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ON HALPHEN'S THEOREM AND SOME GENERALIZATIONS

by Alcides LINS NETO (*)

ABSTRACT. — Let M^n be a germ at $0 \in \mathbb{C}^m$ of an irreducible analytic set of dimension n, where $n \geqslant 2$ and 0 is a singular point of M. We study the question: when does there exist a germ of holomorphic map $\phi\colon (\mathbb{C}^n, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$? We prove essentialy three results. In Theorem 1 we consider the case where M is a quasi-homogeneous complete intersection of k polynomials $F = (F_1, \ldots, F_k)$, that is there exists a linear holomorphic vector field X on $\mathbb{C}^m$, with eigenvalues $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}_+$ such that $X(F^T) = U \cdot F^T$, where U is a $k \times k$ matrix with entries in $\mathcal{O}_m$. We prove that if there exists a germ of holomorphic map ϕ as above and $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$, then $\lambda_1 + \cdots + \lambda_m > \mathrm{Re}(\mathrm{tr}(U)(0))$. In Theorem 2 we answer the question completely when $n = 2$, $k = 1$ and 0 is an isolated singularity of M. In Theorem 3 we prove that, if there exists a map as above, $k = 1$ and $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$, then $\dim_{\mathbb{C}}(\mathrm{sing}(M)) = n-2$. We observe that Theorems 1 and 2 are generalizations of some results due to Halphen.

RÉSUMÉ. — Soit M^n un germe en $0 \in \mathbb{C}^m$ d'ensemble analytique irréductible de dimension n, où $n \geqslant 2$ et 0 est un point singulier de M. Nous étudions le problème suivant : quand est-ce qu'il existe un germe d'application holomorphe $\phi\colon (\mathbb{C}^m, 0) \to (M, 0)$ telle que $\phi^{-1}(0) = \{0\}$? Nous démontrons essentiellement trois résultats. Dans le théorème 1 nous considérons le cas où M est une intersection complète quasi-homogène de k polynômes $F = (F_1, \ldots, F_k)$, c'est-à-dire il existe un champ de vecteurs linéaire holomorphe X dans $\mathbb{C}^m$, avec valeurs propres $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}_+$ telles que $X(F^T) = U \cdot F^T$, où U est une matrice $k \times k$ d'éléments dans $\mathcal{O}_m$. Nous démontrons que s'il existe un germe d'application ϕ comme précédemment et $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$ alors $\lambda_1 + \cdots + \lambda_m > \mathrm{Re}(\mathrm{tr}(U))(0)$. Dans le théorème 2 nous répondons complètement à la question quand $n = 2$, $k = 1$ et 0 est une singularité isolée de M. Dans le théorème 3 nous démontrons que, s'il existe une application ϕ comme précédemment, $k = 1$ et $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$, alors $\dim_{\mathbb{C}}(\mathrm{sing}(M)) = n-2$. Remarquons que les théorèmes 1 et 2 sont des généralisations de quelques résultats de Halphen.

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1. Introduction

Around 1884 Halphen proved the following result (*cf.* [9] or [10], chap. I, p. 15):

Theorem. — *Let f, g and h be three (non zero) homogeneous polynomials in $\mathbb{C}^3$, two by two without common factors. Suppose that $f^p + g^q + h^r \equiv 0$, where p, q, r are integers, $2 \leqslant p \leqslant q \leqslant r$ and $p\cdot \deg(f) = q\cdot \deg(g) = r\cdot \deg(h)$. Then*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 \tag{1.1}$$

Moreover, for each solution of the inequality (1.1)*, then*

(a) *There exist homogeneous polynomials F, G, H in $\mathbb{C}^2$ such that $F^p + G^q + H^r \equiv 0$.*

(b) *If f, g, h are three homogeneous polynomials in $\mathbb{C}^n$ without common factors which satisfy $f^p+g^q+h^r \equiv 0$, then there exists a homogeneous map $\phi\colon \mathbb{C}^n \to \mathbb{C}^2$ such that $(f, g, h) = (F, G, H) \circ \phi$.*

In other words, we can say that for each solution (p, q, r) of the inequality (1.1), there exists a map $\psi = (F, G, H)\colon \mathbb{C}^2 \to M$, where $M = \{(X, Y, Z) \in \mathbb{C}^3 | X^p + Y^q + Z^r = 0\}$, such that if $M^* = M \setminus \{0\}$ and $\psi_1 := \psi|_{\mathbb{C}^2 \setminus \{0\}}$, then $\psi_1\colon \mathbb{C}^2 \setminus \{0\} \to M^*$ is the holomorphic universal covering of M^*.

The purpose of this paper is to generalize this result in two ways. First of all, we will generalize inequality (1.1) for germs of holomorphic maps $\phi\colon (\mathbb{C}^n, 0) \to (M^n, 0)$, where $M^n \subset \mathbb{C}^m$, $m = n+k$, is a quasi-homogeneous complete intersection defined by polynomials $F_1 = \cdots = F_k = 0$. In order to state our first result, we need some definitions.

Definition 1.1. — *Let $M \neq \{0\}$, be a germ at $0 \in \mathbb{C}^m$ of an analytic set defined by an ideal $\mathcal{I}$ of germs at $0 \in \mathbb{C}^m$ of holomorphic functions. We say that M is* quasi-homogeneous, *if there exists a germ at $0 \in \mathbb{C}^m$ of holomorphic vector field X with the following properties:*

(a) *There exists a local holomorphic coordinate system $(x_1, \ldots, x_m)$ around $0 \in \mathbb{C}^m$ where $X = \sum_{j=1}^m \lambda_j\cdot x_j \frac{\partial}{\partial x_j}$ and $\lambda_j \in \mathbb{Q}_+$ for all $j = 1, \ldots, m$.*

(b) *$X(\mathcal{I}) := \{X(F) | F \in \mathcal{I}\} \subset \mathcal{I}$.*

In this case, we will say that M is quasi-homogeneous with respect to X *(briefly q.h.w.r. to X).*

Remark 1.2. — Condition (b) means that X is tangent to M and M is invariant by the flow X_T of the vector field X: Take a representative $\widetilde{M} \subset B$ of M, where B is a ball around $0 \in \mathbb{C}^m$ and $\widetilde{M}$ is a closed analytic subset of B. If $p \in \widetilde{M}$ and $T \in \mathbb{C}$ is such that $X_T(p) \in B$ then $X_T(p) \in \widetilde{M}$. In fact M is the germ of a global analytic subset of $\mathbb{C}^m$: Since $\lambda_1, \ldots, \lambda_m > 0$, we get that $\mathrm{sat}(B) := \{X_T(p) | p \in B\} = \mathbb{C}^m$. This implies that $\mathrm{sat}(\widetilde{M}) = \{X_T(p) | p \in \widetilde{M}, T \in \mathbb{C}\}$ is an analytic subset of $\mathbb{C}^m$ which extends $\widetilde{M}$ and the germ at 0 of $\mathrm{sat}(M)$ is M. From now on a quasi-homogeneous analytic set will be considered as an analytic subset of $\mathbb{C}^m$, for some m.

Remark 1.3. — The name quasi-homogeneous is motivated by the situation where $\mathcal{I} =< F >$ and F is quasi-homogeneous, that is there are $k_1, \ldots, k_m, \ell \in \mathbb{N}$ such that $F(T^{k_1} \cdot x_1, \ldots, T^{k_m} \cdot x_m) = T^{\ell} \cdot F(x_1, \ldots, x_m)$. In this case, if we take $X = \sum_{j=1}^{m} \frac{k_j}{\ell} x_j \frac{\partial}{\partial x_j}$ then $X(F) = F$ and $M = F^{-1}(0)$ is q.h.w.r. to X. Note that the relation $X(F) = F$ implies that F is a polynomial. An example is $F(x_1, \ldots, x_m) = x_1^{n_1} + \cdots + x_m^{n_m}$ and $X = \sum_{j=1}^{m} \frac{1}{n_j} x_j \frac{\partial}{\partial x_j}$, where $X(F) = F$ and F is q.h.w.r. to X. This example will be used in Corollary 1.7.

In our first result we will consider the following situation: $M^n \subset \mathbb{C}^m$, $m = n + k$, will be an irreducible complete intersection of k polynomials $F_1, \ldots, F_k$. We suppose that M is q.h.w.r. to a diagonal vector field $X = \sum_{j=1}^{m} \lambda_j x_j \frac{\partial}{\partial X_j}$, where $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}_+$. The condiction that M is q.h.w.r. to X means the following: let $F = (F_1, \ldots, F_k)^T$, where $(\cdots)^T$ is the transpose of the vector $(\cdots)$. Then M is q.h.w.r. to X if, and only if,

$$X(F) = U \cdot F \tag{1.2}$$

where $X(F) = (X(F_1), \ldots, X(F_k))^T$ and $U = (u_{ij})_{1 \leqslant i,j \leqslant k}$ is a $k \times k$ matrix with entries $u_{ij} \in \mathcal{O}_{m+k}$. We set $\mathrm{tr}(U) = \sum_{j=1}^{k} u_{jj}$.

Definition 1.4. — *Let M^n be an irreducible analytic subset of dimension n of a ball $B \subset \mathbb{C}^m$. We will denote by $\mathrm{sing}(M)$ the singular set of M. We will say that $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant k$ if, either $\mathrm{sing}(M) = \emptyset$, or $\mathrm{sing}(M) \neq \emptyset$ and all irreducible components of $\mathrm{sing}(M)$ have complex dimension $\leqslant k$. We will say that $\dim_{\mathbb{C}}(\mathrm{sing}(M)) = k$ if $\mathrm{sing}(M) \neq \emptyset$ and all irreducible components of $\mathrm{sing}(M)$ have complex dimension k. Let $p \in M$ and $\phi\colon (\mathbb{C}^n, q) \to (M, p)$ be a germ of holomorphic map. We will say that $\phi^{-1}(p) = \{q\}$ if there exists a representative of ϕ, denoted again by ϕ, say $\phi\colon V \to M$, such that $\phi^{-1}(p) \cap V = \{q\}$.*

The first generalization is the following:

THEOREM 1.5. — *Let $n \geqslant 2$ and $M^n \subset \mathbb{C}^m$, $m = n+k$, be an irreducible complete intersection defined by $(F_1 = \cdots = F_k = 0)$, q.h.w.r. to the linear vector field*

$$X(z) = \sum_{j=1}^{m} \lambda_j z_j \frac{\partial}{\partial z_j}$$

with $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}_+$. Let $X(F) = U \cdot F$ and $\Lambda = \mathrm{Re}(\mathrm{tr}(U)(0))$, where F and U are as in (1.2). *Suppose that $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$ and that there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^n, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$. Then $\sum_{j=1}^m \lambda_j > \Lambda$.*

As a particular case, we get the following:

COROLLARY 1.6. — *Let $M \subset \mathbb{C}^m$, $m = n+k$, be an irreducible complete intersection $(F_1 = \cdots = F_k = 0)$ with $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$. Suppose that there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^n, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$ and a linear vector field X as in Definition 1.1 such that $X(F_j) = \ell_j \cdot F_j$, $\forall j$, where $\ell_j \in \mathbb{Q}_+$, $j = 1, \ldots, k$. Then $\sum_{j=1}^m \lambda_j > \sum_{i=1}^k \ell_i$.*

We observe that the above result is no longer true if $\mathrm{sing}(M)$ has some component of dimension $n-1$ (see Example 1.16).

As a consequence, we obtain a generalization of the first part of Halphen's theorem:

COROLLARY 1.7. — *Let $M_{(p,q,r)} \subset \mathbb{C}^3$ be the surface given by $x^p + y^q + z^r = 0$, where $p, q, r \in \mathbb{N}$ and $p \leqslant q \leqslant r$. Suppose that there exists a holomorphic map $\phi\colon U \to M_{(p,q,r)}$, where U is some neighborhood of $0 \in \mathbb{C}^n$, $n \geqslant 2$, such that $\phi(0) = 0 \in M$ and $\phi^{-1}(0) = \{0\}$. Then $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ and, if $2 \leqslant p \leqslant q \leqslant r$, then $(p, q, r) \in \{(2, 2, r), (2, 3, 3), (2, 3, 4), (2, 3, 5)\}$.*

In the next two results we will consider germs at $0 \in \mathbb{C}^{n+1}$ of hypersurfaces. We need another definition.

DEFINITION 1.8. — *Let M_1, M_2 be two germs at $0 \in \mathbb{C}^m$ of analytic sets. We will say that M_1 and M_2 are* equivalent *if there exists a germ of biholomorphism $\psi\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ such that $\psi(M_1) = M_2$.*

The second generalization is the folowing:

THEOREM 1.9. — *Let M be a germ at $0 \in \mathbb{C}^3$ of hypersurface with an isolated singularity at 0. Suppose there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^2, 0) \to (M, 0)$, such that $\phi^{-1}(0) = \{0\}$. Then M is equivalent to one of the following surfaces:*

(a) *$M_{(p,q,r)}$, where $(p, q, r) \in \{(2, 2, r), (2, 3, 3), (2, 3, 4), (2, 3, 5)\}$.*

(b) $X_m = \{(x,y,z) \in \mathbb{C}^3 | z^2 = xy(y - x^{m+1})\}$, *where* $m \geqslant 1$.
(c) $Y = \{(x,y,z) \in \mathbb{C}^3 | z^2 = y(y^2 + x^3)\}$.
(d) $Z_m = \{(x,y,z) \in \mathbb{C}^3 | z^2 = x(y^2 + x^{2m+1})\}$, *where* $m \geqslant 1$.

Moreover, the surfaces in (a)–(d) *are two by two non-equivalent.*

Concerning the dimension of the singular set of M we have the following result:

THEOREM 1.10. — *Let M be a germ at $0 \in \mathbb{C}^{n+1}$, $n \geqslant 3$, of hypersurface where* $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$. *Suppose there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^n, 0) \to (M, 0)$, such that $\phi^{-1}(0) = \{0\}$. If $0 \in \mathrm{sing}(M)$ then* $\dim_{\mathbb{C}}(\mathrm{sing}(M)) = n-2$.

Observe that Corollary 1.7 of Theorem 1.5 could be stated for hypersurfaces of the form $M_{(n_1,\ldots,n_m)} = \{x_1^{n_1} + \cdots + x_m^{n_m} = 0\}$, for any $m \geqslant 3$ (see Remark 1.3). However, Theorem 1.10 implies that for $m \geqslant 4$ there is no germ of holomorphic map $\phi\colon (\mathbb{C}^{m-1}, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$, because $\mathrm{sing}(M_{(n_1,\ldots,n_m)}) = \{0\}$.

In the next four examples we show that for any one of the surfaces as in (a), (b), (c) or (d), there exists a regular map ϕ like in Theorem 1.9. In all the examples, the map $\phi|_{\mathbb{C}^2 \setminus \{0\}}\colon \mathbb{C}^2 \setminus \{0\} \to M^*$ is a universal covering of $M^* = M \setminus \{0\}$ (see also [16], [12], [9] and [14]).

Example 1.11. — The parametrizations $\phi\colon \mathbb{C}^2 \to M_{(p,q,r)}$, where p, q, r satisfy the inequality (1.1), is closely related with Platonic solids and to the non-cyclic finite subgroups of $PSL(2,\mathbb{C})$. Some of them were known already by Euler, Hoppe, Liouville and others, but the general case was found by Schwarz (*cf.* [16] and also [12], [9], [3] and [14]). If $2 \leqslant p \leqslant q \leqslant r$ then, the possible solutions of inequality (1.1) are $(p,q,r) \in \{(2,2,r), (2,3,3), (2,3,4), (2,3,5)\}$. In each case, the holomorphic map $\phi = (F,G,H)\colon \mathbb{C}^2 \to M_{(p,q,r)}$ can be obtained by considering a finite subgroup of $PSL(2,\mathbb{C})$. These groups were classified by Klein and are the following (*cf.* [5] and [3]):

(a) The Dihedral group of order $2r$. From this group it can be obtained the parametrization of $M_{(2,2,r)}$.
(b) The Tetrahedral group, the group of isometries of $\overline{\mathbb{C}} \simeq S^2 \subset \mathbb{R}^3$ which leaves invariant the regular tetrahedral inscribed in S^2. From this group it can be obtained the parametrization of $M_{(2,3,3)}$.
(c) The Octahedral group, the group of isometries of S^2 which leaves invariant the regular octahedral (or cube) inscribed in S^2. From this group it can be obtained the parametrization of $M_{(2,3,4)}$.

