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TILINGS ASSOCIATED WITH NON-PISOT MATRICES

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ABSTRACT. — Suppose $A \in Gl_d(\mathbb{Z})$ has a 2-dimensional expanding subspace E^u, satisfies a regularity condition, called "good star", and has $A^* \geqslant 0$, where A^* is an *oriented compound* of A. A morphism θ of the free group on $\{1, 2, \ldots, d\}$ is called a *non-abelianization* of A if it has structure matrix A. We show that there is a *tiling substitution* Θ whose "boundary substitution" $\theta = \partial\Theta$ is a non-abelianization of A. Such a tiling substitution Θ leads to a self-affine tiling of $E^u \sim \mathbb{R}^2$ with $A_u := A|_{E_u} \in GL_2(\mathbb{R})$ as its expansion. In the last section we find conditions on A so that A^* has no negative entries.

RÉSUMÉ. — Supposons que $A \in Gl_d(\mathbb{Z})$ ait un sous-espace d'extension bidimensionnel E^u, satisfaisant une condition de régularité, appelée "bonne étoile", et telle que $A^* \geqslant 0$, où A^* est un *composé orienté*. Un morphisme θ du groupe libre sur $\{1, 2, \ldots, d\}$ est une *non-abélianisation* de A si sa matrice de structure est A. Nous prouvons qu'il existe une *substitution de pavage* Θ dont la substitution de frontière $\theta = \partial\Theta$ est une non-abélianisation de A. Une telle substitution de pavage θ donne un pavage "auto-affine" de $E^u \sim \mathbb{R}^2$ avec pour expansion $A_u := A|_{E_u} \in GL_2(\mathbb{R})$. Dans la dernière section nous trouvons des conditions sur A de sorte que A^* n'ait pas de coefficients négatifs.

1. Introduction

A *tiling substitution* Θ is a mapping from tiles in $\mathbb{R}^2$ to finite tiling patches that has enough regularity to be extended to a mapping from tiling patches to tiling patches. This permits the iteration of Θ, starting with a single tile, or a small tiling patch, to obtain larger tiling patches, and ultimately tilings of the plane. A tiling substitution Θ induces a mapping $\theta = \partial\Theta$ on tile boundaries. Assuming there are finitely many polygonal prototiles, having d different boundary segments, this boundary map θ can

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be viewed as a 1-dimensional *substitution* on $\mathcal{B} := \{1, 2, \ldots, d\}$. The *structure matrix* of θ is the matrix A whose i, jth entry counts the number of times the edge j occurs in the substitution $\theta(i)$. Frequently, this matrix A is hyperbolic, and has a 2-dimensional expanding subspace. This corresponds to how Θ "expands" $\mathbb{R}^2$. Iterating Θ yields an uncountable collection X_Θ of tilings of $\mathbb{R}^2$, which can be regarded as a compact metric space (see e.g., [16]), and on which Θ is a "hyperbolic" self homeomorphism. In the best case, the action of Θ on X_Θ is semi-conjugate to the action of A on $\mathbb{T}^d$ (as a toral automorphism). In fact, the tilings $z \in X_\Theta$ are closely related to a Markov partition for A on $\mathbb{T}^d$ (see [12], [10]).

In this paper we seek to reverse the process described above. Starting with a hyperbolic matrix $A \in GL_d(\mathbb{Z})$ having a 2-dimensional expanding subspace, we want: (1) to find a 1-dimensional substitution θ, or more generally a *free group endomorphism* θ, that has A as its structure matrix, and (2) find a tiling substitution Θ with $\partial\Theta = \theta$. As it turns out, not every A works, at least not for the method that we describe here. But we find a sufficient condition for A to work, namely that a certain "oriented compound" of A, denoted A^*, satisfies $A^* \geqslant 0$ (we also give a sufficient condition on A for $A^* \geqslant 0$). Then, given a matrix A with $A^* \geqslant 0$, we construct what we call a *properly ordered* endomorphism θ, which has A as its structure matrix. Finally, we show how to use such a θ to construct a tiling substitution Θ with $\partial\Theta = \theta$.

The idea of starting with a matrix, constructing a corresponding substitution, and using it to define geometric objects first appears in G. Rauzy [15], where the "Rauzy fractal" is introduced (a nice update on this theory appears in [3], highlighting its connections to number theory, dynamics and fractal geometry). The idea of using A and a 1-dimensional substitution to construct a tiling substitution appears many works by P. Arnoux, S. Ito and their co-workers (see e.g., [12], [4], [6], and [7]). An approach related to the one described here appears in [10].

It should be remarked, however, that all of the work mentioned above assumes some form of the *Pisot condition*. Recall that a *Pisot number* (or PV number) is real algebraic integer $\lambda > 1$ whose conjugates λ' satisfy $|\lambda'| < 1$. The companion matrix B for a Pisot number λ is a $d \times d$ hyperbolic integer matrix with a 1-dimensional expanding subspace and contracting subspace of dimension $d - 1$. It is this codimension-1 hyperbolicity that is the crux of the Pisot condition. The importance of this condition for most approaches to tiling substitutions cannot be overstated. However, in this paper we do not need the Pisot condition. We also eliminate another common assumption: that $A \geqslant 0$. A program similar to ours, (in particular, also without the Pisot assumption) appears in the dissertation [13] of R. Kenyon, but with few details. For other approaches to the non-Pisot case, see [11], [10] [8] and [9].

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2. Linear theory

2.1. The non-Pisot property

Let $A \in Gl_d(\mathbb{Z})$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ that satisfy

$$|\lambda_1| \geqslant |\lambda_2| > 1 > |\lambda_3| \geqslant \cdots \geqslant |\lambda_d| > 0. \tag{2.1}$$

In particular, A is *nonsingular* and *hyperbolic* and has a 2-dimensional *expanding subspace*, which we denote E^u. Let E^s be the *contracting subspace*, which has dimension $d-2$, and assume $d-2 \geqslant 1$.

We say A satisfies a *Pisot condition* if $d-2=1$ (i.e., E^u has codimension-1). A matrix A satisfying (2.1) is called *non-Pisot of order* n if $n = d-2 > 1$. The main innovation in this paper is that we do not need to assume the Pisot condition. In particular, our examples typically have $d=4$ and $n = d-2 = 2$. However, we do not need to exclude the Pisot condition either.

2.2. The good star property

Let $\mathbf{v}_1, \mathbf{v}_2$ be an ordered basis for E^u (for example, we could take $\mathbf{v}_1$ and $\mathbf{v}_2$ to be eigenvectors). We call a matrix $P \in \mathbb{R}^{2\times d}$ a "projection" to E^u if

$$P\mathbf{v}_1 = \mathbf{e}_1 \text{ and } P\mathbf{v}_2 = \mathbf{e}_2. \tag{2.2}$$

It will often be convenient to identify E^u with its P-image, which we think of as the plane $\mathbb{R}^2$. If P' is another projection, corresponding to a different basis for E^u, then

$$P' = MP \text{ for } M \in Gl_2(\mathbb{R}). \tag{2.3}$$

Note that a projection P is onto $\mathbb{R}^2 \sim E^u$ and transverse to E^s. By appropriate choice of basis it may be made orthogonal to E^u or parallel to E^s.

Let us write $P = (\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^d)$ in terms of its columns. Then $P\mathbf{e}_j = \mathbf{p}^j$. In other words, the columns $\mathbf{p}^j$ are projections of $\mathbf{e}_j$ to $\mathbb{R}^2$. When we draw these vectors in $\mathbb{R}^2$, we refer to the picture as a *star* of vectors (see Figure 2.1).

Definition 2.1. — *We call* $P = (\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^d)$ a good star *of vectors if*

$$\mathbf{p}^i = \omega \mathbf{p}^j \text{ for some real } \omega \neq 0 \text{ implies } i = j. \tag{2.4}$$

In other words, the vectors $\mathbf{p}^i$ *are pairwise non-parallel. We say a matrix* A *satisfying* (2.1) *has the* good star property *if some projection* $P = (\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^d)$ *to* E^u *is a good star of vectors.*

By (2.3), if (2.4) holds for some projection P, then it holds for any projection. Hence the good star property depends only on A.

Definition 2.2. — *Let* A *satisfy* (2.1) *and* (2.4)*, and let* P *be a projection to* E^u*. Define* $A_u \in \mathbb{R}^{2\times 2}$ *be the matrix conjugate to* $A|_{E^u}$ *via* P*. That is,*

$$A_u := PA(\mathbf{v}_1, \mathbf{v}_2). \tag{2.5}$$

It is easy to see that A_u has eigenvalues λ_1, λ_2.

2.3. The compound

Let $B \in \mathbb{C}^{p\times q}$ (the set of $p \times q$ complex matrices), with $p, q \geqslant n$. Define the n^{th} *compound* of B to be the matrix $C_n(B) \in \mathbb{C}^{\binom{p}{n}\times\binom{p}{n}}$ whose entries are the $n \times n$ minors of B (see [2]). In this paper, $n = 2$. We index $C_2(B)$ by pairs $i \wedge j, k \wedge \ell$, where $i < j$ and $k < \ell$, in lexicographic order.

Theorem 2.3 (The Binet-Cauchy Theorem, see [2]). — *If* $B = B_1 B_2$ *then* $C_n(B) = C_n(B_1)C_n(B_2)$*. If* B *is non-singular then* $C_n(B^{-1}) = C_n(B)^{-1}$.

Let $P \in \mathbb{R}^{2\times d}$ be the projection for a matrix A satisfying (2.1) and (2.4). The compound $C_2(P) \in \mathbb{R}^{1\times\binom{d}{2}}$ has entries

$$p_{i\wedge j} := \det(\mathbf{p}^i, \mathbf{p}^j) = \sin(\angle \mathbf{p}^i \mathbf{p}^j) \neq 0, \tag{2.6}$$

where the last inequality follows $\angle \mathbf{p}^i \mathbf{p}^j \in (-\pi, 0) \cup (0, \pi)$ by (2.4).

For $p \neq 0$, let $\operatorname{sgn}(p) = p/|p|$. For a vector $\mathbf{a} \in \mathbb{C}^n$, let $\operatorname{diag}(\mathbf{a})$ denote the $n \times n$ matrix with the entries of $\mathbf{a}$ along the diagonal, and zeros everywhere else. Define

(2.7)
$$S(A) = \operatorname{diag}\big((\operatorname{sgn}(p_{1\wedge 2}), \operatorname{sgn}(p_{1\wedge 3}), \ldots, \operatorname{sgn}(p_{(d-1)\wedge d})\big) \in \{-1,0,1\}^{\binom{d}{2}\times\binom{d}{2}}.$$

The Binet-Cauchy Theorem and (2.3) imply that $S(A)$ is well defined up to a change of sign. Define

$$A^* = S(A) C_2(A) S(A). \tag{2.8}$$

This is the matrix with entries $a_{i\wedge j,k\wedge \ell}\operatorname{sgn}(p_{i\wedge j})\operatorname{sgn}(p_{k\wedge \ell})$, where $a_{i\wedge j,k\wedge\ell}$ are the entries of $C_2(A)$. We call A^* the *oriented compound* of A.

From now on, in addition to (2.1) and (2.4), we will assume:

$$A^* \geqslant 0, \tag{2.9}$$

i.e., A^* has no negative entries. It is easy to see that this does not depend on the choice of the basis for E^u.

2.4. Example: The Ammann matrix

Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix}, \tag{2.10}$$

which is related to the tiling sometimes called the Ammann-Beenker tiling (see [17], [11]). Note that A is symmetric and has characteristic polynomial $p(x) = (x^2 + 2x - 1)^2$. The eigenvalues are $\lambda_1 = \lambda_2 = -1 - \sqrt{2}$ and $\lambda_3 = \lambda_4 = \sqrt{2} - 1$, so A satisfies (2.1).

Let

$$Q = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = \begin{pmatrix} \sqrt{2} & -1 & -\sqrt{2} & -1 \\ -1 & \sqrt{2} & -1 & -\sqrt{2} \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

where the columns $\mathbf{q}_j$ are eigenvectors for λ_j. Define the projection P to be the the first two rows of Q^{-1}:

$$P = \frac{\sqrt{2}}{4}\begin{pmatrix} 1 & 0 & 1 & \sqrt{2} \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix} = (\mathbf{p}^1, \mathbf{p}^2, \mathbf{p}^3, \mathbf{p}^4), \tag{2.11}$$

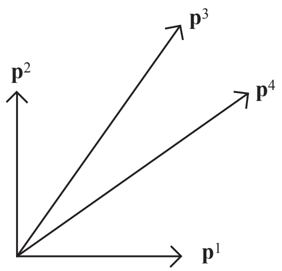


Figure 2.1. The good star of vectors corresponding to the Ammann matrix A in (2.10).

The columns of P are plotted in Figure 2.1. Clearly A satisfies the good star property (2.4).

We have $A_u = \lambda_1 \mathrm{Id} = -(1+\sqrt{2})\mathrm{Id}$, which we interpret as a composition of an expansion of $\mathbb{R}^2$ by $1+\sqrt{2}$, and a rotation by π. Applying (2.7) we find

$$S(A) = \mathrm{diag}(1,1,1,-1,-1,-1), \tag{2.12}$$

so that by (2.8)

$$A^* = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \geqslant 0. \tag{2.13}$$

Thus A satisfies (2.9). We shall return to this example below.

3. Proto-objects and objects

In the next few sections we are going to consider geometric objects g in $\mathbb{R}^2$, including *curves*, *closed curves*, *tiles*, *tiling patches* and *tilings*. Each object g will have a particular *location* in $\mathbb{R}^2$, and we can move an object by a *translation*. In particular, $g + \mathbf{w}$ denotes the translation of g by the vector $\mathbf{w} \in \mathbb{R}^2$. We call two objects g_1 and g_2 *translationally equivalent* if $g_1 = g_2 + \mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^2$. We denote the translational equivalence class of the object g by G, and refer to G as a *proto-object*. Thus, we will refer to *proto-curves*, *prototiles*, etc..

It will be convenient to represent each proto-object G by an arbitrary fixed choice of a geometric object from its translational equivalence class. We think of this object as being "located at the origin" in $\mathbb{R}^2$. Let $\mathcal{G}$ denote the set of all proto-objects G. Each object g is then a translation $g = G + \mathbf{w} := (\mathbf{w}, G)$ of the corresponding proto-object $G \in \mathcal{G}$ by a unique vector $\mathbf{w} \in \mathbb{R}^2$.

3.1. Words and substitutions

Let $\mathcal{B} = \{1, 2, \ldots, d\}$. Let $\mathcal{B}^*$ denote the free semi-group of nonempty finite words in $\mathcal{B}$. Let $\mathcal{B}^\pm = \{i^{\pm 1} : i \in \mathcal{B}\}$, and let $\mathcal{F}\langle\mathcal{B}\rangle$ denote the *free group* on $\mathcal{B}$, which we think of as the set of *reduced* words in $(\mathcal{B}^\pm)^*$, together with the empty word ϵ.

