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DYNAMICAL DIRECTIONS IN NUMERATION

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Abstract. — This survey aims at giving a consistent presentation of numeration from a dynamical viewpoint: we focus on numeration systems, their associated compactification, and dynamical systems that can be naturally defined on them. The exposition is unified by the fibred numeration system concept. Many examples are discussed. Various numerations on rational integers, real or complex numbers are presented with special attention paid to β-numeration and its generalisations, abstract numeration systems and shift radix systems, as well as G-scales and odometers. A section of applications ends the paper.

Résumé. — Le but de ce survol est d'aborder définitions et propriétés concernant la numération d'un point de vue dynamique : nous nous concentrons sur les systèmes de numération, leur compactification, et les systèmes dynamiques qui peuvent être définis dessus. La notion de système de numération fibré unifie la présentation. De nombreux exemples sont étudiés. Plusieurs numérations sur les entiers naturels, relatifs, les nombres réels ou les nombres complexes sont présentées. Nous portons une attention spéciale à la β-numération ainsi qu'à ses généralisations, aux systèmes de numération abstraits, aux systèmes dits "shift radix", de même qu'aux G-échelles et aux odomètres. Un paragraphe d'applications conclut ce survol.

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Keywords: Numeration, fibred systems, symbolic dynamics, odometers, numeration scales, subshifts, f-expansions, β-numeration, sum-of-digits function, abstract number systems, canonical numeration systems, shift radix systems, additive functions, tilings, Rauzy fractals, substitutive dynamical systems.
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1. Introduction

1.1. Origins

Numeration is the art of representation of numbers; primarily natural numbers, then extensions of them - fractions, negative, real, complex numbers, vectors, a.s.o. Numeration systems are algorithmic ways of coding numbers, *i.e.*, essentially a process permitting to code elements of an infinite set with finitely many symbols.

For ancient civilisations, numeration was necessary for practical use, commerce, astronomy, *etc.* Hence numeration systems have been created not only for writing down numbers, but also in order to perform arithmetical operations.

Numeration is inherently dynamical, since it is collated with infinity as potentiality, as already asserted by Aristotle[(1)]: if I can represent some natural number, how do I write the next one? On that score, it is significant that motion (greek $\delta\acute{\upsilon}\nu\alpha\mu\iota\varsigma$) and infinity are treated together in Aristotle's work (*Physics*, third book). Furthermore, the will to deal with arbitrary large numbers requires some kind of invariance of the representation and a recursive algorithm which will be iterated, hence something of a dynamical kind again.

In the sequel, we briefly mention the most important historical steps of numeration. We refer to the book of Ifrah [179] for an amazing amount of information on the subject and additional references.

Numeration systems are the ultimate elaboration concerning representation of numbers. Most early representations are only concerned with finitely many numbers, indeed those which are of a practical use. Some primitive civilisations ignored the numeration concept and only had names for cardinals that were immediately perceptible without performing any action of counting, *i.e.*, as anybody can experiment alone, from one to four. For example, the Australian tribe Aranda say "*ninta*" for one, "*tara*" for two, "*tara-ma-ninta*" for three, and "*tara-ma-tara*" for four. Larger numbers are indeterminate (many, a lot).

Many people have developed a representation of natural numbers with fingers, hands or other parts of the human body. Using phalanxes and articulations, it is then possible to represent (or show) numbers up to ten thousand or more. A way of showing numbers up to 10^{10} just with both hands was implemented in the XVIth century in China (*Sua fa tong zong*, 1593). Clearly, the choice of base 10 was at the origin of these methods. Other bases were attested as well, like five, twelve, twenty or sixty by Babylonians. However, all representations of common use work with a base.

Bases have been developed in Egypt and Mesopotamia, about 5000 years ago. The Egyptians had a special sign for any small power of ten: a vertical stroke for 1, a kind of horseshoe for 10, a spiral for 100, a loto flower for 1000, a finger for 10000, a tadpole for 10^5, and a praying man for a

(1) "The infinite exhibits itself in different ways - in time, in the generations of man, and in the division of magnitudes. For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different. Again, 'being' has more than one sense, so that we must not regard the infinite as a 'this', such as a man or a horse, but must suppose it to exist in the sense in which we speak of the day or the games as existing things whose being has not come to them like that of a substance, but consists in a process of coming to be or passing away; definite if you like at each stage, yet always different." [28] translation from http://people.bu.edu/wwildman/WeirdWildWeb/courses/wphil/readings/wphil_rdg07_physics_entire.htm

million. For 45200, they drew four fingers, five loto flowers and two spirals (hieroglyphic writing). A similar principle was used by Sumerians with base 60. To avoid an over complicated representation, digits (from 1 to 59) were written in base 10. This kind of representation follows an additional logic. A more concise coding has been used by inventing a symbol for each digit from 1 to 9 in base 10. In this modified system, 431 is understood as $4 \times 100 + 3 \times 10 + 1 \times 1$ instead of $100 + 100 + 100 + 100 + 10 + 10 + 10 + 1$. Etruscans used such a system, as did Hieratic and Demotic handwritings in Egypt.

The next crucial step was the invention of positional numeration. It has been discovered independently four times, by Babylonians, in China, by the pre-Columbian Mayas, and in India. However, only Indians had a distinct sign for every digit. Babylonians only had two, for 1 and 10. Therefore, since they used base 60, they represented 157, say, in three blocks: from the left to the right, two times the unit symbol (representing 120), three times the symbol for 10 (for 30), and seven times the unit symbol again (for 7). To avoid any confusion between blocks (does eight times the unit symbol represent 8×1 or $2 \times 60 + 6$, *etc*), they used specific arrangements of the symbols - as one encounters nowadays on the six faces of a dice(2). Positional numeration enabled the representation of arbitrary large numbers. Nevertheless, the system was uncomplete without the most ingenuous invention, *i.e.*, the zero. A sign for zero was necessary and it was known to these four civilisations. To end the story, to be able to represent huge numbers, but also to perform arithmetic operations with any of them, one had to understand that this zero was a quantity, and not “nothing”, *i.e.*, an entity of the same type as the other numbers. Ifrah writes: [Notre] “*numération est née en Inde il y a plus de quinze siècles, de l’improbable conjonction de trois grandes idées ; à savoir :*

- *l’idée de donner aux chiffres de base des signes graphiques détachés de toute intuition sensible, n’évoquant donc pas visuellement le nombre des unités représentées ;*
- *celle d’adopter le principe selon lequel les chiffres de base ont une valeur qui varie suivant la place qu’ils occupent dans les représentations numériques ;*

(2) For pictures and examples, see [179], vol. 1, page 315 *et seq.* or the internet page http://history.missouristate.edu/jchuchiak/HST%20101-Lecture%202cuneiform_writing.htm

- *et enfin celle de se donner un zéro totalement 'opératoire', c'est-à-dire permettant de remplacer le vide des unités manquantes et ayant simultanément le sens du nombre 'nul'."* [179].[3]

After this great achievement, it was possible to become aware of the multiplicative dimension of the numeration system: 431 not only satisfies $431 = 4 \times 100 + 3 \times 10 + 1$ (additive understanding) but also $431 = 1 + 10 \times (3 + 4 \times 10)$. Moreover, the representation could be obtained in a purely dynamical way and had a meaning in terms of modular arithmetic. Finally, the concept of number fits closely with its representation. A mathematical maturation following an increasing abstraction process culminating in the invention of the zero had been necessary to construct a satisfactory numeration system. It turned out to be the key for many further mathematical developments.

1.2. What this survey is (not) about

The subject of representing nonnegative integers, real numbers or any suitable extension, generalisation or analogon of them (complex numbers, integers of a number field, elements of a quotient ring, vectors of a finite-dimensional vector space, points of a function field, and so on) is too vast to be covered in a single paper. Hence we made choices among the most notable ways to think about numbers and their representations. Our standpoint is essentially dynamical: we are more interested in transformations yielding representations than in the representations themselves. We also focus on dynamical systems emerging from these representations as well, since we think that they give some insight into their understanding: as we explained through our historical considerations, numeration is itself a dynamical concept. Usually, papers on numeration deal with numeration on some special set of numbers: $\mathbb{N}$, $\mathbb{Z}$, $[0,1]$, $\mathbb{Z}[i]$,... Our purpose is to introduce a general setting in which these examples can take place. In fact, a suitable concept already exists in the literature, since it turns out that the notion

(3) Our numeration was created in India more than fifteen centuries ago on the basis of the improbable conjunction of three important ideas, namely:
- to give base digits graphic signs unlinked with any sensitive intuition; they thus do not visually indicate the number of represented quantities;
- to adopt a principle whereby base digits have a value that depends on their position in the numerical representation;
- and lastly, to give a totally "operatory" zero, *i.e.*, so that the gap left by missing units can be filled, while also representing a "zero number".

of fibred system, according to Schweiger, is a powerful object to subsume most of the different numerations under a unified concept. Therefore, the concepts we define in Section 2 originate directly from his book [299][(4)] or have been naturally built up from it. More precisely, the key concept will be that of a *fibred numeration system* that we present in Section 2.2. A second conceptual source of inspiration was the survey of Kátai [199].

These notions - essentially fibred systems and numeration systems - are very general and helpful for determining what quite different types of numerations may have in common. Simultaneously, they are flexible since they can be enriched with different structures. According to our needs, that is, describing the classical examples of numeration, we will equip them progressively with a topology, a sigma-algebra or an algebraic structure, giving rise to new questions. In other words, our purpose is not to study properties of fibred numeration systems, but rather to use them as a framework for considering numeration.

This paper is organised as follows. The main definitions are introduced in Sections 2.1, 2.2 and 2.3. Section 2.1 proposes a general definition of a numeration system and introduces the difference between *representation* and *expansion*. The key of Section 2 is Section 2.2, where fibred numeration systems are introduced (Definition 2.4) and where their general properties are discussed. A second important mathematical object of this paper is defined in Section 2.3: the compactification associated with a fibred numeration system. The main notions are illustrated by the most usual expansion, *i.e.*, the q-adic numeration. Section 2.4 presents in detail several well and less known examples from the viewpoint given by the vocabulary we just introduced. Section 2.5 deals with questions we will handle along the paper and presents a series of significative examples.

Each of the next three sections is devoted to a specific direction of generalisation of standard numeration. Section 3 is devoted to canonical numeration systems that originate in numeration in number fields, and to a very recent and promising generalisation of them: shift radix systems. Section 4 deals with sofic numeration systems, Dumont-Thomas numeration and abstract numeration systems. In both sections, the exposition focuses

(4) "The notion of a fibred system arose from successive attempts to extend the so-called metrical number theory of decimal expansions and continued fractions to more general types of algorithms. [...]

Another source for this theory is ergodic theory, especially the interest in providing examples for one-sided subshifts, topological Markov chains, sofic systems and the like." [299], pages 1-2. For other applications of fibred systems and relevant references, see the preface and Chapter 1 of [299], and the subsequent book of the same author [300].

on geometrical aspects and on the connection with β-numeration. The progression towards a higher degree of generalisation is also emphasised. The presentation through fibred numeration systems is new. Section 5 deals with a large family of dynamical systems with zero entropy, called odometers: roughly speaking, they correspond to the addition by 1. These systems are natural generalisations of Hensel's q-adic numbers. These three sections begin with a detailed introduction to which the reader is referred for more details.

Section 6 is concerned with a selection of applications. Section 6.1 gives a partial and short survey on the vast question of the asymptotic distribution of additive functions with respect to numeration systems, especially the sum-of-digits function. Section 6.2 explains how Rauzy fractals (that have been developed in Section 4) can be used to construct bounded remainder sets and to get discrepancy estimates of Kronecker sequences. Section 6.3 deals with computer arithmetics and cryptography, and Section 6.4 is concerned with applications in physics, namely quasicrystals. Note that the current resarch on quasicrystals is very active, as shown in this volume by the contribution [153].

A survey on dynamical aspects of numeration assumes that the reader is already familiar with the underlying basic concepts from dynamical systems, ergodic theory, symbolic dynamics, and formal languages. Only Section 6 here and there requires more advanced notions. As general references on dynamical systems and ergodic theory, see [71, 107, 205, 268, 270, 340]. For symbolic dynamics, see [58, 75, 209, 239]. For references on word combinatorics, automata and formal languages, see [27, 241, 242, 243, 267, 272, 286]. Up to our knowledge, we did not treat subjects that have been already covered in previous surveys or books, even if some of them contain certain dynamical aspects. Let us now briefly mention some of these surveys.

A pedagogical introductive exposition of numeration from a dynamical point of view can be found in [108]. For a related dynamical approach of numeration systems based on the compactification of the set of real numbers, see [196, 197]. This latter approach includes in particular the β-numeration and numerations inspired by weighted substitutions (substitution numeration systems are discussed in Section 4).

Connections between β-expansions, Vershik's adic transformation and codings of hyperbolic automorphisms are extensively presented in Sidorov's survey [304], where the author already studies alternative β-expansions from a probabilistic viewpoint. In the same vein, see also [137, 296]. Let us note that tiling theory has also close connections with numeration (*e.g.*, see [284, 318, 327]).

For the connections between arithmetic properties of numbers and syntaxic properties of their representations, see [282]. The question of renormalisation (or change of alphabet) is motivated by performing arithmetic operations. In [148, 149], Frougny shows among other things that renormalisation is computable by a finite transducer in the case of a G-scale given by a linear recurrence sequence (G-scales are introduced and discussed in Section 5). These survey papers develop the theory of β-representation from the point of view of automata theory.

Numeration systems are also closely related to computer arithmetics such as illustrated in [45, 212, 259, 260]; indeed some numerations can be particularly efficient for algorithms that allow to perform the main mathematical operations and to compute the main mathematical functions; see also Section 6.3.

A wide literature has been devoted to Cobhams's theorem [98] and its connections with numeration systems, *e.g.*, see [87, 130, 131, 133] and the survey [132]. Let us recall that Cobhams's theorem states that if the characteristic sequence of a set of nonneegative integers is recognisable in two multiplicatively independent bases, then it is ultimately periodic.

Let us also quote [6] for spectacular recent results concerning combinatorial transcendence criteria that may be applied to the b-adic expansion of a real number. For more details, see also the survey [5].

2. Fibred numeration systems

2.1. Numeration systems

Let $q \geqslant 2$ be an integer. Then every nonnegative integer n can be uniquely written as

$$n = \varepsilon_\ell(n)q^\ell + \varepsilon_{\ell-1}(n)q^{\ell-1} + \cdots + \varepsilon_1(n)q + \varepsilon_0(n), \tag{2.1}$$

with nonnegative digits $0 \leqslant \varepsilon_k(n) \leqslant q-1$, and $\varepsilon_\ell(n) \neq 0$ for $\ell \neq 0$. Otherwise stated, the word $\varepsilon_0(n)\varepsilon_1(n)\ldots\varepsilon_{\ell-1}(n)\varepsilon_\ell(n)$ *represents* the number n. Similarly, any real number $x \in [0,1)$ can be uniquely written as

$$x = \sum_{k=1}^{\infty} \varepsilon_k(x)q^{-k}, \tag{2.2}$$

with $0 \leqslant \varepsilon_k(n) \leqslant q-1$ again and the further assumption that the sequence $(\varepsilon_k(x))_{k\geqslant 1}$ does not eventually take only the value $q-1$. The sequence

$(\varepsilon_k(x))_{k\geqslant 1}$ represents the real number x. These sequences are called *q-adic representation* of n and x, respectively.

If $(F_n)_n$ is the (shifted) Fibonacci sequence with convention $F_0 = 1$, $F_1 = 2$, $F_{n+2} = F_{n+1} + F_n$, any nonnegative integer can be uniquely written as

$$n = \varepsilon_\ell(n)F_\ell + \varepsilon_{\ell-1}(n)F_{\ell-1} + \cdots + \varepsilon_1(n)F_1 + \varepsilon_0(n), \tag{2.3}$$

with digits $\varepsilon_k(n) \in \{0,1\}$ satisfying the condition $\varepsilon_k(n)\varepsilon_{k+1}(n) = 0$ for all k, and $\varepsilon_\ell(n) \neq 0$ for $\ell \neq 0$. This is called the Zeckendorf expansion (see Example 2.14).

Both ways of writing nonnegative integers characterise integers with a finite sequence of digits satisfying some conditions. For real numbers, the representation is done through an (infinite) sequence (and it has to be so, since the interval $[0,1)$ is uncountable). A *numeration system* is a *coding* of the elements of a set with a (finite or infinite) sequence of digits. The result of the coding - the sequence - is a *representation* of the element.

Definition 2.1. — *A numeration system (resp. a finite numeration system) is a triple (X, I, φ), where X is a set, I a finite or countable set, and φ an injective map $\varphi\colon X \hookrightarrow I^{\mathbb{N}^*}$, $x \mapsto (\varepsilon_n(x))_{n\geqslant 1}$ (resp. $\varphi\colon X \hookrightarrow I^{(\mathbb{N})}$, where $I^{(\mathbb{N})}$ stands for the set of finite sequences over I). The map φ is the representation map, and $\varphi(x)$ is the representation of $x \in X$. Let (X, I, φ) be a numeration system (resp. finite numeration system). The admissible sequences (resp. admissible strings) are defined as the representations $\varphi(x)$, for $x \in X$.*

Let us note that we have chosen the convention $\varphi\colon X \hookrightarrow I^{\mathbb{N}^*}$ for the choice of the index set, *i.e.*, we have chosen to start with index 1. Example 2.1 (resp. 2.2) shows that it can be more natural to begin with index 0 (resp. 1). Therefore, we shall allow us to switch from one convention to the other one according to the context.

Equations (2.1) and (2.2) say actually more than expressing a representation. The equality takes into account the algebraic structure of the set of represented numbers (existence of an addition on $\mathbb{N}$ and $\mathbb{R}$, respectively), and the topological structure as well for (2.2) by considering a convergent series: these structures allow us to understand the *representation* as an *expansion*. These expansions use the sequence of nonnegative (resp. negative) powers of q as a *base*. This can be formulated in an abstract way in the following definition.

Definition 2.2. — *Let (X, I, φ) be a numeration system. An expansion is a map $\psi : I^{\mathbb{N}^*} \to X$ (resp. $\psi : I^{(\mathbb{N})} \to X$) such that $\psi \circ \varphi(x) = x$ for all*

$x \in X$. An expansion of an element $x \in X$ is an equality $x = \psi(y)$; it is a proper expansion if $y = \varphi(x)$, and an improper expansion otherwise.

If X is a subset of an A-module (in the case of a finite number system) or of a topological A-module, an expansion is often of the type $\psi(y) = \sum_{n=1}^{\infty} \nu(y_n)\xi_n$, with $\nu : I \to A$ and $(\xi_n)_{n\geqslant 1} \in X^{\mathbb{N}^*}$. In this case, the sequence $(\xi_{n\geqslant 1})_n$ is called a base or scale.

For example, if one considers the q-adic expansion (2.1), then $X = \mathbb{N}$ is a subset of the $\mathbb{Z}$-module $\mathbb{Z}$, and we have an expansion defined by a finite sum $\psi(y) = \sum_{n\geqslant 0} y_n q^n$, *i.e.*, a base $\xi_j = q^j$ and $\nu(i) = i$. For (2.2), the expansion is given by the series $\psi(y) = \sum_{n\geqslant 1} y_n q^{-n}$.

2.2. Fibred systems and fibred numeration systems

Section 2.1 introduced a useful vocabulary, but the notion of numeration system remains poor. It becomes much more interesting when one asks *how* the digits are produced, that is, how the representation map is constructed. The dynamical dimension of numeration lies precisely there. Therefore, the key concept of Section 2 originates in the observation that, in (2.1), (2.2), (2.3), and many other examples of representations, the digits are (at least, can be) obtained by iteration of a transformation, and that this transformation contains an amount of interesting information on the numeration. This concept is that of fibred numeration system and we will use it along the paper. It is itself constructed from the notion of fibred system, issued from [299], that we recall now.

Definition 2.3. — *A* fibred system *is a set X and a transformation $T\colon X \to X$ for which there exist a finite or countable set I and a partition $X = \biguplus_{i\in I} X_i$ of X such that the restriction T_i of T on X_i is injective, for any $i \in I$. This yields a well defined map $\varepsilon\colon X \to I$ that associates the index i with $x \in X$ such that $x \in X_i$.*

Suppose (X, T) is a fibred system with the associated objects above. Let $\varphi\colon X \to I^{\mathbb{N}^*}$ be defined by $\varphi(x) = (\varepsilon(T^n x))_{n\geqslant 1}$. We will write $\varepsilon_n = \varepsilon \circ T^{n-1}$ for short. Let S stand for the (right-sided) shift operator on $I^{\mathbb{N}^*}$. These definitions yield a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \varphi\downarrow & & \downarrow\varphi \\ I^{\mathbb{N}^*} & \xrightarrow[S]{} & I^{\mathbb{N}^*} \end{array} \tag{2.4}$$

DEFINITION 2.4. — *Let* (X,T) *be a fibred system and* $\varphi\colon X \to I^{\mathbb{N}^*}$ *be defined by* $\varphi(x) = (\varepsilon(T^n x))_{n\geqslant 1}$. *If the function* φ *is injective (i.e., if* (X, I, φ) *is a numeration system), we call the quadruple* $\mathcal{N} = (X,T,I,\varphi)$ *a* fibred numeration system *(FNS for short). Then* I *is the set of digits of the numeration system; the map* φ *is the* representation map *and* $\varphi(x)$ *the* $\mathcal{N}$-representation *of* x.

In general, the representation map is not surjective. The set of prefixes of $\mathcal{N}$*-representations is called the* language $\mathcal{L} = \mathcal{L}(\mathcal{N})$ *of the fibred numeration system, and its elements are said to be admissible. The* admissible sequences *are defined as the elements* $y \in I^{\mathbb{N}^*}$ *for which* $y = \varphi(x)$ *for some* $x \in X$.

Note that we could have taken the quadruple (X,T,I,ε) instead of the quadruple (X,T,I,φ) in the definition. In almost all examples, the set of digits is finite. It may happen that it is countable (*e.g.*, see Example 2.10 and 2.14 below).

Let (X,T) be a fibred system with a partition $(X_i)_{i\in I}$ and a map φ as in the diagram (2.4). By definition of a partition, $X_i \neq \emptyset$ for each $i \in I$; hence all digits are admissible. Moreover, *set of prefixes* and *set of factors* are synonymous here:

$$\begin{aligned}\mathcal{L} &= \big\{(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x))\,;\, n \in \mathbb{N},\, x \in X\big\}\\ &= \big\{(\varepsilon_{m+1}(x), \varepsilon_{m+2}(x), \dots, \varepsilon_{m+n}(x))\,;\, (m,n) \in \mathbb{N}^2,\, x \in X\big\}.\end{aligned} \tag{2.5}$$

However, $\varphi(X)$ is not necessarily shift invariant and it may happen that for some m,

$$\big\{(\varepsilon_{m+1}(x), \varepsilon_{m+2}(x), \dots, \varepsilon_{m+n}(x))\,;\, n \in \mathbb{N}, x \in X\big\} \neq \mathcal{L}.$$

This is due to the lack of surjectivity of the transformation T.

The representation map transports cylinders from the product space $I^{\mathbb{N}^*}$ to X, and for $(i_0, i_1, \dots, i_{n-1}) \in I^n$, one may define the cylinder

$$X \supset C(i_0, i_1, \dots, i_{n-1}) = \bigcap_{0\leqslant j<n} T^{-j}(X_{i_j}) = \varphi^{-1}[i_0, i_1, \dots, i_{n-1}]. \tag{2.6}$$

Moreover, the earlier assumption in Definition 2.3 that the restriction of T to X_i is injective says that the application $x \mapsto (\varepsilon(x), T(x))$ is itself injective. It is a necessary condition for φ to be injective, and $\mathcal{N}$ is an FNS if and only if

$$\forall x \in X \;:\; \bigcap_{n\geqslant 0} C(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x)) = \{x\}. \tag{2.7}$$

If X is a metric space, a sufficient condition for (2.7) to hold is that, for any admissible sequence $(i_1, i_2, \dots, i_n, \dots)$, the diameter of the cylinders

$C(i_1, i_2, \ldots, i_n)$ tends to zero when n tends to infinity. In this case, if F is a closed subset of X, then

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\substack{(i_1,\ldots,i_n)\in\mathcal{L} \\ C(i_1,\ldots,i_n)\cap F\neq\emptyset}} C(i_1, \ldots, i_n),$$

which proves that the σ-algebra $\mathcal{B}$ generated by the cylinders is the Borel algebra. In general, T is $\mathcal{B}$-measurable.

The representations introduced in Definition 2.4 are by nature infinite. It is suitable to have access to finite expansions, in order to have a notion of finite fibred numeration system, as one had finite numeration systems in Section 2.1. For that purpose, we need to look at fixed points of the transformation T. Let (X, T, I, φ) be an FNS. If $x \in X$ satisfies $T(x) = x$, then its representation is constant, *i.e.*, there exists $i \in I$ such that $\varphi(x) = (i, i, i, \ldots)$. By injectivity of φ, the converse is true too, and φ induces a bijection between the set of fixed points and the constant admissible sequences.

Definition 2.5. — *A fibred numeration system $\mathcal{N} = (X, T, I, \varphi)$ is* finite *(FFNS) if there exists a fixed point x_0 under the transformation T with $\mathcal{N}$-representation $\varphi(x_0) = (i_0, i_0, \ldots)$ such that for every element $x \in X$, there exists a nonnegative integer n_0 satisfying $\varepsilon_n(x) = i_0$ for all $n \geqslant n_0$.*

A fibred numeration system $\mathcal{N}$ is quasi-finite *if and only if it is not finite and every $\mathcal{N}$-representation is ultimately periodic.*

The attractor *of the system is defined as the set $\mathbf{A} = \{x \in X \,;\, \exists k \geqslant 1 : T^k(x) = x\}$.*

In other words, an FNS is finite if there exists a unique fixed point and if it belongs to every orbit. By injectivity of φ, the attractor is the set of elements having a purely periodic representation. An FNS is finite or quasi-finite if every orbit falls in the attractor: $\forall x \in X, \exists k$ such that $T^k x \in \mathbf{A}$.

In an FFNS, the representation $\varphi(x) = (\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_{n_0-1}(x), i_0, i_0, \ldots)$ of an element can be identified with the finite representation $(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_{n_0-1}(x))$. With this convention, an FFNS is an FNS where every element has a finite representation. Then the representation map can be considered as a map $\varphi\colon X \to I^{(\mathbb{N})}$. This gives finite expansions (according to Definition 2.2), by defining

$$\psi(y_1, y_2, \ldots, y_n) = \psi(y_1, y_2, \ldots, y_n, i_0, i_0, \ldots).$$

The interest of the notion of FFNS comes from the examples: a lot of expansions are finite (*e.g.*, see Example 2.7 below). Furthermore, dealing with infinite representations whenever the set X is at most countable is irrelevant. Lastly, coding an essentially finite information like $\varphi(x) = (\varepsilon_1(x), \varepsilon_2(x), \ldots, \ \varepsilon_{n_0-1}(x), i_0, i_0, \ldots)$ by an infinite sequence is abusive. This concept is thus a translation into the framework of (infinite) representations of the natural notion of finite expansion.

The difficulty lies in the fact that we deal in full generality only with representations and not with explicit expansions using a zero. Indeed, finite expansions (in the sense above) are usually those whose infinite form ends with zeros: if $n \in \mathbb{N}$ as in (2.1), we have

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_{\ell-1}(n)q^{\ell-1} + \varepsilon_\ell(n)q^\ell + 0 \cdot q^{\ell+1} + 0 \cdot q^{\ell+2} + \cdots .$$

Embedding $\mathbb{N}$ in $\mathbb{Z}_q$, the latter is even the Hensel expansion of n.

We thus do not use a zero to define FFNS. Actually, a zero is needed if one wants to characterise finite expansions in a non-finite FNS, since there is no possibility to differentiate the different fixed points of T in general. Consider for instance $X = \mathbb{Z}_2$ with the usual representation giving the Hensel expansion. The transformation is $T(x) = (x - x \pmod 2)/2$. It has two fixed points, 0 and -1. The set of digits is $\{0,1\}$. According to Definition 2.5, the FNS $(\mathbb{Z}_2, T, \{0,1\}, \varphi)$ is not finite, but it induces two FFNS, on the nonnegative integers, and on the negative integers, respectively. From the formal viewpoint of representations, there is no difference between both subsystems. Nevertheless, as elements of $\mathbb{Z}_2$, only nonnegative integers have finite expansions. This is done by privileging the fixed point 0. Note that if there exist several fixed points and if the representation of every element ends with the representation of some of them, all representations are finitely codable. However, it would be confusing to speak of finite representation in this latter case.

2.3. $\mathcal{N}$- compactification

Endowing I with a suitable topology, one may see the closure of $\varphi(X)$ in the product space $I^{\mathbb{N}^*}$ as a topological space equipped with the product topology. This yields the following definition.

Definition 2.6. — *For a fibred numeration system $\mathcal{N} = (X, T, I, \varphi)$, with a Hausdorff topological space I as digit set, the associated $\mathcal{N}$-compactification $X_\mathcal{N}$ is defined as the closure of $\varphi(X)$ in the product space $I^{\mathbb{N}^*}$.*

By the diagram (2.4) and its consequence (2.5), $X_{\mathcal{N}}$ is stable under the shift action. We will therefore consider in the sequel the subshift $(X_{\mathcal{N}}, S)$. If I is a discrete set, then

$$X_{\mathcal{N}} = \{(i_0, i_1, \ldots)\ ;\ \forall n \geqslant 0\,,\ C(i_0, \ldots, i_{n-1}) \neq \emptyset\}.$$

Lastly note that if (X, T, I, φ) is a finite FNS, and if $(x_n)_n \in X_{\mathcal{N}}$, then $(x_0, \ldots, x_m, i_0, i_0, \ldots) \in X_{\mathcal{N}}$.

2.4. Examples

The present section presents in a detailed way several numeration systems. It illustrates the definitions introduced above and fixes vocabulary and notation used in the rest of the paper. In particular, Example 2.7 and 2.14 generalise systematically (2.1), (2.2), and (2.3). Example 2.8 is central in Section 3 and Section 4.

Example 2.7. — **q-adic representations**

The q-adic numeration is the most usual numeration. There exist several q-adic numeration systems, all fibred, depending on whether one deals with nonnegative integers, integers, real numbers, Hensel's q-adic numbers, or whether one uses the classical set of digits, or else allows other representations.