(d) The Icosahedral group, the group of isometries of S^2 which leaves invariant the regular icosahedral (or dodecahedron) inscribed in S^2. From this group it can be obtained the parametrization of $M_{(2,3,5)}$.

Some explicit formulae for the uniformizations can be found in [3], p. 55–56. We observe that, in all cases, the map ϕ is such that $\phi|_{\mathbb{C}^2\setminus\{0\}}\colon \mathbb{C}^2\setminus\{0\} \to M^*_{(p,q,r)}$ is a universal covering of $M^*_{(p,q,r)} := M_{(p,q,r)} \setminus \{0\}$ (*cf.* [14]). Moreover, we have the following:

(a) In the case (2,2,r), ϕ has topological degree r and $\#(\pi_1(M^*_{(2,2,r)})) = r$.
(b) In the case (2,3,3), ϕ has topological degree 8 and $\#(\pi_1(M^*_{(2,3,3)})) = 8$.
(c) In the case (2,3,4), ϕ has topological degree 24 and $\#(\pi_1(M^*_{(2,3,4)})) = 24$.
(d) In the case (2,3,5), ϕ has topological degree 120 and $\#(\pi_1(M^*_{(2,3,5)})) = 120$.

In the next three examples we will use that $M_{(p,q,r)}$ is equivalent to the surfaces given by $a.x^p + b.y^q + c.z^r = 0$, where $a, b, c \in \mathbb{C}^*$.

Example 1.12. — Let $X_m = \{(x, y, z) \in \mathbb{C}^3 | z^2 = xy^2 - x^{m+1}y\}$ and $M_{(2,2,2m)}$ be given as $\{(u, v, w) \in \mathbb{C}^3 | u^{2m} - v^2 + w^2 = 0\}$. Consider the map $\varphi\colon \mathbb{C}^3 \to \mathbb{C}^3$ defined by $(x, y, z) = \varphi(u, v, w) = (u^2, v^2, u.v.w)$. Note that

$$z^2 - x\cdot y^2 + x^{m+1}\cdot y = u^2\cdot v^2(w^2 - v^2 + u^{2m}) \implies \varphi(M_{(2,2,2m)}) \subset X_m.$$

Let $\psi = \varphi|_{M_{(2,2,2m)}}\colon M_{(2,2,2m)} \to X_m$. It is easy to see that $\psi^{-1}(0) = \{0\}$ and $\#(\psi^{-1}(p_0)) = 4$ for all $p_0 \in X_m \setminus \{0\}$. This implies that

$$\psi|_{M^*_{(2,2,2m)}}\colon M^*_{(2,2,2m)} \to X^*_m$$

is a covering map with four sheets. Therefore, if $\psi_1\colon \mathbb{C}^2 \to M_{(2,2,2m)}$ is as in (a) of Example 1.11, then $\phi = \psi \circ \psi_1\colon \mathbb{C}^2 \to X_m$ satisfies $\phi^{-1}(0) = \{0\}$. Moreover, $\phi|_{\mathbb{C}^2\setminus\{0\}}\colon \mathbb{C}^2 \setminus \{0\} \to X^*_m$ is a (universal) covering map with $8m$ sheets. In particular, we have $\#(\pi_1(X^*_m)) = 8m$. Observe that X_m is q.h.w.r. to the vector field

$$X = \frac{1}{2m+1}x\frac{\partial}{\partial x} + \frac{m}{2m+1}y\frac{\partial}{\partial y} + \frac{1}{2}z\frac{\partial}{\partial z}.$$

Example 1.13. — Let $Y = \{(x, y, z) \in \mathbb{C}^3 | z^2 = y(y^2 + x^3)\}$ and $M_{(2,3,4)}$ be given as $\{(u, v, w) \in \mathbb{C}^3 | u^2 - v^3 - w^4 = 0\}$. Consider the map $\varphi\colon \mathbb{C}^3 \to \mathbb{C}^3$ defined by $(x, y, z) = \varphi(u, v, w) = (u, w^2, u\cdot w)$. It can be checked that $\varphi(M_{(2,3,4)}) \subset Y$ and that, if $\psi := \varphi|_{M_{(2,3,4)}}\colon M_{(2,3,4)} \to Y$ then $\psi^{-1}(0) = \{0\}$ and $\psi|_{M^*_{(2,3,4)}}\colon M^*_{(2,3,4)} \to Y^*$ is a covering with two sheets. Therefore, if $\psi_1\colon \mathbb{C}^2 \to M_{(2,3,4)}$ is as in (c) of Example 1.11, then $\phi = \psi \circ \psi_1\colon \mathbb{C}^2 \to Y$

satisfies $\phi^{-1}(0) = \{0\}$. Moreover, $\phi|_{\mathbb{C}^2 \setminus \{0\}} \colon \mathbb{C}^2 \setminus \{0\} \to Y$ is a (universal) covering map with 48 sheets. In particular, we have $\#(\pi_1(Y^*)) = 48$. Observe that Y is quasi-homogeneous with respect to the vector field

$$X = \frac{2}{9}x\frac{\partial}{\partial x} + \frac{1}{3}y\frac{\partial}{\partial y} + \frac{1}{2}z\frac{\partial}{\partial z}.$$

Example 1.14. — Let $Z_m = \{(x,y,z) \in \mathbb{C}^3 | z^2 = x(y^2 + x^{2m+1})\}$ and $M_{(2,2,2(2m+1))}$ be given as $\{(u,v,w) \in \mathbb{C}^3 | u^{2(2m+1)} + v^2 - w^2 = 0\}$. Consider the map $\varphi \colon \mathbb{C}^3 \to \mathbb{C}^3$ defined by $(x,y,z) = \varphi(u,v,w) = (u^2, v, u\cdot w)$. It can be checked that $\varphi(M_{(2,2,2(2m+1))}) \subset Z_m$ and that, if $\psi := \varphi|_{M_{(2,2,2(2m+1))}} \colon M_{(2,2,2(2m+1))} \to Z_m$ then $\psi^{-1}(0) = \{0\}$. As in Examples 1.12 and 1.13,

$$\psi|_{M^*_{(2,2,2(2m+1))}} \colon M^*_{(2,2,2(2m+1))} \to Z^*_m$$

is a covering with two sheets and if $\psi_1 \colon \mathbb{C}^2 \to M_{(2,2,2(2m+1))}$ is as in (a) of Example 1.11, then $\phi = \psi \circ \psi_1 \colon \mathbb{C}^2 \to Z_m$ satisfies $\phi^{-1}(0) = \{0\}$. Moreover, $\phi|_{\mathbb{C}^2 \setminus \{0\}} \colon \mathbb{C}^2 \setminus \{0\} \to Z^*_m$ is a (universal) covering map with $4(2m+1)$ sheets. In particular, we have $\#(\pi_1(Z^*_m)) = 4(2m+1)$. Observe that Z_m is quasi-homogeneous with respect to the vector field

$$X = \frac{1}{2(m+1)}x\frac{\partial}{\partial x} + \frac{2m+1}{4(m+1)}y\frac{\partial}{\partial y} + \frac{1}{2}z\frac{\partial}{\partial z}.$$

Let us give an example in higher dimension.

Example 1.15. — Let

$$M = \Big\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} | z_0^p = z_1 \cdots z_n\Big\}.$$

We have the following map $\phi \colon \mathbb{C}^n \to M$,

$$\phi = (\phi_0, \dots, \phi_n), \text{ where } \phi_0(u_1, \dots, u_n) = u_1 \cdots u_n,$$

and

$$\phi_j(u_1, \dots, u_n) = u_j^p, j = 1, \dots, n.$$

Observe that $\dim_{\mathbb{C}}(\mathrm{sing}(M)) = n-2$, $\phi^{-1}(0) = \{0\}$ and M is quasi-homogeneous, that is $X(z_0^p - z_1 \cdots z_n) = z_0^p - z_1 \cdots z_n$, where

$$X = \frac{1}{p}z_0\frac{\partial}{\partial z_0} + \frac{1}{n}\sum_{j=1}^{n} z_j\frac{\partial}{\partial z_j}.$$

Example 1.16. — In this example we show that the hypothesis $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$ is the best possible in Theorem 1.5. Let $M = \{(x_0, x_1, x_2, \dots, x_n) \in \mathbb{C}^{n+1} | F(x) = x_0^6 - x_1^3 \cdot x_2^2 \cdots x_n^2 = 0\}$. Then M is irreducible and $\mathrm{sing}(M) = \cup_{j=1}^n S_j$, where $S_j = \{x_0 = x_j = 0\}$ and

$\dim_{\mathbb{C}}(S_j) = n-1$. The reader can easily verify that the map $\phi\colon \mathbb{C}^n \to M$ defined by

$$\phi(u_1, u_2, \ldots, u_n) = (u_1 \cdots u_n, u_1^2, u_2^3, \ldots, u_n^3)$$

satisfies $\phi^{-1}(0) = \{0\}$. On the other hand, let $X = \sum_{j=0}^n \lambda_j x_j \frac{\partial}{\partial x_j}$ be a vector field such that $X(F) = F$ and $\lambda_0, \ldots, \lambda_n \in \mathbb{Q}_+$. Then we must have $\lambda_0 = \frac{1}{6}$ and $3\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_n = 1$. But this implies that $\lambda_1 + \cdots + \lambda_n < \frac{1}{2}$ and so $\sum_{j=0}^n \lambda_j < \frac{1}{6} + \frac{1}{2} < 1$.

Remark 1.17. — We would like to observe that the conclusion of Theorem 1.10 is not true if ϕ is not holomorphic. Indeed, there are examples of hypersurfaces of the form

$$M_p = \left\{ (x_0, \ldots, x_n) \in \mathbb{C}^{n+1} | x_0^{p_0} + \cdots + x_n^{p_n} = 0 \right\},$$

with $n \geqslant 3$ and $p_0, \ldots, p_n \geqslant 2$ such that $K_r = M_p \cap S_r$ is homeomorphic to a sphere S^{2n-1} (*cf.* [11] and [14]), where $S_r = \{(x_0, \ldots, x_n) | |x_0|^2 + \cdots + |x_n|^2 = r^2\}$. Since M_p is homeomorphic to a cone over K_r (*cf.* [14]), then M_p is homeomorphic to $\mathbb{C}^n$ in these cases and there exists a continuous map $\phi\colon \mathbb{C}^n \to M_p$ satisfiyng the hypothesis of Theorem 1.10, but $\dim_{\mathbb{C}}(\mathrm{sing}(M_p)) = 0$. An example of such hypersurfaces is when $p_0 = 3$, $p_1 = \cdots = p_n = 2$ and n is odd (*cf.* [14]).

We would like to state the following problems:

Problem 1.18. — Let M^n be a germ at $0 \in \mathbb{C}^{n+k}$ of an irreducible complete intersection, where $\dim_{\mathbb{C}}(\mathrm{sing}(M)) = n-2$ and, either $n, k \geqslant 2$, or $k = 1$ and $n \geqslant 3$. Suppose that there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^n, 0) \to (M, 0)$. Is M (as germ) equivalent to a quasi-homogeneous analytic set ? We would like to observe that when $n = 2$ and $k = 1$ the answer is yes. This fact will be proved in §3 and it is crucial in our proof of Theorem 1.9. However, our proof works only when the singularities of M are isolated and this is not the case if $n \geqslant 3$ and $k = 1$, by Theorem 1.10.

Problem 1.19. — Is it possible to classify the germs at $0 \in \mathbb{C}^{n+1}$, of hypersurfaces M such that $\dim_{\mathbb{C}}(\mathrm{sing}(M)) = n-2$ and there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^n, 0) \to (M, 0)$, when $n \geqslant 3$? This question seems easier when we restrict to the case where M is quasi-homogeneous.

Another interesting problem, suggested by the referee, is the following:

Problem 1.20. — In the case of a surface M^2 with an isolated singularity at 0, every germ of holomorphic map $\phi\colon (\mathbb{C}^2, 0) \to (M, 0)$ factorizes through the universal covering of M^*. What happens in higher dimensions ? Does a local uniformization of a quasi-homogeneous hypersurface gives rise to a global one by the affine space ?

The next section will be devoted to the proof of Theorem 1.5. The proof of this theorem will be based on the existence of a holomorphic n-form η on $M^* = M \setminus \operatorname{sing}(M)$ such that $\eta(p) \neq 0$ for any $p \in M^*$. This form will be used also in the proofs of Theorems 1.9 and 1.10, which will be done in §3 and in §4, respectively. As a consequence of the proof of Theorem 1.9 we will obtain the following result (see Lemma 3.2):

"Let M be a germ at $0 \in \mathbb{C}^3$ of an irreducible surface with an isolated singularity at 0. *Let η be a holomorphic 2-form on M^* such that $\eta(p) \neq 0$ for all $p \in M^*$. If $\eta = d\omega$, where ω is holomorphic, then M is equivalent to a quasi-homogeneous surface in $\mathbb{C}^3$."*

The converse of this statement is not true (see Remark 3.7).

I would like to aknowledge the referee for many suggestions which have improved a lot the paper. In particular, in the original version of the paper Theorem 1.5 was proved for hypersurfaces and he suggested that it should be true also for complete intersections, which in fact I have done in the final version.

2. Basic facts and proof of Theorem 1.5

2.1. Basic facts

Let M^n be a germ at $0 \in \mathbb{C}^m$, $m = n + k$, of an irreducible complete intersection defined by $(F_1 = \cdots = F_k = 0)$, where $F_1, \ldots, F_k \in \mathcal{O}_m$. We will consider a representative of M, denoted by the same letter, which is an analytic subset of a ball $B \subset \mathbb{C}^m$. It is well known that the singular set of M is given by $\operatorname{sing}(M) = \{p \in M | dF_1(p) \wedge \cdots \wedge dF_k(p) = 0\}$. We will suppose that 0 is effectively a singularity: $0 \in \operatorname{sing}(M)$. We will use the notation $M^* = M \setminus \operatorname{sing}(M)$. Note that, if $p \in M^*$ then

$$T_pM^* = \{v \in T_p\mathbb{C}^m | i_v(dF_1(p) \wedge \cdots \wedge dF_k(p)) = 0\},$$

where i_v denotes the interior product.

We are going now to describe a well-known construction, which proves that there exists a non-vanishing holomorphic n-form on M^*. Let us consider a holomorphic coordinate system in B, say $(x_1, \ldots, x_m)$. The k-form $\Theta := dF_1 \wedge \cdots \wedge dF_k$ can be written as

$$\Theta = \sum_I \Phi_I dx_I,$$

where $I = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, m\}$, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and $\Phi_I = \det(F_{jx_{i_r}})_{1 \leqslant j,r \leqslant k}$. Given $I = \{i_1 < \cdots < i_k\}$, set $U_I = \{z \in U | \Phi_I \neq 0\}$ and $M_I = U_I \cap M$. We observe that $(M_I)_{I \in \mathcal{K}}$ is a covering of M^* by Stein open sets, where $\mathcal{K} = \{\{i_1 < \cdots < i_k\} | 1 \leqslant i_j \leqslant m\}$. For $I \in \mathcal{K}$, let $J(I) = \{1, \ldots, m\} \setminus I = \{j_1 < \cdots < j_n\}$ and η_I be the n-form on U_I defined by

$$\eta_I = \frac{\sigma(I)}{\Phi_I} dx_{j_1} \wedge \cdots \wedge dx_{j_n},$$

where $\sigma(I) \in \{1, -1\}$ is chosen in such a way that $\Theta \wedge \eta_I = dx_1 \wedge \cdots \wedge dx_m$. Given $I, J \in \mathcal{K}$ set $M_{IJ} = M_I \cap M_J$.