A *substitution* is a mapping $\theta : \mathcal{B} \to \mathcal{B}^*$. Since $\mathcal{B}$ is a basis for both $\mathcal{B}^*$ and $\mathcal{F}\langle\mathcal{B}\rangle$, a substitution uniquely defines both a semigroup and a group *endomorphism*. More generally, any mapping $\theta : \mathcal{B} \to \mathcal{F}\langle\mathcal{B}\rangle$ defines an endomorphism of $\mathcal{F}\langle\mathcal{B}\rangle$. We always assume an endomorphism is *non-erasing*, which means $\theta(i) \neq \epsilon$. We think of a non-erasing endomorphism a sort of "generalized substitution", but refrain from using this language to avoid confusion with its other uses.

The *abelianization homomorphism* $f : \mathcal{F}\langle\mathcal{B}\rangle \to \mathbb{Z}^d$ is defined on $\mathcal{B}$ by $f(i) = \mathbf{e}_i$, and extended to $\mathcal{F}\langle\mathcal{B}\rangle$. A endomorphism of $\mathbb{Z}^d$ is given by a matrix $M \in \mathbb{Z}^{d\times d}$. For any endomorphism θ of $\mathcal{F}\langle\mathcal{B}\rangle$, the abelianization f induces an (abelian group) endomorphism of $\mathbb{Z}^d$, denoted by L_θ, satisfying $L_\theta \circ f = f \circ \theta$. In particular,

$$L_\theta = (f(\theta(1)), f(\theta(2)), \ldots, f(\theta(d)))\,. \tag{3.1}$$

This matrix is also called the *structure matrix* of θ, or more formally the *abelianization* of θ. We note that if each $\theta(i)$ is an efficient word (see Definition 5.3 below), then entry $\ell_{i,j}$ of L_θ equals the signed number times that i appears in $\theta(j)$ (in particular, this always holds if θ is a substitution). For a given matrix A, we call any endomorphism θ that satisfies $L_\theta = A$ a *non-abelianization* of A.

3.2. Curves

A *curve* is continuous, piecewise C^1 mapping $w : [a, b] \to \mathbb{R}^2$. Two curves are considered *the same* if they differ by an orientation preserving piecewise

C^1 change of parameterization. The reverse of a curve w, denoted $-w$, is given by $-w(t) = w(-t + a + b)$.

For $\mathbf{x}_0, \mathbf{x}_1 \in \mathbb{R}^2$ the *linear* curve from $\mathbf{x}_0$ to $\mathbf{x}_1$ is defined to be $w(t) = (1-t)\mathbf{x}_0 + t\mathbf{x}_1$, for $t \in [0,1]$. Similarly, for a sequence of points $\mathbf{x}_0, \mathbf{x}_2, \dots, \mathbf{x}_\ell$ we define the *piecewise linear curve* (or "broken line") connecting the points by $w : [0,\ell] \to \mathbb{R}^2$ where

$$w(t) = (1 - (t - j))\mathbf{x}_j + (t - j)\mathbf{x}_{j+1} \text{ for } t \in [j, j+1].$$

We will usually restrict our attention to the following case. Let $P = (\mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^d)$ be a good star of vectors. For each $k = 1, \dots, \ell$, suppose there exists $i_k \in \{1, \dots, d\}$ and a nonzero $a_k \in \mathbb{R}$ so that

$$\mathbf{x}_k - \mathbf{x}_{k-1} = a_k \mathbf{p}^{i_k}. \tag{3.2}$$

In effect, we consider piecewise linear curves whose segments are parallel to the vectors $\mathbf{p}^1, \dots, \mathbf{p}^d$.

Now suppose $W \in (\mathcal{B}^{\pm})^*$ and $\mathbf{x} \in \mathbb{R}^2$. In particular, $W = i_1^{a_1} i_2^{a_2} \dots i_\ell^{a_\ell}$, where $i_k \in \mathcal{B}$ and $a_k \in \mathbb{Z}$, $a_k \neq 0$. Define a finite sequence of points $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\ell$ inductively, by putting $\mathbf{x}_0 = \mathbf{x}$, and applying (3.2) for $k = 1, \dots, \ell$. We define the curve $w := (\mathbf{x}, W)$ to be the piecewise linear curve connecting the points $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\ell$. In this way a word is a *proto-curve*.

A curve $w : [a,b] \to \mathbb{R}^2$ is *closed* if $w(a) = w(b)$ (we sometimes think of it as mapping $w : \mathbb{S}^1 \to \mathbb{R}^2$). A *simple closed curve* is a closed curve that is injective on $\mathbb{S}^1$. We say $W \in (\mathcal{B}^{\pm})^*$ is a *cyclic* if the corresponding curve $w = (\mathbf{x}_0, W)$ is closed. This is equivalent to $\mathbf{x}_\ell = \mathbf{x}_0$. Since

$$\mathbf{x}_\ell - \mathbf{x}_0 = f(W), \tag{3.3}$$

the curve w is closed, or equivalently the word W is a cyclic, if and only if $f(W) = 0$. A cyclic word is a *proto-closed curve*. We define the *commutator* of two words W_1, W_2 by

$$W = [W_1, W_2] := W_1 W_2 W_1^{-1} W_2^{-1}. \tag{3.4}$$

Note that $f([W_1, W_2]) = f(W_1 W_2 W_1^{-1} W_2^{-1}) = 0$, so that a commutator is always cyclic.

3.3. A topological proposition

Unfortunately, it is not easy to tell if a cycle W corresponds to a *simple* close curve. However, in this section we obtain a partial result in the case $W = [W_1, W_2]$.

The *Jordan curve theorem* (see e.g., [1]) says that if $w : \mathbb{S}^1 \to \mathbb{R}^2$ is a simple closed curve then $\mathbb{R}^2 \backslash w(\mathbb{S}^1)$ consists of two open sets: one bounded, called the *inside* of w, and one unbounded, called the *outside*. More generally, if w is piecewise linear closed curve that is not necessarily simple, then $\mathbb{R}^2 \backslash w(\mathbb{S}^1)$ consists of a finite collection of open sets: the *outside*, which is unbounded, and a finite number of bounded open sets, called the *components* of the *inside*.

A *chain* is a finite sum of the form

$$w = n_1 w_1 + n_2 w_2 + \cdots + n_k w_k, \tag{3.5}$$

where each w_k is a different (not necessarily closed) curve in the plane, and $n_k \in \mathbb{Z}$ (see [1]). There is a fairly obvious equivalence relation on chains, which in addition to orientation preserving re-parameterization, allows the combination of curves by following a secession of them. Line integrals of real or complex functions F in the plane are defined over chains. It can be shown that two chains are equivalent if and only if they yield the same line integral for every function F, (see [1]). Equivalent chains are considered to be *equal*.

A chain is called a *cycle* if (up to equivalence) each w_i is a closed curve. We assume in addition that each w_i is piecewise linear satisfying (3.2). A cycle is a generalization of a closed curve, and like in that case, $\mathbb{R}^2 \backslash \cup_{i=1}^k w_k(\mathbb{S}^1)$ consists of a finite collection of open sets: an unbounded *outside*, and a finite number of bounded inside *components*. The union of the components is called the *inside* of the curve or the cycle. The set $\cup_{i=1}^k w_k(\mathbb{S}^1)$ is called the *trace*.

Given a cycle w, and $\mathbf{x} \in \mathbb{R}^2$ not in the trace, the *winding number* is defined

$$n_w(\mathbf{x}) = \frac{1}{2\pi i} \int_w \frac{d\mathbf{z}}{\mathbf{z} - \mathbf{x}}. \tag{3.6}$$

Here we think of $\mathbf{x}$ and $\mathbf{z}$ as a complex numbers. The integral is carried out separately over each closed curve in (3.5), and the results are added.

The basic property of the winding number function n_w is that it is integer valued, constant on each component of the inside of w, and zero outside (see [1]). It can be shown that two cycles are equivalent if and only if they assign the same winding numbers to all non-trace points (this follows from the generalized Cauchy integral formula, [1]). The change of variables formula for integrals shows that $n_{w+\mathbf{y}}(\mathbf{x} + \mathbf{y}) = n_w(\mathbf{x})$, and $n_{-w}(\mathbf{x}) = -n_w(\mathbf{x})$. It is easy to see that $n_{w_1+w_2}(\mathbf{x}) = n_{w_1}(\mathbf{x}) + n_{w_2}(\mathbf{x})$ provided $\mathbf{x}$ is not in the trace of w_1 or w_2.

Let w be a simple closed curve. Then $n_w(\mathbf{x}) = \pm 1$ for $\mathbf{x}$ inside w (see [1]). If $n_w(\mathbf{x}) = 1$ for $\mathbf{x}$ inside w, we say w is *positive* (i.e., it is positively oriented). More generally, we say a (not necessarily simple) closed curve w, or even a cycle w, is *positive* if $n_w(\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^2$ not in the trace. We call a closed curve or cycle w *positive semi-simple* if $n_w(\mathbf{x}) \in \{0, 1\}$ for all $\mathbf{x} \in \mathbb{R}^2$ not in the trace.

Proposition 3.1. — *Let $P = (\mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^d)$ be a good star of vectors and let $f : \mathcal{F}\langle\mathcal{B}\rangle \to \mathbb{Z}^d$ be the abelianization homomorphism. Suppose $W = [W_1, W_2]$ is the commutator of $W_1, W_2 \in \mathcal{F}\langle\mathcal{B}\rangle$, and let $w = (0, W)$ be the corresponding closed curve. Assume w is positive and that*

$$\det\big([Pf(W_1), Pf(W_2)]\big) > 0. \tag{3.7}$$

Then w is positive semi-simple.

Proof. — For simplicity we identify $W = w$. Let $\mathbf{v}_1 = Pf(W_1)$ and $\mathbf{v}_2 = Pf(W_2)$, which are linearly independent vectors by (3.7). Let R be the piecewise linear curve in $\mathbb{R}^2$ connecting the points $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2, \mathbf{0}$, and let R_1, R_2, R_3, and R_4 denote the four linear segments that make up R. It follows from (3.7) that R is a positive simple closed curve, and thus $n_R(\mathbf{x}) = 1$ for $\mathbf{x}$ inside R and $n_R(\mathbf{x}) = 0$ outside. We also divide W into

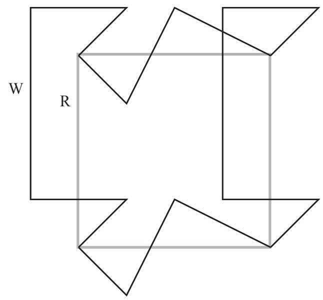


Figure 3.1. The curve $W = [W_1, W_2]$ and its "linearization" R.

four segments: W_1, W_2, W_1^{-1}, and W_2^{-1}, corresponding to the four factors of W with the same names. Note that for each i, the curves W_i and R_i start and end at the same place.

Starting at 0, follow W_1. For a while it may follow R_1, but assuming, $W_1 \neq R_1$, the two curves eventually part. Continue following W_1 until its first return to R_1. Call this point $\mathbf{z}_1$. Then follow R_1 back to $\mathbf{0}$. Call the resulting closed curve Z_1. Let $Z_1' = -Z_1 + \mathbf{v}_2$ be the *reflection* of Z_1 across R.

Next, starting at $\mathbf{z}_1$, repeat the previous construction. Follow W_1 until it leaves R_1 and then returns for the first time at $\mathbf{z}_2$. Then follow R_1 back to $\mathbf{z}_1$. Call the resulting closed curve Z_2, and define its reflection $Z_2' = -Z_2 + \mathbf{v}_2$. Continuing in this fashion we get a sequence $Z_1, Z_2, \dots, Z_{\ell_1}$ of closed curves. We stop when $\mathbf{z}_{\ell_1} = \mathbf{v}_1$. We also obtain their reflections $Z_1', Z_2', \dots, Z_{\ell_1}'$.

Repeat construction with W_2 and R_2 to obtain simple closed curves $Z_{\ell_1+1}, \dots, Z_{\ell_2}$, and their reflections $Z_{\ell_1+1}', \dots, Z_{\ell_2}'$, where $Z_k' = -Z_k - \mathbf{v}_1$ for $k = \ell_1 + 1, \dots, \ell_2$.

Let $\Lambda = \{n\mathbf{v}_1 + m\mathbf{v}_2 : n, m \in \mathbb{Z}\}$ be the lattice in $\mathbb{R}^2$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$. Consider the tessellation $\tilde{R}$ of $\mathbb{R}^2$ by the parallelograms $\Lambda + R$. We can assume without loss of generality that each closed curve Z_i lies inside one of the parallelograms in $\tilde{R}$. If not, we can subdivide Z_i into a sum of closed curves (i.e., a cycle) that follow Z_i and the lines in $\tilde{R}$. Simultaneously, we do the same thing to each Z_i' in reverse.

Now for convenience, we renumber all the reflections

$$Z_{\ell_1}', Z_{\ell_1-1}', \dots, Z_1',\ Z_{\ell_2+\ell_1}', Z_{\ell_2+\ell_1-1}', \dots, Z_{\ell_1+1}'$$

as

$$Z_{\ell_1+\ell_2+1}, Z_{\ell_1+\ell_2+2}, \dots, Z_{2(\ell_1+\ell_2)}.$$

Thus the closed curves $Z_1, \dots, Z_{2(\ell_1+\ell_2)}$ come in pairs of reflections (with opposite orientations), and each lies in just one parallelogram of $\tilde{R}$.

Define the chain

$$Z := \sum_{j=1}^{2(\ell_1+\ell_2)} Z_j. \tag{3.8}$$

Because of the way the curves Z_i were constructed, $W = R + Z$. It follows that

$$n_W(\mathbf{x}) = n_R(\mathbf{x}) + n_Z(\mathbf{x}), \tag{3.9}$$

and by (3.8),

$$n_Z(\mathbf{x}) = \sum_{j=1}^{2(\ell_1+\ell_2)} n_{Z_j}(\mathbf{x}), \tag{3.10}$$

for (non-trace) $\mathbf{x} \in \mathbb{R}^2$.