(1) Let $X = \mathbb{N}$, $I = \{0, 1, \ldots, q-1\}$, $X_i = i + q\mathbb{N}$. According to Definition 2.3, $\varepsilon(n) \equiv n \pmod q$ and let $T\colon X \to X$ be defined by $T(n) = (n - \varepsilon(n))/q$. Then 0 is the only fixed point of T. We have an FFNS, with language, set of representations and compactification

$$\mathcal{L}_q = \bigcup_{n \geqslant 0} \{0, 1, \ldots, q-1\}^n,$$

$$\varphi(X) = I^{(\mathbb{N})} = \big\{(i_0, \ldots, i_{n-1}, 0, 0, 0, \ldots)\ ; n \in \mathbb{N}, i_j \in \{0, 1, \ldots, q-1\}\big\},$$

$$X_{\mathcal{N}} = \{0, 1, \ldots, q-1\}^{\mathbb{N}}.$$

The addition can be extended to $X_{\mathcal{N}}$, and gives the additive (profinite) group $\mathbb{Z}_q = \varprojlim \mathbb{Z}/q^n\mathbb{Z}$. The coordinates are independent and identically uniformly distributed on I w.r.t. Haar measure μ_q, which fulfills $\mu_q[i_0, \ldots, i_{k-1}] = q^{-k}$. (See Paragraph 6 of the present example below).

(2) Let $X = \mathbb{Z}$, and everything else as in the first example. This is again an FNS, and actually a quasi-FFNS, since

$$\varphi(X) = \{(i_0, \ldots, i_{n-1}, a, a, a, \ldots); n \in \mathbb{N},\ a = 0 \text{ or } a = q-1\}.$$

In other words, there are two T-invariant points, which are 0 and -1. The other sets are as in the first case: $\mathcal{L} = \mathcal{L}_q$ and $X_{\mathcal{N}} = \mathbb{Z}_q$.

(3) Take now $X = \mathbb{Z}$ and $T(n) = (n - \varepsilon(n))/(-q)$. Curiously, this is again an FFNS, with the same language, set of representatives and compactification as in the first example (see Theorem 3.1).

(4) It is possible to generalise the second example by modifying the set of digits and taking any complete set of representants modulo q, with $q \in \mathbb{Z}$, $|q| \geqslant 2$. Then one always gets an FFNS or a quasi-FFNS. This is due to the observation that for

$$L = \max\{|i|\,;\, i \in I\}/(|q| - 1),$$

the interval $[-L, L]$ is stable by T and $|T(n)| < |n|$ whenever $|n| > L$. The compactification is $I^{\mathbb{N}}$, the language and the set of representations hardly depend on the set of digits (see [199] for a detailed study with many examples, and in particular, Lemma 1 therein, for the fact that it is a quasi-FFNS or an FFNS).

(5) $X = [0, 1)$, $I = \{0, 1, \ldots, q - 1\}$, $X_i = [i/q, (i + 1)/q)$, and $T(x) = qx - \lfloor qx \rfloor$. This defines an FNS, which becomes a quasi-FFNS if the space is restricted to $[0, 1) \cap \mathbb{Q}$. The Lebesgue measure is T-invariant.

The language is $\mathcal{L}_q$ and the compactification $\mathbb{Z}_q$ in both cases. The set of representations is the whole product space without the sequences ultimately equal to $q-1$ in the first case (FNS), the subset of ultimately periodic sequences in the second case (quasi-FFNS). The attractor is the set $\mathbf{A} = \{a/b \;;\; a < b \text{ and } \gcd(b, q) = 1\}$. If $x = a/b$, with a and b coprime integers, write $b = b_1 b_2$, with b_1 being the highest divisor of b whose prime factors divide q. Then the length of the preperiod is $\min\{m\,;\, b_1 | q^m\}$ and the length of the period is the order of q in $(\mathbb{Z}/b_2\mathbb{Z})^*$. The continuous extension ψ of φ^{-1} is defined on $\mathbb{Z}_q$ by $\psi(y) = \sum_{n \geqslant 1} y_n q^{-n}$. Elements of X having improper representations are the so-called "q-rationals", *i.e.*, the numbers of the form a/q^m with $a \in \mathbb{N}$, $m \geqslant 0$ and $a/q^m < 1$. If the proper expansion is $(i_1, i_2, \ldots, i_s, 0^\omega)$, then the improper one is

$$(i_1, i_2, \ldots, i_{s-1}, i_s - 1, (q - 1)^\omega).$$

(6) Let $X = \mathbb{Z}_q$, with $X_i = i + q\mathbb{Z}_q$, $I = \{0, 1, \ldots, q - 1\}$, and $T(x) = (x - \varepsilon(x))/q$. It is an FNS equal to its own $\mathcal{N}$-compactification. There are q fixed points: $\mathbf{F} = \{0, -1, 1/(1 - q), 2/(1 - q), \ldots, (q - 2)/(1 - q)\}$. The attractor is $\mathbf{A} = \mathbf{F} + \mathbb{Z}$.

Example 2.8. — **β-representations**

(1) It is possible in Example 2.7 to replace q by any real number $\beta > 1$. Namely, $X = [0,1]$, $I = \{0, 1, \dots, \lceil \beta \rceil - 1\}$, and $T(x) = T_\beta(x) = \beta x - \lfloor \beta x \rfloor$, $\varepsilon(x) = \lfloor \beta x \rfloor$. This way of producing β-representations (which are actually expansions $\sum_{n \geqslant 1} \varepsilon_n(x)\beta^{-n}$ according to Definition 2.2) is traditionally called "greedy", since the digit chosen at step n is always the greatest possible, that is,

$$\max \left\{ \epsilon \in I; \sum_{j=1}^{n-1} \varepsilon_j(x)\beta^{-j} + \epsilon \beta^{-n} < x \right\}.$$

This is according to Rényi [281]. See Example 2.11 for a discussion on this seminal paper.

According to Parry [264], the set of admissible sequences $\varphi(X)$ is simply characterised in terms of one particular β-expansion. For $x \in [0,1]$, set $d_\beta(x) = \varphi(x)$.(5) In particular, let $d_\beta(1) = (t_n)_{n \geqslant 1}$. We then set $d^*_\beta(1) = d_\beta(1)$, if $d_\beta(1)$ is infinite, and

$$d^*_\beta(1) = (t_1 \dots t_{m-1}(t_m - 1))^\omega,$$

if $d_\beta(1) = t_1 \dots t_{m-1} t_m 0^\omega$ is finite ($t_m \neq 0$). The set $\varphi(X)$ of β-representations of real numbers in $[0,1)$ is exactly the set of sequences $(x_n)_{n \geqslant 1}$ with values in I, such that

$$\forall k \geqslant 1,\ (x_n)_{n \geqslant k} <_{\text{lex}} d^*_\beta(1). \tag{2.8}$$

The set $X_{\mathcal{N}} = \overline{\varphi([0,1))}$ is called the (right) *one-sided β-shift*. It is equal to the set of sequences $(x_n)_{n \geqslant 1}$ which satisfy

$$\forall k \geqslant 1,\ (x_n)_{n \geqslant k} \leqslant_{\text{lex}} d^*_\beta(1), \tag{2.9}$$

where $\leqslant_{\text{lex}}$ denotes the lexicographical order.

Definition 2.9. — *Numbers β such that $d_\beta(1)$ is ultimately periodic are called* Parry numbers *and those such that $d_\beta(1)$ is finite are called* simple Parry numbers.

Parry numbers and simple Parry numbers are clearly algebraic integers: Parry showed that they are *Perron numbers* [264]. For example, the golden mean $\varrho = \frac{1+\sqrt{5}}{2}$ is a simple Parry number with $d_\varrho(1) = 110^\omega$. According to Definition 2.2, simple Parry numbers are those that produce improper expansions. With any sequence $(x_n)_{n \geqslant 1} \in X_{\mathcal{N}}$, we can associate the expansion $\psi(x) =$

(5) This notation is redundant with φ, but it is standard in β-numeration, thus we will use it.

$\sum_{n\geqslant 1} x_n\beta^{-n}$. Then $\psi(x) \in [0,1]$ and numbers with two expansions are exactly those with finite expansion:

$$\psi(x_1 \dots x_{s-1}x_s 0^\omega) = \psi((x_1 \dots x_{s-1}(x_s - 1)d^*_\beta(1)).$$

If β is assumed to be a Pisot number, then every element of $\mathbb{Q}(\beta)\cap[0,1]$ admits a ultimately periodic expansion [295, 68], hence β is either a Parry number or a simple Parry number [68]. One deduces from the characterisation (2.9) that the shift $X_{\mathcal{N}}$ is of finite type if and only if β is a simple Parry number, and it is sofic if and only if $X_{\mathcal{N}}$ is a Parry number [192, 68].

Rényi [281] proved that $([0,1), T_\beta)$ has a unique absolutely continuous invariant probability measure $h_\beta(x)d\lambda$, and computed it explicitly when β was the golden mean. Parry [264] extended this computation to the general case and proved that the Radon-Nikodym derivative of the measure is a step function, with a finite number of steps if and only if β is a Parry number.

(2) Note that $\sum_{n\geqslant 1}(\lceil\beta\rceil - 1)\beta^{-n} = (\lceil\beta\rceil - 1)/(\beta - 1) > 1$, if β is not an integer. This leaves some freedom in the choice of the digit. The "lazy" choice corresponds to the smallest possible digit, that is,

$$\min \left\{\epsilon \in I; x - \left(\sum_{j=0}^{n-1} \varepsilon_j(x)\beta^{-j-1} + \epsilon\beta^{-n-1}\right) < (\lceil\beta\rceil - 1)/\beta^n(\beta - 1)\right\}.$$

This corresponds to $\varepsilon(x) = \left\lceil \beta x - \dfrac{\lceil\beta\rceil - 1}{\beta - 1}\right\rceil$ and $T(x) = \beta x - \varepsilon(x)$. These transformations are conjugated: write φ_g and φ_ℓ for greedy and lazy representations, respectively. Then

$$\varphi_\ell\left(\frac{\lceil\beta\rceil - 1}{\beta - 1} - x\right) = (\lceil\beta\rceil - 1, \lceil\beta\rceil - 1, \dots) - \varphi_g(x).$$

(3) It is also possible to make a choice at any step: lazy or greedy. If this choice is made in alternance, we still have an FNS (with transformation T^2 and pairs of digits). More complicated choices (*e.g.*, random) are also of interest (but are not FNS). See [109], [304], and the references therein.

(4) For β, the dominating root of some polynomial of the type

$$X^d - a_0X^{d-1} - a_1X^{d-2} - \dots - a_{d-1}$$

with integral coefficients $a_0 \geqslant a_1 \geqslant \dots \geqslant a_{d-1} \geqslant 1$, the restriction of the first tranformation $(T(x) = \beta x - \lfloor\beta x\rfloor)$ on $\mathbb{Z}[\beta^{-1}]_+ =$

$\mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+$ yields an FFNS. Such numbers β are said to satisfy *Property* (F) (introduced in [150]). They will take a substantial room in this survey (see Section 3.3 and 4.4). An extensively studied question is to find the characterisation of these β (see Section 4). More generally, for detailed surveys on the β-numeration, see for instance [69, 74, 242, 148, 304].

Example 2.10. — **Continued fractions**

Continued fractions have been an important source of inspiration in founding fibred systems [299]. Classical continued fractions, called regular, use $X = [0, 1]$, the so-called Gauß transformation $T(x) = 1/x - \lfloor 1/x \rfloor$, $T(0) = 0$, partition $X_i = (\frac{1}{i+1}, \frac{1}{i}]$, and $\varepsilon(x) = \lfloor 1/x \rfloor$ ($\varepsilon(0) = \infty$). The set of digits is $\mathbb{N}^* \cup \{\infty\}$. The representation map is one-to-one. In fact, the linear maps $h_a : t \mapsto \frac{1}{t+a}$ ($a \in \mathbb{N}^*$) defined on $[0, \infty]$ generate a free monoid to which it is convenient to add the constant map $h_\infty : t \mapsto 0$. The iteration

$$x = h_{\varepsilon(x)}(Tx) = h_{\varepsilon(x)} \circ \cdots \circ h_{\varepsilon(T^n x)}(T^{n+1}(x)) \tag{2.10}$$

ends with h_∞ for any rational number $r \in [0, 1]$, so that $r = h_{\varepsilon(x)} \circ \cdots \circ h_{\varepsilon(T^n x)} \circ h_\infty(r)$ (with $T^{n+1}(x) = 0$). Irrational numbers x have an infinite expansion (according to Definition 2.2) since $T^n(x)$ is never equal to 0.

The restriction to rational numbers yields an FFNS and the restriction to rational and quadratic numbers is a quasi-FFNS (by Lagrange's theorem). In the generic case, passing to the limit in (2.10), we get for any real number in $[0, 1]$ a unique expansion from the representation $\varphi(x)$ (terminated by $(h_\infty \circ h_\infty \circ \ldots$ if x is rational), namely

$$x = \lim_{n \to \infty} h_{\varepsilon(x)} \circ \cdots \circ h_{\varepsilon(T^n x)}(0)$$

and usually denoted by $[0; \varepsilon_1(x), \varepsilon_2(x), \ldots]$. Note that any rational number $r = h_{\varepsilon(x)} \circ \cdots \circ h_{\varepsilon(T^n x)}(0)$ with $\varepsilon(T^n x) \geqslant 2$ has also the expansion $r = h_{\varepsilon(x)} \circ \cdots \circ h_{\varepsilon(T^n x)-1} \circ h_1(0)$ which does not come from a representation (*cf.* Question 2.22 *infra*).

The expansion of special numbers (nothing is known about the continued fraction expansion of $\sqrt[3]{2}$), as well as the distribution properties of the digits (partial quotients) have been extensively studied since Gauß and are still the focus of many publications. For an example of a spectacular and very recent result, see [4]. The regularity of T allows us to use Perron-Frobenius operators, which yields interesting asymptotic results like the Gauß-Kuzmin-Wirsing's result, that we cite as an example [343]:

$$\lambda\{x \,;\, T^n(x) < t\} = \frac{\log(1+t)}{\log 2} + \mathcal{O}(q^n), \text{ with } q = 0.303663...$$

(here λ is the Lebesgue measure). The limit is due to Gauß, the first error term and the first published proof are due to Kuzmin. The bottom line is due to Wirsing [343], who gave the best possible value for q. There is a huge number of variants (with even, odd, or negative digits for example). See Kraaikamp's thesis [224] for a unified approach by using the so-called singularisation process based on matrix identities like

$$\begin{pmatrix}1 & e\\ 1 & a\end{pmatrix}\begin{pmatrix}0 & f\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & b\end{pmatrix}=\begin{pmatrix}0 & e\\ 1 & a+f\end{pmatrix}\begin{pmatrix}1 & -f\\ 1 & b+1\end{pmatrix}$$

with arbitrary a, b, e and f. For further references involving metrical theory, see [181], and for generalisations to higher dimension we refer to [299] and [300]. Due to the huge amount of literature, including books, it is not worthwile to say much more about the theory of continued fractions.

Example 2.11. — **f-expansions**

It is often referred to the paper of Rényi [281] as the first occurrence of β-expansions. It is rarely mentioned that β-expansions only occupy the fourth section of this famous paper and are seen as an example of the today less popular f-expansions.(6)

The idea is to represent the real numbers $x \in [0,1]$ as

$$\begin{aligned}x &= f(a_1 + f(a_2 + f(a_3 + \cdots + f(a_n + \cdots))\cdots), \text{ with } a_i \in \mathbb{N}\\ &= \lim_{n\to\infty} f(a_1 + f(a_2 + f(a_3 + \cdots + f(a_n)\cdots))).\end{aligned} \tag{2.11}$$

It originates in the observation that both continued fractions and q-adic expansions are special cases of the same type, namely an f-expansion, with $f(x) = 1/x$ for the continued fractions and $f(x) = x/q$ for the q-adic expansions. Furthermore, the coefficients are given in both cases by $a_1 = \lfloor f^{-1}(x)\rfloor$ and it is clear that existence of an algorithm and convergence in (2.11) occur under suitable assumptions of general type on f (injectivity and regularity).

(6) The term β-expansion does not even occur in the Thron's AMS review of [281] who just evokes "more general f-expansions" [than the q-adic one]. In Zentralblatt, the one full page long review of Hartman shortly says (in the citation below, g is the upper bound of the interval on which the function f is defined, see *infra*): "*Der schwierige Fall: $g < \infty$, g nicht ganz, wird nicht allgemein untersucht, jedoch kann Verf. für den Sonderfall $f(x) = x/\beta$ (bei $0 \leqslant x \leqslant \beta$) oder 1 (bei $\beta < x$), β nicht ganz, d.h. für die systematischen Entwicklungen nach einer gebrochenen Basis, den Hauptsatz noch beweisen.*" (The difficult case: $g < \infty$, g not integral, is not investigated in general. However, the author is able to prove the principal theorem for the special case $f(x) = x/\beta$ (for $0 \leqslant x \leqslant \beta$) or 1 (for $\beta < x$), β not an integer, that is, for systematic expansions w.r.t. a fractional base.) [This "principal theorem" is concerned with the absolutely continuous invariant measure (see Example 2.8) - the case of g finite and not an integer is not treated in general.]

More precisely, let $f\colon J \to [0,1]$ be a homeomorphism, where $J \subset \overline{\mathbb{R}}_+$. Let $\varepsilon(x) = \lfloor f^{-1}(x) \rfloor$ for $0 \leqslant x \leqslant 1$ and $T\colon [0,1] \to [0,1]$ be defined by $T(x) = f^{-1}(x) - \varepsilon(x)$. For $1 \leqslant k \leqslant n$, let us introduce

$$\begin{aligned} u_{k,n}(x) &= f(\varepsilon_k(x) + f(\varepsilon_{k+1}(x) + \cdots + f(\varepsilon_n(x) + T^n(x))\cdots) \\ v_{k,n}(x) &= f(\varepsilon_k(x) + f(\varepsilon_{k+1}(x) + \cdots + f(\varepsilon_n(x))\cdots). \end{aligned}$$

Then, one has $u_{1,n}(x) = x$, $u_{k,n}(x) = u_{1,n-k+1}(T^{k-1}(x))$, and similarly $v_{k,n}(x) = v_{1,n-k+1}(T^{k-1}(x))$. We are interested in the convergence of $(v_{1,n}(x))_n$ to x. Indeed,

$$x - v_{1,n}(x) = T^n(x) \prod_{k=1}^{n} \frac{f(v_{k,n}) - f(u_{k,n})}{v_{k,n} - u_{k,n}}. \tag{2.12}$$

Provided that for all x, $(v_{1,n}(x))_n$ tends to x, then one gets a fibred numeration system and expansions according to Definition 2.2. They are called *f-expansions*. This question seems to have been raised for the first time by Kakeya [194] in 1924. Independently, Bissinger treated the case of a decreasing function f [73] and Everett the case of an increasing function f two years later [139] before the already cited synthesis of Rényi [281]. Since one needs the function f to be injective and continuous, there are two cases, whenever f is increasing or decreasing.

The usual assumptions are either $f\colon [1,g] \to [0,1]$, decreasing, with $2 < g \leqslant +\infty$, $f(1) = 1$ and $f(g) = 0$, or $f\colon [0,g] \to [0,1]$, increasing, with $1 < g \leqslant +\infty$, $f(0) = 0$, and $f(g) = 1$. In both cases, the set of digits is $I = \{1, \ldots, \lceil g \rceil - 1\}$. In case $g = +\infty$, the set of digits is infinite and there is a formal problem at the extremities of the interval. Let us consider the decreasing case. Then T is not well defined at 0. It is possible to consider the transformation T on $[0,1] \setminus \cup_{j \geqslant 0} T^{-j}\{0\}$. It is also valid to set $T(0) = 0$ and $\varepsilon(0) = \infty$, say. Then, we say that the f-representation of x is finite if the digit ∞ occurs. In terms of expansions, for $(\varepsilon_n(x))_n = (i_1, \ldots, i_n, \infty, \infty, \ldots)$, we have a finite expansion $x = f(i_1 + f(i_2 + \cdots + f(i_n))\cdots)$. For the special case of continued fractions, the first choice considers the Gauß transformation on $[0,1] \setminus \mathbb{Q}$ and the second one obtains the so-called regular continued fraction expansion of rational numbers. The case f increasing is similar.

The convergence to 0 of the righthand side when n tends to infinity in Equation (2.12) is clearly ensured under the hypothesis that f is contracting (s-lipschitz with $s < 1$). There are several results in this direction, which are variants of this hypothesis and depend on the different cases (f decreasing or increasing, g finite or not). For instance, Kakeya proved the convergence under the hypothesis g integral or infinite and $|f'(x)| < 1$ almost everywhere [194]. We refer to the references cited above and to the paper of Parry [265] for more details.

The rest of Rényi's paper is devoted to the ergodic study of the dynamical system $([0,1],T)$. By considering the case of independent digits ($g \in \mathbb{N}$ or $g = \infty$), and by assuming that there exists a constant C such that for all x, one has $\sup_t |H_n(x,t)| \leqslant C \inf_t |H_n(x,t)|$, where

$$H(x,t) = \frac{d}{dt} f(\varepsilon_1(x) + f(\varepsilon_2 + \cdots + f(\varepsilon_n(x) + t)) \cdots),$$

he proves that there exists a unique T-invariant absolutely continuous measure $\mu = h d\lambda$ such that $C^{-1} \leqslant h(x) \leqslant C$. Note that the terminology "independent" is troublesome, since as random variables defined on $([0,1],\mu)$, the digits ε_n are not necessarily independent. They are in the q-adic case, but they are not in the continued fractions case, nor for the β-expansions. Furthermore, there are sometimes infinite invariant measures. In [326], Thaler gives general conditions on f for that and some examples, as $f\colon [0,\infty] \to [0,1]$, $f(x) = x/(1+x)$. See also [1] for more detailed information on these measures, especially wandering rates. For further developments on f-expansions, we refer to [299] and [108].

Example 2.12. — **Rational bases**

A surprising question is to ask for a q-adic representation of integers with a rational number $q = r/s > 1$ (r and s being coprime positive integers, $s \geqslant 2$). To do that, we can follow Kátai's approach [199], looking for a map $T : \mathbb{Z} \to \mathbb{Z}$ such that any integer n can be written in the form $n = \frac{r}{s}T(n) + R(n)$. A divisibility reasoning requests $R(n) = \frac{\varepsilon(n)}{s}$, where $\varepsilon(n)$ may play the rôle of the least significant digit. This leads to simultaneous definitions of the maps $\varepsilon : \mathbb{Z} \to \{0,1,\ldots,r\}$ and T from the relation

$$sn = rT(n) + \varepsilon(n), \tag{2.13}$$

where $T(n)$ and $\varepsilon(n)$ stand respectively for the quotient and the remainder in the Euclidean divsion of sn by r. The partial $\frac{r}{s}$-expansion of n is then given by the formula

$$n = \frac{\varepsilon(n)}{s} + \frac{1}{s}\Big(\frac{r}{s}\Big)\varepsilon(Tn) + \cdots + \frac{1}{s}\Big(\frac{r}{s}\Big)^{k-1}\varepsilon(T^{k-1}n) + \Big(\frac{r}{s}\Big)^{k} T^k(n). \tag{2.14}$$

It is easy to check from the definition that $T(0) = 0$, $T(n) < n$ if $n \geqslant 1$, and $-n < T(-n) < 0$, if $n \geqslant r$. Consequently, for any positive integer n, there exists a unique integer $\nu = \nu(n) \geqslant 1$ such that $T^{\nu-1}(n) \neq 0$ and $T^{\nu}(n) = 0$. Choosing $I = \{0,1,\ldots,r-1\}$ and the map $A : \mathbb{Z} \to I^{\mathbb{N}^*}$ defined as $A(n) = (\varepsilon_j(n))_{j \geqslant 1}$, we get a numeration system $(\mathbb{Z},T,A)$. The restriction of T to $\mathbb{N}$ (still denoted by T), gives rise to a finite numeration system with (r/s)-adic expansion $\sum_{j=1}^{\nu(n)} \varepsilon_j(n)s^{-1}(r/s)^{j-1}$. Moreover $(\mathcal{N}, T, I, A)$ is also a finite fibred system.

This representation has been recently studied in [15] where it is announced in particular that the language $\mathcal{L}_{r/s}$ of this representation is neither regular, nor context-free. The authors also show that the (r/s)-expansion is closely connected to Mahler's problem on the distribution of the sequences $n \mapsto t(r/s)^n$ $(t \in \mathbb{R})$.

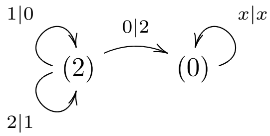

Figure 2.1. The transducer for the addition by 1 for the $\frac{3}{2}$-expansion. The state (a) (for $a = 0, 2$) corresponds to the carry digit a. The label $x|y$ means that x is the current input digit and y the resulting output digit.

The addition by 1 is computed by a transducer which is depicted in Figure 2.1 for $r = 3$, $s = 2$. In this case, adding 1 to n means adding the digit 2 to the string $\varepsilon_1(n)\varepsilon_2(n)\dots$ which is read from left to right by the transducer to produce the output $\varepsilon_1(n+1)\varepsilon_2(n+1)\dots$.

In fact, T is naturally extended to the group $\mathbb{Z}_r$ of r-adic integers by (2.13) where n, now, belongs to $\mathbb{Z}_r$, and $\varepsilon(n)$ is the unique integer in I such that the r-adic valuation of $sn - \varepsilon(n)$ is at least 1. In symbolic notations, $\mathbb{Z}_r$ is identified to $I^{\mathbb{N}}$ and T acts on $I^{\mathbb{N}}$ as the one-sided shift. Note that the map $n \mapsto sn$ is an automorphism of $\mathbb{Z}_r$. By taking the limit in (2.14), the infinite string $\varepsilon_1(n)\varepsilon_2(n)\dots$ $(n \in \mathbb{Z}_r)$ corresponds to the Hensel expansion of n, using the base $s^{-1}(r/s)^j$, $j = 0, 1, 2, \dots$. Hence, $\mathbb{Z}_r$ turns out to be the compactification of $\mathcal{L}_{r/s}$. For the restriction to $X = \mathbb{Z}$, we get a quasi-finite fibred system where the representation of any negative integer is ultimately periodic.

Example 2.13. — **Signed numeration systems**

Such representations have been introduced to facilitate arithmetical operations. To our knowledge, the first appearance of negative digits is due to Cauchy, whose title "*Sur les moyens d'éviter les erreurs dans les calculs numériques*" is significant. Cauchy proposes explicit examples of additions and multiplications of natural numbers using digits i with $-5 \leqslant i \leqslant 5$. He also verbally explains how one performs the conversion between both representations, using what was not yet called a transducer at that time: "*Les nombres étant exprimés, comme on vient de le dire, par des chiffres dont la valeur numérique ne surpasse pas 5, les additions, soustractions, multiplications, divisions, les conversions de fractions ordinaires en fractions décimales et les autres opérations de l'arithmétique, se trouveront notablement simplifiées. Ainsi, en particulier, la table de multiplication pourra être réduite au quart de son étendue, et l'on n'aura plus à effectuer de multiplications partielles que par les seuls chiffres 2, 3,* $4 = 2\times 2$, *et* $5 = 10/2$. *Ainsi, pour être en état de multiplier l'un par l'autre deux nombres quelconques, il suffira de savoir doubler ou tripler un nombre, ou en prendre la moitié.* $[\cdots]$ *Observons en outre que, dans les additions, multiplications, élévations aux puissances, etc, les reports faits d'une colonne à l'autre seront généralement très faibles, et souvent nuls, attendu que les chiffres positifs et négatifs se détruiront mutuellement en grande partie, dans une colonne verticale composée de plusieurs chiffres.*[(7)]" There is a dual interest: considerably reduce the size of the multiplication tables; dramatically decrease the carry propagation.

Nowadays, signed representations have two advantages. The first one is still algorithmic - as for Cauchy, the title of the book in which Knuth mentions them is significant (see [212]). The second interest lies in the associated dynamical systems.

The representation considered by Cauchy is redundant - *e.g.*, $5 = 1\bar{5}$, where $\bar{n} = -n$. In the sequel, we restrict ourselves to base 2 with digits $\{\bar{1}, 0, 1\}$. Reitwiesner proved in [280] that any integer $n \in \mathbb{Z}$ can be uniquely written as a finite sum $\sum_{0\leqslant i\leqslant \ell} a_i 2^i$ with $a_i \in \{-1, 0, 1\}$ and $a_i \cdot a_{i+1} = 0$. This yields the compactification

$$X_{\mathcal{N}} = \big\{x_0x_1x_2\ldots \in \{-1,0,1\}\ ;\ \forall i \in \mathbb{N} : x_ix_{i+1} = 0\big\}.$$

(7) As previously explained, additions, subtractions, multiplications, divisions, conversions of ordinary fractions into decimal fractions, and other arithmetical operations, can be significantly reduced by expressing numbers by digits whose numerical value does not exceed 5. In particular, the multiplication table might be reduced by a quarter, and it will only be necessary to perform partial multiplications using the digits 2, 3, $4 = 2\times 2$, and $5 = 10/2$. Hence, it is just essential to know how to double or triple one number, or to divide it in half in order to be able to multiply any one number by another. Note also that, in additions, multiplications, raisings of numbers to powers, *etc.*, carryovers made from one column to another are generally very weak, and often even equal to zero, since positive and negative digits will gradually and mutually destroy each other in a vertical column made of several digits.

This signed-digit expansion is usually called the nonadjacent form (NAF) or the canonical sparse form (see [166] for more details). Let us note that one of the interests of this numeration is that its redundancy allows sparse representations: this has applications particularly for the multiplication and the exponentiation in cryptography, such as illustrated in Section 6.3.

This numeration system is an FFNS. The elements of this FFNS are $X = \mathbb{Z}$, with partition $X_0 = 2\mathbb{Z}$, $X_{-1} = -1 + 4\mathbb{Z}$ and $X_1 = 1 + 4\mathbb{Z}$, and transformation $T(n) = (n - \varepsilon(n))/2$. Two natural transformations act on $X_{\mathcal{N}}$, the shift S and the addition by 1, denoted as τ, imported from $\mathbb{Z}$ by

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{+1} & \mathbb{Z} \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ X_{\mathcal{N}} & \xrightarrow[\tau]{} & X_{\mathcal{N}}. \end{array} \tag{2.15}$$

Then $(X_{\mathcal{N}}, S)$ is a topological mixing Markov chain whose Parry measure is the Markov probability measure with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}$$

and initial distribution $(1/6, 2/3, 1/6)$. Furthermore, $(X_{\mathcal{N}}, S)$ is conjugated to the dynamical system $([-2/3, 2/3], u)$ by

$$\Psi(x_0x_1x_2\ldots) = \sum_{k=0}^{\infty} x_k 2^{-k-1},$$

where $u(x) = 2x - a(x) \bmod 1$. A realisation of the natural extension is given by $(\overline{X}, \overline{S})$ with

$$\overline{X} = \big([-2/3, -1/3) \times [-1/3, 1/3]\big) \cup \\ \cup \big([-1/3, 1/3) \times [-2/3, 2/3]\big) \cup \big([1/3, 2/3) \times [-1/3, 1/3]\big)$$

and $\overline{S}(x, y) = \big(2x - a(x), (a(x) + y)/2\big)$, where $a(x) = -1$ if $-2/3 \leqslant x < -1/3$, $a(x) = 0$ if $-1/3 \leqslant x < 1/3$ and $a(x) = 0$ if $1/3 \leqslant x \leqslant 2/3$.

