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Nicolas BEDARIDE & Pascal HUBERT

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BILLIARD COMPLEXITY IN THE HYPERCUBE

by Nicolas BEDARIDE & Pascal HUBERT

Abstract. — We consider the billiard map in the hypercube of $\mathbb{R}^d$. We obtain a language by coding the billiard map by the faces of the hypercube. We investigate the complexity function of this language. We prove that n^{3d-3} is the order of magnitude of the complexity.

Résumé. — On considère l'application du billard dans le cube de $\mathbb{R}^d$. On code cette application par les faces du cube. On obtient un langage, dont on cherche à évaluer la complexité. On montre que l'ordre de grandeur de cette fonction est n^{3d-3}.

1. Introduction

A billiard ball, i.e. a point mass, moves inside a polyhedron P with unit speed along a straight line until it reaches the boundary ∂P, then it instantaneously changes direction according to the mirror law, and continues along the new line.

Label the faces of P by symbols from a finite alphabet $\mathcal{A}$ whose cardinality equals the number of faces of P. Either we consider the set of billiard orbits in a fixed direction, or we consider all orbits.

In both cases the orbit of a point corresponds to a word in the alphabet $\mathcal{A}$ and the set of all the words is a language. We define the complexity of the language, $p(n)$, by the number of words of length n that appears in this system. We call the complexity of an infinite trajectory the directional complexity: it does not depend on the initial point under suitable hypotheses. We denote it by $p(n, \omega)$ (where ω is the initial direction of the trajectory), and the other one the global complexity or to short simply the complexity. How complex is the game of billiard inside a polygon or a polyhedron ?

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The only general result about complexity function is that the billiard in a polygon has zero entropy [10], [13], and thus the two complexities grow sub-exponentially. For the convex polyhedron the same fact is true [4].

It is possible to compute the complexity for rational polygons (a polygon is rational if the angles between sides are rational multiples of π). For the directional complexity the first result is in the famous paper of Morse and Hedlund [16], and it has been generalized to any rational polygon by Hubert [12]. This directional complexity is always linear in n.

For the global complexity in the square coded by two letters, Mignosi found an explicit formula see [15], [6]. Then Cassaigne, Hubert and Troubetzkoy [8] proved that $p(n)/n^3$ has a lower and an upper bound, in the case of rational convex polygons, and the first author generalized this result to the case of non convex rational polygons [3]. Moreover, for some regular polygons they showed that $p(n)/n^3$ has a limit, and then calculated it. But even for the hexagon we are not able to obtain an equivalent statement: we must use the result of Masur that gives the order of magnitude of the number of saddle connections [14].

In the polyhedral case much less is known. The directional complexity, in the case of the cube, has been computed by Arnoux, Mauduit, Shiokawa-band Tamura [1] and generalized to the hypercube by Baryshnikov [2]; see also [5] for a generalization. Moreover, in [3] the computation has been done in the case of some right prisms whose bases tile the plane. For those polyhedra the directional complexity is always quadratic in n. In the current article we compute the global complexity for the hypercube of $\mathbb{R}^d$ coded with d letters.

THEOREM 1.1. — *Let $p(n,d)$ be the complexity of the language associated to the hypercubic billiard (coded with d letters). Then there exists $C_1, C_2 \in \mathbb{R}_+$ such that*

$$C_1 n^{3d-3} \leqslant p(n,d) \leqslant C_2 n^{3d-3}.$$

Remark 1.2. — In the proof some constants appear in the inequalities. We denote them all by the same letter C. Moreover we will use the term "cube" even if d is greater than three.

1.1. Overview of the proof

The proof of [8] is based on the fact that the complexity is related to the number of generalized diagonals. A generalized diagonal is an orbit segment which starts and ends on a vertex of the polygon (or an edge of the polyhedron). If we wish to apply this technique to the hypercube, however, a generalized diagonal is not necessarily associated to a single word, so that we must modify the proof. First we show that the complexity is related to the number of words that appear in one diagonal, see Section 3. Next we begin to count the numbers of those words. We split the estimates between several parts. Section 5 is devoted to obtaining the upper bound by a general geometric argument. In Section 6 we establish the lower bound by an induction on the dimension d.