Let $\mathbf{x}_0 \in \mathbb{R}^2$ be such that no $\mathbf{x} \in \Lambda + \mathbf{x}_0$ is in the trace of W, Z, R or $\tilde{R}$. Let

$$I_{\mathbf{x}_0} = \{j : \exists \mathbf{x} \in \Lambda + \mathbf{x}_0 \text{ with } n_{Z_j}(\mathbf{x}) \neq 0\}.$$

For each $j \in I_{\mathbf{x}_0}$ there is a unique $\mathbf{x}_j \in \Lambda + \mathbf{x}_0$ so that $n_{Z_j}(\mathbf{x}_i) \neq 0$. This is because each Z_j lies in a single parallelogram form $\tilde{R}$. Using (3.10) we have

$$\begin{aligned} \sum_{\mathbf{x}\in\Lambda+\mathbf{x}_0} n_Z(\mathbf{x}) &= \sum_{\mathbf{x}\in\Lambda+\mathbf{x}_0} \sum_{j=1}^{2(\ell_1+\ell_2)} n_{Z_j}(\mathbf{x}) \\ &= \sum_{j=1}^{2(\ell_1+\ell_2)} \sum_{\mathbf{x}\in\Lambda+\mathbf{x}_0} n_{Z_j}(\mathbf{x}) \\ &= \sum_{j\in I_{\mathbf{x}_0}} n_{Z_j}(\mathbf{x}_j). \end{aligned} \tag{3.11}$$

For each $j \in I_{\mathbf{x}_0}$ there exists j' so that $Z_{j'}$ is the reflection of Z_j. Since $Z_{j'} = -Z_j + (\mathbf{x}_{j'} - \mathbf{x}_j)$, it follows that $n_{Z_{j'}}(\mathbf{x}_{j'}) = -n_{Z_j}(\mathbf{x}_j)$. This implies the last sum in (3.11) is zero, so that

$$\sum_{\mathbf{x}\in\Lambda+\mathbf{x}_0} n_Z(\mathbf{x}) = 0. \tag{3.12}$$

Suppose that $n_W(\mathbf{x}_0) > 1$, and consider two cases: (i) $\mathbf{x}_0$ is inside R and (ii) $\mathbf{x}_0$ is not inside R.

In case (i), (3.9) and the fact that R is a positive simple closed curve implies $n_Z(\mathbf{x}_0) \geqslant 1$. It also follows that $n_W(\mathbf{x}) = n_Z(\mathbf{x})$ for $\mathbf{x} \in \Lambda + \mathbf{x}_0$, $\mathbf{x} \neq \mathbf{x}_0$. By (3.12) there exists such an $\mathbf{x}$ so that $n_W(\mathbf{x}) = n_Z(\mathbf{x}) \leqslant 1 < 0$. This contradicts the fact that W is positive.

In case (ii), $n_Z(\mathbf{x}_0) = n_W(\mathbf{x}_0) \geqslant 2$. It follows that either: (a) there exists $\mathbf{x} \in \Lambda + \mathbf{x}_0$, $\mathbf{x} \neq \mathbf{x}_0$, so that $n_Z(\mathbf{x}) \leqslant 2$, or (b) there exist two distinct $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda + \mathbf{x}_0$, not equal to $\mathbf{x}_0$, so that $n_Z(\mathbf{x}_1) \leqslant 1$ and $n_Z(\mathbf{x}_2) \leqslant 1$. In case (a), (3.9) implies $n_W(\mathbf{x}) < 1$ since $n_R(\mathbf{x}) \leqslant 1$. In case (b) we have $n_R(\mathbf{x}_i) = 1$ for $i = 1$ or $i = 2$ only if $\mathbf{x}_i$ is inside R. This can happen for at most one of the two, since they differ by Λ. Otherwise $n_R(\mathbf{x}_i) = 0$. Then (3.9) implies that $n_W(\mathbf{x}_i) < 0$ for at least one of the $\mathbf{x}_i$. In both cases we again contradict the fact that W is positive. □

3.4. Tiles, tilings and tiling patches

Let $\mathcal{B}^2$ denote the set of all pairs $i \wedge j$ for $i, j \in \mathcal{B}$. Define

$$\operatorname{sgn}(i \wedge j) := \operatorname{sgn}(p_{i\wedge j}) = \operatorname{sgn}(\sin(\angle \mathbf{p}^i \mathbf{p}^j)). \tag{3.13}$$

By (2.6), $\operatorname{sgn}(i \wedge j) \in \{-1, 1\}$ if $i \neq j$ and $\operatorname{sgn}(i \wedge i) = 0$. For $i < j$, (3.13) shows that the entries of the matrix $S(A)$ in (2.7) are $\operatorname{sgn}(i \wedge j)$. When

$\mathrm{sgn}(i \wedge j) \neq 0$ we write $j \wedge i = -(i \wedge j)$. More generally, for $a, b \in \{-1, 1\}$, we simplify $i^a \wedge j^b = ab(i \wedge j)$. We call $i \wedge j$ a *positive prototile* if $\mathrm{sgn}(i \wedge j) = 1$, and a *negative* prototile if $\mathrm{sgn}(i \wedge j) = -1$. In the latter case, $j \wedge i = -(i \wedge j)$ is positive. The set of all positive prototiles is denoted by $\mathcal{B}^2_+$. Prototiles $i \wedge i$ are neither positive or negative, and are called *trivial*. We simplify $i^a \wedge i^b = i \wedge i$. Thus $\mathcal{B}^2$ denotes the set of *all* prototiles.

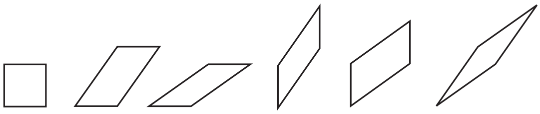


Figure 3.2. The positive prototiles for the matrix A in (2.10): $1 \wedge 2$, $1 \wedge 3$, $1 \wedge 4$, $3 \wedge 2$, $4 \wedge 2$, $4 \wedge 3$.

Definition 3.2. — *Let $W \in \mathcal{F}\langle\mathcal{B}\rangle$ be a positive semi-simple word (i.e., any $(\mathbf{x}, W)$ is a positive semi-simple curve). For $\mathbf{x} \in \mathbb{R}^2$, define the* tile *$t = (\mathbf{x}, \overline{W})$ to be the union of the inside of the curve $(\mathbf{x}, W)$ with its trace. We write $\partial(\mathbf{x}, \overline{W}) = (\mathbf{x}, W)$ for the* boundary *of the tile.*

Now let $i \wedge j \in \mathcal{B}^2$. To define the geometric realization of the tiles corresponding to $i \wedge j$, we first define the boundary of $i \wedge j$ by

$$\partial(i \wedge j) = [i, j] = iji^{-1}j^{-1}.$$

We define the tile

$$t = (\mathbf{x}, i \wedge j) := (\mathbf{x}, \overline{[i, j]}).$$

We define the boundary $\partial(t) = \partial(\mathbf{x}, i \wedge j) := (\mathbf{x}, iji^{-1}j^{-1})$, which is a piecewise linear simple closed curve (i.e., a parallelogram). The four oriented segments of this curve are called the *edges* of the tile, and are denoted $\mathcal{E}\big((\mathbf{x}, i \wedge j)\big)$. Two tiles t_1, t_2 are said to be *adjacent* if $-\mathcal{E}(t_1) \cap \mathcal{E}(t_2) \neq \emptyset$, i.e., the two tiles share an oppositely oriented edge.

Now consider a finite sum[(1)]

$$y = n_1 t_1 + n_2 t_2 + \cdots + n_\ell t_\ell \tag{3.14}$$

of different tiles $t_k = (\mathbf{x}_k, i_k \wedge j_k)$. Assume that if any two tiles in (3.14) intersect at more than a vertex, they are adjacent. Define $\partial(y) = \sum n_k \partial(t_k)$, which is a cycle. If the cycle $\partial(y)$ is positive semi-simple, then we say y is a (positive) *tiling patch*. Note that this implies that $n_1 = n_2 = \cdots = n_k = 1$. Tiling patches, modulo translation are, called *tiling proto-patches*, and are denoted by $(\mathcal{B}^2_+)^*$.

(1) This is in the same spirit as a cycle as a sum of closed curves.

3.5. Positive tiling substitutions

Our goal in this section is to give a definition of a positive tiling substitution analogous to the definition $\theta : \mathcal{B} \to \mathcal{B}^*$ of an ordinary substitution. Basically it will be a mapping $\Theta : \mathcal{B}_+^2 \to (\mathcal{B}_+^2)^*$. But, whereas in the case of a substitution, essentially any mapping works, we have two additional requirements for a tiling substitution: (a) Θ needs to be able to apply to tiles (i.e., translations of prototiles), and (b) Θ needs to apply consistently to adjacent tiles in a tiling patch, and still output a tiling patch. We will achieve these two goals using the next definition.

Definition 3.3. — *A* tiling substitution *is a mapping* $\Theta : \mathcal{B}_+^2 \to (\mathcal{B}_+^2)^*$ *such that there is a morphism* $\theta : \mathcal{B} \to \mathcal{F}\langle\mathcal{B}\rangle$ *with the property that*

$$\partial(\Theta(i \wedge j)) = \theta([i, j]) \text{ for each } i \wedge j \in \mathcal{B}_+^2. \tag{3.15}$$

We abbreviate (3.15) $\theta = \partial\Theta$. In effect, this says that $\Theta(i \wedge j)$ should be a tiling of the inside of the curve $\theta([i, j])$. Since $\Theta(i \wedge j)$ is a positive tiling patch, it follows that its boundary $\theta([i, j])$ is a positive semi-simple curve. More generally, we define $\Theta(\mathbf{x}, i \wedge j) := (A_u\mathbf{x}, \Theta(i \wedge j))$.

Lemma 3.4. — *Let Θ be a positive tiling substitution. Suppose $y = t_1 + t_2 + \cdots + t_\ell \in (\mathcal{B}_+^2)^*$, i.e., y is a positive tiling patch. Define $\Theta(y) = \Theta(t_1) + \Theta(t_2) + \cdots + \Theta(t_\ell)$. Then $\Theta(y) \in (\mathcal{B}_+^2)^*$. i.e., $\Theta(y)$ is a positive tiling patch.*

Corollary 3.5. — *The iterates $\Theta^n(i \wedge j)$ for $i \wedge j \in \mathcal{B}_+^2$, and $\Theta^n(y)$, for $y \in (\mathcal{B}_+^2)^*$, are well defined for all $n \geqslant 0$.*

Proof of Lemma 3.4. — For $i \wedge j$ define $A_u(i \wedge j)$ to be the tile whose boundary is the curve connecting the points $\mathbf{0}, \mathbf{p}^i, \mathbf{p}^i + \mathbf{p}^j, \mathbf{p}^j, \mathbf{0}$. For $t_k = (\mathbf{x}_k, i_k \wedge j_k)$ in y, let $A_u t_k = (A_u\mathbf{x}_k, A_u(i_k \wedge j_k))$, and let $A_u y = A_u t_1 + \cdots + A_u t_\ell$. Then $A_u y$ is a tiling patch.

Similarly, define $\theta(t_k) = \theta(\mathbf{x}_k, i_k \wedge j_k) := (A_u\mathbf{x}_k, \overline{[\theta(i_k), \theta(j_k)]})$ using Definition 3.2, and put $\theta(y) = \theta(t_1) + \cdots + \theta(t_\ell)$. Each of these tiles has four "edges", which along the boundary are labeled $\theta(i_k)$, $\theta(j_k)$, $\theta(i_k^{-1})$ and $\theta(j_k^{-1})$.

We have

$$\begin{aligned} \partial(\theta(t_k)) &= (A_u\mathbf{x}_k, [\theta(i_k), \theta(j_k)]) \\ &= (A_u\mathbf{x}_k, \theta([i_k, j_k])) \\ &= \partial(\Theta(\mathbf{x}_k, i_k \wedge j_k)). \end{aligned} \tag{3.16}$$

Since Θ is a positive tiling substitution, its boundary is a positive semi-simple closed curve. Thus by (3.16), $\partial(\theta(t_k))$ is a positive semi-simple closed curve, and $\theta(t_k)$ is a tile.

If t_h and t_k are adjacent in y then $A_u t_h$ and $A_u t_k$ are adjacent in $A_u y$. Moreover, both endpoints of each edge of $A_u t_k$ are endpoints of the edges of $\theta(t_k)$. Thus $\theta(t_h)$ and $\theta(t_k)$ are adjacent in $\theta(y)$. It follows that $\theta(y)$ is a "positive tiling patch" by the tiles $\theta(t_k)$.

Each term $\Theta(t_k)$ in $\Theta(y)$ is a tiling patch with boundary $\partial(\Theta(\mathbf{x}_k, i_k \wedge j_k))$, which by (3.16) is $\partial(\theta(t_k))$. Since they only overlap on their edges (with opposite orientations), their sum is positive semi-simple closed curve, and it follows that $\Theta(y)$ is a positive tiling patch. □

4. de Bruijn diagrams

4.1. The definition of a de Bruijn diagram

Starting with a cyclic word $W \in (\mathcal{B}^{\pm})^*$, we will construct an object Y, called a *de Bruijn diagram*, which depends (in a non-unique way) on W. We will think of such a diagram as a combinatorial representation of a tiling patch with W as its boundary. However, we will need to drop the restriction that a patch be a tiling by positive tiles, and allow negative and trivial tiles as well.

Suppose $W = i_1^{a_1} i_2^{a_2} \dots i_n^{a_n}$. Let $c : [0,1] \to \mathbb{R}^2$ be a positive simple closed curve. Starting at $c(0)$ we follow c and attach n arrows normal to c, labeled $i_1, i_2, \dots, i_n$. We make the kth arrow *point in* if $a_k = 1$ and *point out* if $a_k = -1$ (see Figure 4.1 (a)). This is called the *frame* of the (still to be defined) de Bruijn diagram Y. We denote the frame by $\partial(Y)$. Up to a diffeomorphism of $\mathbb{R}^2$, $\partial(Y)$ is uniquely defined by W.

Next we describe how to get from the frame $\partial(Y)$ to the diagram Y. Since W is cyclic, $f(W) = 0$, and thus for each $i \in \mathcal{B}$ there are the same number of i labeled in- and out-arrows. We choose some arbitrary matching of these and connect each matched pair by a non-self intersecting curve through the inside of c, called a *pseudo-line*. Assume that at most two pseudo-lines cross at any point, and any two pseudo-lines cross transversally. The resulting picture is a de Bruijn diagram Y (see Figure 4.1).

Each pseudo-line in Y is labeled by some $i \in \mathcal{B}$, and is oriented by its arrows. Two diagrams that differ by an orientation preserving diffeomorphism of $\mathbb{R}^2$ are considered the same. The frame $\partial(Y)$ of a de Bruijn

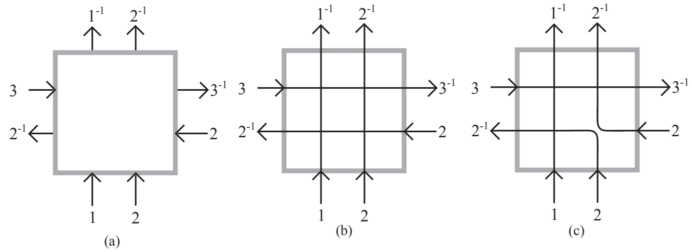


Figure 4.1. (a) The frame for $W = [W_1, W_2] = 1223^{-1}2^{-1}1^{-1}32^{-1}$ *where c is a square. Two corresponding de Bruijn diagrams: (b) the product* $W_1 \wedge W_2$*, and (c) the only other possibility in this case.*

diagram Y determines the cyclic word W up to a cyclic permutation. We write $\partial(Y) = W$.