The odometer $(X_{\mathcal{N}}, \tau)$ (see Section 5) is topologically conjugated to the usual dyadic odometer $(\mathbb{Z}_2, x \mapsto x + 1)$. This FNS and related arithmetical functions are studied by Dajani, Kraaikamp and Liardet [110].

Example 2.14. — **Zeckendorf and Ostrowski representation**

Let $(F_n)_n$ be the (shifted) Fibonacci sequence $F_0 = 1$, $F_1 = 2$ and $F_{n+2} = F_{n+1} + F_n$. Then any nonnegative integer can be represented as a sum $n = \sum_j \varepsilon_j(n) F_j$. This representation is unique if one assumes that $\varepsilon_j(n) \in \{0, 1\}$ and $\varepsilon_j(n)\varepsilon_{j+1}(n) = 0$. It is called *Zeckendorf expansion.* If $\varrho = (1 + \sqrt{5})/2$

is the golden mean, the map f given by $f(n) = \sum_{j\geqslant 0} \varepsilon_j(n)\varrho^{-j-1}$ embeds $\mathbb{N}$ into $[0,1]$, the righthand side of the latter equation being the greedy β-expansion of its sum (for $\beta = \varrho$, see Example 2.8).

Let us note that the representation of the real number $f(n)$ is given by an FNS, but this does not yield an FNS producing the Zeckendorf expansion. Indeed, the Zeckendorf representation of n is required to be able to compute the real number $f(n)$. One obtains it by the greedy algorithm.

The compactification $X_{\mathcal{N}}$ is the set of $(0,1)$-sequences without consecutive 1's. The addition cannot be extended by continuity to $X_{\mathcal{N}}$ as $x + y = \lim(x_n + y_n)$ for integer sequences $(x_n)_n$ and $(y_n)_n$ tending to x and y, respectively (this sequence does not converge in $X_{\mathcal{N}}$), but the addition by 1 can: if $(x_n)_n$ is a sequence of nonnegative integers converging to x, then the sequence $(x_n+1)_n$ converges too. See Example 5.5 for details.

The *Ostrowski representation* of the nonnegative integers is a generalisation of the Zeckendorf expansion (for more details, see the references in [60]). Assume $0 < \alpha < 1/2$, $\alpha \notin \mathbb{Q}$. Let $\alpha = [0; a_1, a_2, \ldots, a_n, \ldots]$ be its continued fraction expansion with convergents $p_n/q_n = [0; a_1, a_2, \ldots, a_n]$. Then every nonnegative integer n has a representation $n = \sum_{j\geqslant 0} \varepsilon_j(n) q_n$, which becomes unique under the condition

$$\begin{cases} 0 \leqslant \varepsilon_0(m) \leqslant a_1 - 1; \\ \forall j \geqslant 1,\ 0 \leqslant \varepsilon_j(m) \leqslant a_{j+1}; \\ \forall j \geqslant 1,\ (\varepsilon_j(m) = a_{j+1} \Rightarrow \varepsilon_{j-1}(m) = 0\,). \end{cases} \tag{2.16}$$

The set $X_{\mathcal{N}}$ accurately describes the representations. Although this numeration system is not fibred, Definition 2.6 gives here

$$\begin{aligned} X_{\mathcal{N}} &= \{(x_n)_{n\geqslant 0} \in \mathbb{N}^{\mathbb{N}}\ ;\ \forall j \geqslant 0 : x_0 q_0 + \cdots + x_j q_j < q_{j+1}\} \\ &= \{(x_n)_{n\geqslant 0} \in \mathbb{N}^{\mathbb{N}}; x_0 \leqslant a_1 - 1 \text{ and} \\ &\qquad\qquad \forall j \geqslant 1 :\ x_j \leqslant a_{j+1} \text{ and } [x_j = a_{j+1} \Rightarrow x_{j-1} = 0]\}. \end{aligned}$$

On $X_{\mathcal{N}}$, the addition by 1 $\tau\colon x \mapsto x+1$ can be performed continuously by extending the addition by 1 for the integers. The map

$$f(n) = \sum_{j=0}^{\infty} \varepsilon_j(n)(q_j\alpha - p_j) \tag{2.17}$$

associates a real number $f(n) \in [-\alpha, 1-\alpha[$ with n.

In particular, if $\alpha = [0; 2, 1, 1, 1, \ldots] = \varrho^{-2} = (3-\sqrt{5})/2$, then the sequence of denominators $(q_n)_n$ of the convergents is exactly the Fibonacci sequence, and the map f coincides with the map given above in the discussion on the Zeckendorf expansion up to a multiplicative constant.

In general, the map f extends by continuity to $X_{\mathcal{N}}$ and realises an almost topological isomorphism in the sense of Denker and Keane [115] between the odometer $(X_{\mathcal{N}}, \tau)$ and $([1-\alpha, \alpha], R_\alpha)$, where R_α denotes the rotation with angle α. Explicitly, we have a commutative diagram

$$\begin{array}{ccc} X_{\mathcal{N}} & \xrightarrow{\tau} & X_{\mathcal{N}} \\ f\big\downarrow & & \big\downarrow f \\ [-\alpha, 1-\alpha] & \xrightarrow[R_\alpha]{} & [-\alpha, 1-\alpha], \end{array} \tag{2.18}$$

where f induces an homeomorphism between $X_{\mathcal{N}} \setminus \mathcal{O}_{\mathbb{Z}}(0^\omega)$ and $[-\alpha, 1-\alpha] \setminus \alpha\mathbb{Z} \pmod 1$, *i.e.*, the spaces without the (countable) two-sided orbit of 0 ($\mathcal{O}_{\mathbb{Z}}(0^\omega)$ denotes the bilateral orbit of 0^ω). In particular, the odometer $(X_{\mathcal{N}}, \tau)$ is strictly ergodic (uniquely ergodic and minimal).

This numeration system is not fibred. Nevertheless, the expansion given by the map f arises from a fibred numeration system too. This latter FNS thus produces Ostrowski expansions of real numbers, and it is defined by introducing a skew product of the continued fraction transformation, according to [183, 187, 321, 334].

Let $X = [0,1) \times [0,1)$, $T(x,y) = (\{1/x\}, \{y/x\})$, $T(0,y) = (0,0)$ (one recognises on the first component the Gauß transformation), $\varepsilon(x,y) = (\lfloor 1/x \rfloor, \lfloor y/x \rfloor)$, and $I = \mathbb{N}^* \times \mathbb{N}^*$. By applying the fibred system (X,T) to the pair (α, y), one recovers an expansion of the real number y in $[0,1)$ as

$$y = \sum_{j=0}^{\infty} \varepsilon_j(y) |q_j \alpha - p_j|,$$

with digits satisfying

$$\begin{cases} \forall j \geqslant 1,\ 0 \leqslant \varepsilon_j(m) \leqslant a_{j+1}; \\ \forall j \geqslant 1,\ (\varepsilon_j(m) = a_{j+1} \Rightarrow \varepsilon_{j+1}(m) = 0\,). \end{cases}$$

Note that this system of conditions is in some sense dual to the system of equations (2.16). It is also possible (see [187]) to recover an expansion of the form $y = \sum_{j=0}^{\infty} \varepsilon_j(y)(q_j\alpha - p_j)$, with digits satisfying constraints (2.16) as a fibred numeration system, but the expression of the map T is more complicated. For their metrical study, see [183, 187]. For more on the connections between Ostrowski's numeration, word combinatorics, and particularly Sturmian words, see the survey [60], the sixth chapter in [272], and the very complete description of the scenery flow given in [31]. In the same vein, see also [193] for similar numeration systems associated with episturmian words.

2.5. Questions

The list of examples above has proposed a medley of fibred numeration systems, with some of their properties. We gather and discuss some recurrent questions brought to light on that occasion that one can ask whenever a fibred system (X, T) and a representation map φ are introduced.

Question 2.15. — First of all, is φ injective? In other words, do we have an FNS? In some cases (nonnegative integers, real numbers or subsets of them), X and I are totally ordered sets and the injectivity of the representation map is a consequence of its monotonicity with respect to the order on X and to the lexicographical order on $I^{\mathbb{N}^*}$.

If we have an FNS, do we have an FFNS, a quasi-FFNS? Are there interesting characterisations of the attractor? The set of elements $x \in X$ whose $\mathcal{N}$-representation is stationary equal to i_0 is stable under the action of T. This also applies to the set of elements with ultimately periodic $\mathcal{N}$-representation. In case we have an FNS, but not an FFNS, this observation interprets the problem of finding elements that have finite or ultimately periodic representations as well as finding induced FFNS and induced quasi-FFNS. This question is discussed, *e.g.*, in Section 3 and particularly in Section 3.3. Note that number theoretists also asked for characterisations of purely periodic expansions (for q-adic expansions of real numbers, continued fractions...) We evoke it in Section 4.4, for instance.

Question 2.16. — The determination of the language is trivial when the representation map is surjective (Examples 2.7 and 2.10). Otherwise, the language can be described with some simple rules (Examples 2.8, 2.13, 2.14) or it cannot (Example 2.11). Hence the question: given an FNS, describe the underlying language. The structure of the language reflects that of the numeration system, and it even often happens that the combinatorics of the language has a translation in terms of arithmetic properties of the numeration (*e.g.*, see the survey [282]). Let us note that it is usual and meaningful to distinguish between different levels of complexity of the language (*e.g.*, *independence* of the digits if $\mathcal{L} = I^{\mathbb{N}^*}$, *Markovian structure*, *finite type*, or *sofic type*). We refer to Section 4 for relevant results and examples. In the case of shift radix systems (See Section 3.4), the language of the underlying number system is described *via* Theorem 3.12 for parameters corresponding to canonical number systems (see Section 3.1) and β-expansions. The stucture of this language for all the other parameters remains to be investigated.

Question 2.17. — The list of properties of the language above corresponds to properties of the subshift $(X_{\mathcal{N}}, S)$. The dynamical structure of this subshift is an interesting question as well. It is not independent of the previous one: suppose $X_{\mathcal{N}}$ is endowed with some S-invariant measure. Then the digits can be seen as random variables $E_n(\omega) = \omega_n$ (the n-th projection). Their distribution can be investigated and reflects the properties of the digits — *e.g.*, a Markovian structure of digits *versus* the sequence of coordinates as a Markov chain. Let us note that the natural extension of the transformation T (in the fibred case) is a useful tool to find explicitly invariant measures. It is standard for continued fractions; see for example [90] for more special continued fractions, and [111] for the β-transformation (see also Question 2.19 below). See also [78] (and the bibliography therein) for recent results on the camparison between the distribution of the number of digits determined when comparing two types of expansions in integer bases produced by fibred systems (*e.g.*, continued fractions and decimal expansions).

Question 2.18. — Let us consider the transfer of some operations on X. This question does not necessarily address numeration systems. More precisely, if X is a group or a semi-group $(X, *)$, is it possible to define an inner law on $X_{\mathcal{N}}$ by $x \hat{*} y = \lim(\varphi(x_n * y_n))$, where $\lim \varphi(x_n) = x$ and $\lim \varphi(y_n) = y$? Or if T' is a further transformation on X, does it yield a transformation $\mathcal{T}$ on $X_{\mathcal{N}}$ by

$$\mathcal{T}(x) = \lim_{x_n \to x} \varphi(T'(x_n))?$$

According to these transformations on $X_{\mathcal{N}}$, some probability measures may be defined on $X_{\mathcal{N}}$. Then coordinates might be seen as random variables whose distribution also reflects the dependence questions asked in Question 2.17.

Question 2.19. — The dynamical system (X, T) is itself of interest. The precise study of the commutative diagram issued from (2.4) by replacing

$I^{\mathbb{N}^*}$ by $X_{\mathcal{N}}$

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ X_{\mathcal{N}} & \xrightarrow[S]{} & X_{\mathcal{N}} \end{array} \tag{2.19}$$

can make (X, T) a factor or even a conjugated dynamical system of $(X_{\mathcal{N}}, S)$. As mentioned above, other transformations (like the addition by 1) or algebraic operations on X can also be considered and transferred to the $\mathcal{N}$-compactification, giving commutative diagrams similar to (2.19):

$$\begin{array}{ccc} X & \xrightarrow{T'} & X \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ X_{\mathcal{N}} & \xrightarrow[\mathcal{T}]{} & X_{\mathcal{N}} \end{array} \tag{2.20}$$

Results on X can be sometimes proved in this way (*cf.* Section 6).

The shift acting on the symbolic dynamical system $(X_{\mathcal{N}}, S)$ is usually not a one-to-one map. It is natural to try to look for a two-sided subshift that would project onto $(X_{\mathcal{N}}, S)$ (a natural extension, see also Question 2.17). Classical applications are, for instance, the determination of the invariant measure [262], as well as the characterisation of the attractor, and of elements of X having a purely periodic $\mathcal{N}$-representation, *e.g.*, in the β-numeration case, see [191, 287, 189, 65] (the attactor is described in this case in terms of central tile or Rauzy fractal discussed in Section 4.3, see also Question 2.23). More generally, the compactification $X_{\mathcal{N}}$ of X opens a broad range of dynamical questions in connection with the numeration.

Question 2.20. — An important issue in numeration systems is to reccognise rotations (discrete spectrum) among encountered dynamical systems. More precisely, let $(X, T, \mu, \mathcal{B})$ be a dynamical system. We first note that if T has a discrete spectrum, then T has a rigid time, *i.e.*, there exists an increasing sequence $(n_k)_{k\geqslant 0}$ of integers such that the sequence $k \mapsto T^{n_k}$ weakly converges to the identity. In other words, for any f and g in $L^2(X, \mu)$, one has

$$\lim_k (T^{n_k} f | g) = (f|g).$$

Such a rigid time can be selected in order to characterise T up to an isomorphism. In fact, it is proved in [72] that for any countable subgroup G of $\mathbb{U}$, there exists a sequence $(a_n)_n$ of integers such that for any complex number ξ, then the sequence $n \mapsto \xi^{a_n}$ converges to 1 if and only if $\xi \in G$. Such a sequence, called *characteristic* for G, is a rigid time for any dynamical system $(X, T, \mu, \mathcal{B})$ of discrete spectrum such that G is the group of eigenvalues. In case G is cyclically generated by $\zeta = e^{2i\pi\alpha}$, a characteristic sequence is built explicitly from the continued fraction expansion of α (see [72], Theorem 1*). Clearly if $(a_n)_n$ is a rigid time for T and if the group of complex numbers z such that $\lim_n z^{a_n} = 1$ is reduced to $\{1\}$, then T is weakly mixing. The following proposition is extracted from [316]:

PROPOSITION 2.21. — *Let $\mathcal{T} = (X, T, \mu, \mathcal{B})$ be a dynamical system. Assume first that there exists an increasing sequence $(a_n)_n$ of integers such that the group of complex numbers z verifying $\lim_n z^{a_n} = 1$ is countable and second, that there exists a dense subset D of $L^2(X, \mu)$, such that for all $f \in D$, the series*

$$\sum_{n \geqslant 0} ||f \circ T^{a_n} - f||_2^2$$

converges, then T has a discrete spectrum.

Question 2.22. — An FNS produces the following situation:

$$X \xrightarrow[\varphi]{\sim} \varphi(X) \overset{i}{\hookrightarrow} X_{\mathcal{N}}.$$

Assume furthermore that X is a Hausdorff topological space and that the map $\varphi^{-1}\colon \varphi(X) \to X$ admits a continuous extension $\overline{\psi}\colon X_{\mathcal{N}} \to X$. We note $\overline{\varphi} = i \circ \varphi$. We have $\overline{\psi} \circ \overline{\varphi} = \mathrm{id}_X$. Elements y of $X_{\mathcal{N}}$ distinct from $\varphi(x)$ such that $\overline{\psi}(y) = x$ (if any) are called *improper representations* of x. Natural questions are to characterise the $x \in X$ having improper $\mathcal{N}$-representations, to count the number of improper representations, to find them, and so on. In other words, study the equivalence relation $\mathcal{R}$ on $X_{\mathcal{N}}$ defined by $u\mathcal{R}v \Leftrightarrow \overline{\psi}(u) = \overline{\psi}(v)$. In many cases (essentially the various expansions of real numbers), X is connected, $X_{\mathcal{N}}$ is completely disconnected, φ is not continuous, but $\overline{\psi}$ (by definition) is continuous and X is homeomorphic to the quotient space $X_{\mathcal{N}}/\mathcal{R}$. The improper representations are naturally understood as expansions.

Question 2.23. — To many numeration systems (see for instance those considered in Section 3 and Section 4) we can attach a set, the so-called *central tile* (or *Rauzy fractal*), which is often a fractal set. The central tile is usually defined by renormalizing the iterations of the inverse T^{-1} of the underlying fibred system (see for instance Sections 3.6 and 4.3). We are interested in properties of these sets. For instance, their boundaries usually have fractional dimension and their topological properties are difficult do describe. In general, we are interested in knowing whether these sets inherit a natural iterated function system structure from the associated number system. One motivation for the introduction of such sets is to exhibit explicitly a rotation factor of the associated dynamical system (see also Question 2.20 and Section 4.4).

There are further questions, which only make sense in determined types of numeration systems and require further special and accurate definitions. They will be stated in the corresponding sections.

3. Canonical numeration systems, β-expansions and shift radix systems

The present section starts with a description of two well-known notions of numeration systems: canonical number systems in residue class rings of polynomial rings, and β-expansions of integers. At a first glance, these two notions of numeration system are quite different. However — and for this reason we treat them both in the same section — they can be regarded as special cases of so-called shift radix systems. Shift radix systems (introduced in Section 3.4) are families of quite simple dynamical systems.

All these notions of number systems admit the definition of fundamental domains. These sets often have fractal structure and admit a tiling of the space. Fundamental domains of canonical number systems are discussed at the end of the present section (Section 3.6), whereas tiles associated with β-expansions (so-called Rauzy fractals) are one of the main topics of Section 4.

3.1. Canonical numeration systems in number fields

This subsection is mainly devoted to numeration systems located in a residue class ring

$$X = A[x]/p(x)A[x]$$

where $p(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_1x + p_0 \in A[x]$ is a polynomial over the commutative ring A. By reduction modulo p, we see that each element $q \in X$ has a representative of the shape

$$q(x) = q_0 + q_1x + \cdots + q_{d-1}x^{d-1} \qquad (q_j \in A)$$

where d is the degree of the polynomial $p(x)$. In order to define a fibred numeration system on X, we consider the mapping

$$\begin{array}{cccc} T: & X & \to & X, \\ & q & \mapsto & \frac{q-\varepsilon(q)}{x} \end{array}$$

where the *digit* $\varepsilon(q) \in X$ is defined in a way that

$$T(q) \in X. \tag{3.1}$$

Note that this requirement generally leaves some freedom for the definition of ε. In the cases considered in this subsection, the image I of ε will always be finite. Moreover, the representation map $\varphi = (\varepsilon(T^n x))_{n\geqslant 0}$ defined in Section 2.2 will be surjective, *i.e.*, all elements of $I^{\mathbb{N}}$ are admissible.

If we iterate T for ℓ times starting with an element $q \in X$, we obtain the representative

$$q(x) = \varepsilon(q) + \varepsilon(Tq)x + \cdots + \varepsilon(T^{\ell-1}q)x^{\ell-1} + T^{\ell}(q)x^{\ell}. \tag{3.2}$$

According to Definition 2.4, the quadruple $\mathcal{N} = (X, T, I, \varphi)$ is an FNS. Moreover, following Definition 2.5, we call $\mathcal{N}$ an FFNS if for each $q \in X$, there exists an $\ell \in \mathbb{N}$ such that $T^k(q) = 0$ for each $k \geqslant \ell$.

Once we have fixed the ring A, the definition of $\mathcal{N}$ only depends on p and ε. Moreover, in what follows, the image I of ε will always be chosen to be a complete set of coset representatives of A/p_0A (recall that p_0 is the constant term of the polynomial p). With this choice, the requirement (3.1) determines the value of $\varepsilon(q)$ uniquely for each $q \in X$. In other words, in this case $\mathcal{N}$ is determined by the pair (p, I). Motivated by the shape of the representation (3.2) we will call p the *base* of the numeration system (p, I), and I its *set of digits*.

The pair (p, I) defined in this way still provides a fairly general notion of numeration system. By further specialization, we will obtain the notion of *canonical numeration systems* from it, as well as a notion of digit systems over finite fields that will be discussed in Section 3.5.

Historically, the term *canonical numeration system* is from the Hungarian school (see [204], [202], [218]). They used it for numeration systems defined in the ring of integers of an algebraic number field.[8] Meanwhile, Pethő [269] generalized this notion to numeration systems in certain polynomial rings, and this is this notion of numeration system to which we will attach the name *canonical numeration system* in the present survey.

Before we precisely define Kovács' as well as Pethő's notion of numeration system and link it to the general numeration systems in residue classes of polynomial rings, we discuss some earlier papers on the subject.

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[8] With the word "canonical" the authors wanted to emphasize the fact that the digits he attached to these numeration systems were chosen in a very simple "canonical" way.

In fact, instances of numeration systems in rings of integers were studied long before Kovács' paper. The first paper on these objects seems to be Grünwald's treatise [162] dating back to 1885 which is devoted to numeration systems with negative bases. In particular Grünwald showed the following result.

Theorem 3.1. — *Let $q \geqslant Z$. Each $n \in \mathbb{Z}$ admits a unique finite representation w.r.t. the base number $-q$, i.e.,*

$$n = c_0 + c_1(-q) + \cdots + c_\ell(-q)^\ell$$

where $0 \leqslant c_i < q$ for $i \in \{0, \ldots, \ell\}$ and $c_\ell \neq 0$ for $\ell \neq 0$.

We can say that Theorem 3.1 describes the bases of number systems in the ring of integers $\mathbb{Z}$ of the number field $\mathbb{Q}$. It is natural to ask whether this concept can be generalised to other number fields. Knuth [210] and Penney [266] observed that $b = -1 + \sqrt{-1}$ serves as a base for a numeration system with digits $\{0, 1\}$ in the ring of integers $\mathbb{Z}[\sqrt{-1}]$ of the field of Gaussian numbers $\mathbb{Q}(\sqrt{-1})$, *i.e.*, each $z \in \mathbb{Z}[\sqrt{-1}]$ admits a unique representation of the shape

$$z = c_0 + c_1 b + \cdots + c_\ell b^\ell$$

with digits $c_i \in \{0, 1\}$ and $c_\ell \neq 0$ for $\ell \neq 0$. Knuth [212] also observed that this numeration system is strongly related to the famous twin-dragon fractal which will be discussed in Section 3.6. It is not hard to see that Grünwald's as well as Knuth's examples are special cases of FFNS.

We consider the details of this correspondence for a more general definition of numeration systems in the ring of integers Z_K of a number field K. In particular, we claim that the pair $(b, \mathcal{D})$ with $b \in Z_K$ and $\mathcal{D} = \{0, 1, \ldots, |N(b)| - 1\}$ defines an FFNS in Z_K if each $z \in Z_K$ admits a unique representation of the shape

$$z = c_0 + c_1 b + \cdots + c_\ell b^\ell \qquad (c_i \in \mathbb{N}) \tag{3.3}$$

if $c_\ell \neq 0$ for $\ell \neq 0$ (note that this requirement just ensures that there are no leading zeros in the representations). To see this, set $X = Z_K$ and define $T : Z_K \to Z_K$ by

$$T(z) = \frac{z - \varepsilon(z)}{b}$$

where $\varepsilon(z)$ is the unique element of $\mathcal{D}$ with $T(z) \in Z_K$. Note that $\mathcal{D}$ is uniquely determined by b. The first systematic study of FFNS in rings of integers of number fields was done by Kátai and Szabó [204]. They proved that the only canonical bases in $\mathbb{Z}[i]$ are the numbers $b = -n + \sqrt{-1}$ with

$n \geqslant 1$. Later Kátai and Kóvacs [202, 203] (see also Gilbert [154]) characterised all (bases of) canonical numeration systems in quadratic number fields. A. Kovács, B. Kovács, Pethő and Scheicher [218, 219, 222, 288, 216] studied numeration systems in rings of integers of algebraic number fields of higher degree and proved some partial characterisation results (some further generalised concepts of numeration systems can be found in [221, 220]). In [223] an estimate for the length ℓ of the CNS representation (3.3) of z w.r.t. base b in terms of the modulus of the conjugates of z as well as b is given.

Pethő [269] observed that the notion of numeration systems in number fields can be easily extended using residue class rings of polynomials. In particular, he gave the following definition.

Definition 3.2. — *Let*

$$p(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_1 x + p_0 \in \mathbb{Z}[x], \quad \mathcal{D} = \{0, 1, \ldots, |p_0| - 1\}$$

and $X = \mathbb{Z}[x]/p(x)\mathbb{Z}[x]$ *and denote the image of* x *under the canonical epimorphism from* $\mathbb{Z}[x]$ *to* X *again by* x. *If every non-zero element* $q(x) \in X$ *can be written uniquely in the form*

$$q(x) = c_0 + c_1 x + \cdots + c_\ell x^\ell \tag{3.4}$$

with $c_0, \ldots, c_\ell \in \mathcal{D}$, *and* $c_\ell \neq 0$, *we call* $(p, \mathcal{D})$ *a canonical number system (CNS for short).*

Let p be irreducible and assume that b is a root of p. Let $K = \mathbb{Q}(b)$ and assume further that $Z_K = \mathbb{Z}[b]$, *i.e.*, Z_K is monogenic. Then $\mathbb{Z}[x]/p(x)\mathbb{Z}[x]$ is isomorphic to Z_K, and this definition is easily seen to agree with the above definition of numeration systems in rings of integers of number fields.

On the other hand, canonical numeration systems turn out to be a special case of the more general definition given at the beginning of this section. To observe this, we choose the commutative ring A occurring there to be $\mathbb{Z}$. The value of $\varepsilon(q)$ is defined to be the least nonnegative integer meeting the requirement that

$$T(q) = \frac{q - \varepsilon(q)}{x} \in X.$$

Note that this definition implies that $\varepsilon(X) = \mathcal{D}$, as required. Indeed, if

$$q(x) = q_0 + q_1 x + \cdots + q_{d-1}x^{d-1} \qquad (q_j \in \mathbb{Z})$$

is a representative of q, then T takes the form

$$T(q) = \sum_{i=0}^{d-1} (q_{i+1} - cp_{i+1}) x^i, \tag{3.5}$$

where $q_d = 0$ and $c = \lfloor q_0/p_0 \rfloor$. Then

$$q(x) = (q_0 - cp_0) + xT(q), \text{ where } q_0 - cp_0 \in \mathcal{D}. \tag{3.6}$$

Thus the iteration of T yields exactly the representation (3.4) given above. The iteration process of T can become divergent (*e.g.*, $q(x) = -1$ for $p(x) = x^2 + 4x + 2$), ultimately periodic (*e.g.*, $q(x) = -1$ for $p(x) = x^2 - 2x + 2$) or can terminate at 0 (*e.g.*, $q(x) = -1$ for $p(x) = x^2 + 2x + 2$). For the reader's convenience, we will give the details for the last constellation.

Example 3.3. — Let $p(x) = x^2 + 2x + 2$ be a polynomial. We want to calculate the representation of $q(x) = -1 \in \mathbb{Z}[x]/p(x)\mathbb{Z}(x)$. To this matter we need to iterate the mapping T defined in (3.5). Setting $d_j = \varepsilon(T^j(q))$ this yields

$$\begin{array}{rclrrll}
q & = & -1, & & & c=-1, & \\
T(q) & = & (0-(-1)\cdot 2)+(0-(-1)\cdot 1)x & = & 2+x, & c=1, & d_0=1, \\
T^2(q) & = & (1-1\cdot 2)+(0-1\cdot 1)x & = & -1-x, & c=-1, & d_1=0, \\
T^3(q) & = & (-1-(-1)\cdot 2)+(0-(-1)\cdot 1)x & = & 1+x, & c=0, & d_2=1, \\
T^4(q) & = & (1-0\cdot 2)+(0-0\cdot 1)x & = & 1, & c=0, & d_3=1, \\
T^5(q) & = & 0, & & & c=0, & d_4=1, \\
T^k(q) & = & 0 \quad \text{for } k \geqslant 6. & & & &
\end{array}$$

Thus

$$-1 = d_0 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 = 1 + x^2 + x^3 + x^4$$

is the unique finite representation (3.4) of -1 with respect to the base $p(x)$.

Note that $(p, \mathcal{D})$ is a canonical numeration system if and only if the attractor of T is $\mathbf{A} = \{0\}$. Indeed, if the attractor of T is $\{0\}$ then for each $q \in X$ there exists a $k_0 \in \mathbb{N}$ with $T^{k_0}(q) = 0$. This implies that $T^k(q) = 0$ for each $k \geqslant k_0$. Iterating T we see in view of (3.6) that the k-th digit c_k of q is given by

$$T^k(q) = c_k + xT^{k+1}(q).$$

If $k \geqslant k_0$ this implies that $c_k = 0$. Thus q has finite CNS expansion. Since $q \in X$ was arbitrary this is true for each $q \in X$. Thus $(p, \mathcal{D})$ is a CNS. The other direction is also easy to see.

The fundamental problem that we want to address concerns exhibiting all polynomials p that give rise to a CNS. There are many partial results on this problem. Generalizing the above-mentioned results for quadratic number fields, Brunotte [84] characterised all quadratic CNS polynomials. In particular, he obtained the following result.

THEOREM 3.4. — *The pair* $(p(x), \mathcal{D})$ *with* $p(x) = x^2 + p_1 x + p_0$ *and set of digits* $\mathcal{D} = \{0, 1, \ldots, |p_0| - 1\}$ *is a CNS if and only if*

$$p_0 \geqslant 2 \qquad \text{and} \qquad -1 \leqslant p_1 \leqslant p_0. \tag{3.7}$$

For CNS polynomials of general degree, Kovács [218] (see also the more general treatment in [12]) proved the following theorem.