2. Background

2.1. Billiard

In this section we recall some definitions: Let P be a polyhedron, the billiard map is called T and it is defined on a subset of $\partial P \times \mathbb{R}P^{d-1}$. This space is called the *phase space*.

- We will call a *face of the cube* a face of dimension $d-1$. If we use a face of smaller dimension we will state the dimension.

We define a partition $\mathcal{P}$ of the phase space on d sets by the following method: the boundary of P is partitioned into d sets by identifying the parallel faces of the cube. Then we consider the partition

$$\mathcal{P}_n = \bigvee_{i=0}^{n} T^{-i}\mathcal{P}.$$

Definition 2.1. — *The* complexity of the billiard map, *denoted* $p(n,d)$, *is the number of atoms of* $\mathcal{P}_n$.

- The unfolding of a billiard trajectory: Instead of reflecting the trajectory in the face we reflect the cube and follow the straight line. Thus we consider the tiling of $\mathbb{R}^d$ by $\mathbb{Z}^d$, and the associated partition into cubes of edges of length one. In the following when we use the term “face” we mean a face of one of those cubes.

The following lemma is very useful in the following.

Lemma 2.2. — *Consider an orthogonal projection on a face of the cube. The orthogonal projection of a billiard map is a billiard map inside a cube of dimension equal to the dimension of the face.*

2.2. Combinatorics

Definition 2.3. — *Let $\mathcal{A}$ be a finite set called the alphabet.*

- *By a* language L over $\mathcal{A}$ *we always mean a factorial extendable language: a language is a collection of sets $(L_n)_{n\geqslant 0}$ where the only element of L_0 is the empty word, and each L_n consists of words of the form $a_1a_2\cdots a_n$ where $a_i \in \mathcal{A}$ and such that for each $v \in L_n$ there exist $a, b \in \mathcal{A}$ with $av, vb \in L_{n+1}$, and for all $v \in L_{n+1}$ if $v = au = u'b$ with $a, b \in \mathcal{A}$ then $u, u' \in L_n$.*
- *The* complexity function $p : \mathbb{N} \to \mathbb{N}$ *is defined by $p(n) = \mathrm{card}(L_n)$.*

First we recall a well-known result of Cassaigne [7] concerning combinatorics of words.

Definition 2.4. — *Let $\mathcal{L}(n)$ be a factorial extendable language. For any $n \geqslant 1$ let*

$$s(n) := p(n+1) - p(n).$$

For $v \in \mathcal{L}(n)$ let

$$\begin{aligned} m_\ell(v) &= \mathrm{card}\big\{u \in \Sigma,\, uv \in \mathcal{L}(n+1)\big\},\\ m_r(v) &= \mathrm{card}\big\{w \in \Sigma,\, vw \in \mathcal{L}(n+1)\big\},\\ m_b(v) &= \mathrm{card}\big\{u \in \Sigma,\, w \in \Sigma,\, uvw \in \mathcal{L}(n+2)\big\}. \end{aligned}$$

A word is called right special *if $m_r(v) \geqslant 2$,* left special *if $m_\ell(v) \geqslant 2$ and* bispecial *if it is right and left special. Let $\mathcal{BL}(n)$ be the set of the bispecial words.*

Cassaigne [7] has shown:

Lemma 2.5. — *Let $\mathcal{L}$ be a language such that $m_\ell(v) \geqslant 1$ and $m_r(v) \geqslant 1$ for all words $v \in \mathcal{L}$. Then the complexity satisfies*

$$\forall n \geqslant 1, \quad s(n+1) - s(n) = \sum_{v \in \mathcal{BL}(n)} i(v),$$

where $i(v) = m_b(v) - m_r(v) - m_\ell(v) + 1$.

For the proof of the lemma we refer to [7] or [8].

Remark 2.6. — If we code the billiard map by the sequence of faces hit in a trajectory, and if we associate the same letter to the parallel faces of the cube the two definitions of complexity coincide, i.e. $p(n, d) = p(n)$.

2.3. Geometry

We recall the Euler's formula.

Lemma 2.7 (see [9]). — *Let P be a simply connected polyhedron of $\mathbb{R}^d$. Let N_i be the number of faces of P of dimension i; then we have*

$$\sum_{i=0}^{d-1}(-1)^i N_i = 1-(-1)^d.$$

Remark 2.8. — In the following sections, we will use this formula for some algebraic manifolds of degree 2, but for simplicity we will always mean a polyhedron and hyperplanes, since the proofs are the same.