Claim 2.1. — *If $I, J \in \mathcal{K}$ then $\eta_I|_{M_{IJ}} = \eta_J|_{M_{IJ}}$. In particular, there exists a holomorphic n-form η_1 on M^* such that $\eta_1|_{M_I} = \eta_I|_{M_I}$ for all $I \in \mathcal{K}$. Moreover, $\eta_1(p) \neq 0$ for all $p \in M^*$. In particular, the (n,n)-form*

$$\mu_1 = c \cdot \eta_1 \wedge \overline{\eta_1}$$

where $c = i^n \cdot (-1)^{n(n+1)/2}$, is a volume form on M^*.

Proof. — We will use the following fact: let θ be a holomorphic m-form defined in an open set $V \subset B$, $m \leqslant n$. Then $\theta|_{M^* \cap V} \equiv 0$ if, and only if, $\Theta(p) \wedge \theta(p) = 0$ for all $p \in M^* \cap V$. Given $I, J \in \mathcal{K}$ we have $\Theta \wedge \eta_I = dx_1 \wedge \cdots \wedge dx_m = \Theta \wedge \eta_J$, which implies $\Theta \wedge (\eta_I - \eta_J) = 0$. Hence, $(\eta_I - \eta_J)|_{M_{IJ}} = 0$, which proves the first part of the claim. Now, let $p \in M^*$ and $\{v_1, \ldots, v_n\}$ be a base of T_pM^*. Since $M^* = \cup_I M_I$ then $p \in M_I$ for some I. Therefore, $\eta_1(p) = \eta_I(p)|_{T_pM^*}$ and $\eta_1(p)(v_1, \ldots, v_n) = \eta_I(p)(v_1, \ldots, v_n)$. Let $u_1, \ldots, u_k \in T_p\mathbb{C}^m$ be such that $\{u_1, \ldots, u_k, v_1, \ldots, v_n\}$ is a base of $T_p\mathbb{C}^m$. A straightforward computation using that $i_{v_j}(\Theta(p)) = 0$ for all $j = 1, \ldots, n$, gives

$$\begin{aligned} 0 &\neq dx_1 \wedge \cdots \wedge dx_m(u_1, \ldots, u_k, v_1, \ldots, v_n) \\ &= \Theta(p) \wedge \eta_I(p)(u_1, \ldots, u_k, v_1, \ldots, v_n) \\ &= \Theta(p)(u_1, \ldots, u_k) \cdot \eta_I(p)(v_1, \ldots, v_n) \implies \eta_1(p)(v_1, \ldots, v_n) \neq 0. \end{aligned}$$

□

Now, let M be quasi-homogeneous with respect to the vector field $X(x) = \sum_{j=1}^m \lambda_j x_j \frac{\partial}{\partial x_j}$, where $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}_+$. Set $X(F) = U \cdot F$, where F and U are as in (1.2). Let η_1 be the n-form on M^* considered in claim 1.

Claim 2.2. — *We have $L_X(\eta_1) = f \cdot \eta_1$, where $f = \sum_{j=1}^m \lambda_j - \mathrm{tr}(U)|_{M^*}$ and L_X denotes the Lie derivative along X. Moreover, there exists $h \in \mathcal{O}^*(M^*)$ such that if $\eta := h \cdot \eta_1$ then $L_X(\eta) = a \cdot \eta$, where $a = f(0)$.*

Proof. — Since $\eta_1|_{M_I} = \eta_I|_{M_I}$ and $M^* = \cup_I M_I$, it is sufficient to prove that $L_X(\eta_I)|_{M_I} = f\cdot \eta_I|_{M_I}$ for all $I \in \mathcal{K}$. Set $\mathrm{tr}(X) = \sum_{j=1}^m \lambda_j$. Given $I \in \mathcal{K}$, we have:

$$\begin{aligned}\mathrm{tr}(X)\cdot dx_1 \wedge \cdots \wedge dx_m &= L_X(dx_1 \wedge \cdots \wedge dx_m)\\ &= L_X(\Theta \wedge \eta_I) = L_X(\Theta) \wedge \eta_I + \Theta \wedge L_X(\eta_I).\end{aligned}$$

On the other hand,

$$L_X(\Theta) = L_X(dF_1 \wedge \cdots \wedge dF_k) = \sum_{j=1}^k dF_1 \wedge \cdots \wedge d(X(F_j)) \wedge \cdots \wedge dF_k.$$

Since $X(F_j) = \sum_{i=1}^k u_{ji} F_i$, given $p \in M^*$ we get:

$$L_X(\Theta)(p) = \sum_{j=1}^k u_{jj}(p)\cdot (dF_1 \wedge \cdots \wedge dF_j \wedge \cdots \wedge dF_k)(p) = \mathrm{tr}(U)(p)\cdot \Theta(p).$$

Therefore,

$$\mathrm{tr}(X)\cdot \Theta(p) \wedge \eta_I(p) = \mathrm{tr}(U)(p)\cdot \Theta(p) \wedge \eta_I(p) + \Theta(p) \wedge L_X(\eta_I)(p) \implies$$

$$\Theta(p) \wedge [L_X(\eta_I)(p) - (\mathrm{tr}(X) - \mathrm{tr}(U)(p))\eta_I(p)] = 0. \tag{2.1}$$

Since $\eta_I|_{M_I} = \eta_1|_{M_I}$ and X is tangent to M_I, we have $L_X(\eta_I)|_{M_I} = L_X(\eta_1)|_{M_I}$. Hence, (2.1) implies that $L_X(\eta_1)(p) = (\mathrm{tr}(X) - \mathrm{tr}(U)(p))\eta_1(p)$, $p \in M^*$.

Let $f_1 = f - f(0)$ and $-f_1(x) = \sum_{|\sigma|>0} a_\sigma \cdot x^\sigma$ be Taylor series of $-f_1$ at 0, where $\sigma = (\sigma_1, \ldots, \sigma_m) \in (\mathbb{N} \cup \{0\})^m$, $|\sigma| = \sum_j \sigma_j$, $a_\sigma \in \mathbb{C}$ and $x^\sigma = x_1^{\sigma_1} \cdots x_m^{\sigma_m}$. If we set $\psi(x) = \sum_\sigma b_\sigma \cdot x^\sigma$, where $b_\sigma = (\sum_{i=1}^m \lambda_i \cdot \sigma_i)^{-1} \cdot a_\sigma$, then the series ψ has positive radius of convergence and satisfies $X(\psi) = -f_1$ (recall that $\lambda_j \in \mathbb{Q}_+$ for all j). Therefore, if $h_1 = \exp(\psi)$ then $h_1 \in \mathcal{O}_m^*$ and $X(h_1) = -h_1 \cdot f_1$. On the other hand, if $h_2 = h_1|_{M^*}$ and $\eta = h_2 \cdot \eta_1$ then,

$$\begin{aligned}L_X(\eta) &= L_X(h_2\cdot \eta_1) = X(h_2)\cdot \eta_1 + h_2 \cdot X(\eta_1)\\ &= h_2(f - f_1)\cdot \eta_1 = f(0)\cdot \eta := a\cdot \eta.\end{aligned}$$

□

Let us prove that the form η can be extended to M^*. We need the following:

Claim 2.3. — *$\mathrm{sing}(M)$ and M^* are invariant for the flow X_T, $T \in \mathbb{C}$, of X.*

Proof. — We have seen that $L_X(\Theta) = \mathrm{tr}(U)\cdot \Theta$ on M. This implies that

$$\Theta \circ X_T(p) = \exp\Big(\int_0^T \mathrm{tr}(U) \circ X_s(p)ds\Big)\cdot \Theta(p), \forall p \in M$$
$$\implies \mathrm{sing}(M) = \{p \in M | \Theta(p) = 0\}$$

is invariant by X_T. Since $M^* = M \setminus \mathrm{sing}(M)$, M^* is also invariant for X_T. □

Denote by X_t, $t \in \mathbb{R}$, the real flow of X,

$$X_t(x_1, \ldots, x_m) = (e^{\lambda_1 t}\cdot x_1, \ldots, e^{\lambda_m t}\cdot x_m).$$

Consider a ball B around $0 \in \mathbb{C}^{n+k}$ such that η is defined in $B \cap M^*$. Since $L_X(\eta) = a\cdot \eta$, $a = f(0)$, $f = \mathrm{tr}(X) - \mathrm{tr}(U)|_{M^*}$ (Claim 2.2), we have

$$X_t^*(\eta)(p) = e^{at}\cdot \eta(p) \tag{2.2}$$

for all $t \in \mathbb{R}$ and $p \in M^*$ such that both members of (2.2) are defined. Note that (2.2) and the fact that M^* is invariant for X_t imply that η can be extended to M^*. In fact, given $q \in M^*$, since $\lambda_1, \ldots, \lambda_m > 0$, and M^* is invariant for X_t (Claim 2.3), there exists $t \in \mathbb{R}_-$ such that $X_t(q) \in M^* \cap B$. By (2.2), given a base $\{v_1, \ldots, v_n\}$ of T_qM^*, we can define

$$\eta(q)\cdot (v_1, \ldots, v_n) = e^{-at}\cdot \eta(X_t(q))\cdot (DX_t(q)\cdot v_1, \ldots, DX_t(q)\cdot v_n)$$

and this definition does not depends on t. This finishes the proof of Claim 2.2.

2.2. Proof of Theorem 1.5

From now on, we fix a representative of the germ $\phi\colon (\mathbb{C}^n, 0) \to (M, 0)$, denoted again by ϕ, and some open ball of $\mathbb{C}^n$, $W \ni 0$, such that $\phi^{-1}(0) \cap W = \{0\}$. We will use the following well known result (*cf.* [8] vol. II, p. 56):

Lemma 2.4. — *If $W \subset \mathbb{C}^n$ is sufficiently small, then $\phi(W)$ is an open neighborhood of 0 in M and $\phi\colon W \to \phi(W)$ satisfies the following properties:*

(a) *ϕ is a proper and open map.*
(b) *If $A \subset M$ is an irreducible analytic subset of complex dimension k then any irreducible component of $\phi^{-1}(A)$ has complex dimension k. In particular $\mathrm{cod}_{\mathbb{C}}(\phi^{-1}(\mathrm{sing}(M))) \geqslant 2$.*

(c) *There exists* $d \in \mathbb{N}$ *such that* $\#(\phi^{-1}(p)) \leqslant d$ *for any* $p \in V$*. In fact, it is possible to find arbitrarily small neighborhoods* U *of* 0 *in* $\mathbb{C}^n$ *and* V *of* 0 *in* $\mathbb{C}^{n+1}$ *such that* $\phi\colon U \to V \cap M$ *is defined,* $\phi^{-1}(V \cap M) = U$ *and* $\phi|_U$ *is a finite ramified covering with* d*-sheets.*

For $r > 0$ set

$$M_r := \Big\{z \in M \,\Big|\, ||x|| := \Big(\sum_{j=1}^{m} |x_j|^2\Big)^{1/2} \leqslant r\Big\} = M \cap \overline{B}_r(0)$$

and $M_r^* = M_r \cap M^*$. Let η be as in the Claim 2.2, that is such that $L_X(\eta) = a\cdot\eta$, $a = f(0)$, and μ be the volume form in M^* given by $\mu = c\cdot\eta\wedge\overline{\eta}$. The main fact is the following:

Lemma 2.5. — *If* $r > 0$ *is small and* ϕ *is as in Lemma 2.4, then* $\mathrm{vol}_\mu(M_r^*) < +\infty$, *where*

$$\mathrm{vol}_\mu(M_r^*) = \int_{M_r^*} \mu.$$

Proof. — Let $\nu = \phi^*(\eta)$, which is a holomorphic n-form on $W \setminus \phi^{-1}(\mathrm{sing}(M))$. It follows from (b) of Lemma 2.4 and Hartogs' theorem, that ν can be extended to a holomorphic n-form on W. This implies that the (n,n)-form $\phi^*(\mu)$ can be extended to a real analytic (n,n)-form on W. Since ϕ is proper and M_r is a compact subset of $\phi(W)$, if $r > 0$ is small, it follows that $\phi^{-1}(M_r)$ is a compact subset of W. This implies that, for $r > 0$ small, we have:

$$\int_{\phi^{-1}(M_r)} \phi^*(\mu) < +\infty.$$

Let $C(\phi) \subset W$ and $CV(\phi) = \phi(C(\phi))$ be the sets of critical points and critical values of ϕ, respectively. Choose open sets $0 \in U \subset W$ and $0 \in V \subset \mathbb{C}^{n+1}$ such that $\phi^{-1}(V \cap M) = U$ and $\phi|_U\colon U \to V \cap M$ is a ramified covering with d-sheets, $d \geqslant 1$. The Lemma is a consequence of the following fact: if $\overline{B}_r(0) \subset V$ then:

$$\mathrm{vol}_\mu(M_r^*) \leqslant \int_{\phi^{-1}(M_r)} \phi^*(\mu) < +\infty. \tag{2.3}$$

Let us prove (2.3). Since $CV(\phi)$ has measure zero (Sard's theorem), we have

$$\int_{M_r^*} \mu = \int_{M_r^* \setminus CV(\phi)} \mu.$$

In order to prove (2.3), it is sufficient to prove that for any open subset $A \subset M_r^* \setminus CV(\phi)$, with closure $\overline{A} \subset M_r^* \setminus CV(\phi)$, then

$$\int_A \mu \leqslant \int_{\phi^{-1}(M_r)} \phi^*(\mu).$$

Let us fix A as above. Note that $\phi|_{\phi^{-1}(A)} \colon \phi^{-1}(A) \to A$ is a regular covering with d-sheets. Therefore,

$$d \int_A \mu = \int_{\phi^{-1}(A)} \phi^*(\mu) \leqslant \int_{\phi^{-1}(M_r)} \phi^*(\mu) \implies \int_A \mu \leqslant \int_{\phi^{-1}(M_r)} \phi^*(\mu).$$

This finishes the proof of the lemma. □

The following lemma implies Theorem 1.5:

Lemma 2.6. — *Let M_r be as before and $\Lambda = \mathrm{Re}(\mathrm{tr}(U)(0))$. If $\mathrm{tr}(X) - \Lambda \leqslant 0$, then $\mathrm{vol}_\mu(M_r^*) = +\infty$.*

Proof. — The proof will be by contradiction. It follows from (2.2) that $X_t^*(\eta)(p) = e^{at} \cdot \eta(p)$ for all $t \in \mathbb{R}$. Hence, the (n,n)-volume form $\mu = c \cdot \eta \wedge \overline{\eta}$ satisfies:

$$X_t^*(\mu) = e^{2\,\mathrm{Re}(a)t} \cdot \mu \tag{2.4}$$

for all $t \in \mathbb{R}$. On the other hand, if $t > 0$ then $X_t(B_r(0)) \supset \overline{B}_r(0)$, because $\lambda_1, \dots, \lambda_m > 0$. This implies that, if $t > 0$ then $M_r^* \subset int(X_t(M_r^*)) \subset M^*$. Therefore, if $\mathrm{vol}_\mu(M_r^*) < +\infty$ then

$$\mathrm{vol}_\mu(M_r^*) < \mathrm{vol}_\mu(X_t(M_r^*)), \forall t > 0.$$

Now (2.4) and the theorem of change of variables imply that,

$$\begin{aligned} \mathrm{vol}_\mu(X_t(M_r^*)) &= \int_{X_t(M_r^*)} \mu = \int_{M_r^*} X_t^*(\mu) \\ &= e^{2\,\mathrm{Re}(a)t} \int_{M_r^*} \mu = e^{2\,\mathrm{Re}(a)t} \cdot \mathrm{vol}_\mu(M_r^*). \end{aligned}$$

Therefore, if $\mathrm{tr}(X) - \Lambda = \mathrm{Re}(a) \leqslant 0$, $t > 0$ and $\mathrm{vol}_\mu(M_r^*) < +\infty$ then

$$\mathrm{vol}_\mu(M_r^*) < \mathrm{vol}_\mu(X_t(M_r^*)) = e^{2\,\mathrm{Re}(a)t} \cdot \mathrm{vol}_\mu(M_r^*) \leqslant \mathrm{vol}_\mu(M_r^*),$$

a contradiction. This finishes the proof of Lemma 2.6 and of Theorem 1.5. □

3. Proof of Theorem 1.9

The proof will be divided in three steps:

1st-step. — We will prove that there exists a germ of holomorphic vector field at $0 \in \mathbb{C}^3$, say X, such $X(F) = F$, where $F = 0$ is a reduced equation of M. In this case, F belongs to its Jacobian ideal and it follows from a theorem of Saito (*cf.* [15]), that there exists a linearizable germ of holomorphic vector field Y on $\mathbb{C}^3$ such that $Y(F) = F$. This vector field can be written in a suitable coordinate system (x, y, z) in a neighborhood of $0 \in \mathbb{C}^3$ as,

$$Y = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y} + \lambda_3 z \frac{\partial}{\partial z}, \text{ where } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}_+. \tag{I}$$

2nd-step. — We will prove that if F is a quasi-homogeneous polynomial with respect to $X = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y} + \lambda_3 z \frac{\partial}{\partial z}$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}_+$ and $\lambda_1 + \lambda_2 + \lambda_3 > 1$, then F is equivalent to one of the forms in (a), (b), (c) or (d) in the statement of Theorem 1.9.