THEOREM 3.5. — *The polynomial*

$$p(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_1 x + p_0$$

gives rise to a CNS if its coefficients satisfy the "monotonicity condition"

$$p_0 \geqslant 2 \quad \text{and} \quad p_0 \geqslant p_1 \geqslant \cdots \geqslant p_{d-1} > 0. \tag{3.8}$$

More recently, Akiyama and Pethő [18], Scheicher and Thuswaldner [293] as well as Akiyama and Rao [19] showed characterisation results under the condition

$$p_0 > |p_1| + \cdots + |p_{d-1}|.$$

Moreover, Brunotte [84, 85] has results on trinomials that give rise to CNS.

It is natural to ask whether there exists a complete description of all CNS polynomials. This characterisation problem has been studied extensively for the case $d = 3$ of cubic polynomials. Some special results on cubic CNS are presented in Körmendi [213]. Brunotte [86] characterised cubic CNS polynomials with three real roots. Akiyama *et al.* [13] studied the problem of describing all cubic CNS systematically. Their results indicate that the structure of cubic CNS polynomials is very irregular.

Recently, Akiyama *et al.* [11] invented a new notion of numeration system, namely, the so-called shift radix systems. All recent developments on the characterisation problem of CNS have been done in this new framework. Shift radix systems will be discussed in Section 3.4.

3.2. Generalisations

There are some quite immediate generalisations of canonical numeration systems. First, we mention that there is no definitive reason for studying only the set of digits $\mathcal{D} = \{0, 1, \ldots, |p_0| - 1\}$. More generally, each set $\mathcal{D}$

containing one of each coset of $\mathbb{Z}/p_0\mathbb{Z}$ can serve as set of digits. Numeration systems of this more general kind can be studied in rings of integers of number fields as well as in residue class rings of polynomials. For quadratic numeration systems, Farkas, Kátai and Steidl [142, 198, 319] showed that for all but finitely many quadratic integers, there exists a set of digits such that each element of the corresponding number field has a finite representation. In particular, Steidl [319] proves the following result for numeration systems in Gaussian integers.

Theorem 3.6. — *If $K = \mathbb{Q}(i)$ and b is an integer of Z_K satisfying $|b| > 1$ with $b \neq 2, 1 \pm i$, then one can effectively construct a residue system $\mathcal{D} \pmod b$ such that each $z \in Z_K$ admits a finite representation*

$$z = c_0 + c_1 b + \cdots + c_\ell b^\ell$$

with $c_0, \ldots, c_\ell \in \mathcal{D}$.

Another way of generalising canonical numeration systems involves an embedding into an integer lattice. Let $(p(x), \mathcal{D})$ be a canonical numeration system. As mentioned above, each $q \in X = \mathbb{Z}[x]/p(x)\mathbb{Z}[x]$ admits a unique representation of the shape

$$q_0 + q_1 x + \cdots + q_{d-1} x^{d-1}$$

with $q_0, \ldots, q_{d-1} \in \mathbb{Z}$ and $d = \deg(p)$. Thus the bijective group homomorphism

$$\begin{array}{rccc} \Phi: & X & \to & \mathbb{Z}^d \\ & q & \mapsto & (q_0, \ldots, q_{d-1}) \end{array}$$

is well defined. Besides being a homomorphism of the additive group in X, Φ satisfies

$$\Phi(xq) = B\Phi(q)$$

with

$$B = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & & \vdots & -p_1 \\ 0 & \ddots & \ddots & & \vdots & -p_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & -p_{d-1} \end{pmatrix}. \tag{3.9}$$

Exploiting the properties of Φ we easily see the following equivalence. Each $q \in X$ admits a CNS representation of the shape

$$q = c_0 + c_1 x + c_2 x^2 + \cdots + c_\ell x^\ell \qquad (c_0, \ldots, c_\ell \in \mathcal{D})$$

if and only if each $z \in \mathbb{Z}^d$ admits a representation of the form

$$z = d_0 + Bd_1 + B^2 d_2 + \cdots + B^\ell d_\ell \qquad (d_0, \ldots, d_\ell \in \Phi(\mathcal{D})).$$

Thus $(B, \Phi(\mathcal{D}))$ is a special case of the following notion of numeration system.

Definition 3.7. — *Let $B \in \mathbb{Z}^{d\times d}$ be an expanding matrix (i.e., all eigenvalues of B are greater than 1 in modulus). Let $\mathcal{D} \subset \mathbb{Z}^d$ be a complete set of cosets in $\mathbb{Z}^d/B\mathbb{Z}^d$ such that $0 \in \mathcal{D}$. Then the pair $(B, \mathcal{D})$ is called a matrix numeration system if each $z \in \mathbb{Z}^d$ admits a unique representation of the shape*

$$z = d_0 + Bd_1 + \cdots + B^\ell d_\ell$$

with $d_0, \ldots, d_\ell \in \mathcal{D}$ and $d_\ell \neq 0$ for $\ell \neq 0$.

It is easy to see (along the lines of [222], Lemma 3) that each eigenvalue of B has to be greater than or equal to one in modulus to obtain a number system. We impose the slightly more restrictive expanding condition to guarantee the existence of the attractor T in Definition 3.18.

Matrix numeration systems have been studied, for instance, by Kátai, Kovács and Thuswaldner in [200, 215, 217, 328]. Apart from some special classes it is quite hard to obtain characterisation results because the number of parameters to be taken into account (namely the entries of B and the elements of the set $\mathcal{D}$) is very large. However, matrix number systems will be our starting point for the definition of lattice tilings in Section 3.6.

3.3. On the finiteness property of β-expansions

At the beginning of the present section we mentioned that so-called shift radix systems form a generalization of CNS as well as β-expansions. Thus, before we introduce shift radix systems in full detail, we want to give a short account on β-expansions in the present subsection.

The β-expansions have already been defined in Example 2.8. They represent the elements of $[0, \infty)$ with respect to a real base number β and with a finite set of nonnegative integer digits. It is natural to ask when these representations are finite. Let $\mathrm{Fin}(\beta)$ be the set of all $x \in [0, \infty)$ having a finite β-expansion. Since finite sums of the shape

$$\sum_{j=m}^{n} c_j \beta^{-j} \qquad (c_j \in \mathbb{N})$$

are always contained in $\mathbb{Z}[\beta^{-1}] \cap [0,\infty)$, we always have

$$\mathrm{Fin}(\beta) \subseteq \mathbb{Z}[\beta^{-1}] \cap [0,\infty). \tag{3.10}$$

According to Frougny and Solomyak [150], we say that a number β satisfies *property (F)* if equality holds in (3.10). Using the terminology of the introduction property (F) is equivalent to the fact that (X, T) with

$$X = \mathbb{Z}[\beta^{-1}] \cap [0,\infty) \quad \text{and} \quad T(x) = \beta x - \lfloor \beta x \rfloor$$

is an FFNS (see Definition 2.5).

In [150, Lemma 1] it was shown that (F) can hold only if β is a Pisot number. However, there exist Pisot numbers that do not fulfill (F). This raises the problem of exhibiting all Pisot numbers having this property. Up to now, there has been no complete characterisation of all Pisot numbers satisfying (F). In what follows, we would like to present some partial results that have been achieved.

In [150, Proposition 1] it is proved that each quadratic Pisot number has property (F). Akiyama [9] could characterise (F) for all cubic Pisot units. In particular, he obtained the following result.

Theorem 3.8. — *Let $x^3 - a_1x^2 - a_2x - 1$ be the minimal polynomial of a cubic Pisot unit β. Then β satisfies (F) if and only if*

$$a_1 \geqslant 0 \qquad \textit{and} \qquad -1 \leqslant a_2 \leqslant a_1 + 1. \tag{3.11}$$

If β is an arbitrary Pisot number, the complete characterisation result is still unknown. Recent results using the notion of shift radix system suggest that even characterisation of the cubic case is very involved (*cf.* [11, 12]). We refer to Section 3.4 for details on this approach. Here we just want to give some partial characterisation results for Pisot numbers of arbitrary degree. The following result is contained in [150, Theorem 2].

Theorem 3.9. — *Let*

$$x^d - a_1x^{d-1} - \cdots - a_{d-1}x - a_d \tag{3.12}$$

be the minimal polynomial of a Pisot number β. If the coefficients of (3.12) *satisfy the "monotonicity condition"*

$$a_1 \geqslant \cdots \geqslant a_d \geqslant 1 \tag{3.13}$$

then β fulfills property (F).

Moreover, Hollander [169] proved the following result on property (F) under a condition on the representation $d_\beta(1)$ of 1.

THEOREM 3.10 ([169, Theorem 3.4.2]). — *A Pisot number β has property (F) if $d_\beta(1) = d_1 \dots d_\ell$ with $d_1 > d_2 + \cdots + d_\ell$.*

Let us also quote [20, 54] for results in the same vein.

In Section 3.4, the most important concepts introduced in this section, namely CNS and β-expansions, will be unified.

3.4. Shift radix systems

At a first glance, canonical numeration systems and β-expansions are quite different objects: canonical numeration systems are defined in polynomial rings. Furthermore, the digits in CNS expansions are independent. On the other hand, β-expansions are representations of real numbers whose digits are dependent. However, the characterisation results of the finiteness properties of CNS and β-expansions resemble each other. As an example, we mention (3.7) and (3.8) on the one hand, and (3.11) and (3.13) on the other.

The notion of shift radix system which is discussed in the present subsection will shed some light on this resemblance. Indeed, it turns out that canonical numeration systems in polynomial rings over $\mathbb{Z}$ as well as β-expansions are special instances of a class of very simple dynamical systems. The most recent studies of canonical numeration systems as well as β-expansions make use of this more general concept which allows us to obtain results on canonical numeration systems as well as β-expansions at once. We start with a definition of shift radix systems (*cf.* Akiyama *et al.* [11, 12]).

DEFINITION 3.11. — *Let $d \geqslant 1$ be an integer, $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ and define the mapping $\tau_\mathbf{r}$ by*

$$\begin{array}{rccl} \tau_\mathbf{r} : & \mathbb{Z}^d & \to & \mathbb{Z}^d \\ & \mathbf{a} = (a_1, \dots, a_d) & \mapsto & (a_2, \dots, a_d, -\lfloor \mathbf{ra} \rfloor), \end{array}$$

where $\mathbf{ra} = r_1 a_1 + \cdots + r_d a_d$, i.e., the inner product of the vectors $\mathbf{r}$ and $\mathbf{a}$. Let $\mathbf{r}$ be fixed. If

$$\text{for all } \mathbf{a} \in \mathbb{Z}^d, \text{ then there exists } k > 0 \text{ with } \tau_\mathbf{r}^k(\mathbf{a}) = 0 \tag{3.14}$$

we will call $\tau_\mathbf{r}$ a shift radix system (SRS for short). For simplicity, we write $0 = (0, \dots, 0)$.

Let

$$\mathcal{D}_d^0 = \left\{ \mathbf{r} \in \mathbb{R}^d \ ; \ \forall \mathbf{a} \in \mathbb{Z}^d \, \exists k > 0 : \tau_\mathbf{r}^k(\mathbf{a}) = 0 \right\}$$

be the set of all SRS parameters in dimension d and set

$$\mathcal{D}_d = \left\{\mathbf{r} \in \mathbb{R}^d \; ; \; \forall \mathbf{a} \in \mathbb{Z}^d \text{ the sequence } (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k \geqslant 0} \text{ is ultimately periodic}\right\}.$$

It is easy to see that $\mathcal{D}_d^0 \subseteq \mathcal{D}_d$.

In [11] (*cf.* also Hollander [169]), it was noted that SRS correspond to CNS and β-expansions in the following way.

Theorem 3.12. — *The following correspondences hold between CNS as well as β-expansions and SRS.*

- *Let $p(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_1 x + p_0 \in \mathbb{Z}[x]$. Then $p(x)$ gives rise to a CNS if and only if*

$$\mathbf{r} = \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}\right) \in \mathcal{D}_d^0. \tag{3.15}$$

- *Let $\beta > 1$ be an algebraic integer with minimal polynomial $X^d - a_1 X^{d-1} - \cdots - a_{d-1}X - a_d$. Define $r_1, \ldots, r_{d-1}$ by*

$$r_j = a_{j+1}\beta^{-1} + a_{j+2}\beta^{-2} + \cdots + a_d \beta^{j-d} \quad (1 \leqslant j \leqslant d-1). \tag{3.16}$$

  Then β has property (F) if and only if $(r_{d-1}, \ldots, r_1) \in \mathcal{D}_{d-1}^0$.

In particular $\tau_{\mathbf{r}}$ is conjugate to the mapping T defined in (3.5) *if $\mathbf{r}$ is chosen as in* (3.15) *and conjugate to the β-transformation $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$ for $\mathbf{r}$ as in* (3.16).

Remark 3.13. — The conjugacies mentioned in the theorem are described in [11, Section 2 and Section 3, respectively]. In both cases they are achieved by certain embeddings of the according numeration system in the real vector space, followed in a natural way by some base transformations.

This theorem highlights the problem of describing the set $\mathcal{D}_d^0$. Describing this set would solve the problem of characterizing all bases of CNS as well as the problem of describing all Pisot numbers β with property (F). We start with some considerations on the set $\mathcal{D}_d$. It is not hard to see (*cf.* [11, Section 4]) that

$$\mathcal{E}_d \subseteq \mathcal{D}_d \subseteq \overline{\mathcal{E}_d} \tag{3.17}$$

where

$$\begin{aligned}\mathcal{E}_d \quad = \quad & \big\{(r_1, \ldots, r_d) \in \mathbb{R}^d \; ; \; x^d + r_d x^{d-1} + \cdots + r_1 \\ & \text{has only roots } y \in \mathbb{C} \text{ with } |y| < 1\big\}\end{aligned}$$

denotes the Schur-Cohn region (see Schur [298]). The only problem in describing $\mathcal{D}_d$ involves characterising its boundary. This problem turns out to be very hard and contains, as a special case, the following conjecture of Schmidt [295, p. 274].

CONJECTURE 3.14. — *Let β be a Salem number and $x \in \mathbb{Q}(\beta) \cap [0,1)$. Then the orbit $(T_\beta^k(x))_{k\geqslant 0}$ of x under the β-transformation T_β is eventually periodic.*

This conjecture is supported by the fact that if each rational in $[0,1)$ has a ultimately periodic β-expansion, then β is either a Pisot or a Salem number. Up to now, Boyd [80, 81, 82] could only verify some special instances of Conjecture 3.14 (see also [14] where the problem of characterising $\partial\mathcal{D}_d$ is addressed).

As quoted in [74], note that there exist Parry numbers which are neither Pisot nor even Salem; consider, *e.g.*, $\beta^4 = 3\beta^3 + 2\beta^2 + 3$ with $d_\beta(1) = 3203$; a *Salem number* is a Perron number, all conjugates of which have absolute value less than or equal to 1, and at least one has modulus 1. It is proved in [80] that if β is a Salem number of degree 4, then β is a Parry number; see [81] for the case of Salem numbers of degree 6. Note that the algebraic conjugates of a Parry number $\beta > 1$ are smaller than $\frac{1+\sqrt{5}}{2}$ in modulus, with this upper bound being sharp [146, 317].

We would like to characterise $\mathcal{D}_d^0$ starting from $\mathcal{D}_d$. This could be achieved by removing all parameters $\mathbf{r}$ from $\mathcal{D}_d$ for which the mapping $\tau_\mathbf{r}$ admits nontrivial periods. We would like to do this "periodwise". Let

$$\mathbf{a}_j = (a_{1+j}, \dots, a_{d+j}) \qquad (0 \leqslant j \leqslant L-1) \tag{3.18}$$

with $a_{L+1} = a_1, \dots, a_{L+d} = a_d$ be L vectors of $\mathbb{Z}^d$. We want to describe the set of all parameters $\mathbf{r} = (r_1, \dots, r_d)$ that admit the period $\pi(\mathbf{a}_0, \dots, \mathbf{a}_{L-1})$, *i.e.*, the set of all $\mathbf{r} \in \mathcal{D}_d$ with

$$\tau_\mathbf{r}(\mathbf{a}_0) = \mathbf{a}_1,\ \tau_\mathbf{r}(\mathbf{a}_1) = \mathbf{a}_2,\ \dots, \tau_\mathbf{r}(\mathbf{a}_{L-2}) = \mathbf{a}_{L-1},\ \tau_\mathbf{r}(\mathbf{a}_{L-1}) = \mathbf{a}_0.$$

According to the definition of $\tau_\mathbf{r}$, this is the set given by

$$0 \leqslant r_1 a_{1+j} + \cdots + r_d a_{d+j} + a_{d+j+1} < 1 \qquad (0 \leqslant j \leqslant L-1). \tag{3.19}$$

To see this, let $j \in \{0, \dots, L-1\}$ be fixed. The equation $\tau_\mathbf{r}(\mathbf{a}_j) = \mathbf{a}_{j+1}$ can be written as

$$\begin{aligned} \tau_\mathbf{r}(\mathbf{a}_j) &= \tau_\mathbf{r}(a_{1+j}, \dots, a_{d+j}) \\ &= (a_{2+j}, \dots, a_{d+j}, -\lfloor r_1 a_{1+j} + \cdots + r_d a_{d+j} \rfloor) \\ &= (a_{2+j}, \dots, a_{d+1+j}), \end{aligned}$$

i.e., $a_{d+1+j} = -\lfloor r_1 a_{1+j} + \cdots + r_d a_{d+j} \rfloor$. Thus (3.19) holds and we are done.

We call the set defined by the inequalities in (3.19) $\mathcal{P}(\pi)$. Since $\mathcal{P}(\pi)$ is a (possibly degenerate or even empty) convex polyhedron, we call it the *cutout polyhedron* of π. Since 0 is the only permitted period for elements of

$\mathcal{D}_d^0$, we obtain $\mathcal{D}_d^0$ from $\mathcal{D}_d$ by cutting out all polyhedra $\mathcal{P}(\pi)$ corresponding to non-zero periods, *i.e.*,

$$\mathcal{D}_d^0 = \mathcal{D}_d \setminus \bigcup_{\pi \neq 0} \mathcal{P}(\pi). \tag{3.20}$$

Describing $\mathcal{D}_d^0$ is thus tantamount to describing the cutout polyhedra coming from non-zero periods. It can be easily seen from the definition that

$$\tau_{\mathbf{r}}(\mathbf{x}) = R(\mathbf{r})\mathbf{x} + \mathbf{v}.$$

Here $R(\mathbf{r})$ is a $d \times d$ matrix whose characteristic polynomial is $x^d + r_d x^{d-1} + \cdots + r_1$. Vector $\mathbf{v}$ is an "error term" coming from the floor function occurring in the definition of $\tau_{\mathbf{r}}$ and always fulfills $||\mathbf{v}||_\infty < 1$ (here $||\cdot||_\infty$ denotes the maximum norm). The further away from the boundary of $\mathcal{D}_d$ the parameter $\mathbf{r}$ is chosen, the smaller are the eigenvalues of $R(\mathbf{r})$. Since for each $\mathbf{r} \in \mathrm{int}(\mathcal{D}_d)$ the mapping $\tau_{\mathbf{r}}$ is contracting apart from the error term $\mathbf{v}$, one can easily prove that the norms of the elements $\mathbf{a}_0, \ldots, \mathbf{a}_{L-1}$ forming a period $\pi(\mathbf{a}_0, \ldots, \mathbf{a}_{L-1})$ of $\tau_{\mathbf{r}}$ can become large only if parameter $\mathbf{r}$ is chosen near the boundary. Therefore the number of periods corresponding to a given $\tau_{\mathbf{r}}$ with $\mathbf{r} \in \mathrm{int}(\mathcal{D}_d)$ is bounded. The bound depends on the largest eigenvalue of $R(\mathbf{r})$.

This fact was used to derive the following algorithm, which allows us to describe $\mathcal{D}_d^0$ in whole regions provided that they are at some distance away from $\partial\mathcal{D}_d$. In particular, the following result was proved in [11].

THEOREM 3.15. — *Let $\mathbf{r}_1, \ldots, \mathbf{r}_k \in \mathcal{D}_d$ and denote by H the convex hull of $\mathbf{r}_1, \ldots, \mathbf{r}_k$. We assume that $H \subset \mathrm{int}(\mathcal{D}_d)$ and that H is sufficiently small in diameter. For $\mathbf{z} \in \mathbb{Z}^d$ take $M(\mathbf{z}) = \max_{1 \leqslant i \leqslant k}\{-\lfloor \mathbf{r}_i \mathbf{z} \rfloor\}$. Then there exists an algorithm to create a finite directed graph (V, E) with vertices $V \subset \mathbb{Z}^d$ and edges $E \in V \times V$ which satisfy*

(1) *each d-dimensional standard unit vector $(0, \ldots, 0, \pm 1, 0, \ldots, 0) \in V$,*
(2) *for each $\mathbf{z} = (z_1, \ldots, z_d) \in V$ and*

$$j \in [-M(-\mathbf{z}), M(\mathbf{z})] \cap \mathbb{Z}$$

we have $(z_2, \ldots, z_d, j) \in V$ and a directed edge $(z_1, \ldots, z_d) \to (z_2, \ldots, z_d, j)$ in E.
(3) *$H \cap \mathcal{D}_d^0 = H \setminus \bigcup_\pi P(\pi)$, where the union is taken over all non-zero primitive cycles of (V, E).*

This result was substantially used in [12] to describe large parts of $\mathcal{D}_2^0$. Since it is fairly easy to show that $\mathcal{D}_2^0 \cap \partial\mathcal{D}_2 = \emptyset$, the difficulties related to the boundary of $\mathcal{D}_2$ do not cause troubles. However, it turned out that $\mathcal{D}_2^0$

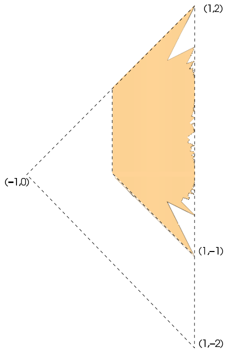

Figure 3.1. An approximation of $\mathcal{D}_2^0$

has a very complicated structure near this boundary. We refer the reader to Figure 3.1 to get an impression of this structure.

The big isoceles triangle is $\mathcal{E}_2$ and thus, by (3.17), apart from its boundary, it is equal to $\mathcal{D}_2$. The grey figure is an approximation of $\mathcal{D}_2^0$ which was constructed using (3.20) and Theorem 3.15. It is easy to see that the periods $(1,1)$ and $(1,0),(0,1)$ correspond to cutout polygons cutting away from $\mathcal{D}_2^0$ the area to the left and below the approximation. Since Theorem 3.15 can be used to treat regions far enough away from $\partial\mathcal{D}_2$, $\mathcal{D}_2^0$ just has to be described near the upper and right boundary of $\partial\mathcal{D}_2$.

Large parts of the region near the upper boundary could be treated in [12, Section 4] showing that this region indeed belongs to $\mathcal{D}_2^0$. Near the right boundary of $\mathcal{D}_2$, however, the structure of $\mathcal{D}_2^0$ is much more complicated.

For instance, in [11] it has been proved that infinitely many different cutouts are needed in order to describe $\mathcal{D}_2^0$. Moreover, the period lengths of $\tau_{\mathbf{r}}$ are not uniformly bounded. The shape of some infinite families of cutouts as well as some new results on $\mathcal{D}_2^0$ can be found in Surer [324]. In view of Theorem 3.12, this difficult structure of $\mathcal{D}_2^0$ implies that in cubic β-expansions of elements of $\mathbf{Z}[\beta^{-1}]\cap[0,\infty)$ periods of arbitrarily large length may occur.

SRS exist for parameters varying in a continuum. In [12, Section 4], this fact was used to exploit a certain structural stability occurring in the orbits of $\tau_{\mathbf{r}}$ when varying $\mathbf{r}$ continuously near the point $(1,-1)$. This leads to a description of $\mathcal{D}_2^0$ in a big area.

Theorem 3.16. — *We have*

$$\{(r_1,r_2)\ ;\ r_1>0,-r_1\leqslant r_2<1-2r_1\}\subset\mathcal{D}_2^0.$$

In view of Theorem 3.12, this yields a large class of Pisot numbers β satisfying property (F).

The description of $\mathcal{D}_2^0$ itself is not as interesting for the characterisation of CNS since quadratic CNS are already well understood (see Theorem 3.4). The set $\mathcal{D}_3^0$ has not yet been well studied. However, Scheicher and Thuswaldner [293] made some computer experiments to exhibit a counterexample to the following conjecture which (in a slightly different form) appears in [222]. It says that

$$p(x) \text{ CNS polynomial } \Longrightarrow p(x)+1 \text{ CNS polynomial.}$$

In particular, they found that this is not true for

$$p(x) = x^3 + 173x^2 + 257x + 198.$$

This counterexample was found by studying $\mathcal{D}_3^0$ near a degenerate cutout polyhedron that cuts out the parameter corresponding to $p(x)+1$ in view of Theorem 3.12, but not the parameter corresponding to $p(x)$. Since Theorem 3.15 can be used to prove that no other cutout polygon cuts out regions near this parameter, the counterexample can be confirmed.

The characterisation of cubic CNS polynomials $p(x) = x^3+p_2x^2+p_1x+p_0$ with fixed large p_0 is related to certain cuts of $\mathcal{D}_3^0$ which very closely resemble $\mathcal{D}_2^0$. In view of Theorem 3.12, this indicates that characterisation of cubic CNS polynomials is also very difficult. In particular, according to the *Lifting theorem* ([11, Theorem 6.2]), each of the periods occurring for two-dimensional SRS also occurs for cubic CNS polynomials. Thus CNS representations of elements of $\mathbb{Z}[x]/p(x)\mathbb{Z}[x]$ with respect to a cubic polynomial $p(x)$ can have infinitely many periods. Moreover, there is no bound for the period length (see [11, Section 7]). For the family of dynamical systems T in (3.5), this means that their attractors can be arbitrarily large if p varies over the cubic polynomials.

Recently, Akiyama and Scheicher [22, 23, 177] studied a variant of $\tau_{\mathbf{r}}$. In particular, they considered the family

$$\begin{array}{rrcl} \tilde{\tau}_{\mathbf{r}} : & \mathbb{Z}^d & \to & \mathbb{Z}^d \\ & (a_1,\dots,a_d) & \mapsto & (a_2,\dots,a_d,-\left\lfloor \mathbf{ra}+\frac{1}{2}\right\rfloor). \end{array}$$

In the same way as above they attach the sets $\tilde{\mathcal{D}}_d$ and $\tilde{\mathcal{D}}_d^0$ to it. However, interestingly, it turns out that the set $\tilde{\mathcal{D}}_2^0$ can be described completely in this modified setting. In particular, it can be shown that $\tilde{\mathcal{D}}_2^0$ is an open triangle together with some parts of its boundary. In Huszti *et al.* ([177]), the set $\tilde{\mathcal{D}}_3^0$ has been characterised completely. It turned out that $\tilde{\mathcal{D}}_3^0$ is the union of three convex polyhedra together with some parts of their boundary. As in the case of ordinary SRS, this variant is related to numeration systems. Namely, some modifications of CNS and β-expansions fit into this framework (see [22]).

3.5. Numeration systems defined over finite fields

In this subsection we would like to present other numeration systems. The first one is defined in residue classes of polynomial rings as follows. Polynomial rings $\mathbb{F}[x]$ over finite fields share many properties with the ring $\mathbb{Z}$. Thus it is natural to ask for analogues of canonical numeration systems in finite fields. Kovács and Pethő [222] studied special cases of the following more general concept introduced by Scheicher and Thuswaldner [291].

Let $\mathbb{F}$ be a finite field and $p(x,y) = \sum b_j(x)y^j \in \mathbb{F}[x,y]$ be a polynomial in two variables, and let $\mathcal{D} = \{p \in \mathbb{F}[x]\,;\, \deg p(x) < \deg b_0(x)\}$. We call $(p(x,y),\mathcal{D})$ a *digit system* with base $p(x,y)$ if each element q of the quotient ring $X = \mathbb{F}[x,y]/p(x,y)\mathbb{F}[x,y]$ admits a representation of the shape

$$q = c_0(x) + c_1(x)y + \cdots + c_\ell(x)y^\ell$$

with $c_j(x) \in \mathcal{D}$ $(0 \leqslant j \leqslant \ell)$.

Obviously these numeration systems fit into the framework defined at the beginning of this section by setting $A = F[y]$ and defining $\varepsilon(q)$ as the polynomial of least degree meeting the requirement that $T(q) \in X$.

It turns out that characterisation of the bases of these digit systems is quite easy. Indeed, the following result is proved in [291].

Theorem 3.17. — *The pair $(p(x,y),\mathcal{D})$ is a digit system if and only if one has $\max_{i=1}^n \deg b_i < \deg b_0$.*

The β-expansions have also been extended to the case of finite fields independently by Scheicher [289], as well as Hbaib and Mkaouar [163]. Let $\mathbb{F}((x^{-1}))$ be the field of formal Laurent series over $\mathbb{F}$ and denote by $|\cdot|$ some absolute value. Choose $\beta \in \mathbb{F}((x^{-1}))$ with $|\beta| > 1$. Let $z \in \mathbb{F}((x^{-1}))$ with $|z| < 1$. A *β-representation* of z is an infinite sequence $(d_i)_{i\geqslant 1}$, $d_i \in \mathbb{F}[x]$ with

$$z = \sum_{i\geqslant 1} \frac{d_i}{\beta^i}.$$

The most important β-representation (called β-expansion) is determined by the "greedy algorithm"

- $r_0 \leftarrow z$,

- $d_j \leftarrow \lfloor \beta r_{j-1} \rfloor$,
- $r_j = \beta r_{j-1} - d_j$.

Here $\lfloor \cdot \rfloor$ cuts off the negative powers of a formal Laurent series.

In [289] several problems related to β-expansions are studied. An analogue of property (F) of Frougny and Solomyak [150] is defined. Contrary to the classical case, all β satisfying this condition can be characterised. In [289, Section 5], it is shown that (F) is true if and only if β is a Pisot element of $\mathbb{F}((x^{-1}))$, *i.e.*, if β is an algebraic integer over $\mathbb{F}[x]$ with $|\beta| > 1$ all whose Galois conjugates β_j satisfy $|\beta_j| < 1$ (see [66]).

Furthermore, the analogue of Conjecture 3.14 could be settled in the finite field setting. In particular, Scheicher [289] proved that all bases β that are Pisot or Salem elements of $\mathbb{F}((x^{-1}))$ admit eventually periodic expansions.