Now we prove the following result.

Lemma 2.9. — *Suppose $(H_i)_{i\leqslant n}$ is a sequence of hyperplanes of $\mathbb{R}^x$ and let $(\mathcal{Q}_i)_{i\in I}$ be the connected components of $\mathbb{R}^x \setminus H_1\cup\ldots\cup H_n$. Then there exists $C(x)>0$ such that*

$$\operatorname{card} I \leqslant C(x)n^x.$$

Proof. — We will prove the assumption by induction on x. The induction hypothesis states that it is true for all $i<x$.

The hyperplanes (H_i) induce a cellular decomposition of $\mathbb{R}^x$. We will denote N_i the number of cells of dimension i for $0\leqslant i\leqslant x$. We remark that $\operatorname{card} I = N_x$. We begin by obtaining an upper bound for N_i for $0\leqslant i<d$. We will see later that this is sufficient to finish the proof.

• Computation of N_0. We denote $\mathcal{H}=\{H_1,\ldots,H_n\}$, and consider the map $\phi_0 : \mathcal{H}^x \to \{\text{vertices}\}\cup\emptyset$,

$$\phi_0 :\ (H_{i_1},\ldots,H_{i_x}) \longmapsto \begin{cases} H_{i_1}\cap\ldots\cap H_{i_x} & \text{if it is a point,}\\ \emptyset & \text{otherwise.}\end{cases}$$

This map is surjective, thus we deduce $N_0\leqslant n^x$. Hence the induction is true for $x=1$.

• Let $\mathcal{E}_i$ the set of subspaces of dimension i which form the cells of dimension i of the cellular decomposition. We denote $E_i=\operatorname{card}(\mathcal{E}_i)$. Then the map $\mathcal{H}^{x-i}\to\mathcal{E}_i\cup\emptyset$,

$$(H_{j_1},\ldots,H_{j_{x-i}}) \longmapsto \begin{cases} H_{j_1}\cap\ldots\cap H_{j_{x-i}} & \text{if } \dim H_{j_1}\cap\ldots\cap H_{j_{x-i}} = x,\\ \emptyset & \text{otherwise}\end{cases}$$

is surjective by definition of the partition. We deduce $E_i\leqslant n^{x-i}$ for all $i\leqslant x-1$.

Now it remains to know into how many pieces each space of dimension i is cut. Let $F \in \mathcal{E}_i$ and $H \in \mathcal{H}$, we have $F \cap H = F$ or X, where X is a subset of codimension 1 contained in F.

The hyperplanes which partition F do not contain F, thus their trace on F is of codimension 1. Thus the problem is reduced to compute the number of connected components of the partition of F by $m \leqslant n$ hyperplanes.

The induction hypothesis implies that $N_F \leqslant C(i)n^i$. Then

$$\max_{F \in \mathcal{E}_i} E_F \leqslant c(i)n^i.$$

We deduce $N_i \leqslant n^{x-i}c(i)n^i \leqslant c(i)n^x$. Euler's formula implies

$$N_x \leqslant 1 + \sum_{i=0}^{x-1} C(i)n^x.$$

Therefore $N_x \leqslant Cn^x$. The induction process has been completed. □

Corollary 2.10. — *Let P be a polyhedron of $\mathbb{R}^x$, let $(H_i)_{i \leqslant n}$ be a sequence of hyperplanes, and let $(\mathcal{Q}_i)_{i \in I}$ be the connected components of $P \setminus H_1 \cup \ldots \cup H_n$. Then there exists $C(x,P) > 0$ such that*

$$\operatorname{card} I \leqslant C(x,P)n^x.$$

Proof. — We can apply the same proof in the case where P is a polyhedron: it suffices to add the hyperplanes which form the boundary of P. In this case only the constant C changes. □

Remark 2.11. — If we consider algebraic equations of *bounded* degree (by δ), the same proof works since an intersection of such manifolds has a bounded number of connected components, and since the Euler characteristic takes a finite number of values (see Remark 2.8), only depending on x and δ.

2.4. Number theory

A general reference for this section is [11].