3rd-step. — We will prove that the surfaces in (a), (b), (c) and (d) are two by two non-equivalent.

3.1. Proof of the 1*st* step

We will divide the proof in three Lemmas.

Let M be a germ of hypersurface at $0 \in \mathbb{C}^3$, with an isolated singularity at 0, given by a reduced equation $F = 0$, where $F \in \mathcal{O}_3$. Consider the 2-form η on M^* as defined in §2. Suppose that there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^2, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$. Given a neighborhood V of $0 \in \mathbb{C}^3$ such that F is defined (that is, has a representative $F\colon V \to \mathbb{C}$) and $\text{sing}(F) \cap V = \{0\}$, we will use the notations $M_V = \{p \in V | F(p) = 0\}$ and $M_V^* = M_V \setminus \{0\}$.

Lemma 3.1. — *If V is sufficiently small, then there exists a holomorphic 1-form ω on M_V^*, such that $d\omega = \eta$.*

Proof. — Fix neighborhoods U of $0 \in \mathbb{C}^2$ and V of $0 \in \mathbb{C}^3$ such that F has a representative $F\colon V \to \mathbb{C}$, ϕ has a representative $\phi\colon U \to \mathbb{C}^3$ and $\phi(U) \subset V$. As we have seen before the form $\phi^*(\eta)$ extends to a closed holomorphic 2-form on U, say θ. Since θ is closed, it follows from Poincaré Lemma that $\theta = d\alpha$ in a small neighborhood of $0 \in \mathbb{C}^2$, where α is holomorphic. Therefore, if we take U and V small enough, we can suppose that:

(i) α is defined in U.

(ii) ϕ has a representative $\phi\colon U \to M_V$.

Let $C(\phi, U) = C$ be the set of critical points of $\phi|_U$, $CV = CV(\phi, V) := \phi(C) \subset M_V$ the set of critical values and $D = \phi^{-1}(CV)$. We can choose U and V in such a way that,

(iii) $\phi^{-1}(M_V) = U$ and $\phi\colon U \setminus D \to M_V \setminus CV$ is a covering map with m sheets. We will use the notation $\widehat{M}$ for $M_V \setminus CV$.

We will use α to construct a form ω on M_V such that $d\omega = \eta$. Let us construct ω on $\widehat{M}$.

It follows from (iii) that, given a point $p \in \widehat{M}$, where $\phi^{-1}(p) = \{q_1, \ldots, q_m\}$, there exists a neighborhood $V_p \subset \widehat{M}$ of p and neighborhoods $U_p^1, \ldots, U_p^d$ of $q_1, \ldots, q_m$, such that

(iv) V_p is biholomorphic to a ball in $\mathbb{C}^2$.

(v) $U_p^i \cap U_p^j = \emptyset$, for $i \neq j$.

(vi) $\phi_p^j := \phi|_{U_p^j}\colon U_p^j \to V_p$ is a biholomorphism.

For each $j = 1, \ldots, m$, consider the 1-form β_p^j on V_p defined by $\beta_p^j = ((\phi_p^j)^{-1})^*(\alpha)$. Since $\phi^*(\eta) = d\alpha$, we have $d\beta_p^j = \eta|_{U_p}$. Define a 1-form ω_p on V_p by

$$\omega_p = \frac{1}{m} \sum_{j=1}^{d} \beta_p^j. \tag{3.1}$$

Observe that $d\omega_p = \eta$. By standards arguments, we can construct a covering $\mathcal{V} = \{V_p\}_{p \in \widehat{M}}$ of $\widehat{M}$ by connected open sets, and a collection of holomorphic 1-forms $\{\omega_p\}_{p \in \widehat{M}}$, $\omega_p \in \Omega^1(V_p)$, such that

(vii) If $V_p \cap V_q \neq \emptyset$ then $V_p \cap V_q$ is contractible.

(viii) $d\omega_p = \eta|_{V_p}$ for all p.

By taking the $V_{p's}$ small, we can suppose

(ix) If $V_p \cap V_q \neq \emptyset$, $\phi^{-1}(V_p) = \cup_{j=1}^m U_p^j$ and $\phi^{-1}(V_q) = \cup_{j=1}^m U_q^j$, then for every $1 \leqslant j \leqslant m$, there exists an unique $k = k(j) \in \{1, \ldots, m\}$ such that $U_p^j \cap U_q^k \neq \emptyset$.

We claim that, if $V_p \cap V_q \neq \emptyset$ then $\omega_p = \omega_q$ on $V_p \cap V_q$. This will imply that ω extends to $\widehat{M}$. In fact, let $\phi^{-1}(p) = \{p_1, \ldots, p_m\}$, $\phi^{-1}(q) = \{q_1, \ldots, q_m\}$, $\phi^{-1}(V_p) = \cup_{j=1}^m U_p^j$ and $\phi^{-1}(V_q) = \cup_{j=1}^m U_q^j$, be as in (ix). Given $1 \leqslant j \leqslant m$, let $k \in \{1, \ldots, m\}$ be such that $U_p^j \cap U_q^k \neq \emptyset$. Since $\phi_p^j = \phi_q^k = \phi$ on $U_p^j \cap U_q^k$, we get from the construction that $\beta_p^j = \beta_q^k$ on $V_p \cap V_q$. This implies that $\omega_p = \omega_q$ on $V_p \cap V_q$.

It follows that we can define a 1-form ω on $\widehat{M}$ such that $d\omega = \eta$. It remains to prove that ω extends to M_V^*. We will use the local forms for ϕ near a singular point. Observe first, that if we take V sufficiently small, then $CV = CV(\phi, V)$ and $D = \phi^{-1}(CV)$ are curves such that $\mathrm{sing}(CV) = \{0\}$ and $\mathrm{sing}(D) = \{0\}$. Remark that, if $CV = \cup_j C_j$ and $D = \cup_k D_k$ are the decompositions of CV and D into irreducible components, then for each k there exists an unique j such that $\phi(D_k) = C_j$. Moreover, if $q \in D_k \setminus \{0\}$ and $p = \phi(q)$ then $D\phi|_{T_q D_k} : T_q D_k \to T_p C_j$ is an isomorphism. Therefore we can find holomorphic coordinate systems $(U_q, (u, v))$ and $(V_p, (x, y))$ around q and p respectively, such that

(x) $U_q = \{(u, v) \in \mathbb{C}^2 | |u| < 1, |v| < 1\}$, $V_p = \{(x, y) \in \mathbb{C}^2 | |x| < 1, |y| < 1\}$, $D \cap U_q = D_k \cap U_q = \{v = 0\}$ and $CV \cap V_p = C_j \cap V_p = \{y = 0\}$.

(xi) $\phi(u, v) = (X(u, v), Y(u, v)) = (u, v^n)$, for some $n \geqslant 1$ (Whitney's local forms).

Observe that on V_p we have $\eta = h(x, y) \cdot dx \wedge dy = d(H(x, y)dx)$, where $H_y = -h$. Therefore, $\phi^*(\eta) = d(H(u, v^n)du) = d\alpha$ on U_q and $\alpha|_{U_q} = H(u, v^n)du + dg(u, v)$, where $g \in \mathcal{O}(U_q)$. Let $g(u, v) = \sum_{j=0}^{\infty} g_j(u)v^j$.

Now, fix $p_0 = (x_0, y_0) \in V_p \setminus \{y = 0\}$ and let $\phi^{-1}(p_0) = \{q_1, \ldots, q_d\}$. Since $\phi(u, v) = (u, v^n)$ for $(u, v) \in U_q$, then $U_q \cap \phi^{-1}(p_0)$ contains n points, say $q_1, \ldots, q_n$, where $q_j = (x_0, \delta^j \cdot v_0)$, δ is a primitive n^{th}-root of the unity and $v_0^n = y_0$. Let $D_r \subset \{y | 0 < |y| < 1\}$ be a small disk centered at y_0 and $b(y) = y^{1/n}$ be the branch of the n^{th}-root of y, defined in D_r and such that $b(y_0) = v_0$. It follows from the definition of $\beta_{p_0}^k$ that, for $k = 1, \ldots, n$, we have $\beta_{p_0}^k = H(x, y)dx + dg_k(x, y)$, in a small neighborhood of p_0, where

$$g_k(x, y) = \sum_{j=0}^{\infty} g_j(x)\delta^{kj}(b(y))^j.$$

Hence,

$$\begin{aligned}\sum_{k=1}^{n} \beta_{p_0}^k &= nH(x, y)dx + d\Big[\sum_{j=0}^{\infty}\Big(\sum_{k=1}^{n}\delta^{kj}\Big)g_j(x)(b(y))^j\Big] \\ &= nH(x, y)dx + nd\Big[\sum_{j=0}^{\infty} g_{jn}(x)y^j\Big],\end{aligned}$$

because $\sum_{k=1}^{n} \delta^{kj} = 0$ if n does not divide j. This implies that the form $\sum_{k=1}^{n} \beta_{p_0}^k$ extends to a holomorphic 1-form on V_p, say β, such that $d\beta = n\eta$. Using the same argument in the other points of $\phi^{-1}(p) \subset W$, it is possible to prove that $\sum_{k=n+1}^{m} \beta_{p_0}^k$ (p_0 near p), extends to a holomorphic 1-form defined in a neighborhood of p, say β_1, such that $d\beta_1 = (m-n)\eta$. Since $\frac{1}{m}(\beta+\beta_1)=\omega$, we get that ω extends to a neighborhood of p. $\square$

Lemma 3.2. — *Let M be a germ at $0 \in \mathbb{C}^3$ of an irreducible surface with an isolated singularity at 0. Let η be a holomorphic 2-form on M^* such that $\eta(p) \neq 0$ for all $p \in M^*$. If $\eta = d\omega$, where ω is a holomorphic 1-form, then M is equivalent to the germ of a quasi-homogeneous surface in $\mathbb{C}^3$.*

Proof. — We will prove the lemma in the case that η is given by the construction of §2 and leave the general case for the reader. In this case, if $M_j = \{p \in M | F_{x_j}(p) \neq 0\}$ then,

$$\eta|_{M_1} = \frac{dx_2 \wedge dx_3}{F_{x_1}}|_{M_1},\ \eta|_{M_2} = \frac{dx_3 \wedge dx_1}{F_{x_2}}|_{M_2} \quad \text{and} \quad \eta|_{M_3} = \frac{dx_1 \wedge dx_2}{F_{x_3}}|_{M_3}. \tag{3.2}$$

We will construct a germ at $0 \in \mathbb{C}^3$ of holomorphic vector field X, such that $X(F) = F$. Since $\eta(p) \neq 0$ for all $p \in M^*$, we get that $\omega = i_Y(\eta)$, where Y is a holomorphic vector field on M^*. The vector field Y can be extended to a a holomorphic vector field defined in a neighborhood of $0 \in \mathbb{C}^3$. In fact, if V is a small Stein neighborhood of $0 \in \mathbb{C}^3$ and $Y = \sum_{j=1}^3 Y_j \frac{\partial}{\partial x_j}$, where $Y_j \in \mathcal{O}(M_V)$, then the functions Y_j can be extended to holomorphic functions on V because $H^1(V \setminus \{0\}, \mathcal{O}) = \{0\}$, by [C] (see the proof of Lemma 4.1). We will denote this extension by the same letter. Since M is invariant for Y we have $Y(F) = h\cdot F$, where $h \in \mathcal{O}_3$. If $h(0) \neq 0$, then we set $X = \frac{1}{h}\cdot Y$. In this case, X is a germ of holomorphic vector field at $0 \in \mathbb{C}^3$ for which $X(F) = F$, and we are done. Therefore, we have only to prove that $h(0) = 0$ leads to a contradiction. Remark that $Y(0) = 0$, because 0 is a singular point of M.

Claim 3.3. — *Let $Y = Y_1\frac{\partial}{\partial x_1} + Y_2\frac{\partial}{\partial x_2} + Y_3\frac{\partial}{\partial x_3}$, be such that $i_Y(\eta) = \omega$ on M, where $Y_1, Y_2, Y_3 \in \mathcal{O}_3$, and $L = DY(0)$ be the linear part of Y at 0. Then:*

$$Y_{1x_1} + Y_{2x_2} + Y_{3x_3} = 1 + h$$

on M. In particular, if $h(0) = 0$, then $\mathrm{tr}(L) = 1$ and L has at least one non zero eigenvalue.

Proof. — Observe first that $L_Y(\eta) = i_Y(d\eta) + d(i_Y(\eta)) = d\omega = \eta$, on M^*. It follows from (3.2) and a straightforward computation that on M_3

we have

$$(3.3)\qquad \begin{aligned} L_Y(\eta) &= L_Y\Big(\frac{dx_1\wedge dx_2}{F_{x_3}}\Big) \\ &= \Big(Y_{1x_1}+Y_{2x_2}-\frac{Y(F_{x_3})}{F_{x_3}}-\frac{F_{x_1}Y_{1x_3}}{F_{x_3}}-\frac{F_{x_2}Y_{2x_3}}{F_{x_3}}\Big)\eta. \end{aligned}$$

On the other hand,

$$\begin{aligned} Y(F_{x_3}) &= Y\Big(\frac{\partial}{\partial x_3}(F)\Big) \\ &= \Big[Y,\frac{\partial}{\partial x_3}\Big](F)+\frac{\partial}{\partial x_3}(Y(F)) \\ &= -Y_{1x_3}\cdot F_{x_1}-Y_{2x_3}\cdot F_{x_2}-Y_{3x_3}\cdot F_{x_3}+h_{x_3}\cdot F+h\cdot F_{x_3} \end{aligned}$$

By substituting the above expression in (3.3) and using that $M=\{F=0\}$, we obtain

$$L_Y(\eta)=(Y_{1x_1}+Y_{2x_2}+Y_{3x_3}-h)\eta$$

which implies that $Y_{1x_1}+Y_{2x_2}+Y_{3x_3}-h\equiv 1$ on M. If $h(0)=0$, we get

$$\mathrm{tr}(L)=Y_{1x_1}(0)+Y_{2x_2}(0)+Y_{3x_3}(0)=1.$$

□

Let N be a a germ at $0\in\mathbb{C}^3$ of a holomorphic submanifold of dimension k, $k\in\{1,2,3\}$. We will say that it is an *invariant manifold of Poincaré type* for the vector field Y (briefly i.m.P.t) if

(I) N is smooth at 0 and invariant for Y.

(II) If L is the linear part of Y at 0 then $L|_{T_0N}$ is in the Poincaré domain, in the sense that its eigenvalues are non zero (observe that $L(T_0N)=T_0N$) and there exists a line $\ell\subset\mathbb{C}$, $0\in\ell$, such that all eigenvalues of $L|_{T_0N}$ are contained in one of the components of $\mathbb{C}\setminus\ell$.