In [163], the "representation of 1", which is defined in terms of an analogue of the β-transformation, is studied.

3.6. Lattice tilings

Consider Knuth's numeration system $(-1+\sqrt{-1}, \{0,1\})$ discussed in Section 3.1. We are interested in the set of all complex numbers admitting a representation w.r.t. this numeration system having zero "integer parts", *i.e.*, in all numbers

$$z = \sum_{j \geqslant 1} c_j (-1+\sqrt{-1})^{-j} \qquad (c_j \in \{0,1\}).$$

Define the set (*cf.* [212])

$$\mathcal{T} = \left\{ z \in \mathbb{C}\ ;\ z = \sum_{j \geqslant 1} c_j (-1+\sqrt{-1})^{-j} \qquad (c_j \in \{0,1\}) \right\}.$$

From this definition, we easily see that $\mathcal{T}$ satisfies the functional equation $(b = -1+\sqrt{-1})$

$$\mathcal{T} = b^{-1}\mathcal{T} \cup b^{-1}(\mathcal{T}+1). \tag{3.21}$$

Since $f_0(x) = b^{-1}x$ and $f_1(x) = b^{-1}(x+1)$ are contractive similarities in $\mathbb{C}$ w.r.t. the Euclidean metric, (3.21) asserts that $\mathcal{T}$ is the union of contracted copies of itself. Since the contractions are similarities in our case, $\mathcal{T}$ is a *self-similar set*. From the general theory of self-similar sets (see for instance Hutchinson [178]), we are able to draw several conclusions on $\mathcal{T}$. Indeed, according to a simple fixed point argument, $\mathcal{T}$ is uniquely defined by the

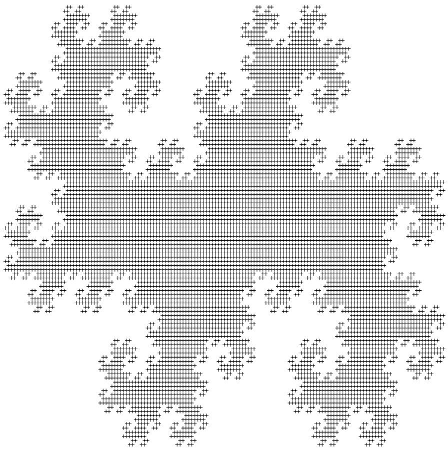

Figure 3.2. Knuth's twin dragon

set equation (3.21). Furthermore, $\mathcal{T}$ is a non-empty compact subset of $\mathbb{C}$. Set $\mathcal{T}$ is depicted in Figure 3.2. It is the well-known twin-dragon.

We now mention some interesting properties of $\mathcal{T}$. It is the closure of its interior ([24]) and its boundary is a fractal set whose Hausdorff dimension is given by

$$\dim_H \partial\mathcal{T} = 1.5236\ldots$$

([155, 184]). Furthermore, it induces a tiling of $\mathbb{C}$ in the sense that

$$\bigcup_{z\in\mathbb{Z}[i]} (\mathcal{T}+z) = \mathbb{C}, \tag{3.22}$$

where $(\mathcal{T}+z_1)\cap(\mathcal{T}+z_2)$ has zero Lebesgue measure if z_1 and z_2 are distinct elements of $\mathbb{Z}[i]$ ([201]). Note that this implies that the Lebesgue measure of $\mathcal{T}$ is equal to 1. We also mention that $\mathcal{T}$ is homeomorphic to the closed unit disk ([25]).

These properties make $\mathcal{T}$ a so-called self-similar lattice tile. Tiles can be associated with numeration systems in a more general way. After Definition 3.7, we already mentioned that matrix numeration systems admit the definition of tiles. Let $(A,\mathcal{D})$ be a matrix numeration system. Since all eigenvalues of A are larger than one in modulus, each of the mappings

$$f_d(x) = A^{-1}(x+d) \qquad (d\in\mathcal{D})$$

is a contraction w.r.t. a suitable norm. This justifies the following definition.

Definition 3.18. — *Let $(A,\mathcal{D})$ be a matrix numeration system in $\mathbb{Z}^d$. Then the non-empty compact set T which is uniquely defined by the set equation*

$$AT = \bigcup_{d\in\mathcal{D}} (T+d) \tag{3.23}$$

is called the self-affine tile *associated with $(A,\mathcal{D})$.*

Since $\mathcal{D} \subset \mathbb{Z}^d$ is a complete set of cosets in $\mathbb{Z}^d/A\mathbb{Z}^d$, these self-affine tiles are often called *self-affine tiles with standard set of digits* (*e.g.*, see [226]). The literature on these objects is vast. It is not our intention here to survey this literature. We just want to link numeration systems and self-affine lattice tiles and give some of their key properties. (For surveys on lattice tiles we refer the reader for instance to [339, 341].)

In [47], it is shown that each self-affine tile with standard set of digits has a positive d-dimensional Lebesgue measure. Together with [227], this implies the following result.

Theorem 3.19. — *Let T be a self-affine tile associated with a matrix numeration system $(A,\mathcal{D})$ in $\mathbb{Z}^d$. Then T is the closure of its interior. Its boundary ∂T has d-dimensional Lebesgue measure zero.*

As mentioned above, the twin-dragon induces a tiling of $\mathbb{C}$ in the sense mentioned in (3.22). It is natural to ask whether all self-affine tiles associated with matrix numeration systems share this property. In particular, let $(A,\mathcal{D})$ be a matrix numeration system. We say that the self-affine tile T associated with $(A,\mathcal{D})$ tiles $\mathbb{R}^n$ with respect to the lattice $\mathbb{Z}^d$ if

$$T+\mathbb{Z}^d = \mathbb{R}^d$$

such that $(T+z_1)\cap(T+z_2)$ has zero Lebesgue measure if $z_1,z_2\in\mathbb{Z}^d$ are distinct.

It turns out that it is difficult to describe all tiles having this property. Lagarias and Wang [226] and independently Kátai [199] found the following criterion.

Proposition 3.20. — *Let $(A,\mathcal{D})$ be a matrix numeration system in $\mathbb{Z}^d$ and set*

$$\Delta(A,\mathcal{D}) = \bigcup_{k\geqslant 1}\left\{\sum_{j=1}^{k} A^j(d_j-d_j')\ ;\ d_j,d_j'\in\mathcal{D}\right\}.$$

The self-affine tile T associated with $(A,\mathcal{D})$ tiles $\mathbb{R}^d$ with respect to the lattice $\mathbb{Z}^d$ if and only if

$$\Delta(A,\mathcal{D}) = \mathbb{Z}^d.$$

In [228], methods from Fourier analysis were used to derive the tiling property for a very large class of tilings. We do not state the theorem in full generality here (see [228, Theorem 6.1]). We just want to give a special case. To state it we need some notation. Let A_1 and A_2 be two $d \times d$ integer matrices. Here $A_1 \equiv A_2$ means A_1 is integrally similar to A_2, *i.e.*, there exists $Q \in \mathrm{GL}(d, \mathbb{Z})$ such that $A_2 = QA_1Q^{-1}$. We say that A is *(integrally) reducible* if

$$A \equiv \begin{pmatrix} A_1 & 0 \\ C & A_2 \end{pmatrix}$$

holds with A_1, and A_2 is non-empty. We call A *irreducible* if it is not reducible. Note that a sufficient condition for the irreducibility of an integer matrix A is the irreducibility of its minimal polynomial over $\mathbb{Q}$.

From [228, Corollary 6.2] the following result follows.

Theorem 3.21. — *Let $(A, \mathcal{D})$ be a matrix numeration system in $\mathbb{Z}^d$ with associated self-affine tile T. If A is irreducible, then T tiles $\mathbb{R}^d$ with respect to the lattice $\mathbb{Z}^d$.*

A special case of this result can also be found in [161]. Theorem 3.21 ensures, for instance, that each canonical numeration system with irreducible base polynomial $p(x)$ yields a tiling of $\mathbb{R}^d$ with $\mathbb{Z}^d$-translates. Indeed, just observe that the matrix A in (3.9) has minimal polynomial $p(x)$.

Many more properties of self-affine tiles associated with number systems have been investigated so far. The boundary of these tiles can be represented as a graph-directed iterated function system (see [141, Chapter 3] for a definition). Indeed, let $(A, \mathcal{D})$ be a matrix numeration system and let T be the associated self-affine tile. Suppose that T tiles $\mathbb{R}^d$ by $\mathbb{Z}^d$-translates. The set of neighbours of the tile T is defined by

$$S = \{s \in \mathbb{Z}^d \; ; \; T \cap (T + s) \neq \emptyset\}.$$

Since T and its translates form a tiling of $\mathbb{R}^d$, we may infer that

$$\partial T = \bigcup_{s \in S \setminus \{0\}} T \cap (T + s).$$

Thus in order to describe the boundary of T, the sets $B_s = T \cap (T + s)$ can be described for $s \in S \setminus \{0\}$. Using the set equation (3.23) for T, we easily derive that (*cf.* [292, Section 2])

$$B_s = A^{-1} \bigcup_{d, d' \in \mathcal{D}} B_{As+d'-d} + d.$$

Here $B_{As+d'-d}$ is non-empty only if the index is an element of S. Now label the elements of S as $S = \{s_1, \ldots, s_J\}$ and define the graph $G(S) = (V, E)$

with a set of states $V = S$ in the following way. Let $E_{i,j}$ be the set of edges leading from s_i to s_j. Then

$$E_{i,j} = \left\{ s_i \xrightarrow{d|d'} s_j \;;\; As_i + d' = s_j + d \text{ for some } d' \in \mathcal{D} \right\}.$$

In an edge $s_i \xrightarrow{d|d'} s_j$, we call d the *input digit* and d' the *output digit*. This yields the following result.

PROPOSITION 3.22. — *The boundary $\partial\mathcal{T}$ is a graph-directed iterated function system directed by the graph $G(S)$. In particular,*

$$\partial\mathcal{T} = \bigcup_{s \in S\setminus\{0\}} B_s$$

where

$$B_s = \bigcup_{\substack{d\in\mathcal{D},\, s'\in S\setminus\{0\} \\ s\xrightarrow{d}s'}} A^{-1}(B_{s'} + d).$$

The union is extended over all d, s' such that $s \xrightarrow{d} s'$ is an edge in the graph $G(S \setminus \{0\})$.

This description of ∂T is useful in several regards. In particular, graph $G(S)$ contains a lot of information on the underlying numeration system and its associated tile. Before we give some of its applications, we should mention that there exist simple algorithms for constructing $G(S)$ (*e.g.*, see [323, 292]).

In [341, 323], the graph $G(S)$ was used to derive a formula for the Hausdorff dimension of ∂T. The result reads as follows.

THEOREM 3.23. — *Let $(A, \mathcal{N})$ be a matrix numeration system in $\mathbb{Z}^d$ and T the associated self-affine tile. Let ρ be the spectral radius of the accompanying matrix of $G(S \setminus \{0\})$. If A is a similarity, then*

$$\dim_B(\partial T) = \dim_H(\partial T) = \frac{d \log \rho}{\log |\det A|}.$$

Similar results can be found in [135, 180, 339, 331, 290]. There they are derived using a certain subgraph of $G(S)$. In [331, 290], there are dimension calculations for the case where A is not a similarity.

In [161], a subgraph of $G(S)$ is used to set up an algorithmic tiling criterion. In [228], this criterion was used as a basis for a proof of Theorem 3.21.

More recently, the importance of $G(S)$ for the topological structure of tile T was discovered. We mention a result of Bandt and Wang [48] that yields a criterion for a tile to be homeomorphic to a disk. Roughly, it says that a self-affine tile is homeomorphic to a disk if it has 6 or 8 neighbours and satisfies some additional easy-to-check conditions. Very recently, Luo and Thuswaldner [244] established criteria for the triviality of the fundamental group of a self-affine tile. Moreover graph $G(S)$ plays an important rôle in these criteria.

At the end, we would like to show the relation of $G(S)$ to the matrix numeration system $(A,\mathcal{D})$ itself. If we change the direction of all edges in $G(S)$, we obtain the transposed graph $G^T(S)$. Suppose we have a representation of an element $z \in \mathbb{Z}^d$ of the shape

$$z = d_0 + Ad_1 + \cdots + A^{\ell} d_{\ell} \qquad (d_j \in \mathcal{D}).$$

To this representation, we associate the digit string $(\ldots 00 d_{\ell} \ldots d_0)$. Select a state s of the graph $G^T(S)$. It can be shown that a walk in $G^T(S)$ is uniquely defined by its starting state and a sequence of input digits. Now we run through the graph $G^T(S)$ starting at s along a path of edges whose input digits agree with the digit string $(\ldots 00 d_{\ell} \ldots d_0)$ starting with d_0. This yields an output string $(\ldots 00 d'_{\ell'} \ldots d'_0)$. From the definition of $G^T(S)$, it is easily apparent that this output string is the A-ary representation of $z + s$, *i.e.*,

$$z + s = d'_0 + Ad'_1 + \cdots + A^{\ell'} d'_{\ell'} \qquad (d'_j \in \mathcal{D}).$$

Thus $G^T(S)$ is an adding automaton that allows us to perform additions of A-ary representations (*e.g.*, see [158, 290]). In [328], the graph $G^T(S)$ was used to get a characterisation of all quadratic matrices that admit a matrix numeration system with finite representations for all elements of $\mathbb{Z}^2$ with a certain natural set of digits.

4. Some sofic fibred numeration systems

This section is devoted to a particular class of FNS for which the subshift $X_{\mathcal{N}}$ is sofic. This class especially includes β-numeration for β assumed to be a Parry number (see Example 2.8), the Dumont-Thomas numeration associated with a primitive substitution (see Section 4.1), as well as some abstract numeration systems (see Section 4.2). We focus on the construction of central tiles and Rauzy fractals in Section 4.3. In the present section, we highly use the algebraicity of the associated parameters of the FNS (*e.g.*, β for the β-numeration). We especially focus on the Pisot case and end this section by discussing the Pisot conjecture in Section 4.4.

4.1. Substitutions and Dumont-Thomas numeration

We now introduce a class of examples of sofic FNS — the Dumont-Thomas numeration. For this purpose, we first recall some basic facts on substitutions and substitutive dynamical systems.

If $\mathcal{A}$ is a finite set with cardinality n, a *substitution* σ is an endomorphism of the free monoid $\mathcal{A}^*$. A substitution naturally extends to the set of two-sided sequences $\mathcal{A}^{\mathbb{Z}}$. A one-sided *σ-periodic point* of σ is a sequence $u = (u_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ that satisfies $\sigma^n(u) = u$ for some $n > 0$. A two-sided *σ-periodic point* of σ is a two-sided sequence $u = (u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ that satisfies $\sigma^n(u) = u$ for some $n > 0$, and $u_{-1}u_0$ belongs to the image of some letter by some iterate σ^m of σ. This notion of σ-periodicity should not be confused with the usual notion of periodicity of sequences.

A substitution over the finite set $\mathcal{A}$ is said to be of *constant length* if the images of all letters of $\mathcal{A}$ have the same length. The *incidence matrix* $\mathbf{M}_\sigma = (m_{i,j})_{1\leqslant i,j\leqslant n}$ of the substitution σ has entries $m_{i,j} = |\sigma(j)|_i$, where the notation $|w|_i$ stands for the number of occurrences of the letter i in the word w. A substitution σ is called *primitive* if there exists an integer n such that $\sigma^n(a)$ contains at least one occurrence of the letter b for every pair $(a,b) \in \mathcal{A}^2$. This is equivalent to the fact that its incidence matrix is primitive, *i.e.*, there exists a nonnegative integer n such that $\mathbf{M}_\sigma^n$ has only positive entries. If σ is primitive, then the *Perron-Frobenius theorem* ensures that the incidence matrix $\mathbf{M}_\sigma$ has a simple real positive dominant eigenvalue β. A substitution σ is called *unimodular* if $\det \mathbf{M}_\sigma = \pm 1$. A substitution σ is said to be *Pisot* if its incidence matrix $\mathbf{M}_\sigma$ has a real dominant eigenvalue $\beta > 1$ such that, for every other eigenvalue λ, one has $0 < |\lambda| < 1$. The characteristic polynomial of the incidence matrix of such a substitution is irreducible over $\mathbb{Q}$, and the dominant eigenvalue β is a Pisot number. Furthermore, it can be proved that Pisot substitutions are primitive [272].

Every primitive substitution has at least one periodic point [274]. If u is a periodic point of σ, then the closure in $\mathcal{A}^{\mathbb{Z}}$ of the shift orbit of u does not depend on u. We thus denote it by X_σ. The *symbolic dynamical system generated by* σ is defined as (X_σ, S). The system (X_σ, S) is *minimal* and *uniquely ergodic* [274]; it is made of all the two-sided sequences whose set of factors coincides with the set of factors u (which does not depend on the choice of u by primitivity). For more results on substitutions, the reader is referred to [27, 272, 274].

There are many natural connections between substitutions and numeration systems (*e.g.*, see [130, 131, 140]). We now describe a numeration system associated with a primitive substitution σ, known as the Dumont-Thomas numeration [125, 126, 279]. This numeration allows to expand prefixes of the fixed point of the substitution, as well as real numbers in a noninteger base associated with the substitution. In this latter case, one gets an FNS providing expansions of real numbers with digits in a finite subset of the number field $\mathbb{Q}(\beta)$, with β being the Perron-Frobenius eigenvalue of the substitution σ.

Let σ be a primitive substitution. We denote by β its dominant eigenvalue. Let $\delta_\sigma : \mathcal{A}^* \to \mathbb{Q}(\beta)$ be the morphism defined by

$$\forall w \in \mathcal{A}^*,\ \delta_\sigma(w) = \lim_{n\to\infty} |\sigma^n(w)|\beta^{-n}.$$

Note that the convergence is ensured by the Perron-Frobenius theorem.

By definition, we have $\delta_\sigma(\sigma(a)) = \beta\delta_\sigma(a)$ and $\delta_\sigma(ww') = \delta_\sigma(w) + \delta_\sigma(w')$ for any $(w, w') \in (\mathcal{A}^*)^2$. Furthermore, the row vector $V^{(n)} = (|\sigma^n(a)|)_{a\in\mathcal{A}}$ satisfies the recurrence relation $V^{(n+1)} = V^{(n)} M_\sigma$. Hence the map δ_σ sends the letter a to the corresponding coordinate of some left eigenvector $\mathbf{v}_\beta$ of the incidence matrix $\mathbf{M}_\sigma$.

Let $a \in \mathcal{A}$ and let $x \in [0, \delta_\sigma(a))$. Then $\beta x \in [0, \delta_\sigma(\sigma(a)))$. There exist a unique letter b in $\mathcal{A}$, and a unique word $p \in \mathcal{A}^*$ such that pb is a prefix of $\sigma(a)$ and $\delta_\sigma(p) \leqslant \beta x < \delta_\sigma(pb)$. Clearly, $\beta x - \delta_\sigma(p) \in [0, \delta_\sigma(b))$.

We thus define the following map T:

$$T : \bigcup_{a\in\mathcal{A}} ([0, \delta_\sigma(a)) \times \{a\}) \to \bigcup_{a\in\mathcal{A}} ([0, \delta_\sigma(a)) \times \{a\})$$
$$(x, a) \mapsto (\beta x - \delta_\sigma(p), b) \text{ with } \begin{cases} \sigma(a) = pbs \\ \beta x - \delta_\sigma(p) \in [0, \delta_\sigma(b)). \end{cases}$$

Furthermore, one checks that (X, T) is a fibred system by setting

$$\begin{aligned} X &= \bigcup_{a\in\mathcal{A}} ([0, \delta_\sigma(a)) \times \{a\}), \\ I &= \{(p, b, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*;\ \exists\, a \in \mathcal{A},\, \sigma(a) = pbs\}, \\ \varepsilon(x, a) &= (p, b, s), \end{aligned}$$

where (p, b, s) is uniquely determined by $\sigma(a) = pbs$ and $\beta x - \delta_\sigma(p) \in [0, \delta_\sigma(b))$.

According to [125], it turns out that $\varphi = (\varepsilon(T^n x))_{n\geqslant 0}$ is injective, hence we get an FNS $\mathcal{N}$. Note that, at first sight, a more natural choice in the numeration framework could be to define ε as $(x, b) \mapsto \delta_\sigma(p)$, but we would lose injectivity for the map φ by using such a definition.

In order to describe the subshift $X_{\mathcal{N}} = \overline{\varphi(X)}$, we need to introduce the notion of prefix-suffix automaton. The *prefix-suffix automaton* $\mathcal{M}_\sigma$ of the substitution σ is defined in [93, 94] as the oriented directed graph that has the alphabet $\mathcal{A}$ as set of vertices and whose edges satisfy the following condition: there exists an edge labeled by $(p, c, s) \in I$ from b to c if $\sigma(b) = pcs$. We then will describe $X_{\mathcal{N}} = \overline{\varphi(X)}$ in terms of labels of infinite paths in the prefix-suffix automaton. Prefix automata have also been considered in the literature by just labelling edges with the prefix p [125, 279], but here we need all the information (p, c, s), especially for Theorem 4.4 below: the main difference between the prefix automaton and the prefix-suffix automaton is that the subshift generated by the first automaton (by reading labels of infinite paths) is only sofic, while the one generated by the second automaton is of finite type. For more details, see the discussion in chapter 7 of [272].

THEOREM 4.1 ([125]). — *Let σ be a primitive substitution on the alphabet $\mathcal{A}$. Let us fix $a \in \mathcal{A}$. Every real number $x \in [0, \delta_\sigma(a))$ can be uniquely expanded as $x = \sum_{n\geqslant 1} \delta_\sigma(p_n)\beta^{-n}$, where the sequence of digits $(p_n)_{n\geqslant 1}$ is the projection on the first component of an infinite path $(p_n, a_n, s_n)_{n\geqslant 1}$ in the prefix-suffix automaton $\mathcal{M}_\sigma$ stemming from a (i.e., $p_1 a_1$ is a prefix of $\sigma(a)$), and with the extra condition that there exist infinitely many integers n such that $\sigma(a_{n-1}) = p_n a_n$, with s_n not equal to the empty word, i.e., $p_n a_n$ is a proper suffix of $\sigma(a_{n-1})$.*

Note that the existence of infinitely many integers n such that $p_n a_n$ is a proper suffix of $\sigma(a_{n-1})$ is required for the unicity of such an expansion (one thus gets proper expansions).

We deduce from Theorem 4.1 that $\varphi(X)$ is equal to the set of labels of infinite paths $(p_n, a_n, s_n)_{n\geqslant 1}$ in the prefix-suffix automaton, for which there exist infinitely many integers n such that $p_n a_n$ is a proper prefix of $\sigma(a_{n-1})$, whereas $X_{\mathcal{N}} = \overline{\varphi(X)}$ is equal to the set of labels of infinite paths in the prefix-suffix automaton (without further condition).

Note that we can also define a Dumont-Thomas numeration on $\mathbb{N}$. Let v be a one-sided fixed point of σ; we denote its first letter by v_0. We assume, furthermore, that $|\sigma(v_0)| \geqslant 2$, and that v_0 is a prefix of $\sigma(v_0)$. This numeration depends on this particular choice of a fixed point, and more precisely on the letter v_0. One checks ([125], Theorem 1.5) that every finite prefix of v can be uniquely expanded as

$$\sigma^n(p_0)\sigma^{n-1}(p_{-1}) \cdots p_{-n},$$

where $p_0 \neq \varepsilon$, $\sigma(v_0) = p_0 a_0 s_0$, and $(p_0, a_0, s_0), \ldots, (p_{-n}, a_{-n}, s_{-n})$ is the sequence of labels of a path in the prefix-suffix automaton $\mathcal{M}_\sigma$ starting from the state v_0; for all i, one has $\sigma(a_i) = p_{i-1} a_{i-1} s_{i-1}$. Conversely, any path in $\mathcal{M}_\sigma$ starting from v_0 generates a finite prefix of v. This numeration works *a priori* on finite words but we can expand the nonnegative integer N as $N = |\sigma^n(p_0)| + \cdots + |p_{-n}|$, where N stands for the length of the prefix $\sigma^n(p_0)\sigma^{n-1}(p_{-1}) \cdots p_{-n}$ of v. One thus recovers a number system defined on $\mathbb{N}$.

Example 4.2. — We consider the so-called *Tribonacci substitution* σ_β : $1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$. It is a unimodular Pisot substitution. Its dominant eigenvalue $\beta > 1$, which is the positive root of $X^3 - X^2 - X - 1$, is called *Tribonacci number*. Its prefix-suffix automaton $\mathcal{M}_\sigma$ is depicted in Figure 4.2.

$$\begin{tikzcd}
1 \arrow[loop left, "(\varepsilon{,}1{,}2)"] \arrow[r, bend left, "(1{,}2{,}\varepsilon)"] & 2 \arrow[l, bend left, "(\varepsilon{,}1{,}3)"] \arrow[r, bend left, "(1{,}3{,}\varepsilon)"] & 3 \arrow[ll, bend left=60, "(\varepsilon{,}1{,}\varepsilon)"]
\end{tikzcd}$$

Figure 4.1. The prefix-suffix automaton for the Tribonacci substitution

The set of prefixes that occur in the labels of $\mathcal{M}_\sigma$ is equal to $\{\varepsilon, 1\}$. One checks that the (finite or infinite) paths with label $(p_n, a_n, s_n)_n$ in $\mathcal{M}_\sigma$, where $\sigma(a_n) = p_{n+1} a_{n+1} s_{n+1}$ for all n, are exactly the paths for which the factor 111 does not occur in the sequence of prefixes $(p_n)_n$. The expansion given in Theorem 4.1, with $a = 1$, coincides, up to a multiplication factor, with the expansion provided by the β-numeration (see Example 2.8), with β being equal to the Tribonacci number. Indeed one has $d_\beta^*(1) = (110)^\omega$. Hence $X_\mathcal{N}$ is equal to the set of sequences $(u_i)_{i \geqslant 1} \in \{0,1\}^{\mathbb{N}^*}$ which do not contain the factor 111, *i.e.*, $X_\mathcal{N}$ is the shift of finite type recognized by the automaton of Figure 4.2 that is deduced from $\mathcal{M}_\sigma$ by replacing the labeled edge (p, a, s) by the length $|p|$ of the prefix p (as in Example 4.2).

The Tribonacci substitution has been introduced and studied in detail in [276]. For more results and references on the Tribonacci substitution, see [34, 35, 186, 243, 252, 253, 272, 285]. Let us also quote [29] and [253, 254] for an extension of the Fibonacci multiplication introduced in [211] to the the Tribonacci case.

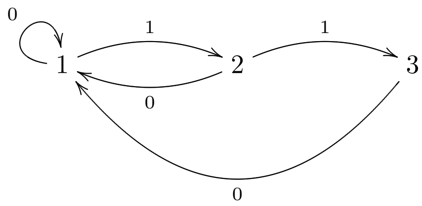

Figure 4.2. The prefix-suffix automaton recognizing the β-shift for the Tribonacci number

Example 4.3. — We continue Example 4.2 in a more general setting. Let $\beta > 1$ be a Parry number as defined in Example 2.8. As introduced, for instance, in [327] and in [140], one can naturally associate with (X_β, S) a substitution σ_β called *β-substitution* defined as follows according to the two cases, β simple and β non-simple:

- Assume that $d_\beta(1) = t_1 \ldots t_{m-1} t_m$ is finite, with $t_m \neq 0$. Thus $d^*_\beta(1) = (t_1 \ldots t_{m-1}(t_m - 1))^\omega$. One defines σ_β over the alphabet $\{1, 2, \ldots, m\}$ as

$$\sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1}2 \\ 2 & \mapsto 1^{t_2}3 \\ \vdots & \vdots \\ m-1 & \mapsto 1^{t_{m-1}}m \\ m & \mapsto 1^{t_m}. \end{cases}$$

- Assume that $d_\beta(1)$ is infinite. Then it cannot be purely periodic (according to Remark 7.2.5 in [242]). Hence one has $d_\beta(1) = d^*_\beta(1) = t_1 \ldots t_m (t_{m+1} \ldots t_{m+p})^\omega$, with $m \geqslant 1$, $t_m \neq t_{m+p}$ and $t_{m+1} \ldots t_{m+p} \neq 0^p$. One defines σ_β over the alphabet $\{1, 2, \ldots, m+p\}$ as

$$\sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1}2 \\ 2 & \mapsto 1^{t_2}3 \\ \vdots & \vdots \\ m+p-1 & \mapsto 1^{t_{m+p-1}}(m+p) \\ m+p & \mapsto 1^{t_{m+p}}(m+1). \end{cases}$$

It turns out that in both cases the substitutions σ_β are primitive and that the dominant eigenvalue of σ_β is equal to β. When β is equal to the Tribonacci number, then one recovers the Tribonacci substitution, since $d_\beta(1) = 111$. The prefix-suffix automaton of the substitution σ_β is strongly connected to the finite automaton $\mathcal{M}_\beta$ recognizing the set of finite factors of the β-shift $X_{\mathcal{N}}$. Indeed, we first note that the prefixes that occur as labeled edges of $\mathcal{M}_\sigma$ contain only the letter 1; it is thus natural to code a prefix by its length; one recovers the automaton $\mathcal{M}_\beta$ by replacing in the prefix-suffix automaton $\mathcal{M}_\sigma$ the labeled edges (p, a, s) by $|p|$.

