_{k,1}\lambda_1, H^1_{k,2} - \gamma^1_{k,2}\lambda_1, H^1_{k,3} - \gamma^1_{k,3}\lambda_1]$, leaving the rest of A unchanged, we still get that (4.2) defines a complex. We can proceed by simultaneously changing the 0 component of $\pi_2 \eta_2$ to λ_2 and the corresponding four columns of the matrix A as described above, etc. Put $\lambda := \Pi_{i=1}^{m_1} \lambda_i$. By Lemma 4.8 we get a flat irreducible family of curves C' over $\operatorname{Spec}(k[\lambda_1, \dots, \lambda_{m_1}]_a)$, for some $a \in A - (\lambda_1, \dots, \lambda_{m_1})$, having the same (not necessarily minimal) resolution, hence the same postulation, as C. Since λ is invertible in $\operatorname{Spec}(k[\lambda_1, \dots, \lambda_{m_1}]_{\lambda \cdot a})$, we can remove redundant factors of the resolution of $I(C')$ in this open set. Since $M(C') \cong \ker \mathrm{H}^3_*(\tilde{\sigma} \oplus 0 \oplus 0)$, we have a generization C' with properties as claimed in Proposition 4.2.

Finally using Remark 3.5 for $v=0$, the assumption on L'_2 shows that (3.5) holds and hence we conclude by the left formula of (3.6).

(b) We will prove (b) by linking C to a C_l via a complete intersection of two surfaces of degrees f and g satisfying $\mathrm{H}^1(\mathcal{I}_C(v))=0$ for $v=f,g,f-4$ and $g-4$, and then apply (a) to C_l. To see that C_l satisfies the assumption of (a), first note that $M(C_l)$ admits a decomposition $M(C_l)\cong M'(C_l)\oplus M_{[f+g-4-t]}$ as R-modules. Indeed $M=M(C)$ satisfies the duality

$$M(C_l)\cong \mathrm{Ext}^4_R(M,R)(-f-g)\cong \mathrm{Hom}_k(M,k)(-f-g+4), \tag{4.6}$$

(*cf.* [37] and [30], p. 133). If we let $M'(C_l):=\mathrm{Ext}^4_R(M',R)(-f-g)$, then the decomposition $M\cong M'\oplus M_{[t]}$ translates to

$$\begin{aligned} M(C_l) &\cong \mathrm{Ext}^4_R(M',R)(-f-g)\oplus \mathrm{Ext}^4_R(M_{[t]},R)(-f-g)\\ &\cong M'(C_l)\oplus M_{[t]}(2t+4-f-g)\end{aligned}$$

since $M_{[t]}(t)\cong \mathrm{Ext}^4_R(M_{[t]}(t),R(-4))$ by the self-duality of the minimal resolution of $M_{[t]}(t)$. Finally since $M_{[t]}(2t+4-f-g)$ is supported in degree $f+g-4-t$, we may write the module $M_{[t]}(2t+4-f-g)$ as $M(C_l)_{[f+g-4-t]}:=M_{[f+g-4-t]}$. Next to see that direct free part F_1 generated in degree t in the resolution of $I(C)$, is equal (at least dimensionally) to the corresponding part in degree $f+g-4-t$ of $F_2(C_l)(4)$ in the minimal resolution of $I(C_l)$ of the linked curve C_l, we remark that since the isomorphism of (2.18) is given by the duality used in (4.6), it must commute with their decomposition as R-modules, *i.e.*, we have

$$ {}_0\mathrm{Hom}_R(I(C),M(C)_{[t]})\cong {}_0\mathrm{Hom}_R(M(C_l)_{[f+g-4-t]},E(C_l)). \tag{4.7}$$

Then we conclude by (3.4) and (3.6) provided we can use Remark 3.5 for $v=0$. Indeed if $L^*:=\mathrm{Hom}_R(L,R)$, we have an exact sequence

$$\to (L'_2)^*\to (L'_3)^*\to (L'_4)^*\to \mathrm{Ext}^4_R(M',R)\cong M'(C_l)(f+g)\to 0.$$

Since L'_2 has no direct free factor of degree t, it follows that $(L'_2)^*(-f-g)$ has no direct free factor of degree $f+g-t$, *i.e.*, we have ${}_{-4}\mathrm{Hom}_R((L'_2)^*(-f-g),M_{[f+g-4-t]})=0$ and Remark 3.5 applies. Now using (a) to the linked curve C_l with $m_2=m_1$, we get a generization of C'_l with constant postulation where $R(-f-g+t)^{m_1}$ is “removed” in its minimal resolution. A further linkage, using a complete intersection of the same type as in the linkage above (such a complete intersection exists by [22], Cor. 3.7) and the formula (4.7) (replacing C and C_l by C' and C'_l respectively), we get the desired generization C', and we are done. □

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