The *vertices* of a de Bruijn diagram Y are defined to be the crossings of its pseudo-lines. A diagram Y with no vertices is called *trivial*. The pseudo-lines in Y divide the inside of c into finitely many *faces* F, the edges E of which are either pseudo-line segments or segments of c. A face with no c segment edges is called an *internal* face. A vertex that is an intersection of two pseudo-lines is called an *internal vertex*.

Given a de Bruijn diagram Y, we can draw a neighborhood of any internal vertex as shown in Figure 4.2, using a (possibly orientation reversing) change of coordinates. In particular, we have a crossing of a straight

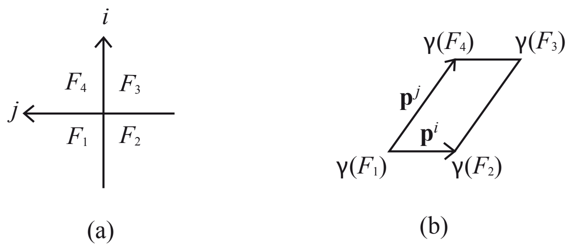


Figure 4.2. (a) A vertex $\mathbf{v}$ *of type* $i \wedge j$ *in canonical coordinates, showing the four faces* F_1, F_2, F_3, F_4*. (b) The* γ *image of* F_1, F_2, F_3, F_4 *in the case* $i \wedge j$ *is positive. In particular, this illustrates the case* $i \wedge j = 1 \wedge 3$ *in the Ammann matrix example.*

pseudo-line segment labeled i, pointing up, and a straight pseudo-line segment labeled j, pointing left. We say this vertex is of type $i \wedge j$.

DEFINITION 4.1. — *A vertex of type $i \wedge j$ in a de Bruijn diagram Y is called* positive *if* $\operatorname{sgn}(i \wedge j) > 0$. *A de Bruijn diagram Y is* positive *if all its vertices are positive.*

4.2. The tiling patch corresponding to a diagram

Here we describe how to go from a nonsingular de Bruijn diagrams Y to the corresponding tiling patch y. The idea—to use planar graph duality—comes from de Bruijn's algebraic theory of Penrose tilings [5]. In particular, an internal type $i \wedge j$ vertex in Y corresponds to a tile t of type $i \wedge j$. Similarly, each face F in Y corresponds, under duality, to a tile vertex in y. We will start with this correspondence.

Let $\mathcal{F}$ denote the set of all faces in Y. Given a vector $\mathbf{x}_0$ and some $F_0 \in \mathcal{F}$, we define $\gamma : \mathcal{F} \to \mathbb{R}^2$ as follows. Put $\gamma(F_0) = \mathbf{x}_0$, and proceed by induction. Suppose γ has been defined on a set $\mathcal{F}_k \subseteq \mathcal{F}$ of $k < \#(\mathcal{F})$ faces. Let $F \in \mathcal{F} \backslash \mathcal{F}_k$ be adjacent to some $F' \in \mathcal{F}_k$ across a pseudo-line segment labeled i. Put $a = 1$ or $a = -1$ depending on whether segment i is oriented to the left or to the right in crossing from F' to F. Define $\gamma(F) = \gamma(F') + a\mathbf{p}^i$, and put $\mathcal{F}_{k+1} = \mathcal{F}_k \cup \{F\}$.

LEMMA 4.2. — *Suppose Y is a nonsingular de Bruijn diagram. Let F_1, F_2, F_3 and F_4 be faces surrounding a type $i \wedge j$ vertex in Y, as shown in* Figure 4.2 (a). *Then $\gamma(F_1)$, $\gamma(F_2)$, $\gamma(F_3)$ and $\gamma(F_4)$ are the vertices of the tile* $(\gamma(F_1), i \wedge j)$ (Figure 4.2 (b)), *and the boundary of this tile is the piecewise linear curve corresponding to the sequence of points $\gamma(F_1)$, $\gamma(F_2)$, $\gamma(F_3)$, $\gamma(F_4)$, $\gamma(F_1)$.*

Proof. — We can assume without loss of generality that $\gamma(F_1) = \mathbf{0}$. Then $\gamma(F_2) = \mathbf{p}^i$, $\gamma(F_4) = \mathbf{p}^j$, $\gamma(F_3) = \mathbf{p}^i + \mathbf{p}^j$ (see Figure 4.2 (b)). □

Lemma 4.2 says that a positive de Bruijn diagram describes a sort of "locally correct" tiling patch. It gives each tile a precise location, and tiles corresponding to adjacent vertices meet across complete edges. However, there is no *a priori* guarantee that different parts of the tiling do not overlap. The next result shows this cannot happen if the boundary choice is correct.

PROPOSITION 4.3. — *Let $P = (\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^d)$ be a good star of vectors and let $f : \mathcal{F}\langle\mathcal{B}\rangle \to \mathbb{Z}^d$ be the abelianization homomorphism. Suppose $W = [W_1, W_2]$, with $W_1, W_2 \in \mathcal{F}\langle\mathcal{B}\rangle$, satisfies*

$$\det\big([Pf(W_1), Pf(W_2)]\big) > 0.$$

If Y is a positive de Bruijn diagram with $\partial(Y) = W$, then Y corresponds to a positive tiling patch y with $\partial(y) = (\mathbf{x}, W)$ for some $\mathbf{x} \in \mathbb{R}^2$.

Proof. — By Lemma 4.2, the vertices of Y correspond to positive tiles $t_1, t_2, \ldots, t_\ell$. Let $T_i = \partial(t_i)$. Then T_i is a simple closed curve, which is positive since Y is positive. Thus $n_{T_i}(\mathbf{x}) = 1$ if $\mathbf{x}$ is inside T_i, and 0 otherwise. Also

$$\sum_{i=1}^{\ell} T_i = \partial(Y) = W, \tag{4.1}$$

since two adjacent tiles have oppositely oriented boundaries.

Let $\mathbf{x}$ be inside W. Then (4.1) implies

$$n_W(\mathbf{x}) = \sum_{i=1}^{\ell} n_{T_i}(\mathbf{x}) \geqslant 0. \tag{4.2}$$

Since $\det\big([Pf(W_1), Pf(W_2)]\big) > 0$, Proposition 3.1 implies $n_W(\mathbf{x}) = 1$, from which it follows that the sum in (4.2) is $\leqslant 1$ for all non-trace $\mathbf{x}$. Thus any $\mathbf{x}$ inside W can be in at most one tile t_i, so that two tiles only can intersect along their boundaries. This means $y = t_1 + t_2 + \cdots + t_\ell$ is a positive tiling patch. □

Conversely, if y is a positive tiling patch with $\partial(y) = (\mathbf{x}, W)$, then there exists a de Bruijn diagram Y corresponding to y such that $\partial(Y) = W$. This fact, which we do not use, is discussed in [5]. In general, we think of a not necessarily positive de Bruijn diagram as a *generalized* tiling proto-patch by positive, negative and trivial tiles. This is illustrated in Section 6.1. We denote the set of all these proto-patches by $(\mathcal{B}^2)^*$.

4.3. Product diagrams and tiling substitutions

From now on we will consider only diagrams Y where $\partial(Y) = [V, W]$ for $V, W \in \mathcal{F}\langle\mathcal{B}\rangle$. For the frame, we take c to be the unit square. We put V along the bottom, put W going up the right side, put V^{-1} going backwards along the top, and put W^{-1} going down along the left side (see Figure 4.1 (a)).

The simplest diagram of this type is a *product diagram*, denoted $Y = V \wedge W$. To obtain it, we start with the frame described above and connect the arrows by vertical and horizontal lines (see Figure 4.1 (b)). It follows that

$$\partial(V \wedge W) = [V, W]. \tag{4.3}$$

The following "matrix notation" for a product diagram will be useful. Suppose $V = v_1 v_1, \ldots v_\ell$ and $W = w_1 w_2 \ldots w_n$. We write

$$V \wedge W = \begin{bmatrix} v_1 \wedge w_n & v_2 \wedge w_n & \ldots & v_\ell \wedge w_n \\ \vdots & \vdots & \ldots & \vdots \\ v_1 \wedge w_1 & v_2 \wedge w_1 & \ldots & v_\ell \wedge w_1 \end{bmatrix}$$

In the matrix above $i^p \wedge j^q$ is the intersection of a vertical pseudo-line labeled i a horizontal pseudo-line labeled j. We have $p = 1$ if the i arrow points up and $p = -1$ if the i arrow points down. Also $q = 1$ if the j arrow points left and $q = -1$ if the j arrow points right. This vertex is equivalent to $pq(i \wedge j)$.

5. Cancellation

5.1. The three moves on de Bruijn diagrams

Let Y be a de Bruijn diagram. In this section we will describe three "moves" μ that can be applied to a diagram Y to obtain a new diagram Y'. Since the moves are reversible denote them $Y \overset{\mu}{\leftrightarrow} Y'$.

The first move μ_1 is called the *flip*. In it, we slide a pseudo-line across a vertex. It is called the flip move because in a (positive) tiling it implements

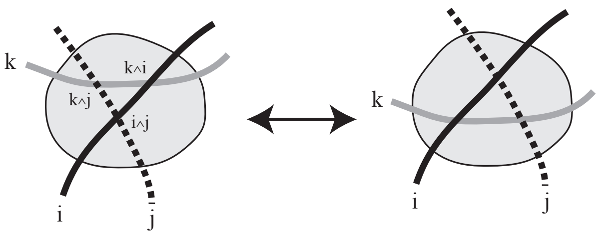


Figure 5.1. The flip move μ_1.

the "Necker cube flip" (see Figure 5.2).

The second move μ_2 is called *cancellation*. It takes a pair of opposite sign adjacent vertices, $i \wedge j$ and $-(i \wedge j)$, and cancels them by uncrossing two loops (see Figure 5.3).

The third move μ_3 is called *trivial tile elimination*. It may be used to cancel a trivial tile $i \wedge i$, which occurs when two pseudo-lines with the same

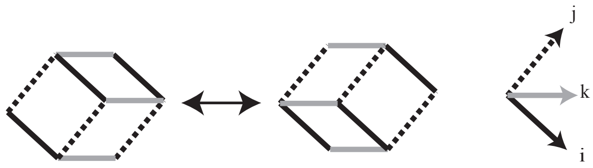


Figure 5.2. The Necker cube flip, implemented by the flip move μ_1.

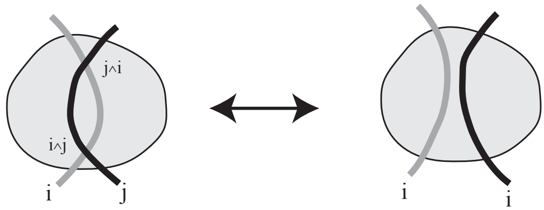


Figure 5.3. The cancellation move μ_2.

label cross (see Figure 5.4). Move μ_3 is different form the other two for two reasons: (i) pseudo-lines are cut and reconnected in a different way, and (ii) it depends on the orientation.

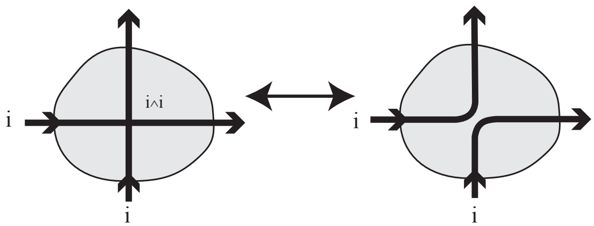


Figure 5.4. Trivial tile elimination move μ_3.

Definition 5.1. — *Two de Bruijn diagrams are called μ-equivalent if one can be obtained form the other by a finite series of moves.*

Lemma 5.2. — *If Y and Y' are μ-equivalent de Bruijn diagrams, then $\partial(Y') = \partial(Y)$.*

5.2. Counting tiles

For a de Bruijn diagram Y based on $\mathcal{B} = \{1, 2, \ldots, d\}$, let $M(Y)$ be the $d \times d$ matrix with i, jth entry $m_{i\wedge j}$ equal to the number of type $i \wedge j$ vertices that occur in Y. Let $\mathbf{e}_{i\wedge j}$ be the $d \times d$ matrix with $(i \wedge j)$th entry equal to 1 and all other entries 0.

Lemma 5.3. — *Let Y and Y' be de Bruijn diagrams with $\partial(Y) = \partial(Y')$.*

(i) *If $Y \overset{\mu_1}{\leftrightarrow} Y'$ (i.e., a flip) then $M(Y') = M(Y)$.*

(ii) *If $Y \overset{\mu_2}{\leftrightarrow} Y'$, where μ_2 implements a cancellation of $i \wedge j$ and $j \wedge i$, then $M(Y') = M(Y) - \mathbf{e}_{i\wedge j} - \mathbf{e}_{j\wedge i}$.*

(iii) *$Y \overset{\mu_3}{\leftrightarrow} Y'$, where μ_3 implements elimination of a trivial tile $i \wedge i$, then $M(Y') = M(Y) - \mathbf{e}_{i\wedge i}$.*

For $i \wedge j \in \mathcal{B}^2$, $i \neq j$ let $|i \wedge j| = \mathrm{sgn}(i \wedge j)(i \wedge j)$. That is to say, for a nontrivial tile $i \wedge j$, $|i \wedge j|$ is its *positive* version. Now suppose Y is a de Bruijn diagram and define the vector $f^*(Y) \in \mathbb{Z}^{\binom{d}{2}}$ to have entries

$$f^*(Y)_{i\wedge j} = \mathrm{sgn}(i \wedge j)(m_{|i\wedge j|} - m_{-|i\wedge j|}), \tag{5.1}$$

$i < j$ in lexicographic order.

Corollary 5.4. — (of Lemma 5.3) *If Y and Y' are μ-equivalent de Bruijn diagrams, then $f^*(Y) = f^*(Y')$.*

Proposition 5.5. — *Suppose θ is an endomorphism on $\mathcal{B} = \{1, \ldots, d\}$ and let $A = L_\theta$. Then the* "product tiling substitution" $\theta \wedge \theta : \mathcal{B}^2_+ \to (\mathcal{B}^2)^*$, *defined by*

$$(\theta \wedge \theta)(i \wedge j) := \theta(i) \wedge \theta(j), \tag{5.2}$$

satisfies

$$A^* = (f^*((\theta \wedge \theta)(1 \wedge 2)), f^*((\theta \wedge \theta)(1 \wedge 3)), \ldots, f^*((\theta \wedge \theta)((d-1) \wedge d))).$$

Remark 5.6. — The product tiling substitution $\theta \wedge \theta$ is essentially the same as $E_1^*(\theta)$ in [4].

