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BOUNDED ALMOST GLOBAL SOLUTIONS FOR NON HAMILTONIAN SEMI-LINEAR KLEIN-GORDON EQUATIONS WITH RADIAL DATA ON COMPACT REVOLUTION HYPERSURFACES

by Jean-Marc DELORT & Jérémie SZEFTEL

Abstract. — This paper is devoted to the proof of almost global existence results for Klein-Gordon equations on compact revolution hypersurfaces with non-Hamiltonian nonlinearities, when the data are smooth, small and radial. The method combines normal forms with the fact that the eigenvalues associated to radial eigenfunctions of the Laplacian on such manifolds are simple and satisfy convenient asymptotic expansions.

Résumé. — Cet article est consacré à la preuve de résultats d'existence presque globale pour des équations de Klein-Gordon sur des hypersurfaces compactes de révolution avec des non-linéarités non hamiltoniennes, lorsque les données sont petites, régulières et radiales. La méthode repose sur l'utilisation de formes normales et sur le fait que les valeurs propres associées à des fonctions propres radiales du Laplacien sont simples et vérifient des propriétés de séparation convenables.

0. Introduction

Let (M,g) be a compact Riemannian manifold without boundary, V a nonnegative potential on M, $m \in]0,+\infty[$, and consider a nonlinear Klein-Gordon equation on M

$$(\partial_t^2 - \Delta_g + V + m^2)u = f(x,u,\partial_t u) \tag{0.1}$$

where f is a polynomial in $(u,\partial_t u)$ with smooth dependence in x. We are interested in questions of almost global existence and H^s-boundedness for (0.1) when (M,V) are rotationally symmetric around some axis, the

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Cauchy data are smooth, small and radial, and m is taken outside an exceptional set of zero measure. This problem has been studied in the case of *Hamiltonian nonlinearities* (*i.e.*, nonlinearities $f(x,u)$ independent of $\partial_t u$) by Bourgain [3], Bambusi [1], Bambusi and Grébert [2] on a bounded interval with boundary conditions, or on the circle. In this case, conservation of H^1 norm implies immediately global existence in H^1, and these authors show for almost all values of m boundedness of H^s norms of the solution (for any s) over intervals of time of length $c_N\epsilon^{-N}$ (for any N), where ϵ is the size of the Cauchy data. Their method relies on construction of approximate action-angle variables for the Hamiltonian formulation of the equation.

On the other hand, a considerable amount of work has been done since the 80's on the problem of long time existence for solutions to wave or Klein-Gordon nonlinear equations on $\mathbb{R}^d$ with data which are smooth, small, and rapidly decaying at infinity. We refer to the introduction of [4] for bibliographical references on that problem. Let us just recall that in this framework, global existence holds true when linear solutions decay rapidly enough at infinity – so that the nonlinearity may be viewed as a short range perturbation of the linear equation. When the nonlinearity is a long range perturbation, global solutions still exist if the nonlinearity satisfies a structure condition, introduced under the name of null condition by Klainerman [8] for the wave equation in three space dimensions. Examples of null conditions for one dimensional Klein-Gordon equations have been found by Moriyama [9], and Delort [4] proved global existence under a general null condition (see also the recent work of Sunagawa [11]). All these null conditions have no relation with a possible Hamiltonian structure of the equation. Actually they appear in most of the above works through a method of normal forms, initially introduced in the framework of nonlinear Klein-Gordon equations by Shatah [10]. This brings the natural question whether problems of type (0.1) have almost global H^s-bounded solutions for more general nonlinearities than the Hamiltonian ones considered by Bourgain, Bambusi, Bambusi-Grébert, and for more general manifolds than the circle or the interval. We prove in this paper that such a result holds true for essentially one dimensional problems – *i.e.*, cases when $M=\mathbb{S}^1$ or M is a revolution hypersurface with radial potential and data – and for nonlinearities satisfying a "null condition" allowing non Hamiltonian contributions. Actually, we consider nonlinearities which contain even powers of $\partial_t u$ (If we accept nonlocal nonlinearities, we can even permit dependence of f in $\sqrt{-\Delta_g+V}u$).

Our method of proof is an extension of the one we used in our papers [6, 5], to study long time existence for equations of type (0.1) when M is a sphere or a Zoll manifold, and when the potential and the data are not assumed rotationally invariant. We introduced a pseudo-Hamiltonian $E_s(u)$, equivalent to the H^s norm on a neighborhood of zero, and we proved, for convenient nonlinearities vanishing at order p at 0 that, for almost all values of the mass m, the solution exists and stays bounded in H^s $(s \gg 1)$ on intervals of time of length at least $c\epsilon^{-2p+1}$ (where ϵ is the size of the data). This is better that the time of existence given by local existence theory, namely $c\epsilon^{-p+1}$, but is far from providing almost global solutions. The point is that the construction of the pseudo-Hamiltonian E_s makes appear new nonlinear contributions, which are of higher order, and whose structure is more involved than the one of the right hand side of (0.1). This prevents one from iterating the method to construct a pseudo-Hamiltonian that can be controlled over intervals of time of arbitrary length. Actually, the difficulties are related to the structure of the spectrum of $-\Delta_g + V$ on a Zoll manifold: the eigenvalues are grouped in well-separated clusters, and the pseudo-Hamiltonian E_s is a multilinear expression in the spectral projections $\Pi_\lambda u$ of u on the different clusters. The right hand side of $\frac{d}{dt}E_s(u(t,\cdot))$ has the same type of structure, except that it involves products of images by spectral projections of products of $\Pi_\lambda u$'s. This prevents one from modifying E_s to cancel out those new contributions.

On the other hand, for rotationally symmetric problems (or one dimensional ones), one has only to cope with those eigenvalues which correspond to symmetric eigenfunctions. Under convenient assumptions, this singles out a sequence of well separated eigenvalues, and allows to construct E_s in terms of Fourier coefficients of u corresponding to the associated eigenfunctions. This greatly simplifies the nonlinear expressions and allows one, exploiting some of the results of our preceding paper [5], to construct a pseudo-Hamiltonian controlled over intervals of arbitrary length.

From a technical point of view, we have to establish some separation properties of the eigenvalues of $-\Delta_g + V$ corresponding to eigenfunctions satisfying convenient symmetry assumptions. For problems on $M = \mathbb{S}^1$ or on a hypersurface of revolution that does not meet the axis of symmetry, these properties follow readily from the well known spectral theory of the Hill operator. When M is a surface of revolution meeting the axis, the spectral problem may be reduced to the study of the eigenvalues for an elliptic second order operator on $[0,1]$, degenerated at the boundary. Since we have been unable to find in the literature references to the spectral results we need, we provide a proof of them in the third section of the paper. Actually, we have to get asymptotics for the eigenvalues of such a degenerate problem. We prove them combining WKB expansions for solutions of the corresponding singular ODE with a quantization condition.

1. Main results

1.1. Statement of the main theorem

Consider (M, g) a compact Riemannian manifold without boundary, of dimension $d \geqslant 1$. Denote by Δ_g its Laplace-Beltrami. Let W be a closed subspace of $L^2(M)$ such that Δ_g restricted to W is a self-adjoint operator. Let $V : M \to \mathbb{R}$ be a smooth nonnegative potential such that $x \to V(x)w(x)$ belongs to W whenever $w \in W$ and set

$$P = \sqrt{-\Delta_g + V}. \tag{1.1}$$

We shall assume that the spectrum of $P|_W$ consists of simple eigenvalues $(\lambda_n)_{n\geqslant 1}$ having the following asymptotic expansion as $n \to +\infty$

$$\lambda_n = \frac{2\pi}{\tau} n + \alpha + \mathcal{O}\left(\frac{1}{n}\right). \tag{1.2}$$

where $\tau > 0, \alpha \in \mathbb{R}$.

If $I \subset \mathbb{R}$ is an interval, u is a smooth real function defined on $I \times M$, denote by f the function defined by

$$f(x, u, \partial_t u, Pu) = \sum_{2\leqslant q\leqslant \bar{q}} f_q(x, u, \partial_t u, Pu), \tag{1.3}$$

where

(1.4) f_q is a sum of homogeneous expressions of degree q of type $a(x)u^\ell(\partial_t u)^k(Pu)^j$ with a in $C^\infty(M)$ and real valued, ℓ, k, j natural integers with $\ell + k + j = q$ and with k even.

We assume furthermore that

$$f(x, u, v, w) \in W \text{ for all } (x, u, v, w) \in M \times (W \cap C^\infty(M))^3. \tag{1.5}$$

We shall look for a solution u defined on $]-T, T[\times M$ of the following problem

$$\begin{aligned} (\partial_t^2 - \Delta_g + V + m^2)u &= f(x, u, \partial_t u, Pu) \\ u|_{t=0} &= \epsilon u_0 \\ \partial_t u|_{t=0} &= \epsilon u_1 \end{aligned} \tag{1.6}$$

where $m > 0, \epsilon > 0$ is a small parameter, $u_0 \in H^{s+1}(M) \cap W, u_1 \in H^s(M) \cap W$ are given real valued functions. Our main result is the following:

Theorem 1.1. — *Assume that condition* (1.2) *holds true. There is a zero measure subset* $\mathcal{N}$ *of* $]0, +\infty[$ *satisfying the following: for any function* f *of form* (1.3) *satisfying* (1.4) *and* (1.5), *for any* $m \in]0, +\infty[- \mathcal{N}$, *for any* $N \in \mathbb{N}$, *there are* $\epsilon_0 > 0, c > 0, s_0 \in \mathbb{N}$ *such that for any* $s \geqslant s_0$, *any pair* (u_0, u_1) *of real valued functions belonging to the unit ball of* $H^{s+1}(M) \times H^s(M)$ *and to* $W \times W$, *any* $\epsilon \in]0, \epsilon_0[$, *problem* (1.6) *has a unique solution*

$$u \in C^0(]-T_\epsilon, T_\epsilon[, H^{s+1}(M)) \cap C^1(]-T_\epsilon, T_\epsilon[, H^s(M)) \tag{1.7}$$

with $T_\epsilon \geqslant c_N \epsilon^{-N}$. *Moreover, the solution is uniformly bounded in* $H^{s+1}(M)$ *on* $]-T_\epsilon, T_\epsilon[$ *and* $\partial_t u$ *is uniformly bounded in* $H^s(M)$ *on the same interval.*

1.2. Application to almost global existence of radial solutions

Let us apply Theorem 1.1 in four situations.

Let M be the circle $\mathbb{S}^1$ identified to $[-\pi, \pi]$ with periodic boundary conditions, let V be a smooth nonnegative odd function, and let g be the canonical metric on $\mathbb{S}^1$. Define W as the set of all the odd functions in $L^2(\mathbb{S}^1)$. Then, the spectrum of $P|_W$ coincides with the spectrum of P on $[0, \pi]$ with Dirichlet boundary conditions. Therefore, the spectrum of $P|_W$ consists of simple eigenvalues and (1.2) holds (see for example chapter 4 of [7]). Thus, we have the following corollary:

Corollary 1.2. — *Let* $M = \mathbb{S}^1$, *let* V *be a smooth nonnegative odd function, let* g *be the canonical metric on* $\mathbb{S}^1$ *and let* W *be the set of all the odd functions in* $L^2(\mathbb{S}^1)$. *Assume* f *satisfies* $f(-x, -u, -v, -w) = -f(x, u, v, w)$ *for all* $(x, u, v, w) \in \mathbb{S}^1 \times \mathbb{R}^3$. *Then Theorem 1.1 holds true.*

Assume now that M and V satisfy the following assumptions:

(1.8) M is a hypersurface of $\mathbb{R}^d, d \geqslant 3,$ with coordinates $(x_1, \ldots, x_d) = (x', x_d)$, given by an equation of the form $\Phi(|x'|, x_d) = 0,$ where $\{(r, x_d) \in]0, +\infty[\times \mathbb{R} : \Phi(r, x_d) = 0\}$ is a simple closed curve, and where $\Phi(r, -x_d) = \Phi(r, x_d)$ for any $r > 0, x_d \in \mathbb{R}$.

(1.9) V is a smooth nonnegative function which is invariant under the action of the rotations with axis x_d, and even with respect to x_d.

PROPOSITION 1.3. — *Let M and V be chosen as in* (1.8) *and* (1.9)*. Let W consist of all functions in $L^2(M)$ which are invariant under the action of the rotations with axis x_d and even with respect to x_d. Then the spectrum of $P|_W$ consists of simple eigenvalues satisfying* (1.2).

The proof of Proposition 1.3 is postponed to section 3. We have the following corollary:

COROLLARY 1.4. — *Let M and V be chosen as in* (1.8) *and* (1.9) *and let W be chosen as in Proposition 1.3. Let f be such that $f(Rx, u, v, w) = f(x, u, v, w)$ for all $(x, u, v, w) \in M \times \mathbb{R}^3$ and all rotations R with axis x_d, and $f((x', -x_d), u, v, w) = f((x', x_d), u, v, w)$ for all $(x, u, v, w) \in M \times \mathbb{R}^3$ where $x = (x', x_d)$. Then Theorem 1.1 holds true.*

PROPOSITION 1.5. — *Let M and V be chosen as in* (1.8) *and* (1.9)*. Let W consist of all functions in $L^2(M)$ which are invariant under the action of the rotations with axis x_d and odd with respect to x_d. Then the spectrum of $P|_W$ consists of simple eigenvalues satisfying* (1.2).

The proof of Proposition 1.5 is postponed to section 3. We have the following corollary:

COROLLARY 1.6. — *Let M and V be chosen as in* (1.8) *and* (1.9) *and let W be chosen as in Proposition 1.5. Let f be such that $f(Rx, u, v, w) = f(x, u, v, w)$ for all $(x, u, v, w) \in M \times \mathbb{R}^3$ and all rotations R with axis x_d, and*

$$f((x', -x_d), -u, -v, -w) = -f((x', x_d), u, v, w) \quad \text{for all } (x, u, v, w) \in M \times \mathbb{R}^3$$

where $x = (x', x_d)$. Then Theorem 1.1 holds true.

Finally, assume that M and V satisfy the following assumptions:

(1.10) M is a hypersurface of $\mathbb{R}^d, d \geqslant 3,$ which is invariant under the action of the rotations with axis x_d. Furthermore, M intersects the x_d axis at two points.

(1.11) V is a smooth nonnegative function which is invariant under the action of the rotations with axis x_d.

PROPOSITION 1.7. — *Let M and V be chosen as in* (1.10) *and* (1.11)*. Let W consist of all functions in $L^2(M)$ which are invariant under the action of the rotations with axis x_d. Then the spectrum of $P|_W$ consists of simple eigenvalues satisfying* (1.2).