Lemma 3.4. — *If $h(0)=0$ and N is an i.m.P.t. for Y, then*

(a) *$N\subset M$. In particular,* $\dim(N)=1$.

(b) *If the eigenvalue of $L|_{T_0N}$ is $\lambda\neq 0$, then the eigenvalues of L are λ, $-k\cdot\lambda$ and $1+(k-1)\lambda$, where $k\in\mathbb{N}$.*

Proof. — Let us prove (a). We denote the local flow of Y by $Y_T=(Y_T^1,Y_T^2,Y_T^3)$, $T\in\mathbb{C}$. Since N is smooth, we can choose a local coordinate system (x,y,z) around 0 such that $N\subset\Sigma\simeq T_0N$, where Σ is a linear subspace of $\mathbb{C}^3$ and $L(\Sigma)=\Sigma$. Since the eigenvalues of $L|_\Sigma$ are in the Poincaré domain, there exists $\alpha\in\mathbb{C}^*$ such that the eigenvalues of $\alpha\cdot L|_\Sigma$ have negative real part. Let $Z=\alpha\cdot Y|_N$, $g=\alpha\cdot h$ and $a>0$ be such that

$a < \min\{-\operatorname{Re}(\lambda)|\lambda$ is an eigenvalue of $Z\}$. It is well known (*cf.* [2]) that there exists a neighborhood U of $0 \in N$ such that

(i) For any $p \in U$, $Z_t(p)$ is defined for all $t > 0$.

(ii) If $t > 0$ and $p \in U$ then $||Z_t(p)|| \leqslant C\cdot e^{-at}$, where $C > 0$.

Since $Z(F) = g\cdot F$, it follows from (i) and (ii) that

(iii) If $p \in U$ and $t > 0$ then

$$F(Z_t(p)) = \exp\Big(\int_0^t g(Z_s(p))ds\Big)\cdot F(p).$$

Now, $g(0) = 0$ and (ii) imply that there exists $A > 0$ such that $|g(Z_s(p))| \leqslant A\cdot e^{-as}$. Hence, $\int_0^\infty g(Z_s(p))ds$ is convergent, say $\int_0^\infty g(Z_s(p))ds = b \in \mathbb{C}$. It follows from (iii) that

$$e^b\cdot F(p) = \lim_{t\to\infty} F(Z_t(p)) = 0 \implies F(p) = 0 \implies p \in M \implies N \subset M.$$

Since $\dim_{\mathbb{C}}(M) = 2$ we must have $\dim_{\mathbb{C}}(N) \leqslant 2$. On the other hand, since M is irreducible and is singular at $0 \in \mathbb{C}^3$, we must have $\dim_{\mathbb{C}}(N) = 1$. This proves (a).

Let us prove (b). We can assume that $N \subset \{(x,0,0)|x \in \mathbb{C}\}$. This implies $F(x,y,z) = y\cdot A(x,y,z) + z\cdot B(x,y,z)$, where $A, B \in \mathcal{O}_3$. It follows from Poincaré's linearization theorem (*cf.* [2]) that we can find a local coordinate system $x \in \mathbb{C}$ such that $Y(x,0,0) = \lambda x\frac{\partial}{\partial x}$ and $Y_T(x,0,0) = (e^{\lambda T}\cdot x,0,0)$. Let us assume that $L = DY(0)$ is in Jordan's canonical form. In this case, the eigenvalues of L are $\frac{\partial Y_1}{\partial x}(0) = \lambda$, $\frac{\partial Y_2}{\partial y}(0)$ and $\frac{\partial Y_3}{\partial z}(0)$.

Observe that $Y(F) = h\cdot F$ implies $L_Y(dF) = d(h\cdot F) = h\cdot dF + F\cdot dh$. Therefore, for $p = (x,0,0)$ we get

$$\text{(3.4)}\quad L_Y(dF)(p) = h(p)\cdot dF(p) \implies [Y_T^*(dF)](p) = \exp\Big(\int_0^T h(Y_s(p))ds\Big)\cdot dF(p).$$

On the other hand, $dF(x,0,0) = A(x,0,0)dy + B(x,0,0)dz$, where either $A(x,0,0) \not\equiv 0$, or $B(x,0,0) \not\equiv 0$, because 0 is an isolated zero of dF. This implies that we can write $dF(x,0,0) = x^k\cdot u(x)dy + x^\ell\cdot v(x)dz$, $k,\ell \geqslant 1$, where either $u \not\equiv 0$ and $u(0) \neq 0$, or $v \not\equiv 0$ and $v(0) \neq 0$. If $u, v \not\equiv 0$, let us suppose, without lost of generality, that $k \leqslant \ell$, and so

$$dF(x,0,0) = x^k\cdot u(x)(dy + x^m v_1(x)dz),$$

where $m = \ell - k \geqslant 0$ and $v_1 = v/u$. The change of variables $\psi(x,y,z) = (x, y + x^m, z) = (x_1, y_1, z_1)$ is a biholomorphism near $0 \in \mathbb{C}^3$ and in these new coordinates we have $dF(x_1,0,0) = x_1^k u(x_1)dy_1$. Returning to the old notation ($x_1 = x, y_1 = y$), we have

$$dF(x,0,0) = x^k u(x)dy. \tag{3.5}$$

Observe that after this change of variables $\frac{\partial Y_2}{\partial y}(0)$ is still an eigenvalue of L. We leave this computation to the reader. We are going to prove that $\frac{\partial Y_2}{\partial y}(0) = -k\lambda$.

If we set $Y_T = (Y_T^1, Y_T^2, Y_T^3)$ and $H(T,x) = \exp\big(\int_0^T h(Y_s(x,0,0))ds\big)$, we get fro (3.4) and (3.5) that

$$e^{k\lambda T} x^k\cdot u(e^{\lambda T}x)\cdot dY_T^2(x,0,0) = H(T,x)\cdot x^k u(x)dy$$
$$\implies dY_T^2(x,0,0) = H(T,x)\frac{u(x)}{u(e^{\lambda T}x)}\cdot e^{-k\lambda T}\cdot dy$$

and

$$\frac{\partial Y_T^2}{\partial y}(x,0,0) = H(T,x)\frac{u(x)}{u(e^{\lambda T}x)}\cdot e^{-k\lambda T}. \tag{3.6}$$

Now, since $\frac{\partial Y_T}{\partial T} = Y(Y_T)$, we have

$$\frac{\partial^2 Y_T^2}{\partial T\partial y}(p) = dY_2(Y_T(p))\cdot \frac{\partial Y_T}{\partial y}(p).$$

Since $Y_0(p) = p$, we have $\frac{\partial Y_0}{\partial y}(x,y,z) = (0,1,0)$. If we set $T = 0$ and $p = (x,0,0)$ in the above relation, we get

$$\frac{\partial^2 Y_T^2}{\partial T\partial y}|_{(T=0,p)} = dY_2(x,0,0)\cdot (0,1,0) = \frac{\partial Y_2}{\partial y}(x,0,0) \implies$$

$$\frac{\partial^2 Y_T^2}{\partial T\partial y}|_{(T=0,0)} = \frac{\partial Y_2}{\partial y}(0). \tag{3.7}$$

Now, (3.6) implies that

$$\frac{\partial^2 Y_T^2}{\partial T\partial y}(x,0,0) = e^{-k\lambda T}\Big\{\frac{\partial}{\partial T}\Big[H(T,x)\frac{u(x)}{u(e^{\lambda T}x)}\Big] - k\lambda H(T,x)\frac{u(x)}{u(e^{\lambda T}x)}\Big\}.$$

Since $H(0,x) = 1$,

$$\frac{\partial H(T,x)}{\partial T} = h(Y_T(x,0,0))\cdot H(T,x)$$

and

$$\frac{\partial u(e^{\lambda T}x)}{\partial T} = \lambda e^{\lambda T} x u'(e^{\lambda T}x),$$

we get for $T = 0$

$$\frac{\partial^2 Y_T^2}{\partial T\partial y}|_{(T=0,(x,0,0))} = h(x,0,0) - \lambda\frac{x\cdot u'(x)}{u(x)} - k\lambda$$

This together with $h(0) = 0$ and (3.7) gives

$$\frac{\partial Y_2}{\partial y}(0) = -k\lambda$$

which implies that $-k\lambda$ is an eigenvalue of L. Since $\mathrm{tr}(L) = 1$, the other eigenvalue of L must be $1 + (k-1)\lambda$. □

We will use the following result, which is a consequence of the stable manifold theorem (*cf.* [13]).

LEMMA 3.5. — *Let Z be a germ at $0 \in \mathbb{C}^n$ of holomorphic vector field such that $Z(0) = 0$. Set $L := DZ(0)$ and let $S = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of L. Suppose that there exists a straight line ℓ through $0 \in \mathbb{C}$ such that $\ell \cap S = \emptyset$ and the components of $\mathbb{C} \setminus \ell$ are A_1 and A_2. Set $S_k = S \cap A_k$, $k = 1, 2$, and let E_k be the invariant subspace of $T_0\mathbb{C}^n$ for L, relative to the eigenvalues in S_k. Then there are germs of i.m.P.t. W_k such that $T_0 W_k = E_k$, $k = 1, 2$.*

The proof of the above result can be found in [1]. Let us suppose by contradiction that $h(0) = 0$. Let $S = \{\lambda_1, \lambda_2, \lambda_3\}$ be the spectrum of $L = DY(0)$, where $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant 0$. It follows from $\mathrm{tr}(L) = 1$ that at least one of the eigenvalues of L is non-zero: $|\lambda_1| > 0$. Let v be an eigenvector of L with eingenvalue λ_1 and set $E_1 = \mathbb{C} \cdot v$.

CLAIM 3.6. — *There exists an i.m.P.t. of dimension one W_1 tangent to E_1.*

We will prove the above claim at the end. Let us finish the proof that $h(0) \neq 0$ by using the claim. If $h(0) = 0$, it follows from Lemma 3.4 that $-k \cdot \lambda_1$ is an eigenvalue of L, where $k \in \mathbb{N}$. Since $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3|$, we must have $k = 1$ and $\lambda_2 = -\lambda_1$, or $\lambda_3 = -\lambda_1$. In both cases, we get $S = \{\lambda_1, -\lambda_1, 1\}$, because $\mathrm{tr}(L) = 1$. Hence, all eigenvalues of L are non-zero and there exists a line ℓ through $0 \in \mathbb{C}$ such that $\mathbb{C} \setminus \ell$ contains two eigenvalues of L in one of its components and one in the other. It follows from Lemma 3.5 that there exists an i.m.P.t. W of dimension two. Therefore, $W \subset M$, by Lemma 3.4, and 0 is not a singularity of M, a contradiction.

Proof. — Proof of claim 3.6 After multiplying Y by a constant, we can suppose that $\lambda_1 = 1$ and $|\lambda_2|, |\lambda_3| \leqslant 1$. Choose coordinates $(x, y) = (x, y_1, y_2) \in \mathbb{C} \times \mathbb{C}^2$, such that $E_1 = \{y = 0\}$ and L is triangular. In this case, the differential equation associated to Y is of the form:

$$\begin{cases} \frac{dx}{dt} = x + \ell(y) + r(x, y) \\ \frac{dy}{dt} = Ay + R(x, y) \end{cases} . \tag{3.8}$$

where ℓ and A are linear, r and R are of order greater than one and the eigenvalues of A are λ_2 and λ_3. After a blowing-up $y = x\cdot z = (x\cdot z_1, x\cdot z_2)$ at $0 \in \mathbb{C}^3$, equation (3.8) is transformed into

$$\begin{cases} \frac{dx}{dt} = x + x\cdot r_1(x,z) \\ \frac{dz}{dt} = x\cdot W + (A-I)z + R_1(x,z) \end{cases} \tag{3.9}$$

where W is a constant vector, r_1 is of order $\geqslant 1$ and R_1 of order $\geqslant 2$. The eigenvalues of the linear part of (3.9) are $\lambda_1' = 1$, $\lambda_2' = \lambda_2 - 1$ and $\lambda_3' = \lambda_3 - 1$. Suppose first that $\lambda_j \neq 1$, $j = 2,3$. In this case, we have $\mathrm{Re}(\lambda_j') < 0$ (because $|\lambda_j| \leqslant 1$), $j = 2,3$. It follows from Lemma 3.5 that (3.9) has an i.m.P.t., W_1, tangent to the eigenspace associated to the eigenvalue 1. If $x \mapsto (x, z(x))$ is a parametrization of W_1, then $x \mapsto (x, x\cdot z(x))$ is a parametrization of an i.m.P.t. for Y, tangent to E_1. In the general case, note first that $\lambda_j' \neq 1 = \lambda_1'$, $j = 2,3$. By a linear change of variables in (3.9), that sends the linear part to the Jordan form, we get $W = 0$. After the blowing-up $z = x\cdot w = (x\cdot w_1, x\cdot w_2)$, equation (3.9) (with $W = 0$) is transformed in an other equation with eigenvalues of the linear part $\lambda_1'' = 1$, $\lambda_2'' = \lambda_2 - 2$, $\lambda_3'' = \lambda_3 - 2$. Since $\mathrm{Re}(\lambda_j'') < 0$, $j = 2,3$, we can apply Lemma 3.5 to obtain an i.m.P.t. tangent to the eigenspace associated to the eigenvalue 1. If $x \mapsto (x, w(x))$ is a parametrization of the i.m.P.t. so obtained, then $x \mapsto (x, x^2.w(x))$ is a parametrization of an i.m.P.t. for Y. This finishes the proof of Lemma 3.2 and of the first step. □

Remark 3.7. — In this remark we analyse the converse of Lemma 3.2. Let $M = \{F = 0\} \subset \mathbb{C}^3$ be q.h.w.r. the vector field $X = \lambda_1 x\frac{\partial}{\partial x} + \lambda_2 y\frac{\partial}{\partial y} + \lambda_3 z\frac{\partial}{\partial z}$, where $\lambda_j \in \mathbb{Q}_+$, $j = 1,2,3$, and $X(F) = F$. The converse of Lemma 3.2 is true when $\mathrm{tr}(X) = \sum_j \lambda_j \neq 1$. In fact, let η be the 2-form constructed as in Claim 2.1. It follows from Claim 2.2 that $L_X(\eta) = a\cdot\eta$, where $a = \mathrm{tr}(X) - 1$. Since $L_X(\eta) = i_X(d\eta)) + d(i_X(\eta)) = d(i_X(\eta))$, if $a \neq 0$ then $\eta = d\omega$ where $\omega = a^{-1}\cdot i_X(\eta)$.

On the other hand, if $\mathrm{tr}(X) = 1$ then the converse of Lemma 3.2 is not true, as was asserted in §1. Consider for instance the surface $M_{(3,3,3)} = \{(x,y,z) | F := x^3 + y^3 + z^3 = 0\}$, which is q.h.w.r. to a vector field X as above with $\lambda_j = \frac{1}{3}$, $j = 1,2,3$. In this case we have $X(F) = F$ and $L_X(\eta) = 0$. If η is as before, then there is no holomorphic 1-form ω on M^* such that $d\omega = \eta$. In fact, suppose by contradiction that there exists ω holomorphic such that $d\omega = \eta$. Let Y be a germ at $0 \in \mathbb{C}^3$ of holomorphic vector field tangent to M, such that $\omega = i_Y(\eta)$ and $Y(F) = h\cdot F$. It follows from the proof of Lemma 3.2 that $h(0) \neq 0$. Therefore, if $Z = h^{-1}\cdot Y$ then $Z(F) = F$, $(X - Z)(F) = 0$ and F is a first integral of $W := X - Z$. The relation $W(F)=0$ implies that the linear part of W at 0 vanishes and $\mathrm{tr}(DZ(0))=1$. On the other hand, it follows from Claim 3.3 that

$$h(0)=h(0)\cdot\mathrm{tr}(DZ(0))=\mathrm{tr}(DY(0))=1+h(0)\implies 1=0,$$

a contradiction.