Proof. — First note that by (2.8) and (3.13) that A^* has entries

$$a^*_{i\wedge j, k\wedge \ell} = \mathrm{sgn}(i \wedge j)\mathrm{sgn}(k \wedge \ell) \det \begin{pmatrix} a_{i,k} & a_{i,\ell} \\ a_{j,k} & a_{j,\ell} \end{pmatrix},$$

where $a_{i,j}$ are the entries of A. By (5.1)

$$\begin{aligned} f^*((\theta \wedge \theta)(k \wedge \ell))_{i\wedge j} &= \mathrm{sgn}(k \wedge \ell)\big(m_{|i\wedge j|} - m_{-|i\wedge j|}\big) \\ &= \mathrm{sgn}(i \wedge j)\mathrm{sgn}(k \wedge \ell)(m_{i\wedge j} - m_{j\wedge i}), \end{aligned}$$

where the $m_{i\wedge j}$ are the entries of $M((\theta\wedge\theta)(k\wedge\ell))$.

Since $L_\theta = A$, it follows that $m_{i\wedge j} = a_{i,k}a_{j,\ell}$ and $m_{j\wedge i} = a_{j,k}a_{i,\ell}$. Thus

$$m_{i\wedge j} - m_{j\wedge i} = \det\begin{pmatrix} a_{i,k} & a_{i,\ell} \\ a_{j,k} & a_{j,\ell}\end{pmatrix},$$

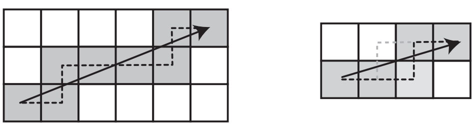
and $f^*((\theta\wedge\theta)(k\wedge\ell))_{i\wedge j} = a^*_{i\wedge j,k\wedge\ell}$. □

5.3. Properly ordered words

Let us write $W = i_1^{a_1} i_2^{a_2} \dots i_\ell^{a_\ell} \in \mathcal{F}\langle\mathcal{B}\rangle$, where $a_j \in \mathbb{Z}$ (i.e., $a_j = 0$ is allowed). We call W *efficient* if whenever $i_j = i_k$, we have that $a_j a_k \geqslant 0$. We say W is positive if $a_j \geqslant 0$ for all j, which implies $f(W) \geqslant 0$. Two positive words are *disjoint* if they have no symbols in common. A positive word has *natural order* if it has the form $W = 1^{a_1}2^{a_2}\dots d^{a_d}$. In effect, we have imposed an arbitrary order on $\mathcal{B}$. In general, a word W has *natural order* if it has the form $W = W_1^{-1}W_2$, where W_1 and W_2 are disjoint, positive natural order words.

The following construction is an elaboration of one from [13]. Let $\mathbf{h} = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{R}^d$. Partition $\mathbb{R}^d$ into unit cubes parallel to the coordinate axes. Each such cube contains a unique vector $\mathbf{a}+\mathbf{h}$, where $\mathbf{a} \in \mathbb{Z}^d$, called its *label*. Given a vector $\mathbf{b} \in \mathbb{Z}^d$ we are going to define a word $W = W(\mathbf{b})$, called the *properly ordered word* for $\mathbf{b}$.

Let ℓ be the line segment connecting $\mathbf{h}$ to $\mathbf{b}+\mathbf{h}$. Consider the sequence of cubes that ℓ intersects, and let $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_n$ be the sequence of their labels. Assume first that any pair of adjacent cubes in the sequence meets in a $d-1$ dimensional face (see Figure 5.5 (a)). Then for each i, $\mathbf{b}_i - \mathbf{b}_{i-1} = (0,0,\dots,0,\pm 1,0,\dots,0) \in \mathbb{Z}^d$, where the ± 1 occurs at position k. We define the properly ordered word $W = W(\mathbf{b})$ to be the word with ith entry $k^{\pm 1}$.



(a) (b)

Figure 5.5. (a) The vector $\mathbf{b} = (5,2) \in \mathbb{Z}^2$ *gives* $W(\mathbf{b}) = 121^3 21$. *(b) For* $\mathbf{b} = (3,1)$, $W' = 1{*}1$, *with* $* = 12$ *proper (i.e.,* $* \neq 21$*), and* $W(\mathbf{b}) = 1^2 21$.

Now suppose there are places in the sequence where cube $i-1$ and cube i meet along face of dimension less than $d-1$. Construct a word W' as above, but insert the symbol $*_i$ as a place-holder at position i. If the two cubes meet along a $d-m$ dimensional face, for $m>1$, then $\mathbf{a}_i = \mathbf{b}_i - \mathbf{b}_{i-1} \in \{0,1,-1\}^d$ has m nonzero entries. We write $\mathbf{a}_i = \mathbf{a}_i^+ - \mathbf{a}_i^-$, where $\mathbf{a}_i^+, \mathbf{a}_i^- \geqslant 0$, and let $\mathbf{a}_i^+ = (a_1, a_2, \ldots, a_d)$ and $\mathbf{a}_i^- = (a'_1, a'_2, \ldots, a'_d)$. Define $V_- = 1^{a'_1} 2^{a'_2} \ldots d^{a'_d}$, $V_+ = 1^{a_1} 2^{a_2} \ldots d^{a_d}$, and $V_i^* = (V_-)^{-1} V_+$. A crucial point is that V_i^* is a naturally ordered word. The properly ordered word $W = W(\mathbf{b})$ is defined to be the word W' with the entries $*_i$ replaced by the words V_i^* (see Figure 5.5 (b)). As above, we can also realize W as a piecewise linear curve with vertices $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$ (see Figure 5.5 (b)). We call this piecewise linear curve, the *broken line* for $\mathbf{b}$.

Lemma 5.7. — *For any $\mathbf{b} \in \mathbb{Z}^d$ the properly ordered word $W = W(\mathbf{b})$ satisfies $f(W) = \mathbf{b}$. Moreover, if $\mathbf{0} = \mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n = \mathbf{b}$ is the corresponding broken line, then for all k, $f(W_k) = \mathbf{b}_k$, where W_k denotes the kth prefix of W.*

Lemma 5.8. — *A properly ordered word W is always efficient. If W is properly ordered and $f(W) \in \{0,1,-1\}^d$, then W is naturally ordered.*

We say an endomorphism θ is *properly ordered* if $\theta(i)$ is a properly ordered word for each $i \in \mathcal{B}$. It is easy to see that for any $d \times d$ integer matrix A, there exists a properly ordered endomorphism θ so that $L_\theta = A$.

Definition 5.9. — *A nonsingular de Bruijn diagram Y is called* efficient *if it has no two vertices equivalent to $i \wedge j$ and $-(i \wedge j) = j \wedge i$.*

Compare this to the definition of an efficient word at the beginning of this section. We think of an efficient de Bruijn diagram as one that has been simplified. The following should be clear.

Lemma 5.10. — *The number of vertices of type $\operatorname{sgn}(f^*(Y)_{i\wedge j})|i \wedge j|$, $i<j$, in an efficient de Bruijn diagram Y is $|f^*(Y)_{i\wedge j}|$. In particular, if $f^*(Y) \geqslant 0$, then Y is a positive de Bruijn diagram.*

5.4. Simplifying diagrams

Theorem 5.11. — *Let $V, W \in \mathcal{F}\langle\mathcal{B}\rangle$ be properly ordered. Then there exists an efficient de Bruijn diagram Y that is μ-equivalent to $V \wedge W$.*

To motivate the proof, we first explain the basic idea (including a cautionary negative example). Let $V = ij$ with $i<j$, so that V is positive and

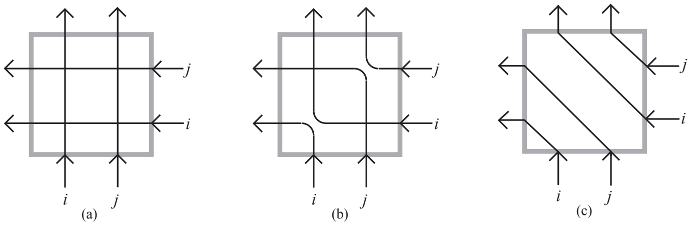


Figure 5.6. (a) The product $V \wedge V$. (b) Elimination of $i \wedge i$ and of $j \wedge j$ using μ_3. (c) Cancellation of $i \wedge j$ and $j \wedge i$ using μ_2.

naturally ordered, and consider the diagram $V \wedge V$. Notice that $f^*(V \wedge V)$ is zero since $j \wedge i = -(i \wedge j)$. The diagram is reduced to a trivial diagram as shown in Figure 5.6. Now let W be one of the other three naturally ordered two letter words. The same idea works for $V \wedge W$ where $W = j^{-1}i^{-1}$. On the other hand, for $W = j^{-1}i$, there is no need for cancellation after elimination, because both remaining vertices are type $-(i \wedge j)$. The same argument works for $W = i^{-1}j$. However, if we take the similar looking diagram $V \wedge W$, where $V = ij$ but $W = ji$, then cancellation fails (see Figure 5.7). This demonstrates the importance of natural order in the words

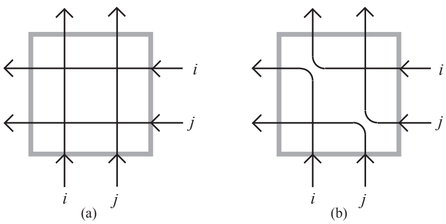


Figure 5.7. Badly ordered words lead to a diagram where cancellation fails.

V_i^* in the definition of proper order.

The remainder of this section constitutes the proof of Theorem 5.11. The proof proceeds by induction.

Let Y_1 be the diagram obtained by using μ_3 on $V \wedge W$ to eliminate all the trivial vertices.

Now suppose we have Y_n, $n \geqslant 1$. Take $i < j$ so that the total number of vertices of type $i \wedge j$ or $j \wedge i$ in Y_n is not zero. Let $r_{i \wedge j} = |f^*(Y_n)_{i \wedge j}| \geqslant 0$ and $s_{i \wedge j} = \operatorname{sgn}(f^*(Y_n))$. There are $r_{i \wedge j}$ more vertices of type $s_{i \wedge j}|i \wedge j|$

than there of type $-s_{i\wedge j}|i\wedge j|$. Call the latter *target vertices*. We would like to cancel each target vertex in Y_n with one of its negatives. Let N_n be the total number of target vertices in Y_n. If $N_n = 0$ then the diagram is effcient and the proof is done.

LEMMA 5.12. — *Let $V, W \in \mathcal{F}\langle\mathcal{B}\rangle$ be properly ordered. Suppose Y is the de Bruijn diagram obtained by eliminating all the trivial vertices from $V \wedge W$. Then for each target vertex v of type $\pm(i\wedge j)$ in Y, $i < j$, there is a unique type $\mp(i \wedge j)$ vertex v' connected to v by a pseudo-line labeled i and a pseudo-line labeled j. Furthermore, we may assume that the i line and j line do not cross between v and v'.*

In other words, the two vertices are connected like the vertices in Figure 5.6 (b). We call v_2 the *partner* of v_1, and call $S = \{v_1, v_2\}$ a *partner vertex pair*.

Proof. — Assume without loss of generality that $i = 1$, $j = 2$. First let V, W be positive, so that $V, W \in \{1, 2\}^*$. The algorithm for constructing properly ordered words commutes with the projection to the $(1, 2)$-plane. We identify this plane with $\mathbb{R}^2$, so that $f(V), f(W) \in \mathbb{Z}^2$.

In the properly ordered word algorithm, the squares surronding $f(V)$ and $f(W)$ also contain the broken lines for V and W (see Figure 5.8 (a)). If $f(V)$ and $f(W)$ are not parallel, then one broken line must lie to one side of the other, although not necessarily strictly. We call this the *separation property*. On the other hand, if $f(V)$ and $f(W)$ are parallel, then because the symbols $*_i$ were all replaced by natural order words V_i^*, the shorter broken line must follow the longer one for its entire length. Again, this is "*separation*", although here the separation is everywhere not strict.

The separation property extends to the two piecewise linear paths defined by VW and WV that approximate the parallelogram spanned by $f(V)$ and $f(W)$, (see Figure 5.8 (a)). Write the words $VW = x_1x_2 \ldots x_n$ and $WV = y_1y_2 \ldots y_n$ symbolically, and denote the vertices of the corresponding broken lines by $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$ and $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n$ respectively. For $\mathbf{z} = (z_1, z_2) \in \mathbb{Z}^2$, define $\nu(\mathbf{z}) = z_2 - z_1$. The separation property implies that either $\nu(\mathbf{x}_k) \geqslant \nu(\mathbf{y}_k)$ or $\nu(\mathbf{y}_k) \geqslant \nu(\mathbf{x}_k)$ for all $k = 1, 2, \ldots, n$. Lemma 5.7 implies that $\nu(f(VW_k)) \geqslant \nu(f(WV_k))$ or $\nu(f(WV_k)) \geqslant \nu(f(VW_k))$ for all k. For concreteness, assume that the former holds. This means that for each k, there are more symbols 1 in VW_k than in WV_k (and more symbols 2 in WV_k).

Let Y be the diagram obtained from $V \wedge W$ by using μ_3 to remove all the trivial tiles, and let Y' be the diagram obtained from Y by stretching all the pseudo-lines in Y into lines (see Figure 5.8 (b)). To do this, it is

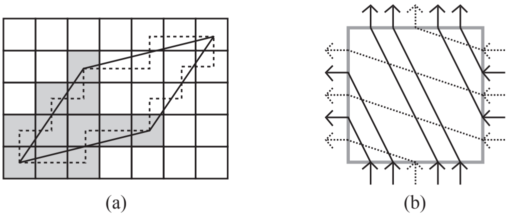


Figure 5.8. (a) Let $V = 1^2 21^2$ *and* $W = 21212$*. Then* WV *is above* VW *since* $f(W)$ *is above* $f(V)$*. (b) The correspinding simplified diagram* Y'*. All vertices are type* $1 \wedge 2$*.*

necessary to use μ_2 to cancel some target vertices v of type $2 \wedge 1$ with partners v' of type $1 \wedge 2$. We can assign these partners in such a way that their 1- and 2-labeled pseudo-lines never cross between v and v'.

In the boundary, the symbols in VW correspond to the in-arrows encountered in going counter-clockwise along the bottom and right, and the symbols in WV correspond to out-arrows, going clockwise around the left and top. By the separation property, all the 2-labeled pseudo-lines cross the 1-labeled pseudo-lines from right to left, so in Y' all vertices are of type $1 \wedge 2$. It follows that all vertices of type $2 \wedge 1$ were cancelled. This shows that all the target vertices in Y were assigned to partners.

So far we have assumed V and W are positive, which means $f(V)$ and $f(W)$ lie in the first quadrant. Essentially the same proof works whenever $f(V)$ and $f(W)$ lie in the same quadrant. If $f(V)$ and $f(W)$ lie in opposite quadrants, replace $V \wedge W$ with $W^{-1} \wedge V$, which is easily seen to be the same diagram, but $f(W^{-1}) = -f(W)$ is in the same quadrant as $f(V)$. Finally, if $f(V)$ and $f(W)$ lie in adjacent quadrants, there is no cancellation necessary.