The proof of Proposition 1.7 is postponed to section 3. We have the following corollary:

Corollary 1.8. — *Let M and V be chosen as in* (1.10) *and* (1.11) *and let W be chosen as in Proposition 1.7. Let f such that $f(Rx,u,v,w) = f(x,u,v,w)$ for all $(x,u,v,w) \in M \times \mathbb{R}^3$ and all rotations R. Then Theorem 1.1 holds true.*

2. Proof of Theorem 1.1

For $\lambda_1, \dots, \lambda_p$ p nonnegative real numbers define respectively the second and third largest elements of this family by

$$
\begin{aligned}
\max_2(\lambda_1,\dots,\lambda_p) &= \max\big(\{\lambda_1,\dots,\lambda_p\} - \{\lambda_{i_0}\}\big) \\
\max_3(\lambda_1,\dots,\lambda_p) &= \max\big(\{\lambda_1,\dots,\lambda_p\} - \{\lambda_{i_0},\lambda_{i_1}\}\big)
\end{aligned} \tag{2.1}
$$

where i_0 and i_1 are indices such that

$$\lambda_{i_0} = \max(\lambda_1,\dots,\lambda_p), \lambda_{i_1} = \max_2(\lambda_1,\dots,\lambda_p).$$

Set also

$$
\begin{aligned}
\mu(\lambda_1,\dots,\lambda_p) &= \max_3(\lambda_1,\dots,\lambda_p) + 1 \\
S(\lambda_1,\dots,\lambda_p) &= \sum_{\ell=1}^{p} \Big[\lambda_\ell - \sum_{j\neq\ell}\lambda_j\Big]_+ + \mu(\lambda_1,\dots,\lambda_p)
\end{aligned} \tag{2.2}
$$

where $[a]_+ = \max(a,0)$ and where, by convention, $\mu(\lambda_1,\lambda_2) = 1$ in the case $p = 2$. If for instance λ_p and λ_{p-1} are the largest two numbers among $\lambda_1,\dots,\lambda_p$ then

$$
\begin{aligned}
\mu(\lambda_1,\dots,\lambda_p) &\sim 1 + \lambda_1 + \cdots + \lambda_{p-2} \\
S(\lambda_1,\dots,\lambda_p) &\sim |\lambda_p - \lambda_{p-1}| + \lambda_1 + \cdots + \lambda_{p-2} + 1.
\end{aligned} \tag{2.3}
$$

2.1. Definition and properties of multilinear forms

For $n \in \mathbb{N}^*$, let φ_n denote a normalized eigenfunction associated to the eigenvalue λ_n of $P|_W$. Denote by $\mathcal{E}$ the span of the φ_n's, $n \in \mathbb{N}^*$.

Definition 2.1. — *Let $s \in [0,+\infty[$, $\nu \in [0,+\infty[, N \in \mathbb{N}, p \in \mathbb{N}, p \geqslant 2$. We denote by $S_p^{s,\nu,N}$ the algebra of all symbols $(\mathbb{N}^*)^p \to \mathbb{R}$ such that there is $C > 0$ satisfying: for every $(n_1,\dots,n_p) \in (\mathbb{N}^*)^p$ one has*

(2.4)

$$|a(n_1,\dots,n_p)| \leqslant C \max(n_1,\dots,n_p)^s \max\nolimits_2(n_1,\dots,n_p)^s \frac{\mu(n_1,\dots,n_p)^{\nu+N}}{S(n_1,\dots,n_p)^N}.$$

The best constant C in (2.4) defines a norm $\|a\|_{S_p^{s,\nu,N}}$. We set $S_p^{s,\nu,+\infty} = \bigcap_{N\in\mathbb{N}} S_p^{s,\nu,N}$.

Definition 2.2. — *Let $s \in [0,+\infty[$, $\nu \in [0,+\infty[, N \in \mathbb{N}, p \in \mathbb{N}, p \geqslant 2$. For a in $S_p^{s,\nu,N}$, we denote by $\mathcal{L}(a)$ the p-linear form on $\mathcal{E}\times\cdots\times\mathcal{E}$ defined for every $u_1,\dots,u_p \in \mathcal{E}$ by*

$$\mathcal{L}(a)(u_1,\dots,u_p) = \sum_{n_1,\dots,n_p} a(n_1,\dots,n_p)\langle u_1,\varphi_{n_1}\rangle\cdots\langle u_p,\varphi_{n_p}\rangle, \tag{2.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product on $L^2(M)$.

Let us give an example of an operator with symbol in $S_p^{s,\nu,N}$.

Example 2.3. — Let $s \in [0,+\infty[$. For every $u_1,u_2 \in \mathcal{E}$, let $M_0(u_1,u_2)$ be defined by

$$M_0(u_1,u_2) = \sum_n n^{2s}\langle u_1,\varphi_n\rangle\langle u_2,\varphi_n\rangle. \tag{2.6}$$

The bilinear form $(u_1,u_2) \to M_0(u_1,u_2)$ is equal to $\mathcal{L}(a)$ where a is in $S_2^{s,0,+\infty}$ and is defined by $a(n_1,n_2) = n_1^s n_2^s \mathbf{1}_{\{n_1=n_2\}}$.

Lemma 2.4. — *Let $s \in [0,+\infty[$, $\nu \in [0,+\infty[, N \in \mathbb{N}, p \in \mathbb{N}, p \geqslant 2, \ell \in \mathbb{N}, 1 \leqslant \ell \leqslant p$. Let $m \in]0,+\infty[$ and set $\Lambda_m = \sqrt{-\Delta_g + V + m^2}$.*

(i) For a in $S_p^{s,\nu,N}$, we denote by M the p-linear form on $\mathcal{E}\times\cdots\times\mathcal{E}$ satisfying: for every $u_1,\dots,u_p \in \mathcal{E}$ one has

$$M(u_1,\dots,u_p) = \mathcal{L}(a)(u_1,\dots,\Lambda_m^{-1}u_\ell,\dots,u_p). \tag{2.7}$$

Then there exists $\tilde a$ in $S_p^{s,\nu,N}$ such that $M = \mathcal{L}(\tilde a)$. Furthermore, $\|\tilde a\|_{S_p^{s,\nu,N}} \leqslant m^{-1}\|a\|_{S_p^{s,\nu,N}}$.

(ii) For a in $S_p^{s,\nu,N}$, we denote by M the p-linear form on $\mathcal{E}\times\cdots\times\mathcal{E}$ satisfying: for every $u_1,\dots,u_p \in \mathcal{E}$ one has

$$M(u_1,\dots,u_p) = \mathcal{L}(a)(u_1,\dots,P\Lambda_m^{-1}u_\ell,\dots,u_p). \tag{2.8}$$

Then there exists $\tilde a$ in $S_p^{s,\nu,N}$ such that $M = \mathcal{L}(\tilde a)$. Furthermore, $\|\tilde a\|_{S_p^{s,\nu,N}} \leqslant \|a\|_{S_p^{s,\nu,N}}$.

Proof.

(i) In this case, an explicit computation shows that

$$\tilde{a}(n_1,\dots,n_p) = \frac{a(n_1,\dots,n_p)}{\sqrt{\lambda_{n_\ell}^2 + m^2}}$$

which yields $\|\tilde{a}\|_{S_p^{s,\nu,N}} \leqslant m^{-1}\|a\|_{S_p^{s,\nu,N}}$.

(ii) In this case, an explicit computation shows that

$$\tilde{a}(n_1,\dots,n_p) = \sqrt{\frac{\lambda_{n_\ell}^2}{\lambda_{n_\ell}^2 + m^2}}\,a(n_1,\dots,n_p)$$

which yields $\|\tilde{a}\|_{S_p^{s,\nu,N}} \leqslant \|a\|_{S_p^{s,\nu,N}}$. □

Proposition 2.5. — *Let $\nu \in [0,+\infty[$, $s \in \mathbb{R}$, $s > \nu+3/2$, $N \in \mathbb{N}$, $N > 1$. Then for any $a \in S_p^{s,\nu,N}$, $\mathcal{L}(a)$ extends as a bounded multilinear form on $H^s(M) \times \cdots \times H^s(M)$. Moreover for any $s_0 \in]\nu+3/2, s]$, there is $C > 0$ such that for any $a \in S_p^{s,\nu,N}$ and any $u_1,\dots,u_p \in H^s(M)$*

$$|\mathcal{L}(a)(u_1,\dots,u_p)| \leqslant C\|a\|_{S_p^{s,\nu,N}} \sum_{1\leqslant j<k\leqslant p}\Big[\|u_j\|_{H^s}\|u_k\|_{H^s}\prod_{\ell\neq j,k}\|u_\ell\|_{H^{s_0}}\Big]. \tag{2.9}$$

Proof. — The proof is a modification of the one of Proposition 4.4 in [6]. We give it for the convenience of the reader. We write

$$|\mathcal{L}(a)(u_1,\dots,u_p)| \leqslant \sum_{n_1,\dots,n_p} |a(n_1,\dots,n_p)||\langle u_1,\varphi_{n_1}\rangle|\cdots|\langle u_p,\varphi_{n_p}\rangle|. \tag{2.10}$$

By (2.4) the above sum is smaller than

$$C\sum_{n_1,\dots,n_p} \max(n_1,\dots,n_p)^s \max\nolimits_2(n_1,\dots,n_p)^s \frac{\mu(n_1,\dots,n_p)^{\nu+N}}{S(n_1,\dots,n_p)^N} \times |\langle u_1,\varphi_{n_1}\rangle|\cdots|\langle u_p,\varphi_{n_p}\rangle|. \tag{2.11}$$

By symmetry we may restrict ourselves to indices satisfying $n_1 \leqslant \cdots \leqslant n_p$. We have

$$\max(n_1,\dots,n_p) = n_p,\ \max\nolimits_2(n_1,\dots,n_p) = n_{p-1}$$

and

$$\mu(n_1,\dots,n_p) \sim 1 + n_{p-2}$$
$$S(n_1,\dots,n_p) \sim |n_p - n_{p-1}| + n_{p-2} + 1.$$

Let $\kappa > 1$ as close to 1 as wanted. We deduce from the above equivalences

$$\sum_{n_p} S(n_1,\dots,n_p)^{-\kappa} \leqslant C,\ \sum_{n_{p-1}} S(n_1,\dots,n_p)^{-\kappa} \leqslant C. \tag{2.12}$$

We estimate the contributions of the sum for $n_1 \leqslant \cdots \leqslant n_p$ to (2.11) by

$$(2.13)\quad C\Bigg(\sum_{n_1\leqslant\cdots\leqslant n_p} n_p^{2s}|\langle u_p,\varphi_{n_p}\rangle|^2\frac{\mu^{\nu+N}}{S^N}\prod_1^{p-2}|\langle u_j,\varphi_{n_j}\rangle|\Bigg)^{1/2} \times\Bigg(\sum_{n_1\leqslant\cdots\leqslant n_p} n_{p-1}^{2s}|\langle u_{p-1},\varphi_{n_{p-1}}\rangle|^2\frac{\mu^{\nu+N}}{S^N}\prod_1^{p-2}|\langle u_j,\varphi_{n_j}\rangle|\Bigg)^{1/2}.$$

Using (2.12) to handle the n_{p-1} sum, we bound the first factor by $C\|u_p\|_{H^s}\prod_1^{p-2}\|u_j\|_{H^{s_0}}^{1/2}$ if $s_0 > \nu+\kappa+1/2$. Using (2.12) to handle the n_p sum, we bound the second factor by $C\|u_{p-1}\|_{H^s}\prod_1^{p-2}\|u_j\|_{H^{s_0}}^{1/2}$ if $s_0 > \nu+\kappa+1/2$. Plugging in (2.13) and (2.11) we get an estimate for the contributions of the sum for $n_1 \leqslant \cdots \leqslant n_p$ in the right hand side of (2.10) by

$$C\|u_p\|_{H^s}\|u_{p-1}\|_{H^s}\prod_1^{p-2}\|u_j\|_{H^{s_0}}.$$

This concludes the proof. □

We shall now compose a multilinear form whose symbol is in the algebra $S_p^{s,\nu,N}$ and a polynomial.

Theorem 2.6. — *Let $p,q\in\mathbb{N}$, $p\geqslant 2$, $q\geqslant 2$, $s\in[0,+\infty[$, $\nu\in[0,+\infty[$, $1\leqslant \ell\leqslant q$. Let $b\in C^\infty(M)$. For $a\in S_q^{s,\nu,N}$ with $N>1+2s+\nu$ and $N>1+2s+(p+1)d/2+(d-1)/2$, define a $(p+q-1)$-linear form on $\mathcal{E}^{p+q-1}$ by*

$$(2.14)\quad M(u_1,\dots,u_{p+q-1}) = \mathcal{L}(a)(u_1,\dots,u_{\ell-1},bu_\ell\cdots u_{\ell+p-1},u_{\ell+p},\dots,u_{p+q-1}).$$

Then there is a symbol $\tilde a$ in $S_{p+q-1}^{s,1+\nu_1+\nu,N-1-2s-\max(\nu_1,\nu)}$ for any ν_1 satisfying $(p+1)d/2+(d-1)/2<\nu_1<N-1-2s$ such that $M=\mathcal{L}(\tilde a)$ and the map $a\to\tilde a$ is bounded from $S_q^{s,\nu,N}$ into these spaces.

Proof. — We shall take $\ell=1$ for the proof. For $u_j\in\mathcal{E}, n_j\in\mathbb{N}^*$ one has

$$u_1\cdots u_p = \sum_{n_1,\dots,n_p}\langle u_1,\varphi_{n_1}\rangle\cdots\langle u_p,\varphi_{n_p}\rangle\varphi_{n_1}\cdots\varphi_{n_p}$$

and therefore

$$\begin{aligned} M(u_1,\ldots,u_{p+q-1}) &= \sum_{n,n_{p+1},\ldots,n_{p+q-1}} a(n,n_{p+1},\ldots,n_{p+q-1})\langle bu_1\cdots u_p,\varphi_n\rangle \\ &\qquad\qquad \times \langle u_{p+1},\varphi_{n_{p+1}}\rangle\cdots\langle u_{p+q-1},\varphi_{n_{p+q-1}}\rangle \\ &= \sum_{n,n_1,\ldots,n_{p+q-1}} a(n,n_{p+1},\ldots,n_{p+q-1})\langle b\varphi_{n_1}\cdots\varphi_{n_p},\varphi_n\rangle \\ &\qquad\qquad \times \langle u_1,\varphi_{n_1}\rangle\cdots\langle u_{p+q-1},\varphi_{n_{p+q-1}}\rangle \\ &= \sum_{n_1,\ldots,n_{p+q-1}} \tilde a(n_1,\ldots,n_{p+q-1})\langle u_1,\varphi_{n_1}\rangle\cdots\langle u_{p+q-1},\varphi_{n_{p+q-1}}\rangle \end{aligned} \tag{2.15}$$

where $\tilde a : (\mathbb{N}^*)^{p+q-1}\to\mathbb{R}$ is defined by

$$\tilde a(n_1,\ldots,n_{p+q-1}) = \sum_n a(n,n_{p+1},\ldots,n_{p+q-1})\langle b\varphi_{n_1}\cdots\varphi_{n_p},\varphi_n\rangle. \tag{2.16}$$

It remains to prove that $\tilde a$ is in $S^{s,1+\nu_1+\nu,N-1-2s-\max(\nu_1,\nu)}_{p+q-1}$ for any $(p+1)d/2+(d-1)/2<\nu_1<N-1-2s$. We shall make use of an estimate for the integral of products of eigenfunctions, which generalizes well known orthogonality properties of products of spherical harmonics. Since we assume that the spectrum of $P|_W$ consists of simple eigenvalues, we may choose for every $n\in\mathbb{N}^*$ an interval of positive length I_n containing the eigenvalue λ_n, such that $I_n\cap I_{n'}=\emptyset$ if $n\neq n'$. If Π_n is the spectral projector determined by I_n we have $\Pi_n\varphi_n=\varphi_n$ for any n. Using (1.2) we see that Proposition 1.2.1 of [5] implies that for any $\nu_1>(p+1)d/2+(d-1)/2$, any $N\in\mathbb{N}$ and any $b\in C^\infty(M)$

$$|\langle b\varphi_{n_1}\cdots\varphi_{n_p},\varphi_n\rangle| \leqslant C\frac{\mu(n,n_1,\ldots,n_p)^{\nu_1+N}}{S(n,n_1,\ldots,n_p)^N}. \tag{2.17}$$

Actually Proposition 1.2.1 of [5] states such an inequality only in the case $b\equiv 1$. But this proposition is deduced from inequality (1.2.10) of Lemma 1.2.3, which allows an arbitrary C^∞ weight.