Definition 2.12. — *Let n be an integer. The invertible elements of $\mathbb{Z}/n\mathbb{Z}$ are denoted by $(\mathbb{Z}/n\mathbb{Z})^*$, and the cardinality of this set is denoted $\phi(n)$, which is called the* Euler's function.

DEFINITION 2.13. — *The* Moebius function μ *is defined by* $\mu(1) = 1$ *and*

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k,\ p_i \in \mathbb{P} \text{ distinct primes,} \\ 0 & \text{if } n \text{ has a square factor.} \end{cases}$$

The multiplicative functions ϕ, μ are linked by the following classical property:

LEMMA 2.14. — *For all positive integer n the following holds:*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad \frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}\cdot$$

Now we use the above lemma to obtain the following result.

LEMMA 2.15. — *For all integer $p \geqslant 1$, there exists a $C > 0$ such that for all n the following holds*

$$\sum_{\ell \leqslant n} S_\ell \geqslant C n^{p+2},$$

where $S_\ell = \sum_{m \leqslant \ell,\, \gcd(m,\ell)=1} m^p$.

We give a proof of the lemma for the sake of the completeness. The integer part is denoted by $E()$.

Proof. — By Lemma 2.14 we have

$$\begin{aligned} S_\ell = \sum_{\substack{m \leqslant \ell \\ \gcd(m,\ell)=1}} m^p &= \sum_{m \leqslant \ell} \sum_{d|m,\, d|\ell} \mu(d) m^p = \sum_{k \leqslant \ell/d} \mu(d) k^p d^p, \\ &= \sum_{d|\ell} \mu(d) d^p \Big[C_{p+1} \Big(\frac{\ell}{d}\Big)^{p+1} + C_p \Big(\frac{\ell}{d}\Big)^p + O\Big(\frac{\ell}{d}\Big)^{p-1} \Big] \\ &= C_{p+1} \ell^{p+1} \sum_{d|\ell} \frac{\mu(d)}{d} + C_p \ell^p \sum_{d|\ell} \mu(d) + \ell^{p-1} \Big[\sum_{d|\ell} \mu(d) O(1/d) \Big]. \end{aligned}$$

Then we have

$$\Big| \sum_{d|\ell} \mu(d) O(1/d) \Big| \leqslant C \ln d,$$

and by Lemma 2.14

$$\sum_{d|\ell} \mu(d) = 0 \quad \text{if} \quad \ell \neq 1.$$

We deduce

$$S_\ell = C_{p+1} \ell^{p+1} \sum_{d|\ell} \frac{\mu(d)}{d} + O(\ell^{p-1} \ln \ell)$$

and

$$\begin{aligned}
\sum_{\ell\leqslant n} S_\ell &= \sum_{\ell\leqslant n} C\ell^{p+1}\sum_{d|\ell}\frac{\mu(d)}{d} + \sum_{\ell\leqslant n} O(\ell^{p-1}\ln\ell),\\
&= \sum_{d\leqslant n} C\frac{\mu(d)}{d}\sum_{n'\leqslant n/d} n'^{\,p+1}d^{p+1} + O(n^p\ln n),\\
&= C\sum_{d\leqslant n}\mu(d)d^p E\Big(\frac{n}{d}\Big)^{p+2} + O(n^p\ln\ell),\\
&= C\sum_{d\leqslant n}\mu(d)d^p\Big[\frac{n}{d}-\Big\{\frac{n}{d}\Big\}\Big]^{p+2} + O(n^p\ln\ell),\\
&= C\sum_{d\leqslant n}\mu(d)d^p\Big[\Big(\frac{n}{d}\Big)^{p+2} + O\Big(\Big(\frac{n}{d}\Big)^{p+1}\Big)\Big] + O(n^p\ln\ell),\\
&= Cn^{p+2}\sum_{d\leqslant n}\frac{\mu(d)}{d^2} + Cn^{p+1}\sum_{d\leqslant n}\frac{\mu(d)}{d} + O(n^p\ln\ell),\\
&= Cn^{p+2}\sum_{d\leqslant n}\frac{\mu(d)}{d^2} + n^{p+1}O(\log n) + O(n^p\ln\ell).
\end{aligned}$$