□

Assume $N_n > 0$. Fix a target vertex v_1 in Y_n and let v_2 be its partner, so that $S = \{v_1, v_2\}$ is a partner vertex pair. If there are no other pseudo-lines crossing the loop connecting v_1 and v_2, we can cancel using μ_2 to obtain a diagram Y_{n+1} with $N_{n+1} = N_n - 1$. However, there may be other pseudo-lines crossing the loop, and there may be other vertices *inside* the loop. Let G be the vertices inside the loop, and let I and J be the vertices from pseudo-lines crossing the i and j sides of the loop. Let H be the remaining vertices, outside the picture (see Figure 5.9 (a)).

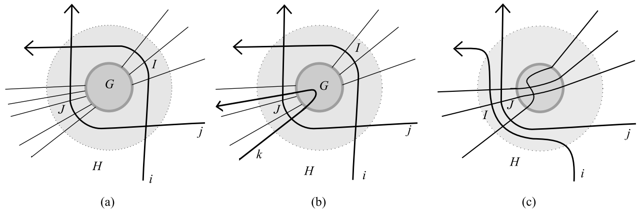


Figure 5.9. The induction step: (a) The vertices in G cross in an unknown way. (b) A pseudo-line k in crosses j twice. (c) Once there are no double-crossings, the i peseudo-line moves across G.

Since all vertices are non-trivial, no pseudo-line labeled i or i^{-1} can cross the i part of the loop, and likewise for j. However, pseudo-lines labeled j or j^{-1} may cross the i part of the loop, and vice versa.

We may assume without loss of generality that there is no vertex v of type $\pm i \wedge j$ in G. If there is, then by the previous paragraph, v must be part of a partner vertex pair $S' = \{v, v'\} \subseteq G$. In this case, we consider the samller loop between v and v' instead of the loop between v_1 and v_2. In a similar way, we can assume that I and J contain no vertices of type $\pm i \wedge j$.

Next, consider a pseudo-line, labeled k (not i or j), that goes through j into G. We may assume without loss of generality that when it comes out of G, it goes through i and not j. For suppose it crosses j twice (see Figure 5.9 (b)). Then J contains a partner vertex pair $S' = \{v, v'\}$ of types $\pm(j \wedge k)$. In this case, we consider the smaller loop between v and v' instead of the loop between v_1 and v_2. Since a similar argument holds for i, we may assume that all the pseudo-lines through G cross both i and j.

To complete the induction, we move the i loop across G using successive μ_1-moves. Then we cancel the pair S using μ_2. The vertices in $G \cup H$ do not change. Moreover, a consequence of the previous paragraph is that the new I has the same vertices as the old I. But since the two vetices in S have been cancelled, and one was a target vertex, $N_{n+1} = N_n - 1$. The proof now follows by induction. □

6. The main result

THEOREM 6.1. — *Suppose A is a $d \times d$ matrix satisfying* (2.1), (2.4) *and* (2.9). *Then there exists a properly ordered endomorphism θ with $L_\theta = A$, such that there is a positive tiling substitution Θ with $\partial\Theta = \theta$.*

Proof. — Let θ be a properly ordered endomorphism with $L_\theta = A$. For each $i \wedge j \in \mathcal{B}_+^2$, Theorem 5.11 gives an efficient de Bruijn diagram $Y(i \wedge j)$ that is μ-equivalent to $\theta(i) \wedge \theta(j)$.

By Corollary 5.4, $f^*(Y(i \wedge j)) = f^*(\theta(i) \wedge \theta(j))$, and by Proposition 5.5, the vector $f^*(\theta(i) \wedge \theta(j))$ is column $i \wedge j$ of A^*. By (2.9) (which requires (2.1) and (2.4)), $A^* \geqslant 0$. It follows that $f^*(Y(i \wedge j)) \geqslant 0$, and since $Y(i \wedge j)$ is efficient, Lemma 5.10 implies that $Y(i \wedge j)$ is positive. Define $\Theta(i \wedge j)$ to be the geometric realization of $Y(i \wedge j)$. Proposition 4.3 shows that $\Theta(i \wedge j) \in (\mathcal{B}_+^2)^*$, i.e., $\Theta(i \wedge j)$ is a positive tiling patch. Finally we have

$$\begin{aligned} \partial(\Theta(i \wedge j)) &= \partial\big((\theta \wedge \theta)(i \wedge j)\big), && \text{by Lemma 5.2,} \\ &= \partial\big(\theta(i) \wedge \theta(j)\big), && \text{by (5.2),} \\ &= [\theta(i), \theta(j)], && \text{by(4.3),} \\ &= \theta([i,j]), \end{aligned}$$

so by (3.15) $\partial\Theta = \theta$, and Θ is a positive tiling substitution. $\square$

We define the *structure matrix* $\mathcal{L}_\Theta$ of a positive tiling substitution Θ to be the $\binom{d}{2} \times \binom{d}{2}$ integer matrix whose $i \wedge j, k \wedge \ell$ entry gives the number of times $k \wedge \ell$ occurs inside $\Theta(i \wedge j)$.

Corollary 6.2. — *The positive tiling substitution Θ constructed above satisfies $\mathcal{L}_\Theta = A^*$, i.e., $(L_\theta)^* = \mathcal{L}_\Theta$.*

6.1. Example. The Ammann matrix (continued)

Let A be the Ammann matrix (2.10), which by Section 2.4 satisfies (2.1), (2.4) and (2.8). Let P be the projection (2.11).

To define a properly ordered endomorphism, we need for each i that $\theta(i)$ is a properly ordered word with $f(\theta(i))$ equal to the ith column of A. Since the columns of A belong to $\{0, 1, -1\}^4$, Lemma 5.8 implies each $\theta(i)$ should be naturally ordered:

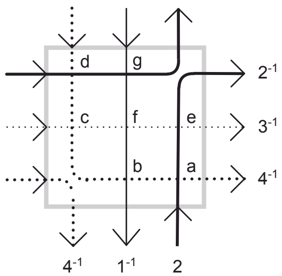
$$\begin{aligned} \theta(1) &= 4^{-1}1^{-1}2 & \theta(2) &= 3^{-1}2^{-1}1 \\ \theta(3) &= 4^{-1}3^{-1}2^{-1} & \theta(4) &= 4^{-1}3^{-1}1^{-1}. \end{aligned}$$

Next we define the product substitution $(\theta \wedge \theta)(i \wedge j) = \theta(i) \wedge \theta(j)$ whose value on each of the six positive prototiles we represent as product de Bruijn diagram.

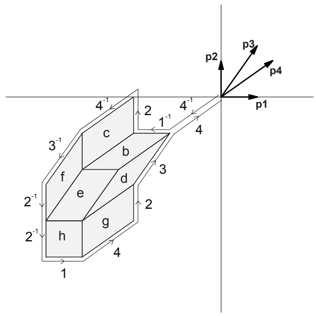
In two cases, $(\theta\wedge\theta)(1\wedge 3)$ and $(\theta\wedge\theta)(4\wedge 2)$, the product diagrams consist of just positive and trivial tiles. For example,

$$(\theta\wedge\theta)(1\wedge 3) = (4^{-1}1^{-1}2)\wedge(4^{-1}3^{-1}2^{-1}) = \begin{bmatrix} 4\wedge 2 & 1\wedge 2 & 2\wedge 2 \\ 4\wedge 3 & 1\wedge 3 & 3\wedge 2 \\ 4\wedge 4 & 1\wedge 4 & 4\wedge 2 \end{bmatrix},$$

in matrix form, where the entries have been simplified to positive prototiles. The resulting μ-equivalent positive de Bruijn diagram $Y(1\wedge 3)$ is shown below (left), as well as its realization $\Theta(1\wedge 3)$ as a tiling patch (right).

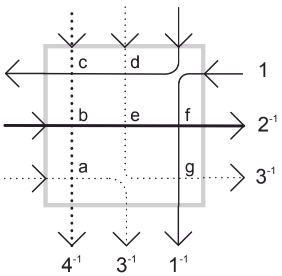


$Y(1\wedge 3)$

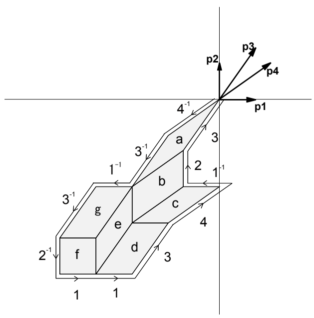


$\Theta\,(1\wedge 3)$

Similarly, for $(\theta\wedge\theta)(4\wedge 2)$ we obtain:



$Y(4\wedge 2)$

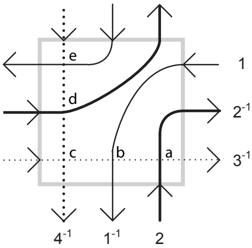


$\Theta\,(4\wedge 2)$

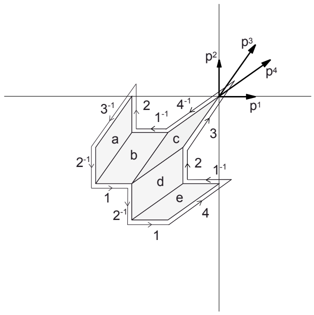
Now consider the case

$$(\theta\wedge\theta)(1\wedge 2) = (4^{-1}1^{-1}2)\wedge(3^{-1}2^{-1}1) = \begin{bmatrix} 1\wedge 4 & 1\wedge 1 & -(1\wedge 2) \\ 4\wedge 2 & 1\wedge 2 & 2\wedge 2 \\ 4\wedge 3 & 1\wedge 3 & 3\wedge 2 \end{bmatrix}.$$

Notice that the upper right 2×2 block has the form shown in Figure 5.6 (a), and can be canceled as shown in Figure 5.6 (b) and (c) to obtain the positive diagram $Y(1 \wedge 2)$ (shown left). The resulting geometric realization $\Theta(1 \wedge 2)$ is shown (right).

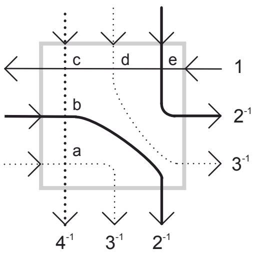


$Y(1 \wedge 2)$

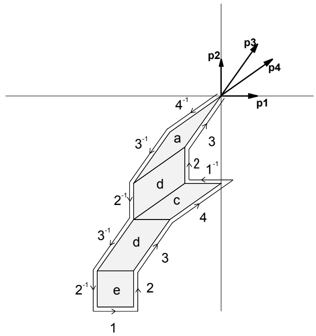


$\Theta\,(1 \wedge 2)$

There are two more cases that work exactly the same way: $(\theta \wedge \theta)(3 \wedge 2)$ and $(\theta \wedge \theta)(4 \wedge 3)$. These are shown below.



$Y(3 \wedge 2)$

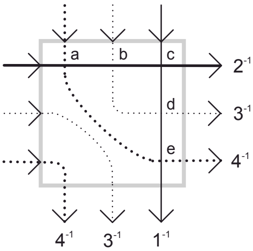


$\Theta\,(3 \wedge 2)$

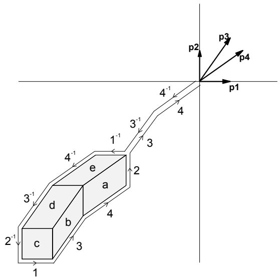
The most complicated case is

$$(\theta \wedge \theta)(1 \wedge 4) = (4^{-1}1^{-1}2) \wedge (3^{-1}2^{-1}1) = \begin{bmatrix} -(1 \wedge 4) & 1 \wedge 1 & 1 \wedge 2 \\ 4 \wedge 2 & 1 \wedge 2 & 2 \wedge 2 \\ 4 \wedge 3 & 1 \wedge 3 & 3 \wedge 2 \end{bmatrix}.$$

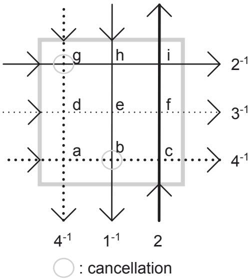
Notice that the required cancellations are not adjacent. We use the method in the proof of Theorem 5.11 as illustrated in Figure 5.6. The steps are shown in detail below.



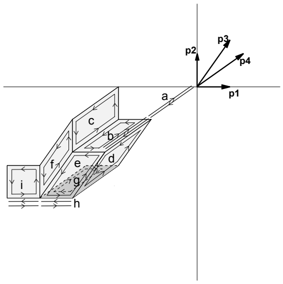
$Y(4 \wedge 3)$



$\Theta\,(4 \wedge 3)$

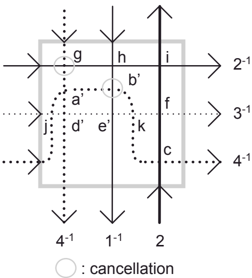


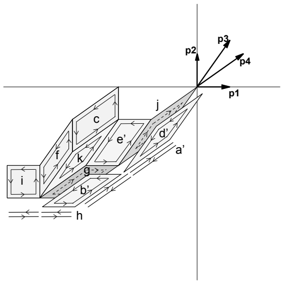
Product diagram
$(\theta \wedge \theta)(1 \wedge 4)$.



$(\theta \wedge \theta)\,(1 \wedge 4)$ as a signed
tiling patch.

↓ ↓





↓ ↓

Finally, we will show how to iterate Θ to obtain tiling patches and tilings. We start with the tiling patch y, shown below, and apply Θ to obtain a sequence of patches $\Theta^N(y)$. The patch y has been carefully chosen so that for each $i \wedge j \in \mathcal{B}^2_+$, a translation of y appears as a sub-patch of $\Theta^N(i \wedge j)$

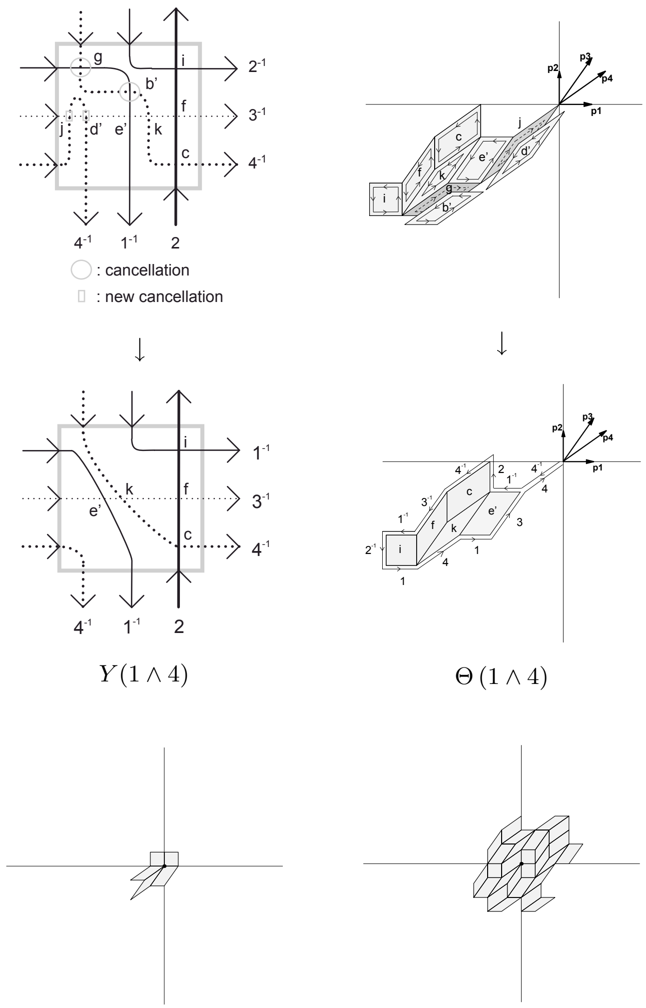


Figure 6.1. The patches y and $\Theta(y)$.

for all N sufficiently large. The existence of such a patch y follows from the fact that $(A^*)^N > 0$ for all $N \geqslant 2$. We say in such a case that Θ is a *primitive* tiling substitution (this means that each $k \wedge \ell$ appears in each $\Theta^N(i \wedge j)$ for all N sufficiently large).