If σ is a constant length substitution of length q, then one recovers the q-adic numeration. If σ is a β-substitution such as defined in Example 4.3, for a Parry number β, then the expansion given in Theorem 4.1, with $a = 1$, coincides with the expansion provided by the β-numeration, up to a multiplication factor. More generally, even when σ is not a β-substitution, then the Dumont-Thomas numeration shares many properties with the β-numeration. In particular, when β is a Pisot number, then, for every $a \in \mathcal{A}$, every element of $\mathbb{Q}(\beta) \cap [0, \delta_\sigma(a))$ admits an eventually periodic expansion, *i.e.*, the restriction to $\mathbb{Q}(\beta)$ yields a quasi-finite FNS. The proof can be conducted exactly in the same way as in [295].
Let $X^l_{\mathcal{N}}$ be the set of labels of infinite left-sided paths $(p_{-m}, a_{-m}, s_{-m})_{m\geqslant 0}$ in the prefix-suffix automaton; they satisfy $\sigma(a_{-m}) = p_{-m+1}a_{-m+1}s_{-m+1}$ for all $m \geqslant 0$. The subshift $X^l_{\mathcal{N}}$ is a subshift of finite type. The set $X^l_{\mathcal{N}}$ has an interesting dynamical interpretation with respect to the substitutive dynamical system (X_σ, S). Here we follow the approach and notation of [93, 94]. Let us recall that substitution σ is assumed to be primitive. According to [258] and [55], every two-sided sequence $w \in X_\sigma$ has a unique decomposition $w = S^\nu(\sigma(v))$, with $v \in X_\sigma$ and $0 \leqslant \nu < |\sigma(v_0)|$, where v_0 is the 0-th coordinate of v, *i.e.*,

$$w = \ldots \mid \underbrace{\ldots}_{\sigma(v_{-1})} \mid \underbrace{w_{-\nu}\ldots w_{-1}.w_0\ldots w_{\nu'}}_{\sigma(v_0)} \mid \underbrace{\ldots}_{\sigma(v_1)} \mid \underbrace{\ldots}_{\sigma(v_2)} \mid \ldots$$

The two-sided sequence w is completely determined by the two-sided sequence $v \in X_\sigma$ and the value $(p, w_0, s) \in I$. The *desubstitution map* $\theta : X_\sigma \to X_\sigma$ is thus defined as the map that sends w to v. We then define $\gamma\colon X_\sigma \to I$ mapping w to (p, w_0, s). It turns out that $(\theta^n(w))_{n\geqslant 0} \in X^l_{\mathcal{N}}$. The *prefix-suffix expansion* is then defined as the map $E_{\mathcal{N}} : X_\sigma \to X^l_{\mathcal{N}}$ which maps a two-sided sequence $w \in X_\sigma$ to the sequence $(\gamma(\theta^n w))_{n\geqslant 0}$, *i.e.*, the orbits of w through the desubstitution map according to the partition defined by γ.

Theorem 4.4 ([93, 94, 172]). — *Let σ be a primitive substitution such that none of its periodic points is shift-periodic. The map $E_{\mathcal{N}}$ is continuous onto the subshift of finite type $X^l_{\mathcal{N}}$; it is one-to-one except on the orbits under the shift S of the σ-periodic points of σ.*

In other words, the prefix-suffix expansion map $E_{\mathcal{N}}$ provides a measure-theoretic isomorphism between the shift map S on X_σ and an adic transformation on $X^l_{\mathcal{N}}$, considered as a Markov compactum, as defined in Section 5.4, by providing set I with a natural partial ordering coming from the substitution.

4.2. Abstract numeration systems

We first recall that the *genealogical order* is defined as follows: if v and w belong to L, then $v \preceq w$ if and only if $|v| < |w|$ or $|v| = |w|$, and v preceds w with respect to the lexicographical order deriving from $<$. One essential feature in the construction of the previous section is that the dynamical system $X_{\mathcal{N}}$ is sofic, which means that the language $\mathcal{L}_{\mathcal{N}}$ is regular. Let us now extend this approach by starting directly with a regular language. According to [230], given an infinite regular language L over a totally ordered alphabet $(A, <)$, a so-called *abstract numeration system* $S = (L, A, \preceq)$ is defined in the following way: enumerating the words of L by increasing genealogical order gives a one-to-one correspondence between $\mathbb{N}$ and L, the nonnegative integer n is then represented by the $(n+1)$-th word of the ordered language $(L, \preceq)$.

Such an abstract numeration system is in line with Definition 2.1, where $X = \mathbb{N}$, $I = A$, and φ is the (injective) map that sends the natural number n to the $(n+1)$-th word of the ordered language L. Abstract numeration systems thus include classical numeration systems like q-adic numeration, β-numeration when β is a Parry number, as well as the Dumont-Thomas numeration associated with a substitution.

Moreover, these abstract numeration systems have been themselves extended to allow the representation of integers and of real numbers [232]: a real number is represented by an infinite word which is the limit of a converging sequence of words in L. Under some ancillary hypotheses, we can describe such a representation thanks to a fibred number system defined as follows, according to [62, 283].

Let L be an infinite regular language over the totally ordered alphabet $(\Sigma, <)$. The trimmed minimal automaton of L is denoted by $\mathcal{M}_L = (Q, q_0, \Sigma, \delta, F)$ where Q is the set of states, $q_0 \in Q$ is the initial state, $\delta : Q \times \Sigma \to Q$ is the (partial) transition function, and $F \subseteq Q$ is the set of final states. We furthermore assume that $\mathcal{M}_L$ is such that $\mathcal{M}_L$ has a loop of label s_0 at the initial state q_0. For any state $q \in Q$, we denote by L_q the regular language accepted by $\mathcal{M}_L$ from state q, and by $\mathbf{u}_q(n)$ the number of words of length n in L_q.

The entry of index $(p,q) \in Q^2$ of the *adjacency matrix* $\mathbf{M}_L$ of the automaton $\mathcal{M}_L$ is given by the cardinality of the set of letters $s \in \Sigma$, such that $\delta(p,s) = q$. An abstract numeration system is said to be *primitive* if the matrix $\mathbf{M}_L$ is primitive. Let $\beta > 1$ be its dominant eigenvalue. We assume moreover that L is a language for which there exist $P \in \mathbb{R}[Y]$, and some nonnegative real numbers a_q, $q \in Q$, which are not simultaneously equal to 0, such that for all states $q \in Q$,

$$\lim_{n\to\infty} \frac{\mathbf{u}_q(n)}{P(n)\beta^n} = a_q. \tag{4.1}$$

The coefficients a_q are defined up to a scaling constant. In fact, vector $(a_q)_{q\in Q}$ is an eigenvector of $\mathbf{M}_L$ [232]; by the Perron-Frobenius theorem, all its entries a_q have the same sign; we normalise it so that $a_{q_0} = 1 - 1/\beta$, according to [283]. Then we have $a_q \geqslant 0$ for all $q \in Q$.

For $q \in Q$ and $s \in \Sigma$, set

$$\alpha_q(s) = \sum_{q'\in Q} a_{q'} \cdot \#\{t < s\,;\, \delta(q,t) = q'\} = \sum_{t<s} a_{\delta(q,t)}.$$

Since $(a_q)_{q\in Q}$ is an eigenvector of $\mathbf{M}_L$ of eigenvalue β, one has for all $q \in Q$:

$$\beta a_q = \sum_{r\in Q} a_r \cdot \#\{s \in \Sigma\,;\, \delta(q,s) = r\} = \sum_{t\in\Sigma} a_{\delta(q,t)}. \tag{4.2}$$

By nonnegativity of the coefficients a_s, we have $0 \leqslant \alpha_q(s) \leqslant \beta a_q$, for all $q \in Q$. Note also that if $s < t$, $s,t \in \Sigma$, then $\alpha_q(s) \leqslant \alpha_q(t)$. We set, for $x \in \mathbb{R}^+$,

$$\lfloor x \rfloor_q = \max\{\alpha_q(s)\,;\, s \in \Sigma,\ \alpha_q(s) \leqslant x\}.$$

Using (4.2) one checks that for $x \in [0, a_q)$, then $\beta x - \lfloor \beta x \rfloor_q \in [0, a_{q'})$, with $\lfloor \beta x \rfloor_q = \alpha_q(s)$ and $\delta(q,s) = q'$. We can define

$$\begin{array}{rccc} T : \bigcup_{q\in Q} ([0,a_q) \times \{q\}) & \longrightarrow & \bigcup_{q\in Q} ([0,a_q) \times \{q\}) \\ (x,q) & \longmapsto & (\beta x - \lfloor \beta x \rfloor_q, q'), \end{array}$$

where q' is determined as follows: let s be the largest letter such that $\alpha_q(s) = \lfloor \beta x \rfloor_q$; then $q' = \delta(q,s)$. One checks that $\mathcal{N} = (X, T, I, \varphi)$ is a fibred number system by setting

$$\begin{aligned} X &= \cup_{q\in Q} ([0,a_q) \times \{q\}), \\ I &= \{(s,q,q') \in \Sigma \times Q \times Q\,;\, q' = \delta(q,s)\}, \\ \varepsilon(x,q) &= (s,q,q'), \end{aligned}$$

where s is the largest letter such that $\alpha_q(s) = \lfloor \beta x \rfloor_q$, and $q' = \delta(q, s)$. One checks furthermore that φ is injective.

We thus can expand any real number $x \in [0, a_{q_0}) = [0, 1 - 1/\beta)$ as follows: let $(x_n, r_n)_{n \geqslant 1} = (T^n(x, q_0))_{n \geqslant 1} \in X^{\mathbb{N}^*}$, and let $(w_0, r_0) = (s_0, q_0)$; for every $n \geqslant 1$, let w_n be the first component of $\varepsilon_n = \varepsilon \circ T^{n-1}$ where $x_0 = x$; then one has $x = \sum_{n=1}^{\infty} \alpha_{r_{n-1}}(w_n)\beta^{-n}$, according to [231].

Abstract numeration systems lead to the generalisation of various concepts related to the representation of integers like summatory functions of additive functions [160], or like the notion of odometer [63].

4.3. Rauzy fractals

We have seen in Section 3.6 that it is possible to naturally associate self-affine tiles and lattice tilings with matrix number systems (see also Question 2.23). Such self-affine tiles are compact sets, they are the closure of their interior, they have a non-zero measure and a fractal boundary that is the attractor of some graph-directed iterated function system. The aim of this section is to show how to associate in the present framework similar tiles, called *Rauzy fractals*, with Pisot substitutions and β-shifts.

Rauzy fractals were first introduced in [276] in the case of the Tribonacci substitution (see Example 4.2), and then in [327], in the case of the β-numeration associated with the Tribonacci number. One motivation for Rauzy's construction was to exhibit explicit factors of the substitutive dynamical system (X_σ, S), under the Pisot hypothesis, as rotations on compact abelian groups.

Rauzy fractals can more generally be associated with Pisot substitutions (see [55, 93, 94, 190, 253, 254, 306, 307] and the surveys [65, 272]), as well as with Pisot β-shifts under the name of *central tiles* (see [7, 8, 9, 10]), but they also can be associated with abstract numeration systems [62], as well as with some automorphisms of the free group [30], namely the so-called irreducible with irreducible powers automorphisms [70].

There are several definitions associated with several methods of construction for Rauzy fractals.

- We detail below a construction based on formal power series in the substitutive case. This construction is inspired by the seminal paper [276], by [252, 253], and by [93, 94].
- A different approach via graph-directed iterated function systems (in the same vein as Proposition 3.22) and generalised substitutions

has been developed on the basis of ideas from [186], and [32, 33]. Indeed, Rauzy fractals can be described as attractors of some graph-directed iterated function system, as in [171], where one can find a study of the Hausdorff dimension of various sets related to Rauzy fractals, and as in [309, 310, 312] with a special focus on the self-similar properties of Rauzy fractals.

- Lastly, they can be defined in case σ is a Pisot substitution as the closure of the projection on the contracting plane of $\mathbf{M}_\sigma$ along its expanding direction of the images by the abelianisation map of prefixes of a σ-periodic point [55, 190, 272], where the abelianization map, also called Parikh map, is defined as $\mathbf{l}\colon \mathcal{A}^* \to \mathbb{N}^n$, $\mathbf{l}(W) \mapsto (|W|_k)_{k=1,\dots,n} \in \mathbb{N}^n$.

For more details on these approaches, see Chapters 7 and 8 of [272], and [65].

Let us describe how to associate a Rauzy fractal with a Pisot substitution that is not necessarily unimodular, as a compact subset of a finite product of Euclidean and p-adic spaces following [306]. We thus consider a primitive substitution σ that we assume furthermore to be Pisot. We then consider the FNS $\mathcal{N}$ provided by the Dumont-Thomas numeration, such as described in Section 4.1. We follow here [64, 65, 306]. Let us recall that the set $X^l_{\mathcal{N}}$ is the set of labels of infinite left-sided paths $(p_{-n}, a_{-n}, s_{-n}) \in I^{\mathbb{N}}$ in the prefix-suffix automaton $\mathcal{M}_\sigma$, with the notation of Section 4.1. (By analogy with Section 3.6, this amounts to work with representations having zero "integer part" w.r.t. the FNS $\mathcal{N}$.) Let β stand for the dominant eigenvalue of the primitive substitution σ.

We first define the map Γ on $X^l_{\mathcal{N}}$ as

$$\Gamma((p_{-n}, a_{-n}, s_{-n})_{n\geqslant 0}) = \sum_{n\geqslant 0} \delta_\sigma(p_{-n}) Y^n;$$

hence Γ takes its values in a finite extension of the ring of formal power series with coefficients in $\mathbb{Q}$; we recall that the coefficients $\delta_\sigma(p_{-n})$ take their values in a finite subset of $\mathbb{Q}(\beta)$.

Let us specialise these formal power series by giving the value β to the indeterminate Y, and by considering all the Archimedean and non-Archimedean metrizable topologies on $\mathbb{Q}(\beta)$ in which all the series $\sum_{n\geqslant 0} \delta_\sigma (p_{-n})\beta^n$ would converge for $(p_{-n}, a_{-n}, s_{-n})_{n\geqslant 0} \in X^l_{\mathcal{N}}$.

We recall that β is a Pisot number of degree d, say. Let $\beta_2, \dots, \beta_r$ be the real conjugates of β, and let $\beta_{r+1}, \overline{\beta_{r+1}}, \dots, \beta_{r+s}, \overline{\beta_{r+s}}$ be its complex conjugates. For $2 \leqslant j \leqslant r$, let $\mathbb{K}_{\beta_j}$ be equal to $\mathbb{R}$, and for $r+1 \leqslant j \leqslant r+s$, let $\mathbb{K}_{\beta_j}$ be equal to $\mathbb{C}$, with $\mathbb{R}$ and $\mathbb{C}$ being endowed with the usual topology.

Let $\mathcal{I}_1, \ldots, \mathcal{I}_\nu$ be the prime ideals in the integer ring $\mathcal{O}_{\mathbb{Q}(\beta)}$ of $\mathbb{Q}(\beta)$ that contain β, *i.e.*,

$$\beta\mathcal{O}_{\mathbb{Q}(\beta)} = \prod_{i=1}^{\nu} \mathcal{I}_i^{n_i}.$$

Let $\mathcal{I}$ be a prime ideal in $\mathcal{O}_{\mathbb{Q}(\beta)}$. We denote by $\mathbb{K}_{\mathcal{I}}$ the completion of $\mathbb{Q}(\beta)$ for the $\mathcal{I}$-adic topology. The field $\mathbb{K}_{\mathcal{I}}$ is a finite extension of the $p_{\mathcal{I}}$-adic field $\mathbb{Q}_{p_{\mathcal{I}}}$, where $\mathcal{I} \cap \mathbb{Z} = p_{\mathcal{I}}\mathbb{Z}$. The primes which appear as p-adic spaces are the prime factors of the norm of β. One then defines the representation space of $X_{\mathcal{N}}^l$ as

$$\mathbb{K}_\beta = \mathbb{K}_{\beta_2} \times \ldots \mathbb{K}_{\beta_{r+s}} \times \mathbb{K}_{\mathcal{I}_1} \times \ldots \mathbb{K}_{\mathcal{I}_\nu} \simeq \mathbb{R}^{r-1} \times \mathbb{C}^s \times \mathbb{K}_{\mathcal{I}_1} \times \cdots \times \mathbb{K}_{\mathcal{I}_\nu}.$$

Endowed with the product of the topologies of each of its elements, $\mathbb{K}_\beta$ is a metric abelian group. If σ is unimodular, then $\mathbb{K}_\beta = \mathbb{R}^{r-1} \times \mathbb{C}^s$ is identified with $\mathbb{R}^{d-1}$.

The canonical embedding of $\mathbb{Q}(\beta)$ into $\mathbb{K}_\beta$ is defined by the following morphism $\Phi\colon P(\beta) \in \mathbb{Q}(\beta) \mapsto (\underbrace{P(\beta_2)}_{\in\mathbb{K}_{\beta_2}}, \ldots, \underbrace{P(\beta_{r+s})}_{\in\mathbb{K}_{\beta_{r+s}}}, \underbrace{P(\beta)}_{\in\mathbb{K}_{\mathcal{I}_1}}, \ldots, \underbrace{P(\beta)}_{\in\mathbb{K}_{\mathcal{I}_\nu}}) \in \mathbb{K}_\beta$.

The topology on $\mathbb{K}_\beta$ was chosen so that the series

$$\lim_{n\to+\infty} \Phi(\sum_{n=0}^{j} \delta_\sigma(p_{-n})\beta^n) = \sum_{n\geqslant 0} \Phi(\delta_\sigma(p_{-n})\beta^n)$$

are convergent in $\mathbb{K}_\beta$ for every $(p_n, a_n, s_n)_{n\geqslant 0} \in X_{\mathcal{N}}^l$. One thus defines

$$\Upsilon\colon X_{\mathcal{N}}^l \to \mathbb{K}_\beta,\ (p_{-n}, a_{-n}, s_{-n})_{n\geqslant 0} \mapsto \Phi(\sum_{n\geqslant 0} \delta_\sigma(p_{-n})\beta^n).$$

Definition 4.5. — *Let σ be a Pisot substitution and let $\mathcal{N}$ be the FNS provided by the Dumont-Thomas numeration. The* generalized Rauzy fractal *of $X_{\mathcal{N}}^l$ is defined as $\mathcal{T}_{\mathcal{N}} = \Upsilon(X_{\mathcal{N}}^l)$, with the above notation.*

If σ is unimodular, then it is a compact subset of $\mathbb{R}^{d-1}$, where d is the cardinality of the alphabet $\mathcal{A}$ of the substitution.

It can be divided into d subpieces as follows: for every letter a in $\mathcal{A}$,

$$\mathcal{T}_{\mathcal{N}}(a) = \Upsilon(\{(p_{-n}, a_{-n}, s_{-n})_{n\geqslant 0} \in X_{\mathcal{N}}^l;\ (p_{-n}, a_{-n}, s_{-n})_{n\geqslant 0} \text{ is the label of an infinite left-sided path in } \mathcal{M}_\sigma \text{ arriving at state } a_0 = a\}).$$

For every letter a the sets $\mathcal{T}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}(a)$ have non-empty interior [306], hence they have non-zero measure. Moreover, they are the closure of their interior, according to [312]. For examples of Rauzy fractals, see Figure 4.3.

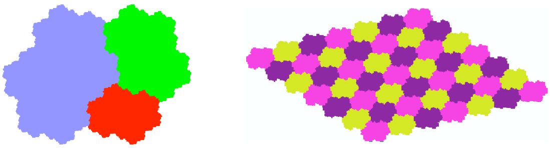

Figure 4.3. Rauzy fractal for the Tribonacci substitution, and Rauzy lattice tiling

Topological properties of Rauzy fractals have aroused a large interest. Their connectedness and homeomorphy to a closed disc are investigated, *e.g.*, in [17, 16, 92, 308]. Let us stress the fact that Rauzy fractals and self-affine tiles associated with matrix numeration systems (such as discussed in Section 3.6) are distinct objects. Nevertheless ideas and methods used for these latter tiles have often been inspirating for the study of the tiling and topological properties of Rauzy fractals. Indeed, Rauzy fractals are solutions of a graph-directed iterated function system directed by the prefix-suffix automaton [313, 65].

Surprisingly enough, the sets $\mathcal{T}_{\mathcal{N}}(a)$ have disjoint interiors provided that the substitution σ satisfies a combinatorial condition, the so-called strong coincidence condition, according to [32] in the unimodular case, and [306], in the general case. A substitution is said to satisfy the *strong coincidence condition* if for any pair of letters (i, j), there exist two integers k, n such that $\sigma^n(i)$ and $\sigma^n(j)$ have the same k-th letter, and the prefixes of length $k-1$ of $\sigma^n(i)$ and $\sigma^n(j)$ have the same image under the abelianisation map $\mathbf{l}$.

The strong coincidence condition has been introduced in [32]. This condition is inspired by Dekking's notion of coincidence [112] which yields a characterisation of constant length substitutions having a discrete spectrum; see also [174]. This notion has lead to the following conjecture:

Conjecture 4.6. — *Every Pisot substitution satisfies the strong coincidence condition.*

The conjecture has been proved for two-letter substitutions in [53]. For more details on the strong coincidence condition, see [55, 190, 272].

4.4. The Pisot conjecture

One of the main incentives behind the introduction of Rauzy fractals is the following result:

THEOREM 4.7 ([276]). — *Let σ be the Tribonacci substitution $\sigma\colon 1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1$. The Rauzy fractal $\mathcal{T}_{\mathcal{N}}$ (considered as a subset of $\mathbb{R}^2$) is a fundamental domain of $\mathbb{T}^2$. Let $R_\beta\colon \mathbb{T}^2 \to \mathbb{T}^2$, $x \mapsto x + (1/\beta, 1/\beta^2)$. The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the toral translation $(\mathbb{T}^2, R_\beta)$.*

This result can also be restated in more geometrical terms: the Rauzy fractal generates a lattice tiling of the plane, as illustrated in Figure 4.3, *i.e.*, $\mathbb{R}^2 = \cup_{\gamma\in\Gamma}(\mathcal{R}_{\mathcal{N}} + \gamma)$, the union being disjoint in measure, and $\Gamma = \mathbb{Z}(\delta_\sigma(1) - \delta_\sigma(3)) + \mathbb{Z}(\delta_\sigma(2) - \delta_\sigma(3))$. More generally, one gets:

THEOREM 4.8. — *Let σ be a Pisot substitution that satisfies the strong coincidence condition. The following conditions are equivalent:*

(1) *(X_σ, S) is measure-theoretically isomorphic to a translation on the torus;*
(2) *(X_σ, S) has a pure discrete spectrum;*
(3) *the associated Rauzy fractal $\mathcal{T}_{\mathcal{N}}$ generates a lattice tiling, i.e.,*

$$\mathbb{K}_\beta = \cup_{\gamma\in\Gamma}(\mathcal{R}_{\mathcal{N}} + \gamma),$$

the union being disjoint in measure, and $\Gamma = \sum_{b\in\mathcal{A}, b\neq a} \mathbb{Z}(\delta_\sigma(b) - \delta_\sigma(a))$, for $a \in \mathcal{A}$.

The equivalence between (1) and (2) is a classical result in spectral theory (*e.g.*, see [340]). The equivalence between (2) and (3) is due to Barge and Kwapisz [55].

CONJECTURE 4.9. — *The equivalent conditions of Theorem 4.8 are conjectured to hold if σ is a Pisot unimodular substitution.*

Here again the conjecture holds true for two-letter alphabets [53, 170, 174]. Substantial literature is devoted to Conjecture 4.9 which is reviewed in [272], Chap.7. See also [55, 54, 46, 65, 190] for recent results.

Let us stress the fact that we have assumed the irreducibility of the characteristic polynomial of the incidence matrix of the substitution: indeed, the incidence matrix of a Pisot substitution has an irreducible characteristic polynomial, by definition. Nevertheless, it is possible to define a Rauzy fractal even if the substitution is not assumed to be irreducible but primitive, with its dominant eigenvalue being a Pisot number (*e.g.*, see [46, 65, 136]).

In this latter case, the substitutive dynamical system might not have a pure discrete spectrum, as illustrated, *e.g.*, by Example 5.3 in [46].

There exist several sufficient conditions that imply the equivalent assertions of Theorem 4.8 inspired by Property (F), as defined in Example 2.8 and Section 3.4. Indeed, similar finiteness properties have been introduced in [65] for substitutive dynamical systems, see also [62, 151] for abstract numeration systems.

There exist also effective combinatorial characterisations for pure discrete spectrum based either on graphs [307, 329], or on the so-called balanced pair algorithm [246, 311], or else conditions inspired by the strong coincidence condition [46, 55, 54, 190]. More generally, for more on the spectral study of substitutive dynamical systems, see [173, 145].

Sofic covers. Analogously, a Rauzy fractal (usually called *central tile*) can be associated with the left one-sided β-shift (*e.g.*, see [7, 8, 9, 10, 21, 327]) for β a Pisot unit. From a dynamical point of view, the transformation corresponding to (X_σ, S) in Theorem 4.8 is an odometer (or an adic transformation [315, 316]) acting on the left one-sided β-shift (for more details, see Section 5). This zero entropy transformation is in some sense not as natural as the shift S acting on X_σ. Nevertheless, if one considers the natural extension of the β-transformation, then one gets an interesting interpretation of Theorem 4.8 by performing a similar construction for the whole set $\widetilde{X_{\mathcal{N}}}$ of two-sided $\mathcal{N}$-representations; one thus gets geometric realisation of the natural extension of the transformation T of the FNS $\mathcal{N}$.

This construction is used, for instance, in [64] to characterise numbers that have a purely periodic β-expansion, producing a kind of generalised Galois' theorem on classical continuous fractions, for β Pisot (see also the references in Question 2.19). This construction also has consequences for the effective construction of Markov partitions for toral automorphisms, the main eigenvalue of which is a Pisot number. See, for instance, [67, 54, 188, 271]. Based on the approach of [207, 305, 336, 334], an algebraic construction of Markov symbolic almost one-to-one covers of hyperbolic toral automorphisms provided by the two-sided β-shift is similarly exhibited in [240, 297] (see also [302, 303, 304, 54]):

Definition 4.10. — *Let α be an automorphism of the torus. A point x is said to be* homoclinic *if* $\lim_{|n|\to\infty} \alpha^n(x) = 0$. *Homoclinic points form a subgroup of the torus, that we denote by Δ_α. A point x is said to be a* fundamental homoclinic point *if* $\{\alpha^n(x) \ ; \ n \in \mathbb{Z}\}$ *generates the additive group* Δ_α.

Example 4.11. — The homoclinic group of the automorphism $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ of $\mathbb{T}^2$ is equal to $\frac{1}{\sqrt{5}}(\mathbb{Z} + \varrho\mathbb{Z}) \begin{pmatrix} 1/\varrho \\ 1 \end{pmatrix}$ (see [305], for example), where $\varrho = \frac{1+\sqrt{5}}{2}$.

Theorem 4.12 ([297]). — *Let $\beta > 1$ be a Pisot number. Let $\alpha \in GL(n, \mathbb{Z})$ be an automorphism of the torus $\mathbb{T}^n$ that its conjugate within $GL(n, \mathbb{Z})$ to the companion matrix of the minimal polynomial of β. Then α admits a fundamental homoclinic point x^Δ. Let $\widetilde{X_\mathcal{N}}$ be the two-sided β-shift and let*

$$\xi \colon \widetilde{X_\mathcal{N}} \to \mathbb{T}^n, \ v \mapsto \sum_{i \in \mathbb{Z}} v_i \alpha^i x^\Delta.$$

Then $\xi(\widetilde{X_\mathcal{N}}) = \mathbb{T}^n$ and ξ is bounded-to-one.

Furthermore, if β satisfies property (F), then ξ is almost one-to-one.

The following question is addressed in [297]: assume that α is conjugate (in $GL(n, \mathbb{Z})$) to the companion matrix of its characteristic polynomial, that α has a single eigenvalue $\beta > 1$, and all other eigenvalues have absolute value < 1; then, is the restriction of ξ to $\widetilde{X_\mathcal{N}}$ almost one-to-one? This question is strongly related to Conjecture 4.9.

5. G-scales and odometers

The present section goes back to the representation of nonnegative integers. As always, our first model is the q-adic expansion. Within the context of an FNS, the algorithm produces the less significant digit first, then the second less significant one, a.s.o. The transformation is based on modular arithmetic, and one has $n = \varepsilon_0(n) + \cdots + \varepsilon_{k-1}(n)q^{k-1} + q^k T^k(n)$ (see Example 2.7-1). A popular extension consists in changing q at any step: it is the Cantor expansion that we present in Example 5.1. The q-adic expansion can also be obtained in the other way round, *i.e.*, beginning with the most significant digit, using the greedy algorithm. One still has a numeration system in sense of Definition 2.1 but it is not fibred anymore. Nevertheless, this way of producing expansions of nonnegative integers yields a more general concept than the Cantor expansion, the G-scale, which is the most general possible way of representing nonnegative integers based on the greedy algorithm (see Definition 5.2).

According to Definition 2.6, the compactification corresponding to the q-adic expansion (Example 2.7-1) is $\mathbb{Z}_q$, where the ring structure broads addition and multiplication on $\mathbb{N}$, giving a rich answer to Question 2.18. For a G-scale, a compactification can be also built, but it is not possible in general to extend the addition from $\mathbb{N}$ to it in a reasonable way. Nevertheless, the addition by 1 on $\mathbb{N}$ extends naturally and gives a dynamical system called *odometer*, that is constructed in Section 5.1. This dynamical system especially reflects how carries are performed when adding 1. In that direction, the *carries tree* introduced and discussed in Section 5.2 is a combinatorial object which describes the carry propagation. Section 5.3 constructs a bridge between the odometer and some subshift. Its interest is double: it allows to understand the odometer as an FNS (Corollary 5.12) despite the greedy construction and it gives indirectly answers to Question 2.17 with results on invariant measures on the odometer (Theorem 5.15). Section 5.4 briefly discusses the relation between odometers and adic transformations on Markov compacta. Section 5.5 presents some cases where it can be proved that the odometer is conjugate to a rotation on a compact group. It partially answers Question 2.20.

5.1. G-scales. Building the odometer

Fibred numeration systems consist in consecutive iterations of a transformation and give rise to infinite representations given by a sequence of digits. A simple generalisation is obtained by changing the transformation at any step. They are still numeration systems in the sense of Definition 2.1. Cantor (also called mixed radix) expansions are the most popular examples in that direction:

Example 5.1. — Let $G = (G_n)_n$ be an increasing sequence of positive integers such that $G_0 = 1$ and $G_n | G_{n+1}$. Let $q_n = G_n / G_{n-1}$, with $T^{(n)}$ being the transformation of Example 2.7-1 for $q = q_n$ and $\varepsilon^{(n)}$ the corresponding digit function. Take $n \in \mathbb{Z}$. Then $\varphi(n) = (\varepsilon^{(1)}(n), \varepsilon^{(2)}(T^{(1)}(n)), \varepsilon^{(3)}(T^{(2)} \circ T^{(1)}(n)), \ldots)$, which gives rise to an expansion $n = \sum_{j \geqslant 0} \varepsilon^{(j)}(T^{(j-1)} \circ \cdots \circ T^{(1)}(n)) G_j$. In other words, the sequence of digits $(\varepsilon_k(n))_{k\geqslant 1}$ is characterised by the two conditions

$$\sum_{1\leqslant j\leqslant k-1} \varepsilon_j(n) G_j \equiv n \pmod{G_k} \qquad \text{and} \qquad 0 \leqslant \sum_{1\leqslant j<k} \varepsilon_j(n) G_j < n. \tag{5.1}$$

This expansion makes sense in the compatification $\mathbb{Z}_G = \varprojlim \mathbb{Z}/G_m\mathbb{Z}$ for general n, and in $\mathbb{N}$ for nonnegative n, since the corresponding representation is finite (the digits are ultimately 0). Everything is similar to Example 2.7, including the variants discussed in that example. The compactification is known as the set of q-adic integers (see [168], chapter 2).