Formulas (2.4), (2.16) and (2.17) yield

$$\begin{aligned} |\tilde a(n_1,\ldots,n_{p+q-1})| &\leqslant C\sum_n \max(n,n_{p+1},\ldots,n_{p+q-1})^s \max\nolimits_2(n,n_{p+1},\ldots,n_{p+q-1})^s \\ &\qquad \times \frac{\mu(n,n_1,\ldots,n_p)^{\nu_1+N}}{S(n,n_1,\ldots,n_p)^N}\,\frac{\mu(n,n_{p+1},\ldots,n_{p+q-1})^{\nu+N}}{S(n,n_{p+1},\ldots,n_{p+q-1})^N}. \end{aligned} \tag{2.18}$$

We will use the following inequality obtained in formulas (2.1.10), (2.1.11) of [5] for $\nu' \geqslant 0, \nu'' \geqslant 0$ and $N > 1 + \max(\nu', \nu'')$:

$$\sum_n \frac{\mu(n, n_1, \dots, n_p)^{\nu'+N}}{S(n, n_1, \dots, n_p)^N} \frac{\mu(n, n_{p+1}, \dots, n_{p+q-1})^{\nu''+N}}{S(n, n_{p+1}, \dots, n_{p+q-1})^N} \leqslant C \frac{\mu(n_1, \dots, n_{p+q-1})^{1+\nu'+\nu''+(N-1-\max(\nu',\nu''))}}{S(n_1, \dots, n_{p+q-1})^{N-1-\max(\nu',\nu'')}}. \tag{2.19}$$

1st case. Assume $n \leqslant 2\max(p, q-1)\max_2(n_1, \dots, n_{p+q-1})$. We have either

$$\max(n, n_{p+1}, \dots, n_{p+q-1}) \max_2(n, n_{p+1}, \dots, n_{p+q-1}) = n \max(n_{p+1}, \dots, n_{p+q-1})$$

or

$$\max(n, n_{p+1}, \dots, n_{p+q-1}) \max_2(n, n_{p+1}, \dots, n_{p+q-1}) = \max(n_{p+1}, \dots, n_{p+q-1}) \max_2(n_{p+1}, \dots, n_{p+q-1}).$$

In each case we have using the assumption of the first case

$$\max(n, n_{p+1}, \dots, n_{p+q-1})^s \max_2(n, n_{p+1}, \dots, n_{p+q-1})^s \leqslant C \max(n_1, \dots, n_{p+q-1})^s \max_2(n_1, \dots, n_{p+q-1})^s$$

which together with (2.18) implies

$$|\tilde{a}(n_1, \dots, n_{p+q-1})| \leqslant C \max(n_1, \dots, n_{p+q-1})^s \max_2(n_1, \dots, n_{p+q-1})^s \times \sum_n \frac{\mu(n, n_1, \dots, n_p)^{\nu_1+N}}{S(n, n_1, \dots, n_p)^N} \frac{\mu(n, n_{p+1}, \dots, n_{p+q-1})^{\nu+N}}{S(n, n_{p+1}, \dots, n_{p+q-1})^N}. \tag{2.20}$$

Inequalities (2.20) and (2.19) yield

$$|\tilde{a}(n_1, \dots, n_{p+q-1})| \leqslant C \max(n_1, \dots, n_{p+q-1})^s \max_2(n_1, \dots, n_{p+q-1})^s \times \frac{\mu(n_1, \dots, n_{p+q-1})^{1+\nu_1+\nu+(N-1-\max(\nu_1,\nu))}}{S(n_1, \dots, n_{p+q-1})^{N-1-\max(\nu_1,\nu)}}. \tag{2.21}$$

2nd case. Assume $n > 2\max(p, q-1)\max_2(n_1, \dots, n_{p+q-1})$. In this case

$$\begin{aligned}\max(n, n_{p+1}, \dots, n_{p+q-1})\max_2(n, n_{p+1}, \dots, n_{p+q-1}) &= n\max(n_{p+1}, \dots, n_{p+q-1}) \\ &\leqslant n\max(n_1, \dots, n_{p+q-1}).\end{aligned} \tag{2.22}$$

Estimates (2.18) and (2.22) imply

$$|\tilde{a}(n_1, \dots, n_{p+q-1})| \leqslant C\max(n_1, \dots, n_{p+q-1})^s \times \sum_n n^s \frac{\mu(n, n_1, \dots, n_p)^{\nu_1+N}}{S(n, n_1, \dots, n_p)^N} \frac{\mu(n, n_{p+1}, \dots, n_{p+q-1})^{\nu+N}}{S(n, n_{p+1}, \dots, n_{p+q-1})^N}. \tag{2.23}$$

We shall obtain a bound on n. We have

$$\begin{aligned} S(n, n_1, \dots, n_p) &\geqslant n - n_1 - \dots - n_p \geqslant n - p\max(n_1, \dots, n_p), \\ S(n, n_{p+1}, \dots, n_{p+q-1}) &\geqslant n - n_{p+1} - \dots - n_{p+q-1} \\ &\geqslant n - (q-1)\max(n_{p+1}, \dots, n_{p+q-1}). \end{aligned} \tag{2.24}$$

As

$$\max(n_1, \dots, n_p) \leqslant \max_2(n_1, \dots, n_{p+q-1})$$

or

$$\max(n_{p+1}, \dots, n_{p+q-1}) \leqslant \max_2(n_1, \dots, n_{p+q-1})$$

(2.24) and the assumption of the second case yield

$$S(n, n_1, \dots, n_p) \geqslant n - p\max_2(n_1, \dots, n_{p+q-1}) \geqslant n/2 \tag{2.25}$$

or

$$S(n, n_{p+1}, \dots, n_{p+q-1}) \geqslant n - (q-1)\max_2(n_1, \dots, n_{p+q-1}) \geqslant n/2. \tag{2.26}$$

Therefore

$$n \leqslant 2S(n, n_1, \dots, n_p) + 2S(n, n_{p+1}, \dots, n_{p+q-1}) \tag{2.27}$$

which together with (2.23) implies

$$\begin{aligned}|\tilde{a}(n_1, \dots, n_{p+q-1})| &\leqslant C\max(n_1, \dots, n_{p+q-1})^s \\ &\times \sum_n \Bigg[\frac{\mu(n, n_1, \dots, n_p)^{\nu_1+N}}{S(n, n_1, \dots, n_p)^{N-s}} \frac{\mu(n, n_{p+1}, \dots, n_{p+q-1})^{\nu+N}}{S(n, n_{p+1}, \dots, n_{p+q-1})^N} \\ &\qquad + \frac{\mu(n, n_1, \dots, n_p)^{\nu_1+N}}{S(n, n_1, \dots, n_p)^N} \frac{\mu(n, n_{p+1}, \dots, n_{p+q-1})^{\nu+N}}{S(n, n_{p+1}, \dots, n_{p+q-1})^{N-s}}\Bigg].\end{aligned} \tag{2.28}$$

Formulas (2.19) and (2.28) yield, remembering the assumptions on N made in the statement of the theorem

(2.29)

$$\begin{aligned}|\tilde{a}(n_1,\dots,n_{p+q-1})| &\leqslant C\max(n_1,\dots,n_{p+q-1})^s\mu(n_1,\dots,n_{p+q-1})^s\\ &\quad\times\frac{\mu(n_1,\dots,n_{p+q-1})^{1+\nu_1+\nu+(N-1-2s-\max(\nu_1,\nu))}}{S(n_1,\dots,n_{p+q-1})^{N-1-2s-\max(\nu_1,\nu)}}\\ &\leqslant C\max(n_1,\dots,n_{p+q-1})^s\max{}_2(n_1,\dots,n_{p+q-1})^s\\ &\quad\times\frac{\mu(n_1,\dots,n_{p+q-1})^{1+\nu_1+\nu+(N-1-2s-\max(\nu_1,\nu))}}{S(n_1,\dots,n_{p+q-1})^{N-1-2s-\max(\nu_1,\nu)}}.\end{aligned}$$

Finally, we deduce from (2.21) and (2.29) that $\tilde{a}$ is in

$$S^{s,1+\nu_1+\nu,N-1-2s-\max(\nu_1,\nu)}_{p+q-1}$$

which concludes the proof. □

2.2. Proof of long-time existence

We first introduce some notations. Define

$$\mathcal{H}=\{\lambda_n;\ n\in\mathbb{N}^*\}. \tag{2.30}$$

If $m\in]0,+\infty[$, $\rho:\{1,\dots,p\}\to\{-1,1\}$ define

$$F^\rho_m(\xi_1,\dots,\xi_p)=\sum_{j=1}^p\rho(j)\sqrt{m^2+\xi_j^2} \tag{2.31}$$

and

$$\begin{aligned}Z(p,\rho)=\{(\xi_1,\dots,\xi_p);\ \xi_j\in\mathcal{H};\ \exists\sigma\in\mathfrak{S}_p\\ \text{with } \sigma^2=\mathrm{Id},\rho\circ\sigma=-\rho \text{ and } \forall j=1,\dots,p,\xi_j=\xi_{\sigma(j)}\}.\end{aligned} \tag{2.32}$$

Remark that by definition, $Z(p,\rho)\neq\emptyset\Rightarrow\sum_1^p\rho(j)=0$, so p must be even and among $1,\dots,p$ there are $\ell=p/2$ indices for which $\rho(j)=1$ and ℓ indices for which $\rho(j)=-1$. In other words, $Z(p,\rho)$ is the set of p-tuples of eigenvalues that can be coupled in such a way that each eigenvalue appears with both signs in (2.31). Therefore, F^ρ_m vanishes identically on $Z(p,\rho)$.

Proposition 2.7. — *Let $p\in\mathbb{N},p\geqslant 2,\rho:\{1,\dots,p\}\to\{-1,1\}$ be given. There is a zero measure subset $\mathcal{N}$ of $]0,+\infty[$ and for every $m\in]0,+\infty[-\mathcal{N}$, there are $c>0,N_1\in\mathbb{N}$, such that for every $(\lambda_{n_1},\dots,\lambda_{n_p})\in\mathcal{H}^p-Z(p,\rho)$ we have*

$$|F^\rho_m(\lambda_{n_1},\dots,\lambda_{n_p})|\geqslant c\mu(n_1,\dots,n_p)^{-N_1} \tag{2.33}$$

with μ defined by (2.2).

We shall prove (2.33) using general estimates we obtained in [6]. Alternatively one could make use of inequalities obtained in more elementary ways by Bambusi [1] and Bambusi and Grébert [2].

Before beginning the proof, let us simplify somewhat notations. Up to a permutation on the indices, we may assume $\rho(j) = 1$ for $j = 1, \dots, \ell, \rho(j) = -1$ for $j = \ell + 1, \dots, p$. Then F_m^ρ may be written

$$F_m^{p,\ell}(\xi_1, \dots, \xi_p) = \sum_{j=1}^{\ell} \sqrt{m^2 + \xi_j^2} - \sum_{j=\ell+1}^{p} \sqrt{m^2 + \xi_j^2}. \tag{2.34}$$

Define an auxiliary function

$$G_m^{p,\ell}(\xi_1, \dots, \xi_{p+1}) = F_m^{p,\ell}(\xi_1, \dots, \xi_p) + \xi_{p+1} \tag{2.35}$$

and set

$$\begin{aligned} &D_F = \mathcal{H}^p,\ D_G = \mathcal{H}^p \times \mathbb{Z} \\ &L_F = Z(p, \rho),\ L_G = Z(p, \rho) \times \{0\} \end{aligned} \tag{2.36}$$

these last two sets being empty if p is odd or p is even and $\ell \neq p/2$. We shall denote by ξ either $(\xi_1, \dots, \xi_p)$ or $(\xi_1, \dots, \xi_{p+1})$ according to the context and set $D_* = D_F$ or $D_* = D_G$.

Lemma 2.8.

(i) *There is $C > 0$ with*

$$\#\{\xi \in D_*;\ |\xi| < \lambda\} \leqslant C\lambda \quad \forall \lambda \geqslant 1. \tag{2.37}$$

(ii) *When p is even and $\ell = p/2$, there is $c > 0$ such that for any ξ belonging respectively to $D_F - L_F$ and $D_G - L_G$, and any $\sigma \in \mathfrak{S}_\ell$*

$$\sum_{j=1}^{\ell} (\xi_{\sigma(j)}^2 - \xi_{p/2+j}^2)^2 \geqslant c, \quad \sum_{j=1}^{\ell} (\xi_{\sigma(j)}^2 - \xi_{p/2+j}^2)^2 + \xi_{p+1}^2 \geqslant c. \tag{2.38}$$

Proof. — Assertion (i) follows from (1.2). Let us prove that (ii) holds true. Let $(\xi_1, \dots, \xi_p) \in D_F - L_F$. Then for every $\sigma \in \mathfrak{S}_\ell$ there is $j, 1 \leqslant j \leqslant \ell$ with $\xi_{\sigma(j)} \neq \xi_{\ell+j}$. Since the distance between two eigenvalues is bounded from below by a fixed positive constant by (1.2), we get the conclusion in the case of $D_F - L_F$. The case of $D_G - L_G$ is similar. □

Proof of Proposition 2.7. — Conditions (2.37) and (2.38) are conditions (A), (B_F) and (B_G) of [6] section 5.2. The proof of Theorem 4.7 in that paper applies – it relies only on these conditions – and shows that there is $\mathcal{N} \subset]0, +\infty[$ of zero measure and for any $m \in]0, +\infty[- \mathcal{N}$, there are constants $c > 0, N_0 \in \mathbb{N}$ such that for any $(\xi_1, \dots, \xi_p) \in D_F - L_F$, $(\xi_1, \dots, \xi_{p+1}) \in D_G - L_G$ respectively one has

$$\tag{2.39} \begin{aligned} |F_m^{p,\ell}(\xi_1, \dots, \xi_p)| &\geqslant c\Big(1 + \sum_1^p |\xi_j|\Big)^{-N_0} \\ |G_m^{p,\ell}(\xi_1, \dots, \xi_{p+1})| &\geqslant c\Big(1 + \sum_1^{p+1} |\xi_j|\Big)^{-N_0}. \end{aligned}$$

We may assume that the largest two indices among $n_1, \dots, n_p$ are n_1 and n_j for some j between 2 and p. Then the quantity by which we want to bound from below the absolute value of (2.34) is a negative power of

$$\tag{2.40} \mu(n_1, \dots, n_p) \sim \sum_{k \neq 1, k \neq j} n_k + 1.$$

If $\mu(n_1, \dots, n_p) \geqslant \delta(1 + \sum_1^p n_k)^\delta$ for some $\delta > 0$, then the first inequality of (2.39) implies the conclusion. Assume now

$$\tag{2.41} \mu(n_1, \dots, n_p) < \delta\Big(1 + \sum_1^p n_k\Big)^\delta.$$

It follows that n_1 and n_j are much larger than n_k for any $k \neq 1, j$ if δ is small enough. If $j \in \{2, \dots, \ell\}$ a lower bound of type (2.33) is then trivial. Assume now $j > \ell$, for instance $j = p$ and write

$$\tag{2.42} F_m^{p,\ell}(\lambda_{n_1}, \dots, \lambda_{n_p}) = \sqrt{m^2 + \lambda_{n_1}^2} - \sqrt{m^2 + \lambda_{n_p}^2} + F_m^{p-2,\ell-1}(\lambda_{n_2}, \dots, \lambda_{n_{p-1}}).$$

Using (1.2), we can write expanding the two square roots in (2.42) this expression as

$$\tag{2.43} \frac{2\pi}{\tau}(n_1 - n_p) + F_m^{p-2,\ell-1}(\lambda_{n_2}, \dots, \lambda_{n_{p-1}}) + O(1/n_1) + O(1/n_p).$$

If $|n_1 - n_p| \gg \mu(n_1, \dots, n_p)$, (2.43) trivially implies (2.33). Assume now there exists $C > 0$ such that

$$\tag{2.44} |n_1 - n_p| \leqslant C\mu(n_1, \dots, n_p).$$

The sum of the first two terms in (2.43) is $G_m^{p-2,\ell-1}(\lambda_{n_2}, \dots, \lambda_{n_{p-1}}, n_1 - n_p)$ so its absolute value is larger by (2.39), (2.40) and (2.44) than $c\mu(n_1, \dots, n_p)^{-N_0}$. Since for δ small enough, (2.41) implies that n_1 and n_p are larger than $C\delta^{-1/\delta}\mu(n_1, \dots, n_p)^{1/\delta}$, the remainders in (2.43) do not perturb this estimate for $\delta > 0$ small enough. This gives the conclusion. □

Let us define convenient subspaces of the algebra of Definition 2.1.