Another important property of the patch y is that y appears as a sub-patch of $\Theta^2(y)$, completely surrounded by other tiles. This is shown in Figure 6.1. It follows that $\Theta^{2N}(y)$ is a sub-patch of $\Theta^{2M}(y)$ for all $M > N$.

Thus the *increasing limit*

$$z := \lim_{N\to\infty} \Theta^{2N}(y)$$

defines a *tiling* of $\mathbb{R}^2$. A swatch of this tiling is shown in Figure 6.4.

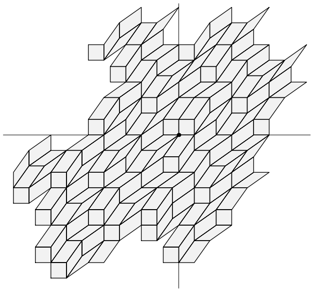


Figure 6.2. The patch $\Theta^2(y)$, showing y as a sub-patch around $\mathbf{0}$.

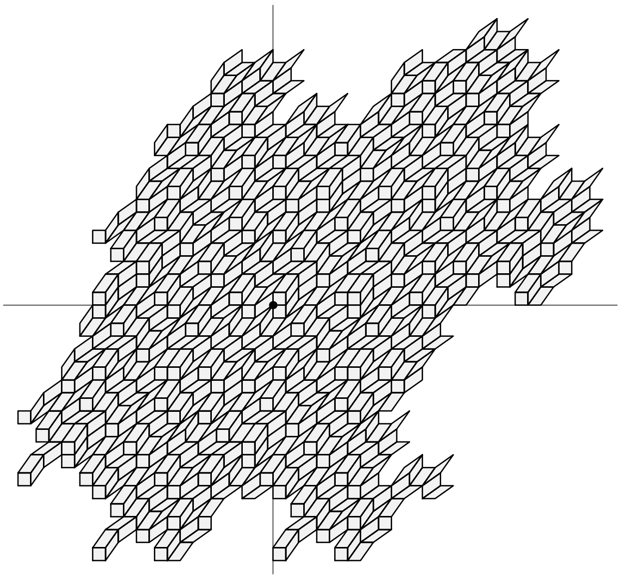


Figure 6.3. The patch $\Theta^3(y)$, which has the same patch y' around $\mathbf{0}$ as $\Theta(y)$.

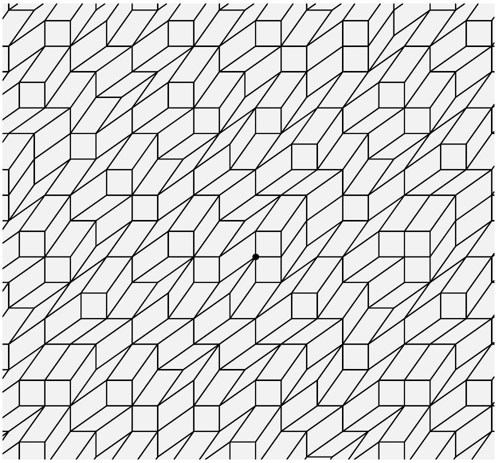


Figure 6.4. A swatch of the "Ammann" tiling y.

It is interesting to note that one obtains a different tiling $z' = \lim_{N\to\infty} \Theta^{2N+1}(y)$ by taking odd iterates. In particular, there is a different patch y' around the origin. However, since Θ is primitive, both of these belong to the same "tiling space", X_Θ in the notation of [16] .

We call z and z' "Ammann tilings" although they are much less symmetric than the classical Ammann-Beenker tiling (see [17]). In fact, there are many different choices non-abelianizations θ of A (i.e., not necessarily properly ordered) that result in completely different tiling substitutions Θ and different tiling spaces X_Θ. None of them seem to be the classical Ammann-Beenker tilings.

7. The positivity of $(A^*)^N$

In addition to (2.1) and (2.4), all of the tiling results in this paper require that A satisfy (2.9): $A^* \geqslant 0$. We begin by noting that A and A^N, $N > 1$, have the same expanding subspace E^u and the same projections P to E^u. Thus A^N satisfies (2.1) and (2.4) if and only if A does. In general, we are not so concerned with the difference between A and A^N. If θ is a non-abelianization of A, then θ^N is a non-abelianization of A^N, and we tend to think of θ^N and θ as the "same substitution". The main result in this section is that A^N may satisfy (2.9) (or rather a slightly stronger condition) even if A does not.

It follows from the Binet-Cauchy Theorem (Theorem 2.3) that if A satisfies (2.1) and (2.4) then

$$(A^*)^N = (A^N)^* \text{ for all } N \geqslant 0. \tag{7.1}$$

Instead of looking for matrices A where $A^* \geqslant 0$, it will suffice to find A so that $(A^*)^N = (A^N)^* \geqslant 0$. Unfortunately this is not so easy. However, it turns out to be possible to find good conditions on A for a stronger conclusion: $(A^*)^N > 0$. In this case, we say A^* is *eventually positive*.

Even in the case where $A^* \geqslant 0$, the eventual positivity of A^* is important because by Corollary 6.2 it implies the corresponding tiling substitution Θ is primitive. In any case, if a matrix A satisfying (2.1) and (2.4) is such that $(A^*)^N > 0$, then (7.1) implies that $B = A^N$ satisfies (2.1), (2.4) and (2.9). This provides lots of new examples.

7.1. The eventual positivity of A^*

A matrix A is called *eventually positive* if $A^N > 0$ for all N sufficiently large. A matrix A is *primitive* if $A \geqslant 0$ and A is eventually positive. Our main result in this section is the following.

THEOREM 7.1. — *Let A satisfy* (2.1) *and* (2.4). *The matrix A^* is eventually positive if and only if $S(A) = \pm S(A^T)$, where $S(A)$ is the matrix defined in* (2.7).

Since A and A^T have the same eigenvalues, A satisfies (2.1) if and only if A^T does. The proof of Theorem 7.1 will come at the end of this section.

COROLLARY 7.2. — *If $A^T = A$, then A^* is eventually positive.*

If $A = A^T$, then A satisfies (2.4) if and only if A^T does (they share the projection P). As we will show, however, there are also many non-symmetric examples. In the 3×3 case we get the following.

COROLLARY 7.3. — *Let $B \in Sl_3(\mathbb{Z})$. Suppose B is eventually positive and satisfies the following Pisot condition:* $\dim(E^u(B)) = 1$. *Then $A := B^{-1}$ satisfies* (2.1), (2.4) *and* (2.9).

Combining this Theorem 6.1, we obtain a new proof of the result from [10] that under the conditions of Corollary 7.3, there exists a tiling substitution Θ with boundary θ whose structure matrix is A. (In [10] an additional connectivity result is also obtained.)

Proof of Corollary 7.3. — Assume without loss of generality that $B > 0$. The Pisot property for B implies that B has three distinct eigenvalues satisfying $\omega_3 > 1$ and $|\omega_2|, |\omega_1| < 1$. It follows that the corresponding eigenvalues of A, $\lambda_i = \omega_i^{-1}$, satisfy $|\lambda_1|, |\lambda_2| > 1$ and $\lambda_3 < 1$, so A satisfies (2.1).

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the corresponding eigenvectors. Since $B > 0$ we can assume $\mathbf{v}_3 = (1, a, b)$ where $a, b > 0$. Let $\mathbf{w}_1 = (-a, 1, 0)$ and $\mathbf{w}_2 = (-b, 0, 1)$. Since $\mathbf{w}_1 \cdot \mathbf{v}_3 = \mathbf{w}_2 \cdot \mathbf{v}_3 = 0$,

$$P = (\mathbf{w}_1, \mathbf{w}_2)^T = \begin{pmatrix} -a & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} \tag{7.2}$$

and $C_2(P) = (b, -a, 1)$. It follows that $S(A) = \operatorname{diag}(1, -1, 1)$. Since $B^T > 0$, and since $A^T = (B^T)^{-1}$, the same argument shows $S(A^T) = S(A)$, and the corollary follows from Theorem 7.1. $\square$

7.2. Subspaces and projections

We say $\mathbf{v} \in \mathbb{C}^d$ is *real* if $\gamma\mathbf{v} \in \mathbb{R}^d$ for some nonzero $\gamma \in \mathbb{C}$; otherwise we say $\mathbf{v}$ is *complex*. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^d$ be linearly independent over $\mathbb{C}$. If both $\mathbf{v}_1$ and $\mathbf{v}_2$ are real then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \mathbb{R}^d$ denotes span over $\mathbb{R}$. Otherwise $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \mathbb{C}^d$ denotes span over $\mathbb{C}$.

In general, we let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. Given a two dimensional subspace $E = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \mathbb{F}^d$, we call a $2 \times d$ matrix P satisfying (2.2) a *projection* to E. As before, (2.3) holds, except now for $M \in Gl_2(\mathbb{F})$.

A subspace E is called *real* if it has a real basis, or equivalently, a real projection P. In particular, this means there is a nonsingular M so that MP has all real entries. Defining $R = (\mathbf{v}_1, \mathbf{v}_2)$, we have that (2.2) implies $PR = I$. We put $A_E = PAR$, which satisfies $P|_E A = A_E P|_E$.

7.3. Jordan forms

Let $V \in GL_d(\mathbb{C})$ be such that $VJ = AV$, where J is the *upper triangular Jordan Canonical Form* of A, with $\operatorname{diag}(J) = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d$. The columns $V = (\mathbf{v}_1, \ldots, \mathbf{v}_d)$ are a basis of *ordered generalized eigenvectors* for A. If A is diagonalizable, these are actually eigenvectors.

Proposition 7.4. — *Let A satisfy* (2.1) *and let $\mathbf{w}_1$ and $\mathbf{w}_2$ be the first two ordered generalized eigenvectors for A^T (i.e., corresponding to $E^u(A^T)$). Then $P = (\mathbf{w}_1, \mathbf{w}_2)^T$ is a projection to $E^u(A)$.*

Proof. — The unique *dual basis* $\mathbf{w}'_1, \ldots, \mathbf{w}'_d$ corresponding to $\mathbf{v}_1 \ \ldots \ \mathbf{v}_d$ is defined to satisfy $\mathbf{v}_i \cdot \mathbf{w}'_j = \delta_{i,j}$. Thus $W' := (\mathbf{w}'_1, \mathbf{w}'_2, \ldots, \mathbf{w}'_d)^T$ satisfies $W' = V^{-1}$.

We have $A^T (W')^T = (W')^T J^T$ because $W' A (W')^{-1} = V^{-1} A V = J$ implies $W'A = JW'$. But J^T is the lower-triangular Jordan form for A.

There exists a permutation matrix U so that $U^T J^T U = J$ where the first two rows (and columns) of U are either: (a) $\mathbf{e}_1, \mathbf{e}_2$, or (b) $\mathbf{e}_2, \mathbf{e}_1$, depending on whether or not λ_1 and λ_2 are in a non-trivial Jordan block (this can only happen if $\lambda_1 = \lambda_2$).

Let $W = U^T W'$, and express $W = (\mathbf{w}_1, \ldots, \mathbf{w}_d)^T$. Then

$$W^{-1} A^T W = (U^T W')^{-1} A^T (U^T W') = J.$$

This shows $\mathbf{w}_1, \ldots, \mathbf{w}_d$ are the ordered generalized eigenvectors for A^T. Let $P = (\mathbf{w}_1, \mathbf{w}_2)^T$. Then $P = MP'$, where $M = I$ in case (a), or $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the case (b). □

COROLLARY 7.5. — *The expanding subspace $E^u(A)$ is real.*

Proof. — This is clear if λ_1 is real, since this is equivalent to λ_2 also being real.

If λ_1 is complex, then $\lambda_2 = \overline{\lambda_1}$. Put $P = (\mathbf{w}_1, \overline{\mathbf{w}_1})^T$. Let $P' = MP = (\mathrm{Re}\{\mathbf{w}_1\},\ \mathrm{Im}\{\mathbf{w}_1\})^T$ where $M = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. Then P' is a real projection to E^u. □

7.4. Perron-Frobenius theory for A^*

Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^d$. Define $\mathbf{v}_1 \wedge \mathbf{v}_2 \in \mathbb{R}^{\binom{d}{2}}$ by

$$\mathbf{v}_1 \wedge \mathbf{v}_2 = C_2\big((\mathbf{v}_1, \mathbf{v}_2)\big). \tag{7.3}$$

LEMMA 7.6. — *If A is $d \times d$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^d$ then*

$$C_2(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) = (A\mathbf{v}_1) \wedge (A\mathbf{v}_2).$$

Proof. —

$$\begin{aligned} C_2(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) &= C_2(A) C_2\big((\mathbf{v}_1, \mathbf{v}_2)\big) \\ &= C_2\big((A\mathbf{v}_1, A\mathbf{v}_2)\big) \\ &= (A\mathbf{v}_1) \wedge (A\mathbf{v}_2). \end{aligned}$$

□

A matrix $B \in \mathbb{Z}^{p\times p}$ is called *spectrally Perron* (see [14]) if it has a real eigenvalue $\omega > 0$, such that for any other eigenvalue ω' of B, $|\omega'| < \omega$. The eigenvector $\mathbf{u}$ corresponding to ω is called a *Perron eigenvector*.

THEOREM 7.7. — (Lind and Marcus, [14]) *A matrix B is eventually positive if and only if it is spectrally Perron and the Perron eigenvectors $\mathbf{u}$ and $\mathbf{u}'$ for B and B^T respectively are both positive.*

The "only if" direction follows from the the Perron-Frobenius Theorem (see [14]). The "if" direction is Lind and Marcus [14], Exercise 11.1.9.