As (5.1) shows, this kind of expansion is based on the divisibility order relation. A different approach is given by expansions of natural numbers which are essentially based on the usual total ordering. It is called *greedy*, since it first looks for the most significant digit.(9)

Definition 5.2. — *A G-scale is an increasing sequence of positive integers $(G_n)_n$ with $G_0 = 1$.*

Note that in the literature following [159], G-scales are called "systems of numeration". We modified the terminology to avoid any confusion with numeration systems and FNS. Given a G-scale, any nonnegative integer n can be written in the form

$$n = \sum_{k\geqslant 0} \varepsilon_k(n) G_k \tag{5.2}$$

with $\varepsilon_k(n) \in \mathbb{N}$. This representation is unique, provided that the following so-called Yaglom condition

$$\sum_{k=0}^{m} \varepsilon_k(n) G_k < G_{m+1} \quad (\forall\, m \in \mathbb{N}) \tag{5.3}$$

is satisfied, which is always assumed in the sequel. The digits are obtained by the so-called *greedy algorithm*: let $N \in \mathbb{N}$

- find the unique n such that $G_n \leqslant N < G_{n+1}$,
- $\varepsilon_n(N) \leftarrow \lfloor N/G_n \rfloor$,
- $N \leftarrow N - \varepsilon_n(N) G_n$, go to the first step.

Formally, we get the expansion (5.2) by writing $\varepsilon_n(N) = 0$ for all values of n that have not been assigned during the performance of the greedy algorithm. In particular, this expansion is finite in the sense that $\varepsilon_n(N) = 0$ for all but finitely many n.

(9) The word *greedy* stresses that, at any step, the representation algorithm chooses in an appropriate sense the digit that gives the greatest possible contribution. In the present situation, it is not fibred. Conversely, what is called "greedy β-representation" is fibred, since the greedyness is thought to be inside an imposed fibred framework: a fibred system is given (β-transformation on the unit interval). The digit is then chosen in this greedy way.

The infinite word $J_G(n) = \varepsilon_0(n)\varepsilon_1(n)\varepsilon_2(n)\cdots$ is by definition the G-representation of n. In particular, $J_G(0) = 0^\omega$. Some examples and general properties can be found in [147]. The first study from a dynamical point of view is due to Grabner *et al.* ([159]). It is a numeration system according to Definition 2.1. Although it is not fibred, one may consider, as in Definition 2.6, its compactification, *i.e.*, the closure of the language in the product space $\Pi(G)$ below. By property (5.3), $\mathcal{K}_G$ is the set of sequences $e = e_0e_1e_2\cdots$ belonging to the infinite product

$$\Pi(G) = \prod_{m=0}^{\infty} \{0, 1, \ldots, \lceil G_{m+1}/G_m \rceil - 1\},$$

satisfying (5.3). The usual notation in the literature is $\mathcal{K}_G$ and we will use it in the sequel. The set of nonnegative integers $\mathbb{N}$ is embedded in $\mathcal{K}_G$ by the canonical injection $n \mapsto J_G(n)$, with n and $J_G(n)$ being freely identified (except if they could be source of confusion). Their image forms a dense subset of $\mathcal{K}_G$. The natural ordering on the nonnegative integers yields a partial order on $\mathcal{K}_G$ by $x \preccurlyeq y$ if $x_n = y_n$ for $n > n_0$ and $x_{n_0} < y_{n_0}$ or $x = y$. This order is called *antipodal*. In particular, the map $n \mapsto J_G(n)$ is increasing with respect to the usual order on $\mathbb{N}$ and the antipodal order on $\mathcal{K}_G$.

From a topological standpoint, the compact space $\mathcal{K}_G$ is almost always a Cantor set:

Proposition 5.3 ([49], Theorem 2). — *If the sequence $(G_{n+1} - G_n)_n$ is not bounded, then $\mathcal{K}_G$ is homeomorphic to the triadic Cantor space. Otherwise, it is homeomorphic to a countable initial segment of the ordinals.*

The addition by 1 naturally extends to $\mathcal{K}_G$ (see Question 2.18):

$$\forall x = x_0x_1\cdots \in \mathcal{K}_G \;:\; \tau(x) = \lim_{n\to\infty} J(x_0 + x_1G_1 + \cdots + x_nG_n + 1). \tag{5.4}$$

According to (5.3), this limit exists. It is 0^ω if and only if there are infinitely many integers n such that

$$x_0 + x_1G_1 + \cdots + x_nG_n = G_{n+1} - 1. \tag{5.5}$$

Definition 5.4. — *The dynamical system $(\mathcal{K}_G, \tau)$ is called an odometer.*

There is no universal terminology concerning the meaning of odometer. By common sense, the word "odometer" is concerned with counting from a dynamical viewpoint, especially how one goes from n to $n+1$. Most of the authors restrict this term to the Cantor case (for instance [120] or [2], or even to the dyadic case [261]). One usually encounters "adding machine" for the q-adic case. It seems that the term "odometer" regularly occurred from the late 1970's on, as a source of constructions in ergodic theory. Osikawa [263] built flows over an odometer to produce singular flows with given spectrum. Ito [182] built in the same spirit flows preserving the measure of maximal entropy. Constructions over odometers have proved to be generic: Katznelson [206] proved that if f is a C^2-orientation-preserving diffeomorphism of the circle whose rotation number has unbounded continued fraction coefficients, then the system (f, μ), where μ is the Lebesgue measure, is orbit equivalent to an odometer of product type (*i.e.*, a Cantor odometer endowed with a product probability measure). Host *et al.* showed in [175] that every rank one system may be written as a Rokhlin-Kakutani tower over an odometer. See also [144] for such constructions and an overview of rank one systems.

Example 5.5. — Example 2.14, continued. For the Zeckendorf representation, we have $G_{2n} - 1 = (01)^n$ and $G_{2n+1} - 1 = (10)^n 1$ (immediate verification by induction). Therefore, 0^ω possesses two preimages by τ, namely $(01)^\omega$ and $(10)^\omega$. This shows that there is no chance to extend the addition on $\mathbb{N}$ to a group on the compactification (as it is the case for the q-adic or even for the Cantor representation). Indeed, even the monoid law cannot be naturally extended: take $x = (01)^\omega$ and $y = (10)^\omega$. Then $(01)^n + (10)^n = (G_{2n} - 1) + (G_{2n-1} - 1) = G_{2n+1} - 2 = 0(01)^n$ but $(01)^n 0 + (10)^n 1 = (G_{2n} - 1) + (G_{2n+1} - 1) = G_{2n+2} - 2 = 10(01)^n$. We have two cluster points, which are the two elements of $\tau^{-2}\{0^\omega\}$. The same phenomenon occurs for the general Ostrowski representation.

5.2. Carries tree

As outlined in the above paragraph, the structure of the words $G_n - 1$ contains important information concerning the odometer. A tree of carries was introduced in [49], which gives some visibility to this information. The nodes of the tree are $\mathbb{N} \cup \{-1\}$, and -1 is the root of the tree. There is an edge joining -1 and n if $G_{k+1} - 1$ is a prefix of $G_{n+1} - 1$ for no $k < n$. For $0 \leqslant m < n$, there is an edge joining k and n if k is the greatest integer such that $G_{k+1} - 1$ is a prefix of $G_{n+1} - 1$ (for simplicity, the integers are here identified with their representation).

To give the tree is equivalent with giving a *descent function* $D\colon \mathbb{N} \to \mathbb{N} \cup \{-1\}$ verifying $D(n) < n$ for all n and: $\forall n \in \mathbb{N}$, $\exists k \in \mathbb{N} : D^k(n) = -1$. We have $D(n) = m$ if and only if m and n are joined by an edge. Given a tree of this type, there exist infinitely many G-scales having this tree as a carries tree. The smallest one with respect to the lexicographical order is unique and called a *low scale*. It is given by $G_{m+1} = G_m + G_{n+1}$ for $D(m) = n$. Therefore, one may imagine any type of tree one wishes.

Example 5.6. — *cf.* [49]

(1) **Linear tree.** If $(G_n)_n$ is a Cantor scale, then the edges join n and $n+1$ for all n.
(2) **Fibonacci tree.** For the Zeckendorf expansion, we have edges between 1 and 0 and between n and $n+2$ for all n.
(3) **Hedgehog tree**. Let $q \geqslant 2$ and $G_{n+1} = qG_n + 1$. Then the language is the set of words with letters in $\{0, 1, \ldots, q\}$ such that $x_n = q$ implies $x_j = 0$ for $j < n$. We have an edge joining -1 and n for all nodes $n \neq -1$.
(4) **Comb tree.** It is given by $D(0) = -1$, and $D(2n+2) = D(2n+1) = 2n$ for any $n \geqslant 0$. The corresponding low scale is $G_{2n+j} = 2^j 3^n$ for $n \geqslant 0$ and $j = 1, 2$.

The image below shows respectively the linear tree, the Fibonacci tree and the comb tree.

$$-1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

Figure 5.1. The linear tree

$$\begin{tikzcd}
 & 2 \arrow[r] & 4 \arrow[r] & 6 \arrow[r] & \cdots \\
-1 \arrow[ur] \arrow[dr] & & & & \\
 & 1 \arrow[r] & 3 \arrow[r] & 5 \arrow[r] & \cdots
\end{tikzcd}$$

Figure 5.2. The Fibonacci tree

The preimages of 0^ω correspond to the infinite branches of the carries tree: to an infinite branch $(n_0, n_1, n_2, \ldots)$, where $n_0 = -1$ and $D(n_{k+1}) = n_k$ corresponds $x = \lim(G_{n_k+1} - 1)$. In particular, for the scale with a hedgehog tree, the preimage of 0^ω is empty. The following proposition indicates a further property of the odometer which can be read on the carries tree.

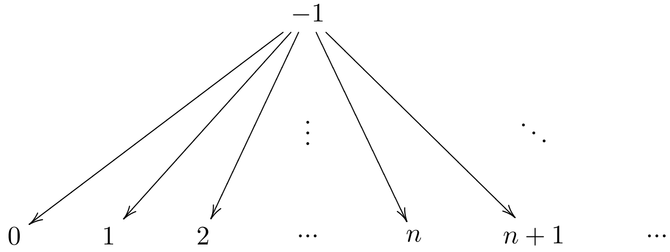

Figure 5.3. The hedgehog tree

$$\begin{tikzcd}
-1 \arrow[r] & 0 \arrow[r] \arrow[d] & 2 \arrow[r] \arrow[d] & 4 \arrow[r] \arrow[d] & \cdots \\
 & 1 & 3 & 5 &
\end{tikzcd}$$

Figure 5.4. The comb tree

DEFINITION 5.7. — *A tree is of finite type if all nodes have finitely many neighbours.*

PROPOSITION 5.8. — *A carries tree is of finite type if and only if*

$$\forall n \in \mathbb{N} \cup \{-1\},\ \{m > n\,;\, n \text{ and } m \text{ are joined by an edge}\} \text{ is finite.}$$

Then the set of discontinuity points of τ is $\omega(G) \setminus \tau^1(0^\omega)$, where the omega limit set $\omega(G)$ is the set of limit points in $\mathcal{K}_G$ of the sequence $(G_n - 1)_n$. Furthermore, the carries tree is of finite type if and only if τ is continuous.

Proof. — By construction, $\omega(G)$ is not empty and compact. For $x \in \mathcal{K}_G$, let $m(x) = \max\{k\,;\, G_{k+1} - 1 \text{ is a prefix of } x\} \in \{-1, 0, \ldots, +\infty\}$. Clearly, τ is continuous at any point $x \in \tau^{-1}\{0^\omega\}$. If $x \notin \omega(G)$, then there is a cylinder C containing x and not intersecting $\omega(G)$, hence $m(x)$ is bounded on C. Therefore, τ is continous at x. If $x \in \omega(G)$, then $x = \lim(G_{n_k} - 1)$. If $\tau(x) \neq 0^\omega$, then τ is not continous at x, since $\tau(G_{n_k} - 1) = G_{n_k}$, which tends to zero in $\mathcal{K}_G$.

Let $x = \lim(G_{n_{k+1}} - 1)$. Then the characterisation follows from the equivalence of the following statements:

- $G_{m(x)+1} - 1$ is a prefix of $G_{n_{k+1}} - 1$ for k large enough and $\lim D(n_k) = \infty$;
- $m(x)$ is finite;
- x is a discontinuity point of τ. □

For a low scale, the discontinuity points are exactly the $G_{n+1} - 1$ such that the node n is not of finite type.

Example 5.9. — G-scales arising from β-numeration. Let $\beta > 1$ and $d^*_\beta(1) = (a_n)_{n\geqslant 0}$ (see Example 2.8). Then define $G_0 = 1$ and $G_{n+1} = a_0G_n + a_1G_{n-1} + \cdots + a_nG_0 + 1$. Sequence $(G_n)_n$ is a scale of numeration whose compactification coincides (up to a mirror symmetry) with the compactification X_β (*cf.* [69]): the lexicographical condition defining the language is

$$(\varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_0) <_{\text{lex}} (a_0, a_1, \ldots) \ \ (\forall n \in \mathbb{N}).$$

Furthermore, set

$$\mathbb{Z}^+_\beta = \{w_m\beta^m + \cdots + w_0\,;\, m \in \mathbb{N}, w_m \cdots w_0 \in \mathcal{L}_\beta\}.$$

If $\mathcal{S}$ is the successor function $\mathcal{S}\colon \mathbb{Z}^+_\beta \to \mathbb{Z}^+_\beta$ given by $\mathcal{S}(x) = \min\{y; y > x\}$, and if $\varphi(\sum \varepsilon_n G_n) = \sum \varepsilon_n \beta^n$, then the diagram

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\tau} & \mathbb{N} \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ \mathbb{Z}^+_\beta & \xrightarrow[\mathcal{S}]{} & \mathbb{Z}^+_\beta \end{array}$$

is commutative and φ is bijective. It has been proved in [159] that the odometer is continuous if and only if the sequence $(a_n)_n$ is purely periodic, that is, if β is a simple Parry number. If the sequence is ultimately periodic with period $b_1 \cdots b_s$ (β is a Parry number), then

$$\omega(G) = \{(b_k b_{k+1} \cdots b_{k+s-1})^\omega; 1 \leqslant k \leqslant s-1\}.$$

The preimage of 0^ω is either empty if β is not a simple Parry number, or equal to $\omega(G)$ otherwise. If we have $a_0 = 2$ and if $a_1a_2\cdots$ is the Champernowne number in base two [95], then $\tau^{-1}(0^\omega) = \emptyset$ and $\omega(G) = \{0,1\}^{\mathbb{N}}$.

5.3. Metric properties. *Da capo al fine* subshifts

If $(G_n)_n$ is a Cantor scale, the odometer is a translation on a compact group for which all orbits are dense. In particular, $(\mathcal{K}_G, \tau)$ is uniquely ergodic and minimal. In general, a natural question is whether there exists at least one τ-invariant measure on $\mathcal{K}_G$ and whether it is unique. Since $\mathcal{K}_G$ is compact, the Krylov-Bogoliubov Theorem (see for instance [340]) asserts that there exists an invariant measure, provided that τ is continuous. But the question remains open without this assumption.

Results on invariant measures concerning special families can be found in [334] for Ostrowski expansions (as in Example 2.14) and in [159] for linear recurrent numeration systems arising from a simple β-number (as in

Example 5.9). For Ostrowski scales $(G_n)_n$, it is proved that the odometer is metrically isomorphic to a rotation, hence in particular uniquely ergodic (see for instance [128], [334] or [52]). Furthermore, Vershik and Sidorov give in [334] the distribution of the coordinates and show that they form a non-homogenous Markov chain with explicit transitions.

For linear recurrent numeration systems, Grabner *et al.* [159] use the following characterisation of unique ergodicity: the means

$$N \mapsto N^{-1} \sum_{m \leqslant n < m+N} f \circ \tau^n$$

converge uniformly w.r.t. m when N tends to infinity for any continuous function $f\colon \mathcal{K}_G \to \mathbb{C}$. A standard application of the Stone-Weierstraß theorem allows us to consider only G-multiplicative functions f depending on finitely many coordinates (see Section 6.1 for the definition). A technical lemma reduces the problem again to the study of convergence of the sequence $n \mapsto G_n^{-1} \sum_{k<G_n} f(k)$. However, the sequence $n \mapsto \sum_{k<G_n} f(k)$ ultimately satisfies the same recurrence relation as $(G_n)_n$, from which the desired convergence is derived. The unique invariant probability measure is explicitly given.

Example 5.10. — We consider the Zeckendorf expansion again (continuation of Example 5.5). The golden ratio is denoted by $\varrho = (1+\sqrt{5})/2$, the unique τ invariant measure on $\mathcal{K}_G$ is $\mathbf{P}$. By unique ergodicity of the odometer, if X_n is the n-th projection on the compactification, *i.e.*, $X_n\colon \mathcal{K}_G \to \{0,1\}$, $X_n(x_0, x_1, \ldots) = x_n$, then

$$\begin{aligned}\mathbf{P}(X_n = 0) &= \lim_{s\to\infty} \frac{1}{F_{n+s}} \#\{k < F_{n+s}\,;\, \varepsilon_n(k) = 0\}\\ &= \lim_{s\to\infty} \frac{F_n F_{s-1}}{F_{n+s}} = \frac{F_n}{\varrho^{n+1}}.\end{aligned}$$

Similarly (or by computing $1 - \mathbf{P}(X_n = 0)$), one finds $\mathbf{P}(X_n = 0) = F_{n-1}\varrho^{-n-2}$ and the transition matrix is $\left(\begin{smallmatrix} 1/\varrho & 1 \\ 1/\varrho^2 & 0 \end{smallmatrix}\right)$. In particular, the sequence $(X_n)_n$ is a homogenous Markov chain. For the most general case of scales arising from a simple Parry β-number of degree d, one gets a homogenous Markov chain of order $d-1$. We refer to [123] and [320] for more information, especially applications to asymptotic studies of related arithmetical functions.

The arguments of [159] can be extended, but with some technical difficulties, to more general odometers. However, a quite different approach turns out to be more powerful. We expose it below.

From now on, $(G_n)_n$ is a G-scale and $(\mathcal{K}_G, \tau)$ the associated odometer. For $x \in \mathcal{K}_G$, we define a "valuation" $\nu(x) = \nu_G(x) = \min\{k; x_k \neq 0\}$, if $x \neq 0^\omega$, and $\nu(0^\omega) = \omega$. Note that ω stands here for the first infinite ordinal and not for the ω-limit set. Denote by $\Lambda = \mathbb{N} \cup \{\omega\}$ the one-point compactification of $\mathbb{N}$. The valuation yields a map $A_G \colon \mathcal{K}_G \to \Lambda^{\mathbb{N}}$ defined by $A_G(x) = (\nu(\tau^n x))_{n \geqslant 0}$ (*e.g.*, $\nu(G_m) = m$). Let $A_m = \nu(1)\nu(2) \cdots \nu(G_m - 1)$ and A be the infinite word defined by the concatenation of the sequence $A_G(1)$. Then, for $n = \sum_{k \leqslant \ell} \varepsilon_k(n) G_n$, the prefix of length n of A is

$$(A_\ell \ell)^{\varepsilon_\ell(n)} (A_{\ell-1}(\ell-1))^{\varepsilon_{\ell-1}(n)} \cdots (A_0 0)^{\varepsilon_0(n)}. \tag{5.6}$$

Let (X_G, S) be the subshift associated with A and $X_G^{(0)} = X_G \cap \mathbb{N}^{\mathbb{N}}$.

Proposition 5.11 ([50], Proposition 2). — *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{K}_G & \xrightarrow{\tau} & \mathcal{K}_G \\ {\scriptstyle A_G}\big\downarrow & & \big\downarrow{\scriptstyle A_G} \\ X_G & \xrightarrow[S]{} & X_G \end{array} \tag{5.7}$$

The map A_G is Borelian. It is continuous if and only if τ is continuous. It induces a bijection between $\mathcal{K}_G^\infty = \mathcal{K}_G \setminus \mathcal{O}_{\mathbb{Z}}(0^\omega)$ and $X_G^{(0)}$, whose inverse map is continuous. If τ is continous, this bijection is a homeomorphism ($\mathcal{O}_{\mathbb{Z}}(0^\omega)$ is the two-sided orbit of 0^ω).

The precise study of A_G is not simple. Some elements can be found in [50]. For example, the equality $A_G(\mathcal{K}_G) = X_G$ holds if and only if $\mathbb{N} \cap \omega(G) = \emptyset$.

Proposition 5.11 has important consequences.

Corollary 5.12. — *The quadruples*

$$(\mathbb{N} \setminus \{0\}, \tau, \mathbb{N}, A_G) \quad \textit{and} \quad (\mathcal{K}_G^\infty, \tau, \mathbb{N}, A_G)$$

are fibred numeration systems. In the sense of Definition 2.6, they have the same compactification X_G on which the shift operator acts. The dynamical systems $(\mathcal{K}_G, \tau)$ and (X_G, S) are metrically conjugated. In particular, A_G transports shift-invariant measures supported by $X_G^{(0)}$ to τ-invariant measures on $\mathcal{K}_G$.

The subshift (X_G, S) is called the *valumeter*. It is conjugated to the odometer. As noted before, G-scales cannot be immediatly associated with a fibred numeration system. But the odometer is conjugated to such a numeration system. Therefore, Proposition 5.11 is a way to understand G-scales and the corresponding numeration as fibred. Moreover, the dynamical study of the odometer reduces to that of the valumeter, which is a more pleasing object. For instance, the shift operator is always continuous, even if the addition is not.

PROPOSITION 5.13. — *The following statements are equivalent:*
1. the word A is recurrent (each factor occurs infinitely often);
2. the word A has a preimage in X_G by the shift;
3. the set $X_G^{(0)}$ is not countable;
4. the set $\mathcal{K}_G$ is a Cantor space.
Furthermore, the valumeter is minimal if and only if the letter ω never appears infinitely many times with bounded gaps in elements of X_G.

Example 5.14. — If G is the scale of Example 5.6-(3) (see also Example 5.20, *infra*), X_G contains the element $(\omega, \omega, \ldots)$. Then the valumeter is not minimal.

The main difficulty in proving theorems on invariant measures on (X_G, S) comes from the fact that the alphabet $\Lambda = \mathbb{N} \cup \{\omega\}$ is not discrete, but it has one non-isolated point - ω. The usual techniques lie on the $*$-weak compactness of the set of probability measures. Indeed, $*$-weak convergence of a sequence $(\mu_n)_n$ of probability measures defined on X_G expresses the convergence of the sequence $(\mu_n(U))_n$, for cylinders of the type $U = [a_1, \ldots, a_n]$, with $a_n \in \mathbb{N}$ or $a_n = \{m \in \Lambda\,;\, m \geqslant n_0\}$. But the relevant notion of convergence in this context takes cylinders $U = [a_1, \ldots, a_n]$ into account, with $a_n \in \Lambda$. Hence one has to introduce a so-called *soft topology*, which is finer as the usual $*$-weak topology. However, the following results can be proved.

THEOREM 5.15 ([50], Theorems 7 and 8). — *1. If the series $\sum G_n^{-1}$ converges, then there exists a shift-invariant probability measure on X_G supported by $X_G^{(0)}$.*
2. If the sequence $(G_{n+1} - G_n)_n$ tends to infinity and if the sequence $m \mapsto G_m \sum_{k \geqslant m} G_k^{-1}$ is bounded, then $(X_G^{(0)}, S)$ is uniquely ergodic, and $(\mathcal{K}_G, \tau)$ as well.
3. The odometer $(\mathcal{K}_G, \tau)$ has zero measure-theoretic entropy with respect to any invariant measure. If τ is continous, it has zero topological entropy.

For instance, G-scales satisfying $1 < a < G_{n+1}/G_n < b < \infty$ for all n satisfy the second condition of Theorem 5.15. Example 9 of [50] shows that a continous odometer can have several invariant measures. The construction below is not continuous, but more simple.

Example 5.16. — Suppose that I, J and K realise a partition of $\mathbb{N}$, and assume that I and K have an upper-density of one, that is, $\limsup N^{-1}\#([0, N) \cap I) = \limsup N^{-1}\#([0, N) \cap K) = 1$. Define $G_{n+1} = G_n + 1$ if

$n \in I$, $G_{n+1} = G_n + 2$ if $n \in K$, and $G_{n+1} = a_n G_n + 1$ for $n \in J$, where the a_n are chosen to make the series $\sum G_n^{-1}$ convergent. Then $\omega(G) = \{0, 1\}$, and τ is discontinous at these two points. Furthermore, the sequence $N \mapsto N^{-1} \sum_{n<N} \delta_0 \circ \tau^n$ has at least two accumulation points. Hence, there exist at least two τ-invariant measures.

Remark 5.17. — Downarowicz calls (X_G, S) *da capo al fine* subshift.[10] Consider a triangular array of nonnegative integers $(\varepsilon_j^{(m)})_{0 \leqslant j \leqslant m}$ such that $\varepsilon_j^{(j)} \geqslant 1$ for all j and

$$\varepsilon_j^{(m)} \varepsilon_{j-1}^{(m)} \cdots \varepsilon_0^{(m)} \leqslant_{\mathrm{lex}} \varepsilon_j^{(j)} \varepsilon_{j-1}^{(j)} \cdots \varepsilon_0^{(j)}$$

for all $j \leqslant m$. Define recursively A_0 to be the empty word and

$$A_{m+1} = (A_m m)^{\varepsilon_m^{(m)}} \cdots (A_0 0)^{\varepsilon_0^{(m)}}.$$

The sequence of words $(A_m)_m$ converges to an infinite word A. This sequence is associated with the G-scale $(G_n)_n$ constructed recursively by $G_0 = 1$ and $G_{m+1} - 1 = \sum_{j \leqslant m} \varepsilon_j^{(m)} G_j$ (*i.e.*, $\varepsilon_j^{(m)} = \varepsilon_j(G_{m+1} - 1)$ for short). The last two equations express (5.3) and (5.6), respectively. At each step, the song is played *da capo*, where the mark *fine* is set at position $G_{m+1} - G_m - 1$. If this number is larger than G_m, the above formula instructs us to periodically repeat the entire song until position $G_{m+1} - 1$ is reached (usually the last repetition is incomplete). In all cases, the note $m + 1$ is added at the end.

5.4. Markov compacta

Let $(r_n)_n$ be a sequence of nonnegative integers, $r_n \geqslant 2$ for all n, and a sequence of $0 - 1$ matrices $(M^{(n)})_n$, where $M^{(n)}$ is a $r_n \times r_{n+1}$ matrix. Build the Markov compactum

$$\mathcal{K}(M) = \left\{ (x_0, x_1, \dots) \in \Pi(G) \, ; \, \forall n \in \mathbb{N} : M^{(n)}_{x_n, x_{n+1}} = 1 \right\}. \tag{5.8}$$

$\mathcal{K}(M)$ is an analog of a non-stationary topological Markov chain. According to Vershik, the *adic transformation* $\mathcal{S}$ associates with $x \in \mathcal{K}(M)$ its successor with respect to the antipodal order (the definition is recalled in Section 5.1). Then Vershik has proved in [335] the following theorem (see also [134] for related results).

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[10] "The expression *da capo al fine* is taken from musical terminology. Having played the entire song the musicians must play it again from the start (*da capo*) to a certain spot marked in the score as *fine*. [...] Following musical convention, the elements of Λ will be called *notes*". [119]

THEOREM 5.18. — *Any ergodic automorphism of a Lebesgue space is metrically isomorphic to some adic transformation.*

Usually, the isomorphism is not explicit. If $(G_n)_n$ is a Cantor scale (with the notation of Example 5.1), then the odometer $(\mathcal{K}_G, \tau)$ is an adic transformation, where $M^{(n)}$ is the $q_n \times q_{n+1}$ matrix containing only 1's (lines and columns being indexed from 0 included). Odometers that are adic transformations are not difficult to characterise, they correspond to scales $(G_n)_n$ where expansions of $G_n - 1$ satisfy

$$G_{n+2}-1 = \begin{cases} \varepsilon_{n+1}(G_{n+2}-1)G_{n+1} + \varepsilon_n(G_{n+2}-1)G_n + (G_n-1) \\ \qquad \text{if } \varepsilon_n(G_{n+2}-1) < \varepsilon_n(G_{n+1}-1); \\ \varepsilon_{n+1}(G_{n+2}-1)G_{n+1} + (G_{n+1}-1) \\ \qquad \text{if } \varepsilon_n(G_{n+2}-1) = \varepsilon_n(G_{n+1}-1), \end{cases}$$

with the initial condition $G_1 = \varepsilon_0(G_1 - 1) + 1$. The transition matrices $M^{(n)}$ have $\varepsilon_n(G_{n+1} - 1) + 1$ rows, $\varepsilon_{n+1}(G_{n+2} - 1) + 1$ columns, and have zero coefficients $m_{i,j}$ if and only if $j = \varepsilon_{n+1}(G_{n+2}-1)$ and $i > \varepsilon_n(G_{n+2}-1)$.

Assume now that the odometer does not coincide with an adic transformation. In some simple cases, there is a simple isomorphism with such a dynamical system. It is in particular the case for the so-called Multinacci scale, which generalises the Fibonacci one: $G_k = k + 1$ for $k \leqslant m$ and $G_{n+m} = G_{n+m-1} + G_{n+m-2} + \cdots + G_{n+1} + G_n$ for all n (compare with Example 4.2).

Example 5.19. — Consider the scale of Example 5.6-(3). The Markov compactum $\mathcal{K}(M)$ built from the square $(q+1)$-dimensional matrices with $m_{i,j} = 0$ if $j = q$ and $i \neq q$ is formed by sequences $q^k \varepsilon_k \varepsilon_{k+1} \cdots$ with $0 \leqslant \varepsilon_j < q$. The map $\psi\colon \mathcal{K}(M) \to \mathcal{K}_G$ defined by

$$\psi(0^\omega) = 0^\omega \quad \text{and} \quad \psi(q^k \varepsilon_k \varepsilon_{k+1} \cdots) = 0^{k-1} q \varepsilon_k \varepsilon_{k+1} \cdots$$

realises an isomorphism between the odometer and the adic system.

It should be noticed that Markov compacta are often described in terms of paths in an infinite graph under the name of Bratelli diagrams. We refer to [164], [134] and [117].