3.2. Proof of the $2nd$ step

Let $M=F^{-1}(0)$, where $F\colon\mathbb{C}^3\to\mathbb{C}$ is a polynomial with an isolated singularity at $0\in\mathbb{C}^3$, quasi-homogeneous with respect to the vector field $X=\lambda_1 x\frac{\partial}{\partial x}+\lambda_2 y\frac{\partial}{\partial y}+\lambda_3 z\frac{\partial}{\partial z}$, where $X(F)=F$ and $\lambda_1,\lambda_2,\lambda_3\in\mathbb{Q}_+$. Suppose that there exists a germ of holomorphic map $\phi\colon(\mathbb{C}^2,0)\to(M,0)$ such that $\phi^{-1}(0)=\{0\}$. We are going to prove that M is equivalent to one of the surfaces: $M_{(p,q,r)}$, where $(p,q,r)\in\{(2,2,r),(2,3,3),(2,3,4),(2,3,5)\}$, X_m, where $m\geqslant 1$, Y or Z_m, where $m\geqslant 1$ (see the statement of Theorem 1.9).

Notation. — Let $G,H\in\mathcal{O}_n$. We will use the notation $G\simeq H$ to say that there exist $u\in\mathcal{O}_n$ and a germ ψ of biholomorphism at $0\in\mathbb{C}^n$ such that $u(0)\neq 0$ and $G=u\cdot H\circ\psi$. The following remark will be used several times in the proof:

Remark 3.8. — Let $G\in\mathcal{O}_n$ be of the form $G(u_1,\dots,u_{n-1},u_n):=G(u,v)=a_m\cdot v^m+a_{m-1}(u)\cdot v^{m-1}+\cdots+a_0(u)$, where $a_0,\dots,a_{m-1}\in\mathcal{O}_{n-1}$ and $a_m\in\mathbb{C}^*$. Then there exist $b_0,\dots,b_{m-2}\in\mathcal{O}_{n-1}$ such that $G\simeq v^m+b_{m-2}(u)\cdot v^{m-2}+\cdots+b_0(u):=H(u,v)$. Moreover, if $X(G)=G$, where $X=\sum_{j=1}^n\lambda_j u_j\frac{\partial}{\partial u_j}$, then $\lambda_n=\frac{1}{m}$, $X(H)=H$ and $X(b_j)=(1-\frac{j}{m})b_j$, $j=0,\dots,m-2$.

Note that the change of variables $\psi_1(u,v)=(u,\alpha\cdot v)=(u,v_1)$, where $\alpha^m=a_m$, is such that $G\circ\psi_1^{-1}(u,v_1)=v_1^m+\tilde{a}_{m-1}(u)\cdot v_1^{m-1}+\cdots+a_0(u)$, where $\tilde{a}_j=\alpha^{-j}\cdot a_j$, $j=0,\dots,m-1$. Therefore, we can suppose that $a_m=1$. On the other hand, the germ of biholomorphism $\psi(u,v)=(u,v+\frac{1}{m}a_{m-1}(u))=(u,v_1)$ is such that $G\circ\psi^{-1}(u,v_1)=v_1^m+b_{m-2}(u)\cdot v_1^{m-2}+\cdots+b_0(u):=H$, where $b_0,\dots,b_{m-2}\in\mathcal{O}_{n-1}$. Note that, $\psi_*(X)(H)=\psi_*(X)(G\circ\psi^{-1})=X(G)\circ\psi^{-1}=H$. On the other hand, $X(a_j\cdot v^j)=X(a_j)\cdot v^j+a_j\cdot j\cdot\lambda_n\cdot v^j=a_j\cdot v^j$, $j=0,\dots,m$ $(a_m=1)$, which implies that $\lambda_n=\frac{1}{m}$ and $X(a_j)=(1-\frac{j}{m})a_j$, $j=0,\dots,m-1$. It follows that, $X(v_1)=\frac{1}{m}v_1$ and $\psi_*(X)=\frac{1}{m}v_1\frac{\partial}{\partial v_1}+\sum_{j=1}^{n-1}\lambda_j u_j\frac{\partial}{\partial u_j}$, which has the same expression as X. This proves the remark.

Let $M=F^{-1}(0)$ and $X=\lambda_1 x\frac{\partial}{\partial x}+\lambda_2 y\frac{\partial}{\partial y}+\lambda_3 z\frac{\partial}{\partial z}$, where $\lambda_1,\lambda_2\lambda_3\in\mathbb{Q}_+$ and $X(F)=F$. We will suppose that $\lambda_1\leqslant\lambda_2\leqslant\lambda_3$.

Claim 3.9. — *We claim that* $F_{xyz} \equiv F_{yyz} \equiv F_{xzz} \equiv F_{yzz} \equiv F_{zzz} \equiv 0$. *In particular, we must have*

$$F(x,y,z) = A(x,y) + B(x,y)z + Cz^2$$

where $X(A) = A$, $X(B) = (1-\lambda_3)B$ *and* C *is a constant.*

Proof. — The claim is a consequence of the following fact: if $G \in \mathcal{O}_3$ is such that $X(G) = -a\cdot G$, where $a > 0$, then $G \equiv 0$.

Note that $X(F_{xyz}) = (1-\lambda_1-\lambda_2-\lambda_3)\cdot F_{xyz}$, where $1-\lambda_1-\lambda_2-\lambda_3 < 0$, by Theorem 1.5. Therefore $F_{xyz} \equiv 0$. The other cases are similar.

Now, $F_{zzz} \equiv 0$, implies that $F(x,y,z) = A(x,y) + B(x,y)z + C(x,y)z^2$, where A, B, C are polynomials. Since $F_{xzz} \equiv F_{yzz} \equiv 0$, C is a constant. The fact that $X(F) = F$ implies that $X(A) = A$ and $X(B) = (1-\lambda_3)B$. □

We have two possibilities: either (i) $C \neq 0$, or (ii) $C = 0$.

Case (i): $C \neq 0$. — We claim that in this case, $\lambda_3 = \frac{1}{2}$ and $F \simeq z^2 + E(x,y)$, where $X(z) = \frac{1}{2}z$, $X(E) = E$ and $E_{xxyy} = E_{xyyy} = E_{yyyy} = 0$. In particular,

$$E(x,y) = e_3\cdot y^3 + e_2(x)\cdot y^2 + e_1(x)\cdot y + e_0(x), \tag{3.10}$$

where $e_3 \in \mathbb{C}$ and $\deg(e_2) \leqslant 1$.

In fact, since $C \neq 0$ we get $F \simeq z^2 + B_1(x,y)\cdot z + A_1(x,y)$, where $A_1 = C^{-1}\cdot A$ and $B_1 = C^{-1}\cdot B$. It follows from Remark 3.8 that $\lambda_3 = \frac{1}{2}$ and $F \simeq z^2 + E(x,y)$, where $X(E) = E$. Now, $X(E_{xxyy}) = (1-2(\lambda_1+\lambda_2))\cdot E_{xxyy}$. Since $\lambda_1+\lambda_2 > 1-\lambda_3 = \frac{1}{2}$, we get $1-2(\lambda_1+\lambda_2) < 0$ and so $E_{xxyy} = 0$. Similarly $E_{xyyy} = E_{yyyy} = 0$. This implies that E can be written as in (3.10), where $e_3 \in \mathbb{C}$ and $\deg(e_2) \leqslant 1$.

Case (i.1): $C \neq 0$ *and* $e_3 \neq 0$. — We claim that in this case, M is equivalent to one of the following surfaces: Y, $M_{(2,2,2)}$ or $M_{(2,3,r)}$, where $r \in \{3,4,5\}$.

Proof. — It follows from Remark 3.8 that $\lambda_2 = \frac{1}{3}$ and that $E(x,y) \simeq y^3 + f_1(x)\cdot y + f_0(x)$, where $f_0, f_1 \in \mathbb{C}[x]$ and $X(f_j) = (1-\frac{j}{3})f_j$, $j = 0, 1$. Therefore, $F \simeq z^2 + y^3 + f_1(x)\cdot y + f_0(x)$.

Note that, if $f_j \neq 0$ then $f_j(x)$ is a monomial of the form $a\cdot x^m$, $a \in \mathbb{C}^*$, $j = 0, 1$. We have two possibilities: Either $f_0 \not\equiv 0$, or $f_0 \equiv 0$. Let us consider first the case $f_0 \not\equiv 0$. In this case, f_0 must be a monomial of the form $f_0(x) = a\cdot x^m$, where $a \neq 0$ and $\lambda_1 = \frac{1}{m} \leqslant \frac{1}{3}$. After the change of variables of the form $(x,y,z) \mapsto (b\cdot x, y, z)$, $b^m = a$, we can suppose that $a = 1$. Since $\lambda_1+\lambda_2+\lambda_3 > 1$ we get that $m < 6$ and $m \in \{3,4,5\}$. In any case, we must have $f_1(x) = c\cdot x^k$, where $c \in \mathbb{C}$ and $k\cdot\lambda_1 = \frac{2}{3}$, if

$c \neq 0$, because $X(f_1) = \frac{2}{3} f_1$. This implies that if $m \in \{4,5\}$ then $c = 0$ and $F \simeq z^2 + y^3 + x^m$. Hence, M is equivalent to $M_{(2,3,m)}$, $m \in \{4,5\}$. If $m = 3$ and $c \neq 0$, then $k = 2$ and $F \simeq z^2 + y^3 + c\cdot x^2 \cdot y + x^3$. Note that $E(x,y) = y^3 + c\cdot x^2\cdot y + x^3 = (y - a_1 x)(y - a_2 x)(y - a_3 x)$, where a_1, a_2, a_3 are the roots of $y^3 + cy + 1 = 0$. Therefore, $a_1 + a_2 + a_3 = 0$. Since 0 is an isolated singularity of M, we must have $a_i \neq a_j$ if $i \neq j$. Consider an isomorphism ψ of $\mathbb{C}^2$ sending the lines $y - a_j x$, $j = 1,2,3$, into the lines $y + x$, $y + \delta\cdot x$, $y + \delta^{-1}\cdot x$, where $\delta = e^{2\pi i/6}$. It can be checked that $E \circ \psi^{-1}(x,y) = \alpha(x^3 + y^3)$, where $\alpha \neq 0$. Hence $F \simeq z^2 + \alpha(x^3 + y^3) \simeq z^2 + x^3 + y^3$ and M is equivalent to $M_{(2,3,3)}$.

Consider now the case $f_0 \equiv 0$. In this case $F \simeq z^2 + y^3 + x^m\cdot y = z^2 + y(y^2 + x^m)$, where $m \geqslant 1$. Since $X(x^m\cdot y) = x^m\cdot y$, we get $m\cdot \lambda_1 + \lambda_2 = 1$, and so $\lambda_1 = \frac{2}{3m}$. Note that the inequality $\lambda_1 + \lambda_2 + \lambda_3 > 1$ implies that $1 \leqslant m < 4$. If $m = 3$ then $F \simeq z^2 - y(y^2 + x^3)$ and M is equivalent to the surface Y. If $m = 2$ then $y(y^2 + x^2)$ is homogeneous of degree three and $F \simeq z^2 + x^3 + y^3$, as we have seen before. Hence, M is equivalent to $M_{(2,3,3)}$. If $m = 1$ then $F \simeq z^2 + y(y^2 + x)$. In this last case, after the changes of variables $\psi_1(x,y,z) = (x + y^2, y, z) = (x_1, y, z)$ and $\psi_2(x_1,y,z) = (\frac{x_1+y}{2}, \frac{x_1-y}{2i}, z) = (x_2, y_2, z)$, we get that $F \simeq z^2 + x_1\cdot y \simeq z^2 + x_2^2 + y_2^2$. Therefore M is equivalent to $M_{(2,2,2)}$. □

Case (i.2): $C \neq 0$, $e_3 = 0$ and $e_2 \neq 0$. — We claim that in this case M is equivalent to one of the following surfaces: $M_{(2,2,r)}$, $r \geqslant 2$, $M_{(2,3,3)}$, X_m, $m \geqslant 1$ or Z_m, $m \geqslant 1$.

Proof. — Since $\deg(e_2) \leqslant 1$ we have two possibilities: Either $\deg(e_2) = 0$, or $\deg(e_2) = 1$. If $\deg(e_2) = 0$ then $E(x,y) = e_2\cdot y^2 + e_1(x)\cdot y + e_0(x)$, where $e_2 \in \mathbb{C}^*$. It follows from Remark 3.8 that $\lambda_2 = \frac{1}{2}$ and $E(x,y) \simeq y^2 + f_0(x)$, where $X(f_0) = f_0$. Therefore, $f_0(x)$ must be a monomial of the form $a\cdot x^r$, where $r \geqslant 2$ and $a \neq 0$. Hence $F \simeq z^2 + y^2 + a\cdot x^r \simeq z^2 + y^2 + x^r$ and M is equivalent to $M_{(2,2,r)}$, $r \geqslant 2$. If $\deg(e_2) = 1$ then $e_2(x) = a\cdot x$, where $a \neq 0$. Similarly, $e_1(x)$ and $e_0(x)$ are also monomials, where $\deg(e_1) \geqslant 1$ and $\deg(e_0) \geqslant 2$, because 0 is a singularity of M. Therefore, we can write E as $E(x,y) = x(a\cdot y^2 + b\cdot x^\ell\cdot y + c\cdot x^k)$. It follows from Remark 3.8 that $a\cdot y^2 + b\cdot x^\ell\cdot y + c\cdot x^k \simeq y^2 + \alpha\cdot x^k$. Therefore, $E \simeq x(y^2 + \alpha\cdot x^k)$. Note that $\alpha \neq 0$ because 0 is an isolated singularity of M. We have two possibilities: Either k is odd, or k is even. If k is odd, $k = 2m+1$, then $F \simeq z^2 + x(y^2 + a\cdot x^{2m+1}) \simeq z^2 - x(y^2 + x^{2m+1})$. Therefore, M is equivalent to the surface Z_m, $m \geqslant 1$. If k is even we have two possibilities: Either $k = 2$, or $k > 2$. If $k = 2$, then $E(x,y)$ is homogeneous of degree three and $F \simeq z^2 + x^3 + y^3$, as we have seen before. Therefore, M is equivalent to $M_{(2,3,3)}$. If $k > 2$ then

$k = 2(m+1)$, $m \geqslant 1$, and $F \simeq z^2+x(y^2+a\cdot x^{2m+2}) \simeq z^2-x(y^2-x^{2m+2}) = z^2-x(y-x^{m+1})(y+x^{m+1})$. In this last case, if we set $y_1 = y-x^{m+1}$, then we get that $F \simeq z^2-xy_1(y_1+2x^{m+1}) \simeq z^2-xy(y+x^{m+1})$. Therefore, M is equivalent to X_m, $m \geqslant 1$. □

Case (i.3): $C \neq 0$, $e_3 = e_2 = 0$. — We claim that in this case M is equivalent to $M_{(2,2,2)}$.

Proof. — In this case, $F \simeq z^2+e_1(x)\cdot y+e_0(x) := G(x,y,z)$. Since 0 is an isolated singularity of M we must have $e_1 \neq 0$. We assert that $e_1(x) = a\cdot x$, where $a \neq 0$. In fact, since $X(e_1) = (1-\lambda_2)e_1$, e_1 must be a monomial of the form $a\cdot x^m$, $a \neq 0$, $m \geqslant 1$. Similarly, $e_0(x) = b\cdot x^k$, where $a \in \mathbb{C}$ and $k \geqslant 2$, because 0 is a singularity of M. This implies that $G_y = a\cdot x^m$ and $G_x = ma\cdot x^{m-1}\cdot y+kb\cdot x^{k-1}$. If $m > 1$ then $G_x(0,y,0) = G_y(0,y,0) = G_z(0,y,0) = 0$ for all y and 0 is not an isolated singularity of M. Therefore,

$$F \simeq z^2+x(a\cdot y+b\cdot x^{k-1}) \simeq z^2+xy \simeq z^2+x^2+y^2.$$

Hence, M is equivalent to $M_{(2,2,2)}$. □

Case (ii): $C = 0$. — We claim that in this case M is equivalent to $M_{(2,2,r)}$, $r \geqslant 2$.