Definition 2.9. — *Let $p \in \mathbb{N}, p \geqslant 2, \rho : \{1, \dots, p\} \to \{-1, 1\}$ be given, $s \in [0, +\infty[, \nu \in [0, +\infty[, N \in \mathbb{N}$.*

- *If $\sum_{j=1}^p \rho(j) \neq 0$, we set $\widetilde{S}_p^{s,\nu,N}(\rho) = S_p^{s,\nu,N}$.*
- *If $\sum_{j=1}^p \rho(j) = 0$, we denote by $\widetilde{S}_p^{s,\nu,N}(\rho)$ the closed subspace of $S_p^{s,\nu,N}$ given by those $a \in S_p^{s,\nu,N}$ such that $a(n_1, \dots, n_p) \equiv 0$ for any $(n_1, \dots, n_p)$ such that there is $\sigma \in \mathfrak{S}_p$ with $\sigma^2 = 1, \rho \circ \sigma = -\rho$ and for any $j = 1, \dots, p$, $n_j = n_{\sigma(j)}$.*

In other words, we assume the vanishing of $a(n_1, \dots, n_p)$ every time the p indices $n_1, \dots, n_p$ can be grouped in $p/2$ pairs, each pair containing an element associated by ρ to sign $+1$ and an element associated by ρ to sign -1.

Proposition 2.7 implies immediately the following:

Proposition 2.10. — *Let $p \in \mathbb{N}, p \geqslant 2, \rho : \{1, \dots, p\} \to \{-1, 1\}$ be given, $s \in [0, +\infty[, \nu \in [0, +\infty[, N \in \mathbb{N}$. There is a zero measure subset $\mathcal{N}$ of $]0, +\infty[$ and for every $m \in]0, +\infty[- \mathcal{N}$, there are $c > 0, N_1 \in \mathbb{N}$, such that for every a in $\widetilde{S}_p^{s,\nu,N}(\rho)$, the symbol $\tilde{a}$ defined by*

$$\tilde{a}(n_1, \dots, n_p) = \frac{a(n_1, \dots, n_p)}{F_m^\rho(\lambda_{n_1}, \dots, \lambda_{n_p})} \tag{2.45}$$

is in $S_p^{s,\nu+N_1,N}$.

We begin the proof of Theorem 1.1. If u is a real valued function on $I \times M$, where I is some interval, we define $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $\Lambda_m = \sqrt{-\Delta + V + m^2}$ and

$$u_\pm = (D_t \pm \Lambda_m)u, \tag{2.46}$$

so that

$$u = \frac{1}{2}\Lambda_m^{-1}(u_+ - u_-),\ D_t u = \frac{1}{2}(u_+ + u_-). \tag{2.47}$$

By (2.46) and (2.47), equation (1.6) with real Cauchy data is equivalent to

$$(D_t - \Lambda_m)u_+ = -f\Big(x, \frac{1}{2}\Lambda_m^{-1}(u_+ - u_-), \frac{i}{2}(u_+ + u_-), \frac{1}{2}P\Lambda_m^{-1}(u_+ - u_-)\Big) \tag{2.48}$$

with Cauchy data

$$u_+|_{t=0} = \epsilon w \tag{2.49}$$

where w is in a fixed ball of $H^s(M)$ and in W.

We shall use the notation for $p, \ell \in \mathbb{N}$

$$e_\ell(p) = + \text{ if } p \leqslant \ell,\ e_\ell(p) = - \text{ if } p > \ell. \tag{2.50}$$

LEMMA 2.11. — *Let* $s \in [0,+\infty[$, $\nu \in [0,+\infty[, p \in \mathbb{N}, p \geqslant 2, \ell \in \mathbb{N}, 0 \leqslant \ell \leqslant p$. *Define* $\rho_\ell : \{1,\dots,p\} \to \{-1,1\}$ *by* $\rho_\ell(j) = 1, j = 1,\dots,\ell, \rho_\ell(j) = -1, j = \ell+1,\dots,p$. *For* a *in* $S_p^{s,\nu,+\infty}$, *we may decompose* $a = a' + a''$ *where:*

• $a' \in \widetilde{S}_p^{s,\nu,+\infty}(\rho_\ell)$

• *For any* $u_\pm$ *defined by* (2.46) *in terms of a real valued smooth enough* u, *we have, using notation* (2.50)

$$\operatorname{Im} \mathcal{L}(a'')(u_{e_\ell(1)},\dots,u_{e_\ell(p)}) \equiv 0. \tag{2.51}$$

Proof. — If p is odd or p is even and $\ell \neq p/2$, then $S_p^{s,\nu,+\infty} = \widetilde{S}_p^{s,\nu,+\infty}(\rho_\ell)$. Therefore, the conclusion follows in this case by taking $a' = a$ and $a'' = 0$. Assume now that p is even and $\ell = p/2$. For $(n_1,\dots,n_p)$ in $(\mathbb{N}^*)^p$, define $a'(n_1,\dots,n_p) = a(n_1,\dots,n_p)\mathbf{1}_{\{(n_1,\dots,n_p)\notin Z(p,\rho_\ell)\}}$. Then $a' \in \widetilde{S}_p^{s,\nu,+\infty}(\rho_\ell)$ by Definition 2.9 and

$$\mathcal{L}(a'')(u_{e_\ell(1)},\dots,u_{e_\ell(p)}) = \sum_{(n_1,\dots,n_p)\in Z(p,\rho_\ell)} a(n_1,\dots,n_p) \times \langle u_{e_\ell(1)},\varphi_{n_1}\rangle \cdots \langle u_{e_\ell(p)},\varphi_{n_p}\rangle. \tag{2.52}$$

For $(n_1,\dots,n_p) \in Z(p,\rho_\ell)$, the definition (2.32) of $Z(p,\rho_\ell)$ and the fact that $\overline{u_+} = -u_-$ imply that every term $\langle u_+,\varphi_{n_j}\rangle$, $j = 1,\dots,\ell$, can be coupled with a term $\overline{\langle u_+,\varphi_{n_j}\rangle}$, $j = \ell+1,\dots,p$. So

$$\langle u_{e_\ell(1)},\varphi_{n_1}\rangle \cdots \langle u_{e_\ell(p)},\varphi_{n_p}\rangle = (-1)^\ell |\langle u_+,\varphi_{n_1}\rangle|^2 \cdots |\langle u_+,\varphi_{n_\ell}\rangle|^2 \tag{2.53}$$

which together with the fact that $a(n_1,\dots,n_p) \in \mathbb{R}$ for all $(n_1,\dots,n_p)$ in $(\mathbb{N}^*)^p$ implies

$$a(n_1,\dots,n_p)\langle u_{e_\ell(1)},\varphi_{n_1}\rangle \cdots \langle u_{e_\ell(p)},\varphi_{n_p}\rangle \in \mathbb{R} \tag{2.54}$$

when $(n_1,\dots,n_p) \in Z(p,\rho_\ell)$. This together with (2.52) gives the conclusion. □

Let $\bar{p} \in \mathbb{N}$, $\bar{p} \geqslant \bar{q}+1$ (where $\bar{q}$ is defined in (1.3)) and let $b_p^\ell \in S_p^{s,\nu_p,+\infty}$, $0 \leqslant \ell \leqslant p$, $3 \leqslant p \leqslant \bar{p}$, with $\nu_p \geqslant 0$. We set for $s \in \mathbb{R}_+$

$$E_s(u_+)(t) = \sum_{n\in\mathbb{N}^*} n^{2s}\frac{1}{2}|\langle u_+,\varphi_n\rangle|^2 + \operatorname{Re}\sum_{3\leqslant p\leqslant\bar{p}}\sum_{\ell=0}^{p}\mathcal{L}(b_p^\ell)(u_{e_\ell(1)},\dots,u_{e_\ell(p)}). \tag{2.55}$$

Remark that for $s \geqslant s_0$ large enough, Proposition 2.5 implies

$$E_s(u_+)(t) \geqslant c\|u_+(t,\cdot)\|_{H^s}^2 - C\sum_{3\leqslant p\leqslant\bar{p}}\|u_+(t,\cdot)\|_{H^s}^p. \tag{2.56}$$

To conclude the proof of Theorem 1.1, it is enough to get $c > 0$ small enough so that if $\epsilon > 0$ is small enough and $T_\epsilon = c\epsilon^{-\bar{p}+1}$, $\|u_+(t,\cdot)\|_{H^s}$ has a uniform a priori bound for $t \in]-T_\epsilon, T_\epsilon[$ (s being fixed large enough). This is a consequence of (2.56) and the following proposition.

Proposition 2.12. — *For any $\bar{p} \geqslant \bar{q}+1$ and any m fixed outside an exceptional subset $\mathcal{N} \subset]0,+\infty[$ of zero measure, there are $s_0 > 0, \nu_p > 0, 3 \leqslant p \leqslant \bar{p}$, large enough, $b_p^\ell \in S_p^{s,\nu_p,+\infty}$, $0 \leqslant \ell \leqslant p$, $3 \leqslant p \leqslant \bar{p}$, and $C > 0$ such that for any $s > s_0$, any interval $]-T,T[$ over which (2.48), (2.49) has a solution staying in the unit ball of $H^s(M)$ and in W, one has for any $t \in]-T,T[$*

$$E_s(u_+)(t) \leqslant E_s(u_+)(0) + C\left|\int_0^t \|u_+(\tau,\cdot)\|_{H^s}^{\bar{p}+1}\,d\tau\right|. \tag{2.57}$$

Before giving the proof of this proposition, let us explain the idea on the following simple example

$$(\partial_t^2 - \Delta_g + V + m^2)u = (\partial_t u)^2,$$

which by (2.48) is equivalent to

$$(D_t - \Lambda_m)u_+ = \frac{1}{4}(u_+ + u_-)^2.$$

One wants, in the right hand side of (2.55), to choose the corrections b_p^ℓ in such a way that $\frac{d}{dt}E_s(u_+)(t)$ vanishes at some large order $\bar{p}+1$ at $u_+ = 0$. Let us explain the construction of these terms when $\bar{p} = 3$. If one computes $\frac{d}{dt}\frac{1}{2}\sum_{n\in\mathbb{N}^*} n^{2s}|\langle u_+, \varphi_n\rangle|^2$ and expresses $\partial_t u_+$ using the equation, one gets

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\sum_{n\in\mathbb{N}^*} n^{2s}|\langle u_+,\varphi_n\rangle|^2 &= \operatorname{Re} i \sum_{n\in\mathbb{N}^*} n^{2s}\langle (D_t-\Lambda_m)u_+,\varphi_n\rangle\langle\overline{u_+},\varphi_n\rangle\\ &= \frac{1}{4}\operatorname{Im}\sum_{n\in\mathbb{N}^*} n^{2s}\left\langle (u_++u_-)^2,\varphi_n\right\rangle\langle u_-,\varphi_n\rangle.\end{aligned} \tag{2.58}$$

We decompose $\langle u_{\epsilon_1}u_{\epsilon_2},\varphi_n\rangle = \sum_{n_1}\sum_{n_2}\langle u_{\epsilon_1},\varphi_{n_1}\rangle\langle u_{\epsilon_2},\varphi_{n_2}\rangle\langle\varphi_{n_1}\varphi_{n_2},\varphi_n\rangle$ ($\epsilon_1,\epsilon_2 = \pm$) to get expressions of type

$$\begin{aligned}&\frac{1}{4}\operatorname{Im}\sum_{n_1,n_2,n_3} n_3^{2s}\langle u_-,\varphi_{n_1}\rangle\langle u_-,\varphi_{n_2}\rangle\langle u_-,\varphi_{n_3}\rangle\langle\varphi_{n_1}\varphi_{n_2},\varphi_{n_3}\rangle\\ &\quad+\frac{1}{2}\operatorname{Im}\sum_{n_1,n_2,n_3} n_3^{2s}\langle u_+,\varphi_{n_1}\rangle\langle u_-,\varphi_{n_2}\rangle\langle u_-,\varphi_{n_3}\rangle\langle\varphi_{n_1}\varphi_{n_2},\varphi_{n_3}\rangle\\ &\quad+\frac{1}{4}\operatorname{Im}\sum_{n_1,n_2,n_3} n_3^{2s}\langle u_+,\varphi_{n_1}\rangle\langle u_+,\varphi_{n_2}\rangle\langle u_-,\varphi_{n_3}\rangle\langle\varphi_{n_1}\varphi_{n_2},\varphi_{n_3}\rangle.\end{aligned} \tag{2.59}$$