5.5. Spectral properties

In general, the spectral structure of an odometer associated with a given scale $G = (G_n)_{n \geqslant 0}$, is far from being elucidated. We present in this section both old and recent results. As above, let $(\mathcal{K}_G,\tau)$ be the G-odometer. We assume that the sequence $n\mapsto G_{n+1}-G_n$ is unbounded, so K_G is a Cantor set according to Proposition 5.3. The first step consists in identifying odometers that admit an invariant measure μ. This is done in Theorem 5.15.

The next question is to characterise G-odometers which have a non-trivial eigenvalue. We keep in mind a result due to Halmos which asserts that the family of ergodic dynamical systems with discrete spectrum coincides with the family of dynamical systems which are, up to an isomorphism, translations on compact abelian groups with a dense orbit (and hence are ergodic) [340]. We only look at examples.

In base q (Example 2.7), the odometer corresponds to the scale $G_n = q^n$ and is nothing but the translation $x \mapsto x+1$ on the compact group of q-adic integers. The situation is analogous for Cantor scales (Example 5.1) for which q-adic integers are replaced by very similar groups exposed *supra.*

Example 5.20. — The scale given by $G_{n+1} = qG_n + 1$ (for a positive integer q, $q \geqslant 2$) is the first example of a weak mixing odometer family. Furthermore, these odometers are measure-theoretically isomorphic to a rank one transformation of the unit interval, with the transformation being constructed by a cutting-stacking method [118].

In case $(G_n)_n$ is given by a finite homogeneous linear recurrence coming from a simple Parry number, as in Example 5.9, the odometer is continuous, uniquely ergodic, but little is known about its spectral properties. The following sufficient condition is given in [159]. Let us say that the odometer satisfies hypothesis (B) if there exists an integer $b > 0$ such that for all k and integer N with G-expansion

$$\varepsilon_0(N)\cdots\varepsilon_k(N)0^{b+1}\varepsilon_{k+b+2}(N)\cdots,$$

where the addition by G_m to N (with $m \geqslant k+b+2$) does not change the digits $\varepsilon_0(N),\dots,\varepsilon_k(N)$. Then the odometer is measure-theoretically isomorphic to a group rotation whose pure discrete spectrum is the group

$$\{z\in\mathbb{C}\,;\ \lim_n z^{G_n} = 1\}.$$

This result applies especially to the Multinacci scale. For the Fibonacci numeration system, the measure theoretic conjugation map between the odometer and the translation on the one-dimensional torus $\mathbb{T}$ with angle the golden number ϱ is exhibited in [334, 52]; see also Example 2.14.

Example 5.21. — The study of wild attractors involves some nice odometers. In particular, the following scales $G_{n+1} = G_n + G_{n-d}$ are investigated in [83], where it is proved that for $d \geqslant 4$, the odometer is weakly mixing, but not mixing (Theorem 3). Using the results of Host [173] and Mauduit [248], the authors prove that the non-trivial eigenvalues $e^{2\pi i\rho}$ (if there were any) are such that ρ is irrational or would belong to $\mathbb{Q}(\lambda)$, for any root λ of the characteristic polynomial P of the recurrence with a modulus of at least 1. They first treat the case $d \equiv 4 \pmod 6$, for which $(x^2 - x + 1) \mid P$, hence a contradiction. Case $d \not\equiv 4 \pmod 6$ is more complicated, since P is then irreducible. But the Galois group of P is the whole symmetric group $\mathfrak{S}_{d+1}$ in this case, and they can conclude with an argument of [315].

6. Applications

6.1. Additive and multiplicative functions, sum-of-digits functions

By analogy with the classical additive and multiplicative functions studied in number theory, whose structure is based on the prime number decomposition, one defines arithmetical functions constructed from their expansion with respect to a numeration system. We only deal in the sequel with functions defined on $\mathbb{N}$.

Definition 6.1. — *For a G-scale $G = (G_n)_n$ and the corresponding digit maps ε_n, a function $f : \mathbb{N} \to \mathbb{C}$ is called G-*additive *if $f(0) = 0$, and if*

$$f\left(\sum_{k=0}^{\infty} \varepsilon_k(n) G_k\right) = \sum_{k=0}^{\infty} f\left(\varepsilon_k(n) G_k\right). \tag{6.1}$$

G-multiplicative functions g *are defined in a similar way; they satisfy $g(0) = 1$ and*

$$f\left(\sum_{k=0}^{\infty} \varepsilon_k(n) G_k\right) = \prod_{k=0}^{\infty} f\left(\varepsilon_k(n) G_k\right). \tag{6.2}$$

The most popular G-additive function is the sum-of-digits function defined by $s_G(n) = \sum_k \varepsilon_k(n)$. Of course, if f is a G-additive function then, for any real number α, the function $g = \exp(i\alpha f(\cdot))$ is G-multiplicative. For the scale $G_n = q^n$, we speak about *q-additive* and *q-multiplicative* functions. A less immediate example of a q-multiplicative function is given by the Walsh functions: for $x \in \mathbb{Z}_q$, $w_x(n) = \prod_k e(q^{-1} x_k \varepsilon_k(n))$, where $e(x) = \exp(2\pi i x)$. In fact, these functions w_x are characters of the additive group $\mathbb{N}_q$ where the addition is done in base q while ignoring carries. Multiplicative functions have a great interest in harmonic analysis and ergodic theory. First, multiplicative functions of modulus 1 belong to the Wiener vector space [342] of bounded sequences $g : \mathbb{N} \to \mathbb{C}$ having a correlation function $\gamma_g : \mathbb{Z} \to \mathbb{C}$. Recall that,

$$\gamma_g(m) := \lim_N \frac{1}{N} \sum_{n=0}^{N-1} g(m+n)\overline{g(n)}$$

for $m \geqslant 0$ and $\gamma_g(m) = \overline{\gamma_g(-m)}$ if $m < 0$. The correlation γ_g is positive definite so that, by the classical Bochner-Herglotz Theorem, there exists a Borel measure σ_g on the torus $\mathbb{T}$, called *spectral measure of* g, such that $\gamma_g(m)$ is the Fourier transform $\sigma_g(m) = \int_{\mathbb{T}} e(mt)\sigma_g(dt)$ of σ_g. One of the interests of this definition comes from Bertrandias' inequality

$$\limsup_N |N^{-1} \sum_{0 \leqslant n < N} g(n)e(-na)| \leqslant \sqrt{\sigma_g(\{a\})}$$

and from the formula $\lim_N N^{-1} \sum_{0 \leqslant n < N} |\gamma_g(n)|^2 = \sigma_g \otimes \sigma_g(\Delta)$, where Δ is the diagonal of $\mathbb{T}^2$ (see [274] for a general reference on the subject). Secondly, many interesting multiplicative functions are *pseudo-random* (*i.e.*, the spectral measure is continuous). To illustrate this notion, we quote the seminal paper of Mendès France [250] where the following result is proved:

Proposition 6.2. — *If $x \in \mathbb{Z}_q \setminus \mathbb{N}$, the Walsh character w_x is pseudo-random, but it is not pseudo-random in the sense of Bass* [56] *(that is, its correlation function does not converge to 0 at infinity).*

The spectral properties of q-multiplicative functions were extensively studied during the 70's, mainly in connection with the study of uniform distribution modulo 1 (see [251, 100, 101, 104, 105, 102], [273] in the more general setting of Cantor scales, and [106] for Ostrowski numeration). The dynamical approach involving skew products is first developed in [238], where additional references can be found; it is exploited in [237] for the study of regularity of distributions.

The summation of the sum-of-digits function has been extensively studied. It is not our purpose to draw up an exhaustive list of known results on it. We just mention a few results, and then give some examples where the dynamics plays a rôle.

In 1940, Bush [91] proved the asymptotics

$$\sum_{n<N} s_q(n) \sim \frac{q-1}{2} N \log_q N,$$

see also Bellman and Shapiro [59]. Trollope gave in [330] an explicit expression of the error term for the binary expansion. Later on, Delange [114] expressed the error term for arbitrary q as $NF(\log_q N)$, where F is a 1-periodic function, continuous and nowhere differentiable, and described its Fourier coefficients in terms of the Riemann zeta function. The power sum $\tau_d(N) = \sum_{0\leqslant n<N}(s_2(n))^d$ is studied by Stolarsky [322]. He proved that $\tau_d(N) \sim \frac{q-1}{N} 2^{-d}(\log_q N)^d$. Coquet [103] obtained more details on the error term in the vein of Delange.

Many results have been proved about the normal distribution of additive functions along subsequences. We only quote a few papers, proposing different directions [57, 121, 122, 320]. Very recently, Mauduit and Rivat [249] solved a long standing conjecture of Gelfond by proving that the sum-of-digits function s_q is uniformly distributed along the primes in the residue classes mod m, with $(m, q) = 1$.

In [113], Delange proved that a real-valued q-additive function f admits an asymptotic distribution function: the sequence of measures $N \mapsto N^{-1}\sum_{n<N}\delta_{f(n)}$ converges weakly to a probability measure if and only if both series

$$\sum_{j=0}^{\infty}\left(\sum_{\varepsilon=0}^{q-1} f(\varepsilon q^j)\right) \quad \text{and} \quad \sum_{j=0}^{\infty}\sum_{\varepsilon=0}^{q-1} f(\varepsilon q^j)^2 \tag{6.3}$$

converge. Delange used the characterisation of weak convergence due to Lévy: the sequence of characteristic functions converges pointwise to a function, which has to be continuous at 0. Therefore, the most important part of the proof deals with estimates of means of q-multiplicative functions g. Let $M_N(g) = N^{-1}\sum_{n<N} g(n)$. Then

$$M_{q^\ell}(g) = \prod_{j<\ell}\sum_{\varepsilon<q} f(\varepsilon q^k).$$

A typical result proved by Delange in this context is $M_N(g) - M_{q^n}(g) = o(1)$ for $q^n \leqslant N < q^{n+1}$. This has been generalised to further numeration systems by Coquet and others. For Cantor numeration systems, this result is generally not true. For example, if $G_n = n^2 G_{n-1}$ and $g(\varepsilon G_n) = -1$ whenever $\varepsilon = 1$ and 1 otherwise, one checks that $(M_{G_n})_n$ converges to some positive constant, although $(M_{2G_n})_n$ tends to zero. Using a martingale argument, Mauclaire proved the following:

THEOREM 6.3 ([247]). — *Assume $(G_n)_n$ is a Cantor numeration system with compactification $\mathcal{K}_G$. For $x = \sum_{k=0}^{\infty} x_k G_k \in \mathbb{Z}_G$, let $x_n = \sum_{k=0}^{n} x_k G_k$. Then*

$$\frac{1}{x_n}\sum_{k=0}^{x_n-1} g(k) - \frac{1}{G_n}\sum_{k=0}^{G_n-1} g(k) = o(1)$$

holds for almost all $x \in \mathbb{Z}_G$ with respect to Haar measure, when n tends to infinity.

Barat and Grabner [51] observed that if f is a real-valued q-additive function and $f_n \colon \mathbb{Z}_q \to \mathbb{R}$ defined by $f_n(\sum x_k q^k) = f(x_n q^n)$, then the conditions (6.3) can be rewritten as the convergence of $\sum \mathbb{E}(f_n)$ and $\sum \mathbb{E}(f_n^2)$, which is indeed equivalent to the convergence of $\sum \mathbb{E}(f_n)$ and $\sum \sigma^2(f_n^2)$, by $f_n(0) = 0$ and Cauchy-Schwartz inequality. Since the random variables f_n are independent and bounded, further conditions equivalent to (6.3) are almost sure convergence of the series $\sum f_n$ (Kolmogorov's three series theorem) and convergence in distribution of the same series. Finally, convergence in the distribution of $\sum f_n$ is by definition the weak convergence of the sequence $N \mapsto q^{-N} \sum_{n<q^N} \delta_{f(n)}$ to a probability measure, where δ_a denotes the Dirac measure at point a.

After this analysis of the problem, one of the implications in Delange's theorem is trivial. The converse assumes the almost sure convergence of $f = \sum f_n$, and is based on the pointwise ergodic theorem. The whole procedure applies to more general numeration systems, even though the lack of independence for the functions f_n makes the work more involved. In the same direction, Manstavičius developed in [245] a Kubilius model for G-additive functions w.r.t. Cantor numeration systems.

Given a numeration scale $(G_n)_n$ and a unimodular G-multiplicative function g, several authors have investigated the subshift associated with the sequence $(g(n))_n$, that we denote by $\mathcal{F}(g)$. Interesting results concerned with q-adic numeration systems and functions of the type $g(n) = e(\alpha s_q(n))$ from a spectral viewpoint are due to Kamae [195], see also Queffélec [273].

Barat and Liardet consider in [52] the case of Ostrowski numeration (with any Ostrowski scale G) and arbitrary G-multiplicative functions g with values in the unit circle $\mathbb{U} \subset \mathbb{C}$. The odometer plays a key rôle in the whole study. It is first proved that if $\Delta g(n) = g(n+1)g(n)^{-1}$ is the first backward difference sequence, then the subshift $\mathcal{F}(\Delta g)$ is constant or almost topological isomorphic to the odometer $(\mathcal{K}_\alpha, \tau)$. A useful property of the sequence Δg is that it continuously extends (up to a countable set) to the whole compact space $\mathcal{K}_\alpha$. Furthermore, the subset $G_1(g)$ of $\mathbb{U}$ of topological essential values is defined as the decreasing intersection of the sets $\overline{g([0^n])}$. It turns out that $G_1(g)$ is a group that contains the group of essential values $E(\Delta g)$ of K. Schmidt [294], and that $E(\Delta g) = G_1(g)$ if and only if $\mathcal{F}(g)$ is uniquely ergodic. The topological essential values are also characterised in terms of compactification: for a character $\chi \in \hat{\mathbb{U}}$, the restriction of χ to $G_1(g)$ is trivial if and only if $\chi \circ g$ extends continuously to $\mathcal{K}_\alpha$. In general, $\mathcal{F}(g)$ is topologically isomorphic to a skew product $\mathcal{F}(\Delta g)\square G_1(\zeta)$. In case of unique ergodicity, a consequence is the well uniform distribution of the sequence $(g(n))_n$ in $\mathbb{U}$. More precise results were obtained by Lesigne and Mauduit in the case of q-adic numeration systems (see [236]): let g be q-multiplicative and $g(kq^n) = \exp(2i\pi\theta_n^k)$ $(k \leqslant q-1)$. Denote by $\|\cdot\|$ the distance to the nearest integer. It is proved in [236] that the three following statements are equivalent:

• For all q-adic rational number α and all rational integer d, the sequence $N \mapsto N^{-1}\sum_{n=0}^{N-1}(g(n+m))^d \exp(2i\pi(n+m)\alpha))$ converges uniformly w.r.t. m.

• For all integer d, if $\|\theta_n^k\|$ tends to 0 when n tends to infinity, for all k between 1 and $q-1$, then either the series of general term $\sum_{k\leqslant q-1}\|d\theta_n^k\|$ converges or the series of general term $\sum_{k\leqslant q-1}\|d\theta_n^k\|^2$ diverges.

• $\mathcal{F}(g)$ is strictly ergodic.

Let us end this section by going back to substitutions and automata. The Dumont-Thomas numeration has been introduced in [125, 126] in order to get asymptotical estimations of summatory functions of the form $\sum_{1\leqslant n\leqslant N} f(u_n)$, where $(u_n)_n$ is a one-sided fixed point of a substitution over the finite alphabet $\mathcal{A}$, and f is a map defined on $\mathcal{A}$ with values in $\mathbb{R}$. These estimates are deduced from the self-similarity properties of the substitution via the Dumont-Thomas numeration, and are shown to behave like sum-of-digits functions with weights provided by the derivative of f. The sequences $(f(u_n))_n$ are currently called *substitutional sequences* (see also Section 6.4 below).

For some particular substitution cases, such as constant length substitutions, one recovers classical summatory functions associated, *e.g.*, to the number of 1's in the binary expansion of n (consider the Thue-Morse substitution), or the number of 11's (consider the Rudin-Shapiro substitution). A natural generalisation of these sequences has the form $\varepsilon_n = (-1)^{u_n}$, where u_n counts the number of occurrences of a given digital pattern in the q-ary expansion of n. Such sequences are studied in [26] where particular attention is paid to sums of the form $S_N = \sum_{0\leqslant n<N} g(n)\varepsilon_n$, with q-multiplicative functions g. It is also proved that $|S_N| \in \mathcal{O}(N^\alpha)$ with $\alpha \in [1/2, 1]$ depending only on $(\varepsilon_n)_n$. In particular, for the Rudin-Shapiro sequence, one has $\alpha = 1/2$. Note that the spectral measure of the Rudin-shaprio sequence and its generalisations is Lebesgue ([275],[26]). For more details related to uniform distribution, see [127] and the references in [125, 27].

In the same vein, Rauzy studies in [279] the asymptotic behaviour of sums $\sum_{1\leqslant n\leqslant N} \mathbf{1}_{[0,1/2]}(\{n\alpha\})$, where $\mathbf{1}_{[0,1/2]}$ is the indicator function of the interval $[0, 1/2]$, for algebraic/quadratic values of α. His strategy involves introducing a fixed point of a substitution and considering orbits in some dynamical systems.

6.2. Diophantine approximation

We review some applications of Rauzy fractals (see Section 4.3) associated with Pisot β-numeration and Pisot substitutions. Rauzy fractals have, indeed, many applications in arithmetics; this was one of the incentives for their introduction by Rauzy [276, 278, 279].

A subset A of the d-dimensional torus $\mathbb{T}^d$ with (Lebesgue) measure $\mu(A)$ is said to be a *bounded remainder set* for the minimal translation $R_\alpha : x \mapsto x + \alpha$, defined on $\mathbb{T}^d$, if there exists $C > 0$, such that

$$\forall N \in \mathbb{N},\quad |\#\{0 \leqslant n < N;\ n\alpha \in A\} - N\mu(A)| \leqslant C.$$

When $d = 1$, an interval of $\mathbb{R}/\mathbb{Z}$ is a bounded remainder set if and only if its length belongs to $\alpha\mathbb{Z} + \mathbb{Z}$ [208]. In the higher-dimensional case, it is proved in [237] that there are no nontrivial rectangles which are bounded remainder sets for ergodic translations on the torus. According to [143, 277], Rauzy fractals associated either with a Pisot unimodular substitution or with a Pisot unit β-numeration provide efficient ways to construct bounded remainder sets for toral translations, provided discrete spectrum holds.

A second application in Diophantine approximation consists in exhibiting sequences of best approximations. Let β denote the Tribonacci number, *i.e.*, the real root of X^3-X^2-X-1. The Tribonacci sequence $(T_n)_{n\in\mathbb{N}}$ is defined as: $T_0 = 1$, $T_1 = 2$, $T_2 = 4$ and for all $n \in \mathbb{N}$, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$. It is proved in [96] (though this was probably already known to Rauzy) that the rational numbers $(T_n/T_{n+1}, T_{n-1}/T_{n+1})$ provide the best possible simultaneous approximation of $(1/\beta, 1/\beta^2)$ if we use the distance to the nearest integer defined by a particular norm, *i.e.*, the so-called Rauzy norm; recall that if $\mathbb{R}^d$ is endowed with the norm $||\cdot||$, and if $\theta \in \mathbb{T}^d$, then an integer $q \geqslant 1$ is a best approximation of θ if $|||q\theta||| < |||k\theta|||$ for all $1 \leqslant k \leqslant q-1$, where $|||\cdot|||$ stands for the distance to the nearest integer associated with the norm $||\cdot||$. Furthermore, the best possible constant

$$\inf\{c\,;\ q^{1/2}|||q\left(1/\beta, 1/\beta^2\right)||| < c \text{ for infinitely many } q\}$$

is proved in [96] to be equal to $(\beta^2+2\beta+3)^{-1/2}$. This approach is generalised in [176] to cubic Pisot numbers with complex conjugates satisfying the finiteness property (F) (see Section 3.3). See also [185] for closely related results on a class of cubic parameters.

Let α be an irrational real number. The *local star discrepancy* for the Kronecker sequence $(n\alpha)_{n\in\mathbb{N}}$ is defined as

$$\Delta_N^*(\alpha,\beta) = |\sum_{n=0}^{N-1} \chi_{[0,\beta[}(\{n\alpha\}) - N\beta|,$$

whereas the *star discrepancy* is defined as $D_N^*(\alpha) = \sup_{0<\beta<1} \Delta_N^*(\alpha,\beta)$. Most of the discrepancy results concerning Kronecker sequences were obtained by using the Ostrowski numeration system (see Example 2.14); for more details and references, we refer to [225] and [124]. A similar approach was developed in [3], where an algorithm is proposed, based on Dumont-Thomas numeration, which computes $\limsup \frac{\Delta_n^*(\alpha,\beta)}{\log n}$, when α is a quadratic number and $\beta \in \mathbb{Q}(\alpha)$.

6.3. Computer arithmetics and cryptography

The aim of this section is to briefly survey some applications of numeration systems to computer arithmetics and cryptography. We have no claim to exhaustivity, we thus restrict ourselves to cryptographical techniques based on numeration systems already considered above.

We have focused so far on the unicity of representations for positional numeration systems. Redundancy can proved to be very useful in computer arithmetics for the parallelism of some basic operations. Indeed, signed-digit representations in the continuation of Cauchy's numeration (see Example 2.13) are of a high interest in computer arithmetics, mostly because of the fact that the redundancy that they induce allows the limitation of the propagation of the carry when performing additions and subtractions. For more details, see [36], where properties of signed-digit representations are discussed with respect to the operations of addition, subtraction, multiplication, division and roundoff. More precisely, let $B \geqslant 2$ be the base, and take as digit set $\mathcal{D} = \{-a, \dots, a\}$, with $1 \leqslant a \leqslant B-1$. If $2a+1 \geqslant B$, then any integer in $\mathbb{Z}$ can be represented as a finite sum $\sum_{i \geqslant 0} a_i B^i$, with $a_i \in \mathcal{D}$. If, furthermore $2a-1 \geqslant B$, then it is possible to perform additions without carry propagation. The binary signed-digit representation studied in Example 2.13 is a particular case with $B = 2$, $\mathcal{D} = \{-1, 0, 1\}$ (but in this latter case the condition $2a-1 \geqslant B$ does not hold).

The non-adjacent signed binary expansion (see Example 2.13) was used by Booth [76] for facilitating multiplication in base 2. It is known to have, on average, only one third of the digits that are different from zero. As a consequence, it is used in public-key protocols on elliptic curves over finite fields for the scalar mutiplication, *i.e.*, for the evaluation of kP, where P is a point of an elliptic curve (see [77]). Indeed, if k is written in base 2 and if the the curve is defined over $\mathbb{F}_2$, then the cost of the evaluation of kP directly depends on the number of "doublings" and "addings" when performing the classical double-and-add algorithm. It is thus particularly interesting to work with binary representations with digits in a finite set with a minimal Hamming weight, *i.e.*, with a minimal number of non-zero digits. For additional details, see, *e.g.*, [157, 156, 165, 166, 167, 314], and the references therein. Note that the redundancy of the signed binary expansion is used for the protection from power analysis attacks against the computational part of cyphering elliptic curves based algorithms, by infering with power consumption during the calculation.

Redundant systems can also be used for the computation of elementary functions such as the complex logarithm and the exponential [44]. Inspired by the signed-digit numeration, a redundant representation for complex numbers that permits fast carry-free addition is introduced in [129].

Another type of numeration system can have interesting applications, namely the so-called *residue number systems*; these systems are modular systems [212] based on the Chinese remainder lemma. This representation system is particularly efficient for the computation of operations on large integers, by allowing the parallel distribution of integer computations on operators defined on smaller integer values, namely the moduli. Indeed, classical public-key protocols in cryptography (such as RSA, Diffie-Hellman, Fiat-Shamir) use modular multiplication with large integers. Residue number systems can proved to be very efficient in this framework (*e.g.*, see [39]). For applications of residue number systems in signal processing and cryptography, see the survey [45] and [40, 41, 42, 43, 40]. For a general and recent reference on elliptic cryptography, see [99].

The double base number system consists in representing positive integers as $\sum_{i\geqslant 0} 2^{s_i}3^{b_i}$, where $a_i, b_i \geqslant 0$. This numeration system is here again highly redundant, and has many applications in signal processing and cryptography; see for instance [97, 116] for multiplication algorithms for elliptic curves based on this double-base numeration, and the references therein. Let us note that Ostrowki's numeration system (see Example 2.14) is used in [61] to produce a greedy expansion for the double base number system. Similarly, a fast algorithm for computing a lower bound on the distance between a straight line and the points of a regular grid is given in [234], see also [235]. This algorithm is used to find worst cases when trying to round correctly the elementary functions in floating-point arithmetic; this is the so-called Table Maker's Dilemma [235].

6.4. Mathematical crystallography: Rauzy fractals and quasicrystals

A set $X \subset \mathbb{R}^n$ is said to be *uniformly discrete* if there exists a positive real number r such that for any $x \in X$, the open ball located at x of radius r contains *at most* one point of X; a set $X \subset \mathbb{R}^n$ is said to be *relatively dense* if there exists a positive real number R such that, for any x in $\mathbb{R}^n$, the open ball located at x of radius R contains *at least* one point of X. A subset of $\mathbb{R}^n$ is a *Delaunay set* if it is uniformly discrete and relatively dense. A Delaunay set is a *Meyer set* if $X-X$ is a Delaunay set, and if there exists a finite set F such that $X - X \subset X + F$ [255, 256]. This endows a Meyer set with a "quasi-lattice" structure. Meyer sets play indeed the rôle of lattices in the crystalline structure theory. A Meyer set [255, 256] is in fact a mathematical model for quasicrystals [257, 38].

An important issue in β-numeration deals with topological properties of the set $\mathbb{Z}_\beta = \{\pm w_M\beta^M + \cdots + w_0;\ M \in \mathbb{N},\ (w_M \cdots w_0) \in \mathcal{L}\}$, where $\mathcal{L}$ is the β-numeration language. If β is a Parry number, then $\mathbb{Z}_\beta$ is a Delaunay set [327]. More can be said when β is a Pisot number. Indeed, it is proved, in [88] that if β is a Pisot number, then $\mathbb{Z}_\beta$ is a Meyer set. For some families of numbers β (mainly Pisot quadratic units), an internal law can even be produced by formalising the quasi-stability of $\mathbb{Z}_\beta$ under subtraction and multiplication [88]. The β-numeration turns out to be a very efficient and promising tool for the modeling of families of quasicrystals thanks to β-grids [88, 89, 138, 152].

The characterisation of the numbers β for which $\mathbb{Z}_\beta$ is uniformly discrete or even a Meyer set has aroused a large interest. Observe that $\mathbb{Z}_\beta$ is always a discrete set at least. It can easily be seen that $\mathbb{Z}_\beta$ is uniformly discrete if and only if the β-shift X_β is specified, *i.e.*, if the strings of zeros in $d_\beta(1)$ have bounded lengths; note that the set of specified real numbers $\beta > 1$ with a noneventually periodic $d_\beta(1)$ has Hausdorff dimension 1 according to [288]; for more details, see for instance [74, 332] and the discussion in [152]. If $\mathbb{Z}_\beta$ is a Meyer set, then β is a Pisot or a Salem number [256].

If β is a Pisot number, then $\mathbb{Z}_\beta$ is a Meyer set. A proof of this implication is given in [152] by exhibiting a cut and project scheme. A *cut and project scheme* consists of a direct product $\mathbb{R}^k \times H$, $k \geqslant 1$, where H is a locally compact abelian group, and a lattice D in $\mathbb{R}^k \times H$, such that with respect to the natural projections $p_0 : \mathbb{R}^k \times H \to H$ and $p_1 : \mathbb{R}^k \times H \to \mathbb{R}^k$:

(1) $p_0(D)$ is dense in H;
(2) p_1 restricted to D is one-to-one on its image $p_1(D)$.

This cut and project scheme is denoted by $(\mathbb{R}^k \times H, D)$. A subset Γ of $\mathbb{R}^k$ is a *model set* if there exists a cut and project scheme $(\mathbb{R}^k \times H, D)$ and a relatively compact set Ω of H with a non-empty interior, such that $\Gamma = \{p_1(P);\ P \in D,\ p_0(P) \in \Omega\}$. Set Γ is called the *acceptance window* of the cut and project scheme. Meyer sets are proved to be subsets of model set of $\mathbb{R}^k$, for some $k \geqslant 1$, that are relatively dense [255, 256, 257]. For more details, see for instance [38, 138, 153, 233, 229, 301, 332, 333]. Note that there are close connections between such a generation process for quasicrystals and lattice tilings for Pisot unimodular substitutions (*e.g.*, see [65, 337, 338]).

Substitutional sequences (defined in Section 6.1) arising from numeration systems play an interesting rôle in quasicrystal theory, and more precisely, in the study of Schrödinger's difference equation

$$\psi_{n-1} + \psi_{n+1} + v_n\psi_n = e\psi_n,$$

where $v = (v_n)_{n\in\mathbb{Z}}$ is a real sequence called potential, and $e \in \mathbb{R}$ is the energy corresponding to the solution $\psi = (\psi_n)_n$, if any one exists. A detailed account of classical results on this subject is given in [325]. Connected interesting topics are exposed in [37]. Note that for a periodic potential, then the Schrödinger operator $H_v(\psi) := \psi_{n-1} + \psi_{n+1} + v_n\psi_n$ on $\ell^2(\mathbb{Z})$ has a purely absolutely continuous spectrum, which is in contrast with Kotani's Theorem [214]: if v is of finite range, not periodic but ergodic with associated invariant measure ρ on the orbit closure $\mathcal{F}(v)$, then H_w has purely singular spectrum for ρ-almost all $w \in \mathcal{F}(v)$. This leads to consider the particular case of v itself. Sturmian sequences like $v_n = \lfloor n\alpha + b \rfloor - \lfloor (n-1)\alpha + b \rfloor$ (the Fibonacci potential corresponds to $\alpha = (\sqrt{5}-1)/2$) and doubling potential (issued from the substitution $a \mapsto ab$, $b \mapsto aa$) are examples of potentials for which the spectrum of the Schrödinger operator is a Cantor set with zero Lebesgue measure and the operator is purely singular continuous. The Thue-Morse sequence and the Rudin-Shapiro sequence, for examples, are not completely elucidated with respect to the general condition on the underlying substitution exhibited by Bovier and Ghez [79]: this latter condition ensures the spectrum to have zero Lebesgue measure. Combinatorial properties of subtitutional sequences v seem to play a fundamental rôle in the spectral nature of H_v.

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