Proof. — In this case $F(x,y,z) = B(x,y)z+A(x,y)$, where $X(A) = A$ and $X(B) = (1-\lambda_3)B$. Since $F_{xyz} = 0$, B must be of the form $B(x,y) = ax^m+by^n$, where $m,n \geqslant 1$ and either $a \neq 0$, or $b \neq 0$. Since 0 is an isolated singularity of M, then, either $m = 1$, or $n = 1$ (if not, then $F_x(0,0,z) = F_y(0,0,z) = F_z(0,0,z) = 0$). We have two possibilities: Either $b = 0$, or $b \neq 0$. If $b = 0$ then $a \neq 0$ and $m = 1$. Hence, $F = axz+A(x,y)$. Since M is irreducible, x does not divides A, and so $A(x,y) = cy^r+x\cdot A_1(x,y)$, where $c \neq 0$. Therefore

$$F = axz+cy^r+x\cdot A_1(x,y) = cy^r+x(az+A_1(x,y)) \simeq y^r+xz \simeq x^2+z^2+y^r.$$

Therefore M is equivalent to $M_{(2,2,r)}$, $r \geqslant 2$.

If $b \neq 0$ then $n = 1$. In fact, if $a = 0$ then it is clear that $n = 1$. On the other hand, if $a \neq 0$ then $\lambda_3+m\lambda_1 = \lambda_3+n\lambda_2 = 1$, because $X(F) = F$. This implies that $n \leqslant m$, because $\lambda_1 \leqslant \lambda_2$. Therefore, $n = 1$ and $F = z(by+ax^m)+A(x,y)$. After the change of variables $\psi(x,y,z) = (x, by+ax^m, z) = (x_1,y_1,z_1)$, we have

$$\begin{aligned} F\circ\psi^{-1}(x_1,y_1,z_1) = y_1z_1+A(x_1,b^{-1}y_1-b^{-1}ax^m) = y_1z_1+A_1(x_1,y_1) \\ \Longrightarrow\ F \simeq yz+A_1(x,y) \end{aligned}$$

Since M is irreducible, y does not divides A_1 and we can write $A_1 = cx^r + y\cdot B_1(x,y)$, where $c \neq 0$. Hence

$$F \simeq cx^r + y(z + B_1(x,y)) \simeq x^r + yz \simeq x^r + y^2 + z^2.$$

Therefore, M is equivalent to $M_{(2,2,r)}$, $r \geqslant 2$. This finishes the proof of the 2^{nd} step. □

3.3. Proof of the 3rd step

We will use two invariants: the Milnor number of M at 0 and the fundamental group of M^*. The Milnor number of M, denoted by $\mu(M)$, is the complex dimension of $\mathcal{O}_3/ < F_x, F_y, F_z >$ (*cf.* [14]). It is known that $\mu(M) = [F_x, F_y, F_z]_0$ (the intersection number of F_x, F_y, F_z at $0 \in \mathbb{C}^3$). By a direct computation, we have

$$\begin{cases} \mu(M_{(p,q,r)}) = (p-1)(q-1)(r-1) \\ \mu(X_m) = 2(m+2) \\ \mu(Y) = 5 \\ \mu(Z_m) = 2m+3 \end{cases} . \tag{3.11}$$

On the other hand, as we have seen in Examples 1.11, 1.12, 1.13 and 1.14:

$$\begin{cases} \#(\pi_1(M^*_{(2,2,r)})) = r \\ \#(\pi_1(M^*_{(2,3,3)})) = 8 \\ \#(\pi_1(M^*_{(2,3,4)})) = 24 \\ \#(\pi_1(M^*_{(2,3,5)})) = 120 \\ \#(\pi_1(X^*_m) = 8m \\ \#(\pi_1(Y^*) = 48 \\ \#(\pi_1(Z^*_m) = 4(2m+1) \end{cases} . \tag{3.12}$$

As the reader can verify easily, if we take two of the above surfaces then, either they have different Milnor numbers, or different fundamental groups. This ends the proof of Theorem 1.9.

4. Proof of Theorem 1.10

Given a smooth complex manifold N, we will use the following notations:

(I) Ω^k_N = the sheaf of germs of holomorphic k-forms on N.

(II) χ_N = the sheaf of germs of holomorphic vector fields on N.

(III) If $N = F^{-1}(0)$ is a germ at $p \in \mathbb{C}^{n+1}$ of an analytic hypersurface and $V \subset \mathbb{C}^{n+1}$ is an open set where F is defined, then $N_V = N \cap V$ and $N_V^* = N_V \setminus \mathrm{sing}(N)$.

The proof of Theorem 1.10 will be in three steps:

1^st^ step. — Let $M = F^{-1}(0)$ be a germ of hypersurface at $0 \in \mathbb{C}^{n+1}$, $n \geqslant 3$. Suppose that $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$ and that there exists a Stein neighborhood V of $0 \in \mathbb{C}^{n+1}$ where F is defined and such that $H^1(M_V^*, \Omega^{n-1}_{M_V^*}) = \{0\}$. Then $0 \notin \mathrm{sing}(M)$.

2^nd^ step. — Let $M = F^{-1}(0)$ be a germ of hypersurface at $0 \in \mathbb{C}^{n+1}$, $n \geqslant 3$. Suppose that $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-3$ and there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^n, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$. Then there exists a Stein neighborhood V of 0 such that $H^1(M_V^*, \Omega^{n-1}_{M_V^*}) = 0$.

3^rd^ step. — Let M be a germ of hypersurface at $0 \in \mathbb{C}^{n+1}$, $n \geqslant 3$. Suppose that $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$, $0 \in \mathrm{sing}(M)$ and there exists a germ of holomorphic map $\phi\colon (\mathbb{C}^n, 0) \to (M, 0)$ such that $\phi^{-1}(0) = \{0\}$. Then all components of $\mathrm{sing}(M)$ through 0 have dimension $n-2$.

Since the 3^{rd} step implies the theorem and is the easiest one, we will prove it first by using the other two.

4.1. Proof of the 3^{rd} step

Let U and V be connected neighborhoods of $0 \in \mathbb{C}^n$ and $0 \in \mathbb{C}^{n+1}$ such that:

(i) F and ϕ have representatives $F\colon V \to \mathbb{C}$ and $\phi\colon U \to \mathbb{C}^{n+1}$ such that $\phi(U) \subset V$ and $F \circ \phi = 0$.

(ii) $\phi\colon U \to M_V$ is a ramified covering. In particular ϕ is finite to one in U and $\phi^{-1}(M_V) = U$.

Suppose by contradiction that $\mathrm{sing}(M_V)$ has a component, say B, of dimension $k \leqslant n-3$. Let $p \in B \cap V$ be a smooth point of $\mathrm{sing}(M) \cap V$. Note that there exists a (Stein) neighborhood $V_1 \subset V$ of p in $\mathbb{C}^{n+1}$ such that $B \cap V_1 = \mathrm{sing}(M) \cap V_1$ has pure dimension $k \leqslant n-3$. It follows from (ii) that there exists $q \in U$ such that $\phi(q) = p$. Let U_1 be the connected component of $\phi^{-1}(V_1)$ which contains q. Note that $\phi^{-1}(p) \cap U_1 = \{q\}$. Set $\phi_1 = \phi|_{U_1}\colon U_1 \to V_1$. After composing ϕ_1 with translations in both sides, we can suppose that $p = 0 \in \mathbb{C}^{n+1}$, $q = 0 \in \mathbb{C}^n$, $\phi_1\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ and $\phi_1^{-1}(0) = \{0\}$. Since $\mathrm{sing}(M) \cap V_1$ has dimension $k \leqslant n-3$, it follows from steps 1 and 2 that $p = 0$ is not a singularity of M, a contradiction. Therefore all irreducible components of $\mathrm{sing}(M)$ have dimension $n-2$.

4.2. Proof of the 1^{st} step

This step is a consequence of the following:

Lemma 4.1. — *Let M be a germ of hypersurface at $0 \in \mathbb{C}^{n+1}$, $n \geqslant 3$. Suppose that $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2$. Then the following assertions are equivalent:*

(a) *0 is not a singular point of M.*

(b) *There exists a Stein neighborhood V of $0 \in \mathbb{C}^{n+1}$ and a holomorphic section $Y\colon M_V^* \to T\mathbb{C}^{n+1}|_{M_V^*}$ such that $dF_p(Y(p)) \equiv 1$ for all $p \in M_V^*$.*

(c) *There exists a Stein neighborhood V of 0 such that $H^1(M_V^*, \Omega^{n-1}_{M_V^*}) = \{0\}$.*

(d) *There exists a Stein neighborhood V of 0 such that $H^1(M_V^*, \dot{\chi}_{M_V^*}) = \{0\}$.*

Proof. — It is not difficult to see that (a) implies the other assertions. On the other hand, the interior product and the n-form η induce an isomorphism $\delta\colon \dot{\chi}_{M_V^*} \to \Omega^{n-1}_{M_V^*}$ defined by $\delta(Y) = i_Y(\eta)$, where Y is a holomorphic vector field on some open subset of M_V^*. Therefore, (c) is equivalent to (d).

(d) $\implies$ (b) Given V as in (d), consider the covering $\mathcal{I} = (M_j)_{j=1}^n$ of M_V^*, where $M_j = \{p \in M_V | F_{x_j}(p) \neq 0\}$. Let $Z_j\colon M_j \to T\mathbb{C}^{n+1}$ be defined by $Z_j = \frac{1}{F_{x_j}} \frac{\partial}{\partial x_j}$. Note that $dF(Z_j) \equiv 1$. For $i, j \in \{0, \dots, n\}$, set $X_{ij} = Z_j - Z_i$. Since $dF_p(X_{ij}(p)) = 0$ for all $p \in M_{ij} := M_i \cap M_j$, the collection $\{X_{ij}\}_{i,j=0}^n$ can be considered as a cocycle in $Z^1(\mathcal{I}, \chi_{M_V^*})$. Hence, $X_{ij} = X_j - X_i$, where X_j is a holomorphic vector field on M_j for all $j = 0, \dots, n$, because $H^1(M_V^*, \dot{\chi}_{M_V^*}) = \{0\}$. This implies that there exists a holomorphic section $Y\colon M_V^* \to T\mathbb{C}^{n+1}$ such that $Y|_{M_j} = Z_j - X_j$ for all $j = 0, \dots, n$. Observe that $dF(Y)|_{M_j} = dF(Z_j) - dF(X_j) = 1$, which proves (b).

(b) $\implies$ (a) Let $Y\colon M_V^* \to T\mathbb{C}^{n+1}$ be a holomorphic section such that $dF_p(Y(p)) = 1$ for all $p \in M_V^*$. The idea is to prove that Y can be extended to a holomorphic vector field on V, say X. Since $0 \in \overline{M_V^*}$, this implies that $dF_0(X(0)) = 1$. Therefore, $dF_0 \neq 0$ and 0 is not a singular point of M.

We will use the following result (cf [6] p. 133), which is a generalization of [4]: Let N be a Stein manifold of dimension $m \geqslant 3$ and $A \subset N$ be an analytic subset of N such that $\dim_{\mathbb{C}}(A) \leqslant m-3$. Then $H^1(N \setminus A, \mathcal{O}) = 0$.

In particular, since $\dim_{\mathbb{C}}(\mathrm{sing}(M)) \leqslant n-2 = (n+1)-3$, we have $H^1(V \setminus \mathrm{sing}(M), \mathcal{O}) = 0$. Let us prove that the section Y can be extended to V. Since $\mathrm{codim}_V(\mathrm{sing}(M)) \geqslant 2$, by Hartogs' theorem it is sufficient to prove that Y can be extended to a holomorphic vector field on $V \setminus \mathrm{sing}(M) := V^*$. Since $M \subset \mathbb{C}^{n+1}$, we can write $Y = \sum_{j=0}^n Y_j \frac{\partial}{\partial x_j}$, where $Y_j \in \mathcal{O}(M_V^*)$. Hence, it is sufficient to prove that $Y_0, \dots, Y_n$ extend to holomorphic functions on V^*. In fact, any $f \in \mathcal{O}(M_V^*)$ can be extended to a holomorphic function on V^*. Let $\mathcal{U} = (V_j)_{j\in J}$ be a Leray covering of V^* by open sets such that, if $V_j \cap M_V^* \neq \emptyset$ then $f|_{V_j \cap M_V^*}$ can be extended to a holomorphic function, say f_j, on V_j. If $V_j \cap M_V^* = \emptyset$, set $f_j \equiv 0$. In this way we have a collection $(f_j)_{j\in J}$, where $f_j \in \mathcal{O}(V_j)$ and $f_j = f$ on $V_j \cap M_V^*$, if this set is not empty. This implies that, if $V_{ij} := V_i \cap V_j \neq \emptyset$ then $f_j - f_i = f_{ij}.F$, where $f_{ij} \in \mathcal{O}(V_{ij})$. Now, the collection $(f_{ij})_{V_{ij}\neq\emptyset}$ is an additive cocycle. Therefore, if $V_{ij} \neq \emptyset$, we can write $f_{ij} = g_j - g_i$ where $g_j \in \mathcal{O}(V_j)$, because $H^1(V^*, \mathcal{O}) = 0$. This implies that there exists $h \in \mathcal{O}(V^*)$, defined by $h|_{V_j} = f_j - g_j \cdot F$ for all $j \in J$. This implies (a), because $h|_{M_V^*} = f$. □

4.3. Proof of the 2^{nd} step

In this step we will use Dolbeault's theorem (*cf.* [7]): if N is a complex manifold of dimension n then $H^1(N, \Omega_N^{n-1}) \simeq H_{\overline{\partial}}^{n-1,1}(N)$. Hence, we are going to prove that there exists a Stein neighborhood V of $0 \in \mathbb{C}^{n+1}$ such that $H_{\overline{\partial}}^{n-1,1}(M_V^*) = \{0\}$.

Fix Stein neighborhoods U of $0 \in \mathbb{C}^n$ and V of $0 \in \mathbb{C}^{n+1}$ such that:

(i) F has a representative $F\colon V \to \mathbb{C}$ and ϕ a representative $\phi\colon U \to V$.

(ii) $\phi\colon U \setminus \phi^{-1}(\mathrm{sing}(M)) \to M_V^*$ is a ramified covering with d sheets. We will use the notation $U \setminus \phi^{-1}(\mathrm{sing}(M)) = U^*$.

We claim that $H_{\overline{\partial}}^{n-1,1}(U^*) = H^1(U^*, \Omega_{U^*}^{n-1}) = \{0\}$. In fact, since U is Stein and $\dim_{\mathbb{C}}(\phi^{-1}\ (\mathrm{sing}(M))) \leqslant n-3$ (see (ii) of Remark 1.17), we have $H^1(U^*, \mathcal{O}) = 0$ (*cf.* [6]). On the other hand, any holomorphic $(n-1)$-form on an open set $A \subset U^* \subset \mathbb{C}^n$ can be written as

$$\sum_{j=1}^n a_j du_1 \wedge \cdots \wedge \widehat{du_j} \wedge \cdots du_n, a_j \in \mathcal{O}(A).$$

This implies that $H^1(U^*, \Omega_{U^*}^{n-1}) \simeq (H^1(U^*, \mathcal{O}))^n = 0$, which proves the assertion.