Define now

(2.60)

$$\begin{aligned}E_s(u_+)(t) &= \sum_{n\in\mathbb{N}^*} n^{2s}\frac{1}{2}|\langle u_+,\varphi_n\rangle|^2\\ &+\frac{1}{4}\operatorname{Re}\sum_{n_1,n_2,n_3}\frac{n_3^{2s}\langle\varphi_{n_1}\varphi_{n_2},\varphi_{n_3}\rangle}{-\sqrt{m^2+\lambda_{n_1}^2}-\sqrt{m^2+\lambda_{n_2}^2}-\sqrt{m^2+\lambda_{n_3}^2}}\\ &\qquad\times\langle u_-,\varphi_{n_1}\rangle\langle u_-,\varphi_{n_2}\rangle\langle u_-,\varphi_{n_3}\rangle\\ &+\frac{1}{2}\operatorname{Re}\sum_{n_1,n_2,n_3}\frac{n_3^{2s}\langle\varphi_{n_1}\varphi_{n_2},\varphi_{n_3}\rangle}{\sqrt{m^2+\lambda_{n_1}^2}-\sqrt{m^2+\lambda_{n_2}^2}-\sqrt{m^2+\lambda_{n_3}^2}}\\ &\qquad\times\langle u_+,\varphi_{n_1}\rangle\langle u_-,\varphi_{n_2}\rangle\langle u_-,\varphi_{n_3}\rangle\\ &+\frac{1}{4}\operatorname{Re}\sum_{n_1,n_2,n_3}\frac{n_3^{2s}\langle\varphi_{n_1}\varphi_{n_2},\varphi_{n_3}\rangle}{\sqrt{m^2+\lambda_{n_1}^2}+\sqrt{m^2+\lambda_{n_2}^2}-\sqrt{m^2+\lambda_{n_3}^2}}\\ &\qquad\times\langle u_+,\varphi_{n_1}\rangle\langle u_+,\varphi_{n_2}\rangle\langle u_-,\varphi_{n_3}\rangle.\end{aligned}$$

Proposition 2.7 will imply that the denominators do not vanish, and have a good enough control, so that the series converge for s large enough. We compute the time derivative of (2.60), using that

$$\frac{d}{dt}\langle u_\pm,\varphi_{n_j}\rangle = \langle \pm i\Lambda_m u_\pm,\varphi_{n_j}\rangle + \left\langle \frac{i}{4}(u_+ + u_-)^2,\varphi_{n_j}\right\rangle.$$

One checks immediately, using expression (2.59) for the time derivative of the first term in the right hand side of (2.60), that all cubic terms in (u_+,u_-) cancel, and that $\frac{d}{dt}E_s(u_+)(t)$ may be written as a quartic expression given in terms of expressions

$$\sum_{n_1,n_2,n_3,n_4} a(n_1,n_2,n_3,n_4)\langle u_{\epsilon_1},\varphi_{n_1}\rangle\langle u_{\epsilon_2},\varphi_{n_2}\rangle\langle u_{\epsilon_3},\varphi_{n_3}\rangle\langle u_{\epsilon_4},\varphi_{n_4}\rangle \tag{2.61}$$

with convenient coefficients a and where $\epsilon_1,\dots,\epsilon_4=\pm$. The idea to get (2.57) at order $\bar{p}=4$ is then to continue the procedure, adding new corrections to (2.60). Let us point out that there is anyway a new difficulty when one deals with expressions of even order: for example, when $\bar{p}=4$, one gets in the definition of $E_s(u_+)$ denominators of type

$$\sqrt{m^2+\lambda_{n_1^2}}+\sqrt{m^2+\lambda_{n_2^2}}-\sqrt{m^2+\lambda_{n_3^2}}-\sqrt{m^2+\lambda_{n_4^2}}$$

which vanish for any m when $\{n_1,n_2\}=\{n_3,n_4\}$. Consequently, to be able to pursue the procedure, one has to check that the quartic expressions (2.61) do not contain contributions of that type. This will be done using lemma 2.11.

Proof of Proposition 2.12. — We compute $\frac{d}{dt}E_s(u_+)(t)$. Using (2.48) we have

$$\begin{aligned}(2.62)\qquad &\frac{d}{dt}\sum_{n\in\mathbb{N}^*} n^{2s}|\langle u_+,\varphi_n\rangle|^2\\ &= 2\operatorname{Re} i\sum_{n\in\mathbb{N}^*} n^{2s}\langle (D_t-\Lambda_m)u_+,\varphi_n\rangle\langle\overline{u_+},\varphi_n\rangle\\ &= 2\operatorname{Im}\sum_{n\in\mathbb{N}^*} n^{2s}\Big\langle f\Big(x,\frac{1}{2}\Lambda_m^{-1}(u_+-u_-),\frac{i}{2}(u_++u_-),\\ &\qquad\qquad \frac{1}{2}P\Lambda_m^{-1}(u_+-u_-)\Big),\varphi_n\Big\rangle\langle\overline{u_+},\varphi_n\rangle\\ &= 2\operatorname{Im} M_0\Big(f\Big(x,\frac{1}{2}\Lambda_m^{-1}(u_+-u_-),\\ &\qquad\qquad \frac{i}{2}(u_++u_-),\frac{1}{2}P\Lambda_m^{-1}(u_+-u_-)\Big),\overline{u_+}\Big)\end{aligned}$$

where M_0 has been defined in Example 2.3. Using (1.3) and (1.4), we can write $f(x,\frac{1}{2}\Lambda_m^{-1}(u_+-u_-),\frac{i}{2}(u_++u_-),\frac{1}{2}P\Lambda_m^{-1}(u_+-u_-))$ as a real linear combination of expressions

$$(2.63)\qquad b(x)u_+^{\alpha}(\Lambda_m^{-1}u_+)^{\beta}(P\Lambda_m^{-1}u_+)^{\gamma}u_-^{\alpha'}(\Lambda_m^{-1}u_-)^{\beta'}(P\Lambda_m^{-1}u_-)^{\gamma'}$$

where $\alpha,\alpha',\beta,\beta',\gamma,\gamma'\in\mathbb{N}$, $2\leqslant\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'\leqslant\bar{q}$ and b is a smooth real valued function. Using (2.63) and the fact that M_0 is in $\mathcal{L}(S_2^{s,0,+\infty})$ by Example 2.3, Theorem 2.6 and Lemma 2.4 imply the existence of $a_p^{\ell}\in S_p^{s,\widetilde{\nu_p},+\infty}$, $0\leqslant\ell\leqslant p$, $3\leqslant p\leqslant\bar{q}+1$, for all $\widetilde{\nu_p}>pd/2+(d+1)/2$ such that

$$\begin{aligned}(2.64)\quad M_0\left(f\Big(x,\frac{1}{2}\Lambda_m^{-1}(u_+-u_-),\frac{i}{2}(u_++u_-),\frac{1}{2}P\Lambda_m^{-1}(u_+-u_-)\Big),\overline{u_+}\right)\\ =\sum_{3\leqslant p\leqslant\bar{q}+1}\sum_{\ell=0}^{p}\mathcal{L}(a_p^{\ell})(u_{e_\ell(1)},\dots,u_{e_\ell(p)})\end{aligned}$$

which together with (2.62) yields

$$(2.65)\quad \frac{d}{dt}\sum_{n\in\mathbb{N}^*} n^{2s}|\langle u_+,\varphi_n\rangle|^2 = 2\operatorname{Im}\sum_{3\leqslant p\leqslant\bar{q}+1}\sum_{\ell=0}^{p}\mathcal{L}(a_p^{\ell})(u_{e_\ell(1)},\dots,u_{e_\ell(p)}).$$

Let us compute for $p\leqslant\bar{p}, 0\leqslant\ell\leqslant p$

$$(2.66)\quad D_t[\mathcal{L}(b_p^\ell)(u_{e_\ell(1)},\dots,u_{e_\ell(p)})] \\ = \sum_{j=1}^p \mathcal{L}(b_p^\ell)(u_{e_\ell(1)},\dots,e_{\ell(j)}\Lambda_m u_{e_\ell(j)},\dots,u_{e_\ell(p)}) \\ + \sum_{j=1}^p \mathcal{L}(b_p^\ell)(u_{e_\ell(1)},\dots,(D_t-e_\ell(j)\Lambda_m)u_{e_\ell(j)},\dots,u_{e_\ell(p)}).$$

An explicit computation yields

$$(2.67)\quad \sum_{j=1}^p \mathcal{L}(b_p^\ell)(u_{e_\ell(1)},\dots,e_{\ell(j)}\Lambda_m u_{e_\ell(j)},\dots,u_{e_\ell(p)}) \\ = \mathcal{L}(F_m^{\rho_\ell} b_p^\ell)(u_{e_\ell(1)},\dots,u_{e_\ell(p)}),$$

with $F_m^{\rho_\ell}$ defined by (2.31). Using (2.48), the fact that $f(x,\frac{1}{2}\Lambda_m^{-1}(u_+ - u_-),\frac{i}{2}(u_+ + u_-),\frac{1}{2}P\Lambda_m^{-1}(u_+ - u_-))$ is a real linear combination of expressions (2.63), and that b_p^ℓ is in $\mathcal{L}(S_p^{s,\nu_p,+\infty})$, Theorem 2.6 and Lemma 2.4 imply the existence of $b_{pp'}^{\ell\ell'} \in S_{p'}^{s,\nu_{pp'},+\infty}$, $0 \leqslant \ell' \leqslant p'$, $p+1 \leqslant p' \leqslant p+\bar{q}-1$, for any $\nu_{pp'} > 1+\nu_p+(p'-p+2)d/2+(d-1)/2$, such that

$$(2.68)\quad \sum_{j=1}^p \mathcal{L}(b_p^\ell)(u_{e_\ell(1)},\dots,(D_t-e_\ell(j)\Lambda_m)u_{e_\ell(j)},\dots,u_{e_\ell(p)}) \\ = \sum_{p+1\leqslant p'\leqslant p+\bar{q}-1}\ \sum_{\ell'=0}^{p'} \mathcal{L}(b_{pp'}^{\ell\ell'})(u_{e_{\ell'}(1)},\dots,u_{e_{\ell'}(p')}).$$

Formulas (2.66), (2.67) and (2.68) yield

$$(2.69)\quad D_t[\mathcal{L}(b_p^\ell)(u_{e_\ell(1)},\dots,u_{e_\ell(p)})] = \mathcal{L}(F_m^{\rho_\ell} b_p^\ell)(u_{e_\ell(1)},\dots,u_{e_\ell(p)}) \\ + \sum_{p+1\leqslant p'\leqslant p+\bar{q}-1}\ \sum_{\ell'=0}^{p'} \mathcal{L}(b_{pp'}^{\ell\ell'})(u_{e_{\ell'}(1)},\dots,u_{e_{\ell'}(p')}).$$

Now, (2.55), (2.65) and (2.69) yield

$$(2.70)\quad \frac{d}{dt}E_s(u_+)(t) = \mathrm{Im}\Big[\sum_{3\leqslant p\leqslant \bar{q}+1}\ \sum_{\ell=0}^{p} \mathcal{L}(a_p^\ell)(u_{e_\ell(1)},\dots,u_{e_\ell(p)}) \\ - \sum_{3\leqslant p\leqslant \bar{p}}\ \sum_{\ell=0}^{p}\Big(\mathcal{L}(F_m^{\rho_\ell} b_p^\ell)(u_{e_\ell(1)},\dots,u_{e_\ell(p)}) \\ - \sum_{p+1\leqslant p'\leqslant p+\bar{q}-1}\ \sum_{\ell'=0}^{p'} \mathcal{L}(b_{pp'}^{\ell\ell'})(u_{e_{\ell'}(1)},\dots,u_{e_{\ell'}(p')})\Big)\Big]$$

Proposition 2.5 and the fact that $b^{\ell\ell'}_{pp'} \in S^{s,\nu_{pp'},+\infty}_{p'}$ imply

(2.71)
$$-\sum_{3\leqslant p\leqslant \bar p}\sum_{\ell=0}^{p}\sum_{\bar p+1\leqslant p'\leqslant p+\bar q-1}\sum_{\ell'=0}^{p'}\mathcal{L}(b^{\ell\ell'}_{pp'})(u_{e_{\ell'}(1)},\dots,u_{e_{\ell'}(p')}) = G(u_+(t,\cdot))$$

with $|G(u_+(t,\cdot))| \leqslant C\|u_+(t,\cdot)\|^{\bar p+1}_{H^s}$ as long as $u_+(t,\cdot)$ stays in the unit ball of H^s. Therefore we obtain

(2.72)
$$\begin{aligned}\frac{d}{dt}E_s(u_+)(t) = \operatorname{Im}\Big[&\sum_{3\leqslant p\leqslant \bar q+1}\sum_{\ell=0}^{p}\mathcal{L}(a^\ell_p)(u_{e_\ell(1)},\dots,u_{e_\ell(p)})\\ &-\sum_{3\leqslant p\leqslant \bar p}\sum_{\ell=0}^{p}\Big(\mathcal{L}(F^{\rho_\ell}_m b^\ell_p)(u_{e_\ell(1)},\dots,u_{e_\ell(p)})\\ &-\sum_{p+1\leqslant p'\leqslant \min(\bar p,p+\bar q-1)}\sum_{\ell'=0}^{p'}\mathcal{L}(b^{\ell\ell'}_{pp'})(u_{e_{\ell'}(1)},\dots,u_{e_{\ell'}(p')})\Big)\Big]\\ &+G(u_+(t,\cdot)).\end{aligned}$$

For $3 \leqslant p \leqslant \bar q+1, 0 \leqslant \ell \leqslant p$, we decompose a^ℓ_p using Lemma 2.11 in $(a^\ell_p)' + (a^\ell_p)''$, where $(a^\ell_p)'$ is in $\widetilde{S}^{s,\widetilde{\nu_p},+\infty}_p(\rho_\ell)$. For $3 \leqslant p \leqslant \bar p, 0 \leqslant \ell \leqslant p, p+1 \leqslant p' \leqslant \min(\bar p, p+\bar q-1), 0 \leqslant \ell' \leqslant p'$, we decompose $b^{\ell\ell'}_{pp'}$ using Lemma 2.11 in $(b^{\ell\ell'}_{pp'})' + (b^{\ell\ell'}_{pp'})''$, where $(b^{\ell\ell'}_{pp'})'$ is in $\widetilde{S}^{s,\nu_{pp'},+\infty}_{p'}(\rho_{\ell'})$. Using the second point of Lemma 2.11, (2.72) becomes

(2.73)
$$\begin{aligned}\frac{d}{dt}E_s(u_+)(t) = \operatorname{Im}\Big[&\sum_{3\leqslant p\leqslant \bar q+1}\sum_{\ell=0}^{p}\mathcal{L}((a^\ell_p)')(u_{e_\ell(1)},\dots,u_{e_\ell(p)})\\ &-\sum_{3\leqslant p\leqslant \bar p}\sum_{\ell=0}^{p}\Big(\mathcal{L}(F^{\rho_\ell}_m b^\ell_p)(u_{e_\ell(1)},\dots,u_{e_\ell(p)})\\ &-\sum_{p+1\leqslant p'\leqslant \min(\bar p,p+\bar q-1)}\sum_{\ell'=0}^{p'}\mathcal{L}((b^{\ell\ell'}_{pp'})')(u_{e_{\ell'}(1)},\dots,u_{e_{\ell'}(p')})\Big)\Big]\\ &+G(u_+(t,\cdot)).\end{aligned}$$

By Proposition 2.10, we may choose when m is fixed outside a convenient subset $\mathcal{N} \subset\,]0,+\infty[$ of zero measure, for $\nu_p > 0$ large enough b^ℓ_p in $S^{s,\nu_p,+\infty}_p$,

$0 \leqslant \ell \leqslant p$, $3 \leqslant p \leqslant \bar{p}$, defined by

(2.74)

$$b_3^\ell(n_1,n_2,n_3) = \frac{(a_3^\ell)'(n_1,n_2,n_3)}{F_m^{\rho_\ell}(\lambda_{n_1},\lambda_{n_2},\lambda_{n_3})}, 0 \leqslant \ell \leqslant 3$$

$$b_p^\ell(n_1,\dots,n_p) = \frac{(a_p^\ell)'(n_1,\dots,n_p) - \sum_{3\leqslant p'\leqslant p-1}\sum_{\ell'=0}^{p'}(b_{p'p}^{\ell'\ell})'(n_1,\dots,n_p)}{F_m^{\rho_\ell}(\lambda_{n_1},\dots,\lambda_{n_p})},$$
$$4 \leqslant p \leqslant \bar{q}+1, 0 \leqslant \ell \leqslant p,$$

$$b_p^\ell(n_1,\dots,n_p) = -\frac{\sum_{p+1-\bar{q}\leqslant p'\leqslant p-1}\sum_{\ell'=0}^{p'}(b_{p'p}^{\ell'\ell})'(n_1,\dots,n_p)}{F_m^{\rho_\ell}(\lambda_{n_1},\dots,\lambda_{n_p})},$$
$$\bar{q}+2 \leqslant p \leqslant \bar{p}, 0 \leqslant \ell \leqslant p.$$

Remark that in order to define $b_p^\ell, 4 \leqslant p \leqslant \bar{p}$, by (2.74) we need to know $b_{p'p}^{\ell'\ell}, p' \leqslant p-1$. This is indeed the case as $b_{p'p}^{\ell'\ell}, p' \leqslant p-1$, depends only on $b_r^j, 3 \leqslant r \leqslant p-1, 0 \leqslant j \leqslant r$, which have already been constructed.