The series of general term $\mu(d)/d^2$ is absolutely convergent and its sum is $1/\zeta(2)$ which is positive [11]; thus we deduce

$$\sum_{\ell\leqslant n} S_\ell \geqslant Cn^{p+2}. \qquad \square$$

3. Preliminary results

Definition 3.1. — *A* diagonal $\gamma_{A,B}$ *between two faces A, B of the cubic tesselation, of dimensions less than $d-2$, is the set of (oriented) segments which start from A and stop in B.*

Definition 3.2. — *We introduce the following order on the faces: two faces A and B verify $A < B$ if each oriented segment from A to B is such that in the unfolding, the associated vector has positive coefficients.*

The diagonals are of several types due to the dimension of A, B. We call a diagonal between the faces A, B a positive diagonal *if we have $B > A$. If we attach a superscript + to an object, then it will consist of positive diagonals.*

Definition 3.3. — *We say that two faces A, B are at* combinatorial length *n if each orbit segment between A, B passes through n cubes. We denote the length by $d(A, B) = n$ (see Fig. 3.1).*

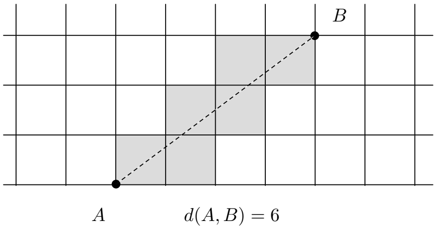

Figure 3.1. Words of billiard

This definition can be made since we are in the hypercube. In other polyhedron it is not well defined.

Definition 3.4. — *If the faces A, B of the cubic tesselation fulfill $d(A,B) = n$ then the diagonal $\gamma_{A,B}$ is of combinatorial length n. We denote the set of these diagonals by* $\mathrm{Diag}(n)$.

Notations. — In the following we only consider diagonals of combinatorial length n whose initial segment is in the cube $[0,1]^d$. If a diagonal is a positive diagonal, it implies that the final edge is in ${\mathbb{R}_+}^d$.

We denote the fact that an orbit in the diagonal γ has code v by $v \in \gamma$. We consider the bispecial words such that, in the unfolding, the associated trajectories are in ${\mathbb{R}_+}^d$, and not in one of the d coordinates planes, we denote these words by $\mathcal{BL}(n,d)^+$.

In the following we call *octant* a proper subspace of $\mathbb{R}^d$ of the form $I_1 \times \cdots \times I_d$ where I_i is equal to $\mathbb{R}^-$ or to $\mathbb{R}^+$.

Definition 3.5. — *Let v be a billiard word, we define the* cell *of v by the subset*

$$\big\{(m,\omega) \in \partial P \times \mathbb{R}P^{d-1}\big\}$$

such that for all $0 \leqslant i \leqslant |v|-1$, $T^i(m,\omega) \cap \partial P$ is in the face labelled by v_i.

The aim of this section is to show:

Proposition 3.6. — *With the preceding notations, there exists $C > 0$ such that*

$$\frac{1}{2^d} \sum_{v \in \mathcal{BL}(n,d)} i(v) = \sum_{v' \in \mathcal{BL}^+(n,d)} i(v') + O\big(|s(n+1,d-1) - s(n,d-1)|\big),$$

and

$$\sum_{\gamma \in \mathrm{Diag}(n)} \sum_{v \in \gamma} 1 \leqslant \sum_{\mathcal{BL}^+(n,d)} i(v') \leqslant C \sum_{\gamma \in \mathrm{Diag}(n)} \sum_{v \in \gamma} 1,$$

where $s(n,d) = p(n+1,d) - p(n,d)$.

For the proof we need the following lemmas.

Lemma 3.7. — *We consider a word v in $\mathcal{L}(n,d)$ with $n \geqslant 2$, consider the unfolding of the billiard trajectories which are coded by v and start inside the cube $[0;1]^d$. Then for all $i, 2 \leqslant i \leqslant n$, there exists only one face corresponding to the letter v_i.*

Proof. — First we consider the intersection of the cell of v with $\mathbb{R}P^{d-1}$. This set is a proper subset of an octant since $n \geqslant 2$. Now we make the proof by contradiction. We consider the first times j where two different faces appear. There exist two lines starting form a face (corresponding to v_{j-1}) which pass through these two different faces. These faces are different but are coded by the same letter, thus they are in two different hypercubes. Thus the two directions are in different octant, contradiction. □