Now, let $\alpha \in \Omega^{n-1,1}(M_V^*)$ be C^∞ and such that $\overline{\partial}\alpha = 0$. We want to prove that $\alpha = \overline{\partial}\omega$ for some $\omega \in \Omega^{n-1,0}(M_V^*)$. Since $\phi^{-1}(M_V^*) = U^*$ and $H_{\overline{\partial}}^{n-1,1}(U^*) = 0$ we have that $\phi^*(\alpha) = \overline{\partial}\beta$, where β is a

$(n-1,0)$-form on U^* of class C^∞. Let η be the holomorphic n-form on M_V^* defined as in §2. As we have seen, $\phi^*(\eta)$ extends to a holomorphic n-form on U which can be written as $f(u)\cdot du_1\wedge\cdots\wedge du_n$, where $f\in\mathcal{O}(U)$. Note that $C=\{p\in U^*|f(p)=0\}$ is the set of critical points of $\phi|_{U^*}$ and $\phi(C)=CV$ is the set of critical values of $\phi|_{U^*}$. In particular, if $W=U^*\smallsetminus\phi^{-1}(CV)$ and $\widehat{M}=M_V^*\smallsetminus CV$, then $\phi|_W\colon W\to\widehat{M}$ is a covering with d-sheets.

Using the method of the proof of Lemma 3.1, it is possible to construct a $(n-1,0)$-form of class C^∞, ω on $\widehat{M}$, such that $\overline{\partial}\omega=\alpha$. The form ω is defined on $\widehat{M}$ by

$$\omega_p=\frac{1}{d}\sum_{q\in\phi^{-1}(p)}(\phi_q)_*(\beta_q), \tag{4.1}$$

where $(\phi_q)_*$ denotes the map induced by the isomorphism $D\phi(q)\colon T_q(\mathbb{C}^n)\to T_p(M_V^*)$. It is well defined because $\phi|_W\colon W\to\widehat{M}$ is a covering map. At this point, we observe that if $f(0)\neq 0$ and V is small then $\widehat{M}=M_V^*$ and the 2^{nd} step is proved in this case. If $f(0)=0$, we must prove that ω extends to CV. The idea is the following: Fix a point $p\in CV$. Since $\overline{\partial}\alpha=0$, it follows from the $\overline{\partial}$-Poincaré Lemma that there exists a $(n-1,0)$-form δ_p of class C^∞, defined in a neighborhood V_p of p in M_V^* such that $\overline{\partial}\delta_p=\alpha$ on V_p. Since $\overline{\partial}(\omega-\delta_p)=0$, the form $\omega-\delta_p$ is holomorphic. Therefore, we have only to prove that $\omega-\delta_p$ extends to V_p, for all $p\in CV$. This will be done as follows: We will divide CV into two parts, say $CV=CV_1\cup CV_2$, such that

(iii) CV_1 is contained in the smooth part of CV, $\dim_{\mathbb{C}}(CV_1)=n-1$ and if $p\in CV_1$ then $\omega-\delta_p$ can be extended to V_p by using a local form of ϕ that will be stated in (v). This will imply that ω can be extended to $\widehat{M}\cup CV_1$.

(iv) $\dim_{\mathbb{C}}(CV_2)\leqslant n-2$. If we suppose that (iii) is proved, we can extend ω to $\widehat{M}\cup CV_1$. Hence, if $p\in CV_2$ then $\omega-\delta_p$ can be extended to V_p by Hartogs' theorem, because $CV_2\cap V_p$ has codimension $\geqslant 2$ in V_p.

Therefore, it is sufficient to prove the existence of such a decomposition.

Construction of CV_1 and CV_2. — It will be done in such a way that:

(v) For any $p\in CV_1$ and $q\in\phi^{-1}(p)$ there exist local coordinate systems $(U_q,(u,v)=(u_1,\dots,u_{n-1},v))$ and $(V_p,(x,y)=(x_1,\dots,x_{n-1},y))$ around q and p respectively, where V_p is a neighborhood of p in M_V^*, such that $u(q)=x(p)=0\in\mathbb{C}^{n-1}$, $v(q)=y(p)=0\in\mathbb{C}$, $\phi(U_q)=V_p$, $x(V_p)=\mathbb{D}^{n-1}$

and $y(V_p) = \mathbb{D}$ ($\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$) and

$$\phi(u,v) = (u, v^m), \tag{4.2}$$

where $m \in \mathbb{N}$ depends only on the irreducible components of CV and $\phi^{-1}(CV)$ which contain p and q respectively.

Since $CV = \phi(C)$ and $C = f^{-1}(0) \cap U^*$, all irreducible components of CV and of $\phi^{-1}(CV)$ have complex dimension $n-1$. Moreover, if Σ is an irreducible component of $\phi^{-1}(CV)$ then $\phi(\Sigma)$ is an irreducible component of CV. Denote by A_1 the singular set of CV. Note that $\dim_{\mathbb{C}}(A_1) < \dim_{\mathbb{C}}(CV)$ and that $CV \setminus A_1$ is smooth of dimension $n-1$. Let Σ and $\phi(\Sigma)$ be as before and denote by ϕ_Σ the restriction $\phi|_\Sigma \colon \Sigma \to \phi(\Sigma)$. Let $A_\Sigma = \{q \in \Sigma \setminus \phi^{-1}(A_1) | \operatorname{rank}(D\phi_\Sigma(q)) \leqslant n-2\}$. Note that $\phi(A_\Sigma)$ and $A'_\Sigma = \phi^{-1}(\phi(A_\Sigma))$ are analytic subsets of $\widehat{M}$ and U^*, both of dimension $\leqslant n-2$. Therefore, the closure in U, $\overline{A}'_\Sigma$ is also an analytic subset of $U \subset \mathbb{C}^n$ of dimension $\leqslant n-2$.

Set $B = \cup_\Sigma \phi(\overline{A}'_\Sigma)$ (in the union Σ runs over all irreducibe components of $\phi^{-1}(CV)$), $CV_2 = A_1 \cup B$ and $CV_1 = CV \setminus CV_2$. Then $\dim_{\mathbb{C}}(CV_1) = n-1$ and $\dim_{\mathbb{C}}(CV_2) \leqslant n-2$. Let us prove (v).

Notice that if $p \in CV_1$ and $q \in \phi^{-1}(p)$, then CV and $\phi^{-1}(CV)$ are smooth of dimension $n-1$ at p and q respectively and $D\phi(q)|_{T_q\phi^{-1}(CV)} \colon T_q\phi^{-1}(CV) \to T_p(CV)$ is an isomorphism. If $q \notin C$ then, in fact $D\phi(q) \colon T_q\mathbb{C}^n \to T_p(M^*_V)$ is an isomorphism and it is clear that we can obtain coordinate systems satisfying (3.11) with $m = 1$. If $q \in C$, it follows from the implicit function theorem that there exist local coordinate systems $(U_q, (u,v) = (u_1, \ldots, u_{n-1}, v))$ and $(V_p, (x,y) = (x_1, \ldots, x_{n-1}, y))$, where $u(q) = x(p) = 0 \in \mathbb{C}^{n-1}$, $v(q) = y(p) = 0 \in \mathbb{C}$, $\phi(U_q) = V_p$ and $\phi(u,v) = (u, f(u,v))$ where $f \in \mathcal{O}(U_q)$ and $f_v(0,0) = 0$. We can assume $\phi^{-1}(CV) \cap U_q = \{v = 0\}$ and $CV \cap V_p = \{y = 0\}$. This implies that $f(u,0) \equiv 0$ and $f(u,v) = g(u,v) \cdot v^m$ for some $m \geqslant 2$, where $g \not\equiv 0$. Now, the set of critical points of ϕ in U_q is defined by $f_v = 0$ and is contained in $\phi^{-1}(CV) \cap U_q = \{v = 0\}$. Since $f_v(u,v) = v^{m-1}(m \cdot g(u,v) + v \cdot g_v(u,v))$, we get that $g(u,0) \neq 0$ for all $(u,0) \in U_q$. Therefore by taking a smaller U_q we can suppose that U_q is simply connected and that $g \in \mathcal{O}^*(U_q)$. Let $h \in \mathcal{O}^*(U_q)$ be such that $h^m = g$ and consider the change of variables in a neighborhood of q given by $u' = u$, $v' = g(u,v) \cdot v$. In the coordinate system (u', v') we have $\phi(u', v') = (u', (v')^m)$. Hence, we get property (v).

In order to prove that ω can be extended to a neighborhood of any point $p \in CV_1$, we need a lemma.

Lemma 4.2. — *For any $p \in CV_1$ there exists a coordinate system $(V_p, (x, y) = (x_1, \dots, x_{n-1}, y))$ such that:*

(a) $x(p) = 0 \in \mathbb{C}^{n-1}$, $y(p) = 0 \in \mathbb{C}$, $x(V_p) = \mathbb{D}^{n-1}$ *and* $y(V_p) = \mathbb{D}$.

(b) $CV_1 \cap V_p = \{y = 0\}$.

(c) *There exist* $0 \leqslant a < 1$, $c > 0$ *and a compact neighborhood* $K_p = K$ *of* p *such that if* $\omega = \omega_n(x, y)dx_1 \wedge \dots \wedge dx_{n-1} + \sum_{j=1}^{n-1} \omega_j(x, y)dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dy$, *then*

$$\max\{|\omega_1(x, y)|, \dots, |\omega_n(x, y)|\} \leqslant c \cdot |y|^{-a}, \ \forall (x, y) \in K' := K \setminus \{y = 0\}. \tag{4.3}$$

Proof. — Fix $p \in CV_1$ and let $\phi^{-1}(p) = \{q_1, \dots, q_r\}$. Since CV_1 is smooth of codimension one in M_V^*, we can find a local coordinate system $(B, (x, y))$ around p such that $x(p) = 0 \in \mathbb{C}^{n-1}$, $y(p) = 0 \in \mathbb{C}$ and $CV_1 \cap B = \{y = 0\}$. Given $q_j \in \phi^{-1}(p)$ fix local coordinate systems $(U_j, (u_j, v_j) = (u_{j1}, \dots, u_{jn-1}, v_j))$ and $(V_j, (x_j, y_j) = (x_{j1}, \dots, x_{jn-1}, y_j))$ as in (v), where $\phi(u_j, v_j) = (u_j, v_j^{m_j})$, $m_j \in \mathbb{N}$. Observe that $m_1 + \dots + m_r = d$. We can suppose without lost of generality that $V_j \subset B$ for all $j = 1, \dots, r$. Let K be a compact neighborhood of p such that $K \subset \cap_j V_j$.

Since $CV_1 \cap V_j = \{y = 0\} = \{y_j = 0\}$, there exists a constant $k_1 > 1$ such that

$$k_1^{-1} \cdot |y(z)| \leqslant |y_j(z)| \leqslant k_1 \cdot |y(z)| \tag{4.4}$$

for all $z \in K$ and all $j = 1, \dots, r$. Now, fix a point $p_0 = (x_0, y_0) \in K \setminus \{y = 0\}$. In the coordinate system V_j we have $p_0 = (x_{0j}, y_{0j})$ and $\phi^{-1}(p_0) \cap U_j = \{(x_{0j}, \gamma^\ell . v_{0j}) := q_{0j}^\ell \mid \ell = 1, \dots, m_j\}$, where γ is a primitive m_j^{th} root of unity and $v_{0j}^{m_j} = y_{0j}$. Let

$$\beta|_{U_j} = \beta_n(u_j, v_j)\, du_{j1} \wedge \dots \wedge du_{jn-1} + \sum_{k=1}^{n-1} \beta_k(u_j, v_j)\, du_{j1} \wedge \dots \wedge du_{jk-1} \wedge du_{jk+1} \wedge \dots \wedge dv_j$$

where $\beta_1, \dots, \beta_n \in C^\infty(U_j)$ (recall that $\phi^*(\alpha) = \overline{\partial}\beta$). The inverse of ϕ from a small neighborhood of p_0 to a small neighborhood of q_{0j}^ℓ can be written as $\psi_{0j}^\ell(x_j, y_j) = (x_j, \gamma^\ell . y_j^{1/m_j})$, where y_j^{1/m_j} is a branch of the m_j^{th} root of y_j. Therefore, the contribution to the sum in (18) which comes from $\phi^{-1}(p) \cap U_j$, in this neighborhood of p_0, is of the form $\frac{1}{d} \sum_{k=1}^{m_j} \theta_k^j$

$$\theta_k^j = (\psi_{0j}^k)_*(\beta) = \theta_{kn}^j dx_{j1} \wedge \dots \wedge dx_{jn-1} + \sum_{m=1}^{n-1} \theta_{km}^j dx_{j1} \wedge \dots \wedge dx_{j,m-1} \wedge dx_{j,m+1} \wedge \dots \wedge dy_j$$

where

$$\theta^j_{kn} = \beta_n(x_j, \gamma^k.y_j^{1/m_j}) \text{ and } \theta^j_{km} = \beta_m(x_j, \gamma^k.y_j^{1/m_j}) \cdot \frac{1}{m_j}\gamma^k.y_j^{1/m_j-1}, \quad m = 1,\dots,n-1$$

Since $\beta_1,\dots,\beta_n$ are bounded in $\phi^{-1}(K)\cap U_j$, we get from the above relations that

$$|\theta^j_{km}(x_j,y_j)| \leqslant c_j\cdot |y_j|^{-(1-1/m_j)}, \tag{4.5}$$

where $c_j > 0$. Now, let us write θ^j_k in the coordinate system (x,y) as

$$\theta^j_k = \theta^{j1}_{kn} dx_1\wedge\dots\wedge dx_{n-1} + \sum_{m=1}^{n-1}\theta^{j1}_{km} dx_1\wedge\dots\wedge dx_{m-1}\wedge dx_{m+1}\wedge\dots\wedge dy.$$

Since $(x,y)\mapsto(x_j,y_j)$ is a diffeomorphism for all $j=1,\dots,r$, we get from (4.4) and (4.5) that

$$|\theta^{j1}_{km}(x,y)| \leqslant c^1_j\cdot |y|^{-(1-1/m_j)}, \tag{4.6}$$

for all $m=1,\dots,n$, where $c^1_j > 0$. We leave this last computation for the reader. Set $a = \max\{1-1/m_j | j=1,\dots,r\}$ and $c = \sum_{j=1}^r c^1_j$. Since $\omega = \sum_{j=1}^r\sum_{k=1}^{m_j}\theta^j_k$, we obtain from (4.6) that

$$|\omega_m(x,y)| = |\sum_j\sum_k \theta^{j1}_{km}(x,y)| \leqslant \sum_j\sum_k |\theta^{j1}_{km}(x,y)| \leqslant c\cdot|y|^{-a},$$

for all $m=1,\dots,n$. □

In order to finish the proof of Theorem 1.10 it is enough to show that ω can be extended to the compact neighborhood of p, $K_p\subset V_p$, like in Lemma 4.2. Let δ_p be a C^∞ form on V_p such that $\overline{\partial}\delta_p = \alpha|_{V_p}$. Note that $\omega-\delta_p$ is holomorphic on $V_p \setminus (V_p\cap CV_1)$. Set

$$\delta_p = \delta_n(x,y)dx_1\wedge\dots\wedge dx_{n-1} + \sum_{j=1}^{n-1}\delta_j(x,y)dx_1\wedge\dots\wedge dx_{j-1}\wedge dx_{j+1}\wedge\dots\wedge dy$$

where $\delta_1,\dots,\delta_n\in C^\infty(V_p)$. Since $\omega-\delta_p$ is holomorphic on $V_p\setminus(V_p\cap CV_1)$, we get that $\omega_m-\delta_m\in\mathcal{O}(V_p\setminus(V_p\cap CV_1))$, for all $m=1,\dots,n$. On the other hand $|\delta_m|$ is bounded in K_p. Hence,

$$|\omega_m(x,y)-\delta_m(x,y)| \leqslant c_1\cdot|y|^{-a}, \forall (x,y)\in K_p, \tag{4.7}$$

for all $m=1,\dots,n$, where c_1 is a positive constant. Since $0\leqslant a<1$, (4.7) implies that, for a fixed x, the holomorphic function $y\mapsto\omega_m(x,y)-\delta_m(x,y)$ has a removable singularity at $y=0$. Hence, $\omega_m(x,y)-\delta_m(x,y)$ extends to

K_p as a holomorphic function, for all $m = 1, \ldots, n$. This finishes the proof of Theorem 1.10. □

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