We finally get from (2.73) and (2.74)

$$\frac{d}{dt}E_s(u_+)(t) = G(u_+(t,\cdot))$$

whence (2.57) and the conclusion of the proof. □

Remark. — When one considers the special case of an *integrable* Hamiltonian equation, like the sine-Gordon equation $\Box u + \sin u = 0$ on $\mathbb{S}^1$, it is classical that the existence of infinitely many conserved quantities allows one to control uniformly all Sobolev norms for all times. Remark that such a result holds true for a *given* value of the mass m^2, while we prove theorem 1.1 only when m is outside an exceptional subset of zero measure of $]0,+\infty[$. This plays an essential role in the above proof, since taking m outside this set prevents $F_m^{\rho_\ell}(\lambda_{n_1},\dots,\lambda_{n_p})$ from vanishing, as soon as the eigenvalues $(\lambda_{n_1},\dots,\lambda_{n_p})$ satisfy $\{\lambda_{n_1},\dots,\lambda_{n_\ell}\} \neq \{\lambda_{n_{\ell+1}},\dots,\lambda_{n_p}\}$. This gives us the possibility of defining in (2.74) b_p^ℓ dividing by $F_m^{\rho_\ell}$. If one were trying to use such a strategy to recover long time boundedness for $E_s(u_+)(t)$, with u_+ coming from a solution u to the sine-Gordon equation, we would have to divide by quantities of type $\sum_{j=1}^\ell \sqrt{1+\lambda_{n_j}^2} - \sum_{j=\ell+1}^p \sqrt{1+\lambda_{n_j}^2}$ which *do* vanish for certain indices $(n_1,\dots,n_p)$ such that $\{\lambda_{n_1},\dots,\lambda_{n_\ell}\} \neq \{\lambda_{n_{\ell+1}},\dots,\lambda_{n_p}\}$. The method we use would thus fail, unless one could prove that every time one wants to divide in (2.74) by a vanishing quantity, the numerator is yet zero. It is possible that such a property hold true, because of the very special structure of the nonlinearity $\sin u - u$, but we have no proof of that.

3. Proof of Propositions 1.3, 1.5 and 1.7

3.1. Proof of Propositions 1.3 and 1.5

As M satisfies (1.8), there are C^∞ 2π-periodic functions r, x_d such that

$$M = \{x = (r(\theta)\omega, x_d(\theta)) : \omega \in \mathbb{S}^{d-1}, \theta \in [0, 2\pi]\} \tag{3.1}$$

and satisfying for any $\theta \in [0, 2\pi]$

$$\begin{aligned} &r(2\pi - \theta) = r(\theta),\ x_d(2\pi - \theta) = -x_d(\theta), \\ &r(\theta) > 0,\ r'(\theta)^2 + x_d'(\theta)^2 \neq 0. \end{aligned} \tag{3.2}$$

The function $V(r(\theta)\omega, x_d(\theta))$ is by (1.9) a function of θ alone, that we shall denote for short by $V(\theta)$. Because of (1.9) it satisfies $V(2\pi - \theta) = V(\theta)$ for any $\theta \in [0, 2\pi]$.

In the cylindrical coordinates (r, ω, x_d), the euclidean metric is given by $dr^2 + r^2 d\omega^2 + dx_d^2$. Its restriction to M is

$$(r'(\theta)^2 + x_d'(\theta)^2)d\theta^2 + r(\theta)^2 d\omega^2 = b(\theta)^2 d\theta^2 + r(\theta)^2 d\omega^2 \tag{3.3}$$

where we have set $b(\theta) = \sqrt{r'(\theta)^2 + x_d'(\theta)^2}$. Thus, the determinant of the metric is $b(\theta)^2 r(\theta)^{2(d-2)}$. As $\Delta_g = (\det g)^{-1/2}\partial_i[g^{ij}(\det g)^{1/2}\partial_j]$, the part of the Laplacian acting only on θ is

$$b(\theta)^{-1} r(\theta)^{-(d-2)}\partial_\theta[b(\theta)^{-2} b(\theta) r(\theta)^{d-2}\partial_\theta]. \tag{3.4}$$

The riemannian measure is $(\det g)^{1/2} d\theta = b(\theta) r(\theta)^{d-2} d\theta$ up to a multiplicative constant. The eigenfunctions of $P = \sqrt{-\Delta_g + V}$ associated to the eigenvalue λ, invariant under the action of the rotations with axis x_d, are functions $\varphi(\theta)$ satisfying

$$\int_0^{2\pi} |\varphi(\theta)|^2 b(\theta) r(\theta)^{d-2} d\theta < +\infty \tag{3.5}$$

and

$$-b(\theta)^{-1} r(\theta)^{-(d-2)}\partial_\theta[b(\theta)^{-2} b(\theta) r(\theta)^{d-2}\partial_\theta \varphi(\theta)] + V(\theta)\varphi(\theta) = \lambda^2 \varphi(\theta). \tag{3.6}$$

It remains to prove that the eigenvalues λ of (3.6) are simple and satisfy (1.2) under the additional hypothesis that φ belongs to the space W defined in the statement of Proposition 1.3 (resp. to the space W defined in the statement of Proposition 1.5).

First, let us write equation (3.6) in a different manner. Set

$$p(\theta) = b(\theta)^{-1} r(\theta)^{d-2}, s(\theta) = b(\theta) r(\theta)^{d-2}, q(\theta) = V(\theta) b(\theta) r(\theta)^{d-2}. \tag{3.7}$$

Then (3.6) becomes

$$\partial_\theta[p(\theta)\partial_\theta\varphi(\theta)] + (\lambda^2 s(\theta) - q(\theta))\varphi(\theta) = 0. \tag{3.8}$$

We obtain a Hill equation and φ is an eigenfunction in $L^2(s(\theta)d\theta)$. Let $(\lambda_n^P)_n$ denote the periodic eigenvalues, $(\lambda_n^A)_n$ the anti-periodic eigenvalues, $(\lambda_n^D)_n$ the Dirichlet eigenvalues and $(\lambda_n^N)_n$ the Neumann eigenvalues. It is known (see for instance chapters 2 and 3 of [7]) that $\lambda_{2n}^A \leqslant \lambda_{2n+1}^A < \lambda_{2n+1}^P \leqslant \lambda_{2n+2}^P$, $\lambda_{2n}^D, \lambda_{2n+1}^N \in [\lambda_{2n}^A, \lambda_{2n+1}^A]$, and $\lambda_{2n+1}^D, \lambda_{2n+2}^N \in [\lambda_{2n+1}^P, \lambda_{2n+2}^P]$. Furthermore, as the functions defined in (3.7) satisfy

$$p(2\pi-\theta) = p(\theta), s(2\pi-\theta) = s(\theta), q(2\pi-\theta) = q(\theta) \tag{3.9}$$

we have

$$\{\lambda_{2n}^D, \lambda_{2n+1}^N\} = \{\lambda_{2n}^A, \lambda_{2n+1}^A\}, \{\lambda_{2n+1}^D, \lambda_{2n+2}^N\} = \{\lambda_{2n+1}^P, \lambda_{2n+2}^P\}. \tag{3.10}$$

Proof of Proposition 1.3. — In this case, W consists of all functions in $L^2(M)$ invariant under the action of the rotations with axis x_d and even in x_d. Therefore, we look for 2π-periodic functions φ which satisfy also $\varphi(2\pi-\theta) = \varphi(\theta)$. In particular, $\varphi'(2\pi) = -\varphi'(0)$. We have also $\varphi'(2\pi) = \varphi'(0)$ as φ is 2π-periodic and thus $\varphi'(2\pi) = \varphi'(0) = 0$. This implies that our eigenfunctions φ are 2π-periodic and satisfy the Neumann boundary conditions. Therefore, the eigenvalues λ of $P|_W$ coincide with $(\lambda_{2n}^N)_n$ which are known to be simple and satisfy (1.2) (see for instance chapter 4 of [7]). □

Proof of Proposition 1.5. — In this case, W consists of all functions in $L^2(M)$ invariant under the action of the rotations with axis x_d and odd with respect to x_d. Therefore, we look for 2π-periodic functions φ which satisfy also $\varphi(2\pi-\theta) = -\varphi(\theta)$. In particular, $\varphi(2\pi) = -\varphi(0)$. We have also $\varphi(2\pi) = \varphi(0)$ as φ is 2π-periodic and thus $\varphi(2\pi) = \varphi(0) = 0$. This implies that our eigenfunctions φ are 2π-periodic and satisfy the Dirichlet boundary conditions. Therefore, the eigenvalues λ of $P|_W$ coincide with $(\lambda_{2n+1}^D)_n$ which are known to be simple and satisfy (1.2) (see for instance chapter 4 of [7]). □

3.2. Proof of Proposition 1.7

As M satisfies (1.10), there are C^∞ functions $r, x_d : [0,1] \to \mathbb{R}$ such that

$$M = \{x = (r(\theta)\omega, x_d(\theta)) : \omega \in \mathbb{S}^{d-1}, \theta \in [0,1]\} \tag{3.11}$$

and satisfying

$$
\begin{aligned}
&r^{(p)}(0) = 0 \text{ if } p \text{ is even,}\\
&r^{(p)}(1) = 0 \text{ if } p \text{ is even,}\\
&x_d^{(p)}(0) = 0 \text{ if } p \text{ is odd,}\\
&x_d^{(p)}(1) = 0 \text{ if } p \text{ is odd,}\\
&r(\theta) > 0 \ \forall\theta \in]0,1[,\\
&r'(\theta)^2 + x_d'(\theta)^2 \neq 0 \ \forall\theta \in [0,1].
\end{aligned}
\tag{3.12}
$$

Since V satisfies (1.11) it may be written as a function of θ.

Let $b(\theta) = \sqrt{r'(\theta)^2 + x_d'(\theta)^2}$. As in section 3.1, the eigenfunctions of $P = \sqrt{-\Delta_g + V}$ invariant under the action of the rotations with axis x_d are functions $\varphi(\theta)$ satisfying

$$\int_0^1 |\varphi(\theta)|^2 b(\theta) r(\theta)^{d-2} d\theta < +\infty \tag{3.13}$$

and

$$- b(\theta)^{-1} r(\theta)^{-(d-2)} \partial_\theta [b(\theta)^{-2} b(\theta) r(\theta)^{d-2} \partial_\theta \varphi(\theta)] + V(\theta)\varphi(\theta) = \lambda^2 \varphi(\theta). \tag{3.14}$$

It remains to prove that the eigenvalues λ of (3.14) are simple and satisfy (1.2).

Multiplying by $-b(\theta)^2$, we can write (3.14) as

$$\partial_\theta^2 \varphi(\theta) + \left[(d-2)\frac{r'(\theta)}{r(\theta)} - \frac{b'(\theta)}{b(\theta)}\right] \partial_\theta \varphi(\theta) + b(\theta)^2 (\lambda^2 - V(\theta))\varphi(\theta) = 0. \tag{3.15}$$

Let

$$g(\theta) = (d-2)\frac{r'(\theta)}{r(\theta)} - \frac{b'(\theta)}{b(\theta)}, \ h = \frac{1}{\lambda} \tag{3.16}$$

and write (3.15) as

$$\partial_\theta^2 \varphi(\theta) + g(\theta)\partial_\theta \varphi(\theta) + b(\theta)^2 (h^{-2} - V(\theta))\varphi(\theta) = 0, \tag{3.17}$$

where g is a smooth function on $]0,1[$. Moreover, as $r(\theta)$ vanishes exactly at order 1 at $\theta = 0$ and $\theta = 1$, $\theta(1-\theta)g(\theta)$ extends as a smooth function on $[0,1]$ and

$$\lim_{\theta \to 0_+} \theta g(\theta) = \lim_{\theta \to 1_-} (1-\theta) g(\theta) = d - 2. \tag{3.18}$$

To simplify notations, we will thus assume in the following that

$$\text{(3.19)}\quad \begin{array}{l} g \text{ is a smooth function on }]0,1[\text{ such that } \theta(1-\theta)g(\theta) \text{ extends} \\ \text{as a smooth function on } [0,1] \text{ and there is } \alpha \in [1,+\infty[\text{ with} \\ \lim_{\theta\to 0_+} \theta g(\theta) = \lim_{\theta \to 1_-} (1-\theta)g(\theta) = \alpha. \end{array}$$

We first look at particular solutions of the ordinary differential equation (3.17).

Proposition 3.1. — *For any $h>0$, there is a unique smooth function φ^h_+ (resp. φ^h_-) defined on $[0,1[$ (resp. on $]0,1]$) solution of (3.17) on $]0,1[$ and satisfying*

$$\lim_{\theta\to 0_+}\Big(\varphi^h_+(\theta), \frac{d}{d\theta}\varphi^h_+(\theta)\Big) = (1,0),\ \Big(\textit{resp.} \lim_{\theta\to 1_-}\Big(\varphi^h_-(\theta), \frac{d}{d\theta}\varphi^h_-(\theta)\Big) = (1,0)\Big).$$

Moreover, any solution φ^h of (3.17) which does not belong to the vector space spanned by φ^h_+ (resp. by φ^h_-) satisfies

$$\begin{gathered}\varphi^h(\theta) \sim c_h\theta^{-\alpha+1},\ \frac{d}{d\theta}\varphi^h(\theta) \sim c'_h\theta^{-\alpha} \textit{ when } \theta\to 0_+ \textit{ and } \alpha>1,\\ \varphi^h(\theta) \sim c_h \ln(\theta),\ \frac{d}{d\theta}\varphi^h(\theta) \sim c'_h\theta^{-1} \textit{ when } \theta\to 0_+ \textit{ and } \alpha=1,\\ \Big(\textit{resp. } \varphi^h(\theta) \sim c_h(1-\theta)^{-\alpha+1},\ \frac{d}{d\theta}\varphi^h(\theta) \sim c'_h(1-\theta)^{-\alpha}\\ \textit{when } \theta\to 1_- \textit{ and } \alpha>1,\\ \varphi^h(\theta) \sim c_h\ln(1-\theta),\ \frac{d}{d\theta}\varphi^h(\theta) \sim c'_h(1-\theta)^{-1} \textit{ when } \theta\to 1_- \textit{ and } \alpha=1\Big)\end{gathered}$$

for nonzero constants c_h and c'_h.