LEMMA 7.8. — *Let A satisfy (2.1) and let $E = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ be any real or complex 2-dimensional A-invariant subspace. Let P be a projection to E, and let A_E be the induced matrix on $\mathbb{F}^2$. Then $\mathbf{u} = S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2)$ is an eigenvector for A^* corresponding to the eigenvalue $\omega = \det(A_E)$. Moreover, every eigenvalue of A^* is obtained this way.*

Proof. — We have

$$RA_E = RPAR = AR.$$

Using this with (7.3) and the Binet-Cauchy Theorem (Theorem 2.3)

$$\begin{aligned} C_2(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) &= C_2(A)C_2(R) \\ &= C_2(AR) = C_2(RA_E) \\ &= \det(A_E)C_2(R) = \omega\,(\mathbf{v}_1 \wedge \mathbf{v}_2), \end{aligned}$$

where $\omega = \det(A_E)$.

The fact that every eigenvalue of $C_2(A)$ is obtained this way follows from the Jordan canonical form for A.

Now put $\mathbf{u} = S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2)$. Then

$$\begin{aligned} A^*\mathbf{u} &= S(A)C_2(A)S(A)^2(\mathbf{v}_1 \wedge \mathbf{v}_2) \\ &= S(A)C_2(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) \\ &= \omega\, S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) = \omega\,\mathbf{u}. \end{aligned}$$

□

PROPOSITION 7.9. — *Let A be non-Pisot of order-2 and satisfy the good star property. Then A^* is spectrally Perron. In particular, if $E^u = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\mathbf{u} = S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2)$ is the Perron eigenvector and $\omega = \det(A_u)$ is the Perron eigenvalue.*

Proof. — Since $A^* = S(A)C_2(A)S(A)$ and $A = V^{-1}JV$, the Binet-Cauchy theorem implies

$$A^* = (C_2(V)S(A))^{-1}C_2(J)(C_2(V)S(A)).$$

Now $C_2(J)$ is upper triangular so its eigenvalues are its diagonal entries. These are pairwise products of the diagonal entries of J, which are the eigenvalues of A.

Since E^u is a real 2-dimensional invariant subspace there are three possibilities for A_u: (i) A_u is diagonalizable over $\mathbb{R}$ and has two real eigenvalues λ_1, λ_2 with $|\lambda_1|, |\lambda_2| > 1$. If necessary, by replacing A with A^2, we can assume $\lambda_1, \lambda_2 > 0$. Then $\omega = \lambda_1\lambda_2 > 1$. (ii) A_u is a nontrivial Jordan block with eigenvalue λ. As above, assume by replacing A with A^2 if necessary $\lambda > 0$. Then $\omega = \lambda^2 > 1$. (iii) A_u has complex eigenvalues λ and $\overline{\lambda}$, and $\omega = \lambda\overline{\lambda} = |\lambda|^2 > 1$.

Now let E be any other real or complex 2-dimensional A-invariant subspace, and let ω' be the corresponding eigenvalue for A^*. If $E \subseteq E^s$ then $\omega' = \det(A_E) = \lambda_i\lambda_j$, where $i, j > 2$, so $|\omega'| = |\lambda_i||\lambda_j| \leqslant |\lambda_3|^2 < 1 < \omega$. Otherwise A_u is diagonalizable and has eigenvalues $\lambda_i, \lambda_j, i \leqslant 2, j > 2$. Then $|\omega'| = |\lambda_i||\lambda_j| < |\lambda_1||\lambda_2| = \omega$. The result now follows from an application of Lemma 7.8. □

Proposition 7.10. — *Suppose A satisfies* (2.1). *Let $\mathbf{v}_1, \mathbf{v}_2$ be a real basis for $E^u(A)$ and let $\mathbf{w}_1, \mathbf{w}_2$ be a real basis for $E^u(A^T)$. Then A^* is eventually positive if and only if $S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2)$ and $S(A^T)(\mathbf{w}_1 \wedge \mathbf{w}_2)$ are positive.*

Proof. — This follows from Theorem 7.7, Proposition 7.9 and Corollary 7.4. □

Proof of Theorem 7.1. — Let

$$\mathbf{u} = S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) \text{ and } \mathbf{u}' = S(A^T)(\mathbf{w}_1 \wedge \mathbf{w}_2).$$

By proposition 7.10, Theorem 7.1 follows once we show

$$\mathbf{u}, \mathbf{u}' > 0 \text{ if and only if } S(A) = \pm S(A^T). \tag{7.4}$$

By Proposition 7.4, we have

$$P = P_{E^u(A)} = (\mathbf{w}_1, \mathbf{w}_2)^T \text{ and } P' = P_{E^u(A^T)} = (\mathbf{v}_1, \mathbf{v}_2)^T$$

where

$$E^u(A) = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \text{ and } E^u(A^T) = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}.$$

By (7.3), $C_2(P) = \mathbf{w}_1 \wedge \mathbf{w}_2$ and $C_2(P') = \mathbf{v}_1 \wedge \mathbf{v}_2$. It follows from (2.7) that

$$S(A) = \operatorname{diag}(\operatorname{sgn}(C_2(P)) = \operatorname{diag}(\operatorname{sgn}(\mathbf{w}_1 \wedge \mathbf{w}_2))$$

and

$$S(A^T) = \operatorname{diag}(\operatorname{sgn}(C_2(P')) = \operatorname{diag}(\operatorname{sgn}(\mathbf{v}_1 \wedge \mathbf{v}_2)).$$

Thus $S(A) = \pm S(A^T)$ if and only if $\operatorname{sgn}(\mathbf{v}_1 \wedge \mathbf{v}_2) = \operatorname{sgn}(\mathbf{w}_1 \wedge \mathbf{w}_2)$.

Now suppose $S(A) = S(A^T)$. Then

$$\begin{aligned}\mathbf{u} &= S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) &&= \operatorname{sgn}(\mathbf{w}_1 \wedge \mathbf{w}_2) * (\mathbf{v}_1 \wedge \mathbf{v}_2)\\ & &&= \operatorname{sgn}(\mathbf{v}_1 \wedge \mathbf{v}_2) * (\mathbf{v}_1 \wedge \mathbf{v}_2) > 0,\end{aligned}$$

and similarly $\mathbf{u}' > 0$, where $*$ denotes entry-wise multiplication.

Conversely, suppose $\mathbf{u}, \mathbf{u}' > 0$. Then

$$\operatorname{sgn}(\mathbf{w}_1 \wedge \mathbf{w}_2) * (\mathbf{v}_1 \wedge \mathbf{v}_2) > 0,$$

which implies

$$\operatorname{sgn}(\mathbf{v}_1 \wedge \mathbf{v}_2) = \pm \operatorname{sgn}(\mathbf{w}_1 \wedge \mathbf{w}_2).$$

It follows that $S(A) = S(A^T)$. $\square$

8. Example: Irreducible characteristic polynomial

Consider the matrix

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with irreducible characteristic polynomial $p(x) = x^4 + x^3 - 4x^2 + 1$. The eigenvalues are $\lambda_1 = -2.5231$, $\lambda_2 = 1.44129$, $\lambda_3 = 0.566889$, and $\lambda_4 = -0.485084$, so A satisfies (2.1). Note that A is symmetric, so by Corollary 7.2, $(A^*)^N \geqslant 0$ for some N sufficiently large. We now find such an N.

Starting with the matrix

$$Q = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4) = \begin{pmatrix} -3.12676 & -0.252532 & -2.19713 & 0.576415 \\ -2.23925 & -0.824788 & 2.87576 & 0.188279 \\ -2.5231 & 1.44129 & 0.566889 & -0.485084 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

of eigenvectors, the projection P to E^u is given by

$$P = \begin{pmatrix} -0.141119 & -0.101064 & -0.113874 & 0.0451327 \\ -0.0660841 & -0.215836 & 0.377166 & 0.261686 \end{pmatrix}, \tag{8.1}$$

(the first two rows of Q^{-1}). The columns of P are plotted in Figure 8.1. The matrix A^* is given by

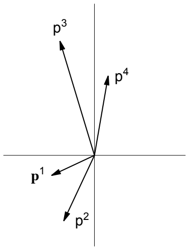


Figure 8.1. The good star of vectors for the projection P in (8.1). The positive prototiles are $1 \wedge 2$, $3 \wedge 1$, $4 \wedge 1$, $3 \wedge 2$, $4 \wedge 2$, *and* $4 \wedge 3$.

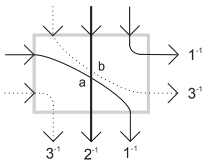
$$A^* = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{pmatrix},$$

which satisfies $A^* \leqslant 0$. This is sufficient for our method. By (7.1), $(A^*)^2 = (A^2)^* \geqslant 0$.

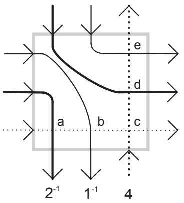
For the matrix A, we find the properly ordered morphism θ to be

$$\theta : \begin{cases} 1 & \rightarrow \quad 3^{-1}2^{-1}1^{-1} \\ 2 & \rightarrow \quad 3^{-1}1^{1} \\ 3 & \rightarrow \quad 2^{-1}1^{-1}4 \\ 4 & \rightarrow \quad 3 \end{cases} .$$

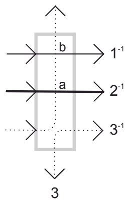
Next, we simplify all six product de Bruijn diagrams $(\theta \wedge \theta)(i \wedge j)$ to obtain the following efficient diagrams (all of which are negative):



$Y(1 \wedge 2)$

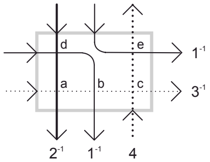


$Y(3 \wedge 1)$

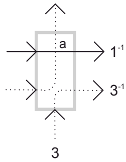


$Y(4 \wedge 1)$

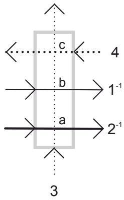
These de Bruijn diagrams have the following realizations as tiling patches. The gray color indicates the tiles are all *negative*.



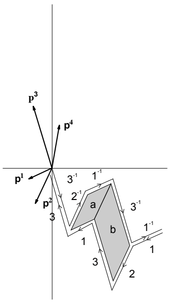
$Y(3 \wedge 2)$



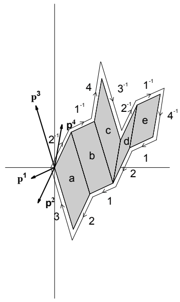
$Y(4 \wedge 2)$



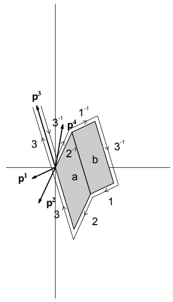
$Y(4 \wedge 3)$



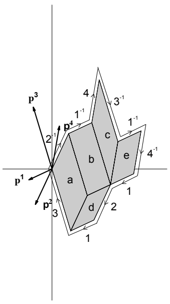
$\Theta(1 \wedge 2)$



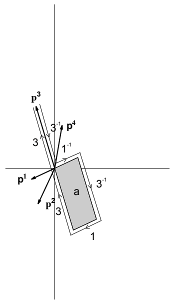
$\Theta(3 \wedge 1)$



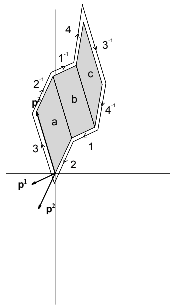
$\Theta(4 \wedge 1)$



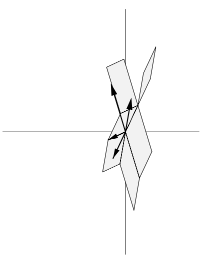
$\Theta(3 \wedge 2)$



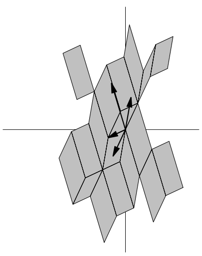
$\Theta(4 \wedge 2)$



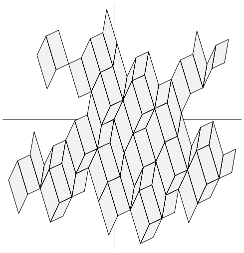
$\Theta(4 \wedge 3)$



The patch y.

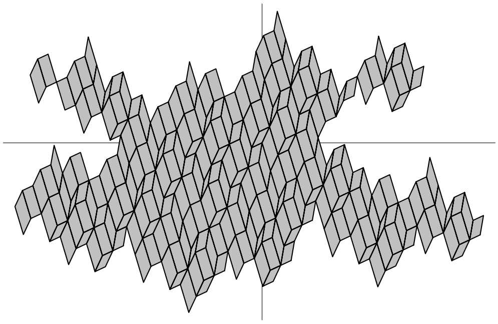


$\Theta(y)$

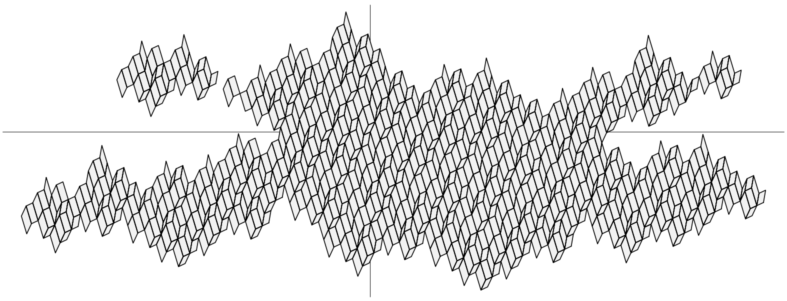


$\Theta^2(y)$

Starting with the tiling patch y shown below, we iterate to get patches $\Theta^N(y)$. Notice that for N even, the patch is a positive tiling patch. Each of these positive patches is a sub-patch of the previous one.



$\Theta^3(y)$



$\Theta^4(y)$

Finally, we define a tiling

$$z = \lim_{N\to\infty} \Theta^{2N}(y).$$

A swatch of this tiling is shown in Figure 8.2.

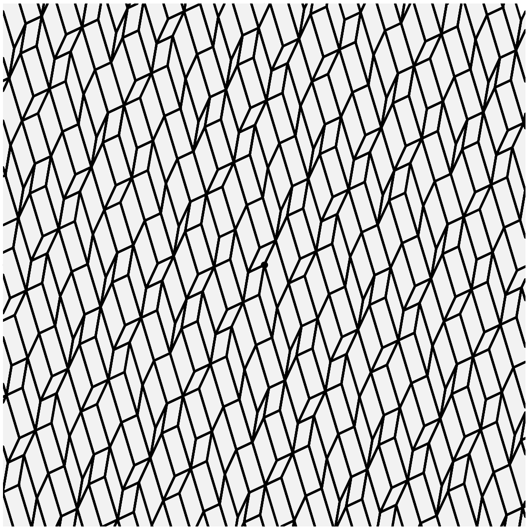


Figure 8.2. The positive tiling $z = \lim_{N\to\infty} \Theta^{2N}(y)$.

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