Finally, if one sets, for x in a compact interval $[0,c]$, $w^h_+(x) = \varphi^h_+(hx)$, $w^h_-(x) = \varphi^h_-(1-hx)$, the functions $w^h_\pm(x)$ are smooth in $(h,x)\in[0,h_0]\times[0,c]$ whenever $ch_0<1$.

Proof. — We shall prove the statements concerning φ^h_+. The assumptions on g allow us to write $g(\theta) = \alpha/\theta + g_1(\theta)$ where g_1 is smooth on $[0,1[$. It is enough to prove that the equation

$$\text{(3.20)}\qquad \Big(\partial_x^2 + \Big(\frac{\alpha}{x} + hg_1(hx)\Big)\partial_x + b(hx)^2(1-h^2V(hx))\Big) w^h_+(x) = 0$$

has a unique solution $w^h_+(x) = \varphi^h_+(hx)$ defined on some interval $[0,c]$ with $c>0$ small enough, smooth as a function of $(h,x)\in[0,1]\times[0,c]$ and satisfying the condition $\lim_{x\to 0_+}(w^h_+(x), \frac{d}{dx}w^h_+(x)) = (1,0)$. Actually, since for $0<x<1/h$, (3.20) has smooth coefficients and smooth h-dependence, that solution extends to a C^∞ function in $(h,x) \in [0,1/c[\times[0,c]$ for any $c>0$.

Write equation (3.20) as a system in $X(x,h) = \begin{pmatrix} w_+^h(x) \\ \left(w_+^h\right)'(x) \end{pmatrix}$

$$X'(x,h) = A(x)X(x,h) + B(x,h)X(x,h) \tag{3.21}$$

with

$$\begin{aligned} A(x) &= \begin{pmatrix} 0 & 1 \\ 0 & -\frac{\alpha}{x} \end{pmatrix} \\ B(x,h) &= \begin{pmatrix} 0 & 0 \\ -b(hx)^2(1-h^2V(hx)) & -hg_1(hx) \end{pmatrix} \end{aligned} \tag{3.22}$$

Define

$$\begin{aligned} S(x,x') &= \begin{pmatrix} 1 & -\frac{xx'}{\alpha-1}({x'}^{\alpha-1}-1) \\ 0 & {x'}^{\alpha} \end{pmatrix} \qquad \text{if } \alpha>1 \\ S(x,x') &= \begin{pmatrix} 1 & -xx'\ln(x') \\ 0 & x' \end{pmatrix} \qquad \text{if } \alpha=1 \end{aligned} \tag{3.23}$$

for $x>0, x'>0$. Then

$$S(x,1) = \mathrm{Id},\ \frac{d}{dx}\left[S\left(x,\frac{x'}{x}\right)\right] = A(x)S\left(x,\frac{x'}{x}\right) \tag{3.24}$$

when $0<x'<x$ and X is a solution of (3.21) with

$$\lim_{x\to 0_+} X(x,h) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

if and only if

$$X(x,h) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x\int_0^1 S(x,x')B(xx',h)X(xx',h)dx'. \tag{3.25}$$

The above equation may be solved by a fixed point for $x \in [0,c]$ with $c>0$ small enough, and the unique solution obtained in that way has smooth h-dependence.

It remains to prove that any solution of (3.17) which is not collinear with the φ_+^h we just constructed has the asymptotics stated in the proposition. Denote by ω^h the Wronskian of the solution w_+^h of (3.20) and of another solution w^h of (3.20), not collinear with w_+^h. We have for a fixed $x_0>0$

$$\begin{aligned} \omega^h(x) &= \omega^h(x_0)\exp\left(\int_{x_0}^x\left(-\frac{\alpha}{t}-hg_1(ht)\right)dt\right) \\ &= \omega^h(x_0)\left(\frac{x_0}{x}\right)^\alpha \exp\left(-\int_{hx_0}^{hx} g_1(t)dt\right). \end{aligned} \tag{3.26}$$

It follows that since

$$w^h(x) = w_+^h(x)\left(\frac{w^h(x_0)}{w_+^h(x_0)} + \int_{x_0}^{x} \frac{\omega^h(y)}{w_+^h(y)^2}dy\right) \tag{3.27}$$

and since $w_+^h(x) \to 1$ when $x \to 0_+$, we have $w^h(x) \sim cx^{-\alpha+1}$ (resp. $w^h(x) \sim c\ln(x)$) when $x \to 0_+$ and $\alpha > 1$ (resp. $\alpha = 1$). We get in the same way the asymptotics of $\frac{d}{dx}w^h$. Coming back to θ variables, we get the asymptotics for $\varphi^h(\theta)$. □

The following proposition gives an asymptotic expansion for $\varphi_\pm^h(\theta)$.

Proposition 3.2. — *Denote $\phi_+(\theta) = \int_0^\theta b(y)dy, \phi_-(\theta) = -\int_\theta^1 b(y)dy$. There are smooth functions $M_\pm^k(\theta), k \in \mathbb{N}$, defined on $]0,1[$ and for any $\delta > 0, N \in \mathbb{N}, \ell \in \mathbb{N}$ there is $C_{N\delta\ell} > 0$ such that when $N > 0, m < N/2 + 1/2$*

$$\left|\partial_h^m\partial_\theta^\ell\left[\varphi_\pm^h(\theta) - \sum_{k=0}^{N} h^{\frac{\alpha}{2}}\left(e^{i\phi_\pm(\theta)/h}h^k M_\pm^k(\theta) + e^{-i\phi_\pm(\theta)/h}h^k\overline{M_\pm^k}(\theta)\right)\right]\right| \leqslant C_{N\delta\ell}h^{N+\frac{\alpha}{2}-\ell-2m+1} \tag{3.28}$$

uniformly for $\theta \in [\delta, 1-\delta]$ and $h \in]0,1]$. Moreover, for any $\delta > 0$ there is $C > 0$ with $C^{-1} \leqslant M_\pm^0(\theta) \leqslant C$ for any $\theta \in [\delta, 1-\delta]$.

We shall prove the proposition only for φ_+^h.

The coefficients $b(\theta), V(\theta), g_1(\theta) = g(\theta) - \alpha/\theta$ in (3.17) are defined only for $\theta \in [0,1[$. We fix $\delta > 0$ small enough, and cut-off V, g_1 with a smooth function compactly supported in $[0, 1-\delta/2[$, equal to 1 over $[0, 1-\delta]$. We may thus assume that V, g_1 are in $C_0^\infty([0,+\infty[)$. Moreover, we may extend b as a smooth function on $[0,+\infty[$ bounded from below by a positive constant and constant for $x \geqslant 1-\delta/2$. Let us prove

Lemma 3.3.

i) *Equation (3.20) has a unique solution $W_\pm^h$ defined on $]0,+\infty[$, and satisfying*

$$x^{\alpha/2}e^{\mp i\phi_+(hx)/h}W_\pm^h(x) \to 1 \text{ and } (x^{\alpha/2}e^{\mp i\phi_+(hx)/h}W_\pm^h(x))' \to 0$$

when $x \to +\infty$.

ii) *Let $N \in \mathbb{N}, N > \alpha/2 - 1$ and $s(x,h)$ be a smooth function defined on $]0,+\infty[$, satisfying $|\partial_x^\ell\partial_h^k s(x,h)| \leqslant C_{k\ell}x^{-N+2k-2}$ when $hx \geqslant \delta$ for any k and ℓ in $\mathbb{N}$. Then*

$$\left(\partial_x^2 + \left(\frac{\alpha}{x} + hg_1(hx)\right)\partial_x + b(hx)^2(1-h^2V(hx))\right)r(x,h) = s(x,h) \tag{3.29}$$

has a unique solution r satisfying $x^{\alpha/2}r(x,h) \to 0$ and $x^{\alpha/2}r'(x,h) \to 0$ when $x \to +\infty$. Moreover one has the estimates

$$|\partial_x^\ell \partial_h^k r(x,h)| \leqslant C_{k\ell} x^{-N+2k-1} \tag{3.30}$$

for $hx \geqslant \delta$, $2k < N - \alpha/2 + 1$ and any $\ell \in \mathbb{N}$.

Proof.

i) Define $v(x) = x^{\alpha/2}w(x)$ for w a solution of (3.20). Then v satisfies

$$\Big(\partial_x^2 + hg_1(hx)\partial_x + \Big[b(hx)^2(1-h^2V(hx)) + \frac{\alpha}{2x^2}\Big(1-\frac{\alpha}{2}\Big) - \frac{\alpha}{2x}hg_1(hx)\Big]\Big)v(x) = 0 \tag{3.31}$$

which may be written

$$(\partial_x^2 + a_1(x,h)\partial_x + [b(hx)^2 + a_2(x,h)])v(x) = 0 \tag{3.32}$$

where a_1, a_2 are smooth functions on $]0,+\infty[$ satisfying

$$\begin{gathered}\textstyle\int_1^{+\infty} |a_1(y,h)|dy \leqslant C,\ |\partial_x^\ell \partial_h^k a_1(x,h)| \leqslant C_{k\ell}x^{-1+k-\ell},\\ |\partial_x^\ell \partial_h^k a_2(x,h)| \leqslant C_{k\ell}x^{-2+k-\ell}\end{gathered} \tag{3.33}$$

for $k,\ell \in \mathbb{N}, x \geqslant 1$, with constants $C, C_{k\ell} > 0$ independent of h. Moreover, $\widetilde{a_2}(\theta) = h^{-2}a_2(\theta/h,h)$ is a smooth function of $\theta > 0$ independent of h, satisfying for any $\ell \in \mathbb{N}$

$$|\widetilde{a_2}^{(\ell)}(\theta)| \leqslant C_\ell \theta^{-2-\ell}. \tag{3.34}$$

We define $X = \begin{pmatrix} v \\ v' \end{pmatrix}$ and v solves (3.32) if and only if

$$X'(x,h) = N(hx)X(x,h) + A(x,h)X(x,h) \tag{3.35}$$

where

$$N(\theta) = \begin{pmatrix} 0 & 1 \\ -b(\theta)^2 & 0 \end{pmatrix},\ A(x,h) = \begin{pmatrix} 0 & 0 \\ -a_2(x,h) & -a_1(x,h) \end{pmatrix}. \tag{3.36}$$

We shall have

$$\int_1^{+\infty} |A(y,h)|dy \leqslant C,\ |\partial_x^\ell \partial_h^k A(x,h)| \leqslant C_{k\ell}x^{-1+k-\ell} \text{ for } k,\ell \in \mathbb{N}. \tag{3.37}$$

Define

$$P_h(\theta) = \begin{pmatrix} e^{i\phi_+(\theta)/h} & e^{-i\phi_+(\theta)/h} \\ ib(\theta)e^{i\phi_+(\theta)/h} & -ib(\theta)e^{-i\phi_+(\theta)/h} \end{pmatrix}. \tag{3.38}$$

Since $\phi_+(hx)/h = \int_0^x b(hy)dy$, and since b is bounded and its derivatives are compactly supported, we have for any k, ℓ,

$$\left|\partial_x^\ell \partial_h^k \left[\frac{\phi_+(hx)}{h}\right]\right| \leqslant C_{k\ell} x^{1+k-\ell} \tag{3.39}$$

whence

$$|\partial_x^\ell \partial_h^k P_h(hx)| \leqslant C_{k\ell} x^{2k},\ |\partial_x^\ell \partial_h^k P_h^{-1}(hx)| \leqslant C_{k\ell} x^{2k},\ \text{ for } x \geqslant 1. \tag{3.40}$$

Moreover

$$\frac{d}{dx}(P_h(hx)) = N(hx)P_h(hx) + A_1(x,h) \tag{3.41}$$

where A_1 satisfies

$$\int_1^{+\infty} |A_1(y,h)|dy \leqslant C,\ |\partial_x^\ell \partial_h^k A_1(x,h)| \leqslant C_{k\ell} x^{-1+2k},\ k,\ell \in \mathbb{N},\ x \geqslant 1 \tag{3.42}$$

with constants independent of h. If we define $Y(x,h) = P_h(hx)^{-1} X(x,h)$, we deduce from (3.35) the equation

$$Y'(x,h) = \widetilde{A}(x,h) Y(x,h) \tag{3.43}$$

where $\widetilde{A}(x,h) = P_h(hx)^{-1} A(x,h) P_h(hx) - P_h(hx)^{-1} A_1(x,h)$. It follows from (3.37), (3.40), (3.42) that

$$\int_1^{+\infty} |\widetilde{A}(y,h)|dy \leqslant C,\ |\partial_x^\ell \partial_h^k \widetilde{A}(x,h)| \leqslant C_{k\ell} x^{-1+2k},\ k,\ell \in \mathbb{N},\ x \geqslant 1. \tag{3.44}$$

The boundary condition at infinity of the statement of the lemma may be written when $v = x^{\alpha/2} W_+$ (resp. $v = x^{\alpha/2} W_-$) on the corresponding vector $Y(x,h) = P_h(hx)^{-1} X(x,h)$ as $Y(+\infty,h) = \begin{pmatrix}1\\0\end{pmatrix}$ (resp. $Y(+\infty,h) = \begin{pmatrix}0\\1\end{pmatrix}$). The Cauchy problem for (3.43) with such data at infinity may be written

$$Y(x,h) = Y(+\infty,h) - \int_x^{+\infty} \widetilde{A}(y,h) Y(y,h) dy. \tag{3.45}$$

The integrability condition of (3.44) implies that this fixed point problem has a unique solution in a neighborhood of $+\infty$, that can be extended to $]0,+\infty[$. Coming back to W_+^h, W_-^h this gives the first statement of the lemma.

ii) Set $\tilde{r}(x,h) = x^{\alpha/2} r(x,h)$, $\tilde{s}(x,h) = x^{\alpha/2} s(x,h)$, $S = \begin{pmatrix}0\\\tilde{s}\end{pmatrix}$. Then $\tilde{r}$ satisfies equation (3.32) with the right hand side 0 replaced by $\tilde{s}$, so

$X = \begin{pmatrix} \tilde{r} \\ \tilde{r}' \end{pmatrix}$ solves

$$X'(x,h) = N(hx)X + A(x,h)X + S(x,h) \tag{3.46}$$

and $X(x,h) \to 0$ if $x \to +\infty$.

If we define again $Y(x,h) = P_h(hx)^{-1}X(x,h)$, we shall have

$$Y'(x,h) = \widetilde{A}(x,h)Y(x,h) + \widetilde{S}(x,h) \tag{3.47}$$

where $\widetilde{S}(x,h) = P_h(hx)^{-1}S(x,h)$ satisfies by (3.40) and the assumptions on $s(x,h)$

$$|\partial_x^\ell \partial_h^k \widetilde{S}(x,h)| \leqslant C_{k\ell} x^{-N+2k-2+\alpha/2},\ k,\ell \in \mathbb{N},\ hx \geqslant \delta. \tag{3.48}$$

Denote by $M(x,h)$ the matrix solving

$$M'(x,h) = \widetilde{A}(x,h)M(x,h), M(1,h) = \mathrm{Id}\,.$$

It follows from (3.44) and Gronwall inequalities

$$|\partial_x^\ell \partial_h^k M(x,h)| \leqslant C_{k\ell} x^{2k},\ |\partial_x^\ell \partial_h^k M^{-1}(x,h)| \leqslant C_{k\ell} x^{2k},\ k,\ell \in \mathbb{N},\ x \geqslant 1. \tag{3.49}$$

If we define $Z(x,h) = M(x,h)^{-1}Y(x,h)$, we shall have

$$Z'(x,h) = M(x,h)^{-1}\widetilde{S}(x,h)$$

whence since $Z(+\infty,h) = 0$,

$$Z(x,h) = -\int_x^{+\infty} M(y,h)^{-1}\widetilde{S}(y,h)dy. \tag{3.50}$$

It follows from (3.49) and (3.48) that if $2k < N - \alpha/2 + 1$, $\ell \in \mathbb{N}$ and $hx \geqslant \delta$

$$|\partial_x^\ell \partial_h^k Z(x,h)| \leqslant C_{k\ell} x^{-N+2k-1+\alpha/2} \tag{3.51}$$

whence a similar estimate for $\partial_x^\ell \partial_h^k Y(x,h)$ and $\partial_x^\ell \partial_h^k X(x,h)$. This gives inequalities (3.30). □

Let us now construct WKB approximations for the functions $W_\pm^h$ defined in Lemma 3.3.

LEMMA 3.4. — *There are for any $k \in \mathbb{N}$ smooth functions $q_k^\pm$ on $]0,+\infty[$ satisfying*

$$|q_k^{\pm\,(\ell)}(\theta)| \leqslant C_{k\ell}\theta^{-k-\ell} \tag{3.52}$$

for any $k, \ell \in \mathbb{N}$, such that the solutions $W_\pm^h$ of Lemma 3.3 satisfy for any $N > 0, \delta > 0$

$$\left|\partial_h^m \partial_x^\ell \left[W_\pm^h(x) - x^{-\alpha/2} e^{\pm i\phi_+(hx)/h} \sum_{k=0}^{N} h^k q_k^\pm(hx)\right]\right| \leqslant C_{N\delta\ell} x^{-N-\alpha/2+2m-1} \tag{3.53}$$

for any $\ell \in \mathbb{N}, m < N/2 + 1/2$ uniformly for $hx \geqslant \delta$. Moreover, $q_0^\pm(\theta) \to 1$ when $\theta \to +\infty$ and there is $C > 0$ with $C^{-1} \leqslant q_0(\theta) \leqslant C$ for any $\theta \in]0, +\infty[$.

Proof. — Let us treat the case of sign +. We look first for an approximate solution v^N to equation (3.32) of the form

$$v^N(x, h) = e^{i\phi_+(hx)/h} \sum_{k=0}^{N} h^k q_k(hx). \tag{3.54}$$

Using that $\phi_+'(\theta) = b(\theta)$, $a_1(x, h) = hg_1(hx)$ and $a_2(x, h) = h^2 \widetilde{a_2}(hx)$, we get

$$\begin{aligned}(\partial_x^2 + a_1(x,h)\partial_x &+ [b(hx)^2 + a_2(x,h)])v^N(x,h) \\ &= e^{i\phi_+(hx)/h} \sum_{k=0}^{N+1} h^{k+1}[2ib(hx)q_k'(hx) \\ &\quad + i(b'(hx) + g_1(hx)b(hx))q_k(hx) + \widetilde{a_2}(hx)q_{k-1}(hx) \\ &\quad + g_1(hx)q_{k-1}'(hx) + q_{k-1}''(hx)]\end{aligned} \tag{3.55}$$

where we have set by convention $q_{-1} = q_{N+1} \equiv 0$. Define

$$B(\theta) = b(\theta)^{1/2} \exp\left[\frac{1}{2}\int_0^\theta g_1(\theta')d\theta'\right].$$

This is a smooth function on $[0, +\infty[$, constant in a neighborhood of infinity. We want v^N to be an approximate solution to equation (3.32). Using (3.55) we set for $k = 0, \ldots, N$

$$\begin{aligned}2ib(hx)q_k'(hx) &+ i(b'(hx) + g_1(hx)b(hx))q_k(hx) \\ &= -\widetilde{a_2}(hx)q_{k-1}(hx) - g_1(hx)q_{k-1}'(hx) - q_{k-1}''(hx)\end{aligned}$$

which may be written

$$\begin{aligned}B(\theta)^{-1}\frac{d}{d\theta}(B(\theta)q_k(\theta)) &= G(q_{k-1}(\theta)),\ k = 0, \ldots, N, \\ G(q(\theta)) &= -\frac{\widetilde{a_2}(\theta)}{2ib(\theta)}q(\theta) - \frac{g_1(\theta)}{2ib(\theta)}q'(\theta) - \frac{1}{2ib(\theta)}q''(\theta).\end{aligned} \tag{3.56}$$

We solve (3.56) defining

$$(3.57)\qquad \begin{aligned} q_0(\theta) &= \frac{B(+\infty)}{B(\theta)} \\ q_k(\theta) &= -B(\theta)^{-1}\int_\theta^{+\infty} B(\theta')G(q_{k-1})(\theta')d\theta',\ k\geqslant 1 \end{aligned}$$

and we obtain (3.52) using that in the definition of G, g_1 is supported for $\theta\leqslant C$ and that we have (3.34). By (3.55) we thus get that v^N is a solution of (3.32) in which the right hand side 0 has been replaced by a function of the form $h^{N+2}e^{i\phi_+(hx)/h}R(hx)$, where $R(\theta)$ is smooth in $]0,+\infty[$ and satisfies

$$(3.58)\qquad |R^{(\ell)}(\theta)|\leqslant C_\ell\theta^{-N-2-\ell}.$$

If we define $r(x,h)=W_+^h(x)-x^{-\alpha/2}v^N$ we thus get a solution to (3.29) for a right hand side $s(x,h)$ given by $-x^{-\alpha/2}h^{N+2}e^{i\phi_+(hx)/h}R(hx)$. Using (3.39) and (3.58), this $s(x,h)$ satisfies the assumptions of ii) of Lemma 3.3, with N replaced by $N+\alpha/2$. Moreover, by assumption on W_+^h and construction of v^N, $x^{\alpha/2}r(x,h)\to 0$, $x^{\alpha/2}r'(x,h)\to 0$ if $x\to+\infty$. We deduce from (3.30) the estimate (3.53). □

Proof of Proposition 3.2. — We shall prove that $\varphi_+^h(\theta)$ has asymptotics given by (3.28). Remind that we defined $w_+^h(x)=\varphi_+^h(hx)$. This is a solution of (3.20). Since W_+^h, W_-^h of Lemma 3.3 are a basis of solutions to this equation, there are constants $\alpha_\pm(h)$ with

$$(3.59)\qquad w_+^h(x)=\alpha_+(h)W_+^h(x)+\alpha_-(h)W_-^h(x).$$

Denote by Ω^h the Wronskian of (W_+^h,W_-^h). By Lemma 3.3,

$$x^{\alpha/2}e^{\mp i\phi_+(hx)/h}W_\pm^h(x)\to 1$$

and

$$(x^{\alpha/2}e^{\mp i\phi_+(hx)/h}W_\pm^h(x))'\to 0$$

when $x\to+\infty$. This implies

$$\lim_{x\to+\infty}x^\alpha\Omega^h(x)=-2ib(+\infty).$$

Thus, as W_+^h and W_-^h are solutions of (3.20), we have

$$x^\alpha\Omega^h(x)=-2ib(+\infty)\exp\left(\int_{hx}^{+\infty}g_1(t)dt\right).$$

Therefore, if we fix $x=x_0$ with $x_0>0$, the Wronskian of (W_+^h,W_-^h) computed at x_0, as well as its inverse, will be a smooth function of h. We then deduce from (3.59), and from the fact that $w_+^h(x_0)$, $\frac{d}{dx}w_+^h(x_0)$ are, by

Proposition 3.1, smooth functions of h in a neighborhood of 0, that $\alpha_\pm(h)$ are smooth functions of h when h is close to 0. Plugging the expansion (3.53) in (3.59) we obtain (3.28), setting $x = \theta/h$. □

Proof of Proposition 1.7. — Remind that $\alpha = d-2$ and $h = 1/\lambda$. As $d \geqslant 3$ by (1.10), we have $\alpha \geqslant 1$.

We first show that λ^2 is an eigenvalue of (3.14) if and only if the functions $\varphi^h_\pm(\theta)$ of Proposition 3.1 are linearly dependent. Since an eigenfunction φ satisfies $-\Delta_g\varphi(x) + V(x)\varphi(x) = \lambda^2\varphi(x)$ on M, multiplying by φ and integrating by parts yields

$$\int_M (|\nabla_g\varphi(x)|^2 + V(x)\varphi(x)^2)d\mathrm{vol}_g(x) = \lambda^2\int_M \varphi(x)^2 d\mathrm{vol}_g(x). \tag{3.60}$$

As φ only depends on θ, (3.60) implies

$$\int_0^1 b(\theta)^{-1}r(\theta)^{d-2}|\partial_\theta\varphi(\theta)|^2 d\theta = \int_0^1 (\lambda^2 - V(\theta))b(\theta)r(\theta)^{d-2}|\varphi(\theta)|^2 d\theta. \tag{3.61}$$

The right hand side of (3.61) is bounded using (3.13) and so

$$\int_0^1 b(\theta)^{-1}r(\theta)^{d-2}|\partial_\theta\varphi(\theta)|^2 d\theta < +\infty \tag{3.62}$$

which yields in particular

$$\int_0^1 |\partial_\theta\varphi(\theta)|^2\theta^{d-2}d\theta < +\infty. \tag{3.63}$$

Assume that $\varphi(\theta)$ does not belong to the vector space spanned by $\varphi^h_+(\theta)$. Then, $\partial_\theta\varphi(\theta) \sim c\theta^{-\alpha}$ when $\theta \to 0_+$ by Proposition 3.1 which is in contradiction with (3.63) as $\alpha = d-2 \geqslant 1$. Thus, $\varphi(\theta)$ belongs to the vector space spanned by $\varphi^h_+(\theta)$. Looking at $\theta \to 1_-$ we deduce in the same manner that $\varphi(\theta)$ belongs to the vector space spanned by $\varphi^h_-(\theta)$. Therefore, the eigenvalues of $P|_W$ are simple and λ^2 is an eigenvalue of (3.14) if and only if $\varphi^h_\pm(\theta)$ are linearly dependent.

It remains to prove that the λ^2 such that $\varphi^h_\pm(\theta)$ are linearly dependent satisfy (1.2). Let $\theta_0 \in]0,1[$ be determined by $\int_0^{\theta_0} b(\theta')d\theta' = 1/2\int_0^1 b(\theta')d\theta'$ so that $\phi_+(\theta_0) + \phi_-(\theta_0) = 0$. By Proposition 3.2, we have an expansion

$$\varphi_h^\pm(\theta) = h^{\frac{d-2}{2}}\Big(2\,\mathrm{Re}[e^{i\phi_\pm(\theta)/h}M_\pm(\theta,h)] + h^N R_\pm(\theta,h)\Big) \tag{3.64}$$

where $M_\pm(\theta,h) = \sum_{k=0}^N h^k M^k_\pm(\theta)$ and where $R_\pm(\theta,h)$ is smooth for $h \in]0,1]$ and satisfies when $\ell \in \mathbb{N}, m \in \mathbb{N}, m < N/2 + 1/2$,

$$|\partial_h^m\partial_\theta^\ell R^k_\pm(\theta,h)| \leqslant C_{\ell m}h^{-\ell-2m+1} \tag{3.65}$$

uniformly for $\theta \in [\delta, 1-\delta]$. Define

$$\Phi_\pm(h) = h^{-\frac{d-2}{2}} \begin{pmatrix} \varphi_h^\pm(\theta_0) \\ h\frac{d}{d\theta}\varphi_h^\pm(\theta)|_{\theta=\theta_0} \end{pmatrix}.$$

By (3.64) we may write

(3.66)

$$\Phi_\pm(h) = e^{\pm i\phi_+(\theta_0)/h} \begin{pmatrix} M_\pm(\theta_0,h) \\ M'_\pm(\theta_0,h) \end{pmatrix} + e^{\mp i\phi_+(\theta_0)/h} \begin{pmatrix} \overline{M_\pm}(\theta_0,h) \\ \overline{M'_\pm}(\theta_0,h) \end{pmatrix} + h\widetilde{R}^\pm(h)$$

where

$$M'_\pm(\theta_0,h) = ib(\theta_0)M_\pm(\theta_0,h) + h\frac{d}{d\theta}M_\pm(\theta_0,h), \tag{3.67}$$

and where by (3.65) $\widetilde{R}^\pm$ is a C^β function of h for any integer $\beta < N/2$. The Wronskian of $(\varphi_h^+(\theta), \varphi_h^-(\theta))$ at $\theta = \theta_0$ multiplied by $h^{-(d-3)}$ may be written

$$\det[\Phi_+(h), \Phi_-(h)] = \operatorname{Re}\mu(h) + \operatorname{Re}(e^{2i\phi_+(\theta_0)/h}\nu(h)) + hr(h), \tag{3.68}$$

where r is a C^β function of h for any integer $\beta < N/2$, and

$$\mu(h) = \begin{vmatrix} M_+(\theta_0,h) & M_-(\theta_0,h) \\ M'_+(\theta_0,h) & M'_-(\theta_0,h) \end{vmatrix},\ \nu(h) = \begin{vmatrix} M_+(\theta_0,h) & \overline{M_-}(\theta_0,h) \\ M'_+(\theta_0,h) & \overline{M'_-}(\theta_0,h) \end{vmatrix}. \tag{3.69}$$

These are smooth functions of h and by (3.67) $\mu(0) = 0$. Moreover, since when h stays small $|M_\pm(\theta_0,h)|$ remains between two positive constants, (3.67) shows that $\nu(h)$ stays also between two positive constants. Consequently we can find a smooth real valued function ω of h, defined on a neighborhood of 0, a C^β function σ of h for any integer $\beta < N/2$, such that (3.68) vanishes if and only if

$$\cos\left[2\frac{\phi_+(\theta_0)}{h} + \omega(h)\right] = h\sigma(h). \tag{3.70}$$

But $\lambda^2 = h^{-2}$ is an eigenvalue if and only if the Wronskian of $(\varphi_h^+, \varphi_h^-)$ vanishes at θ_0 *i.e.,* if and only if (3.70) holds true. The set of all small positive solutions of this equation is given by two families $h_\pm(k)$ indexed by large enough $k \in \mathbb{N}$ and satisfying

$$\frac{1}{h_\pm(k)} = \frac{2k\pi}{a_0} \pm \frac{\pi}{2a_0} + a_1 + \mathcal{O}\Big(\frac{1}{k}\Big),\ k \to +\infty \tag{3.71}$$

for convenient $a_0 > 0, a_1 \in \mathbb{R}$. This concludes the proof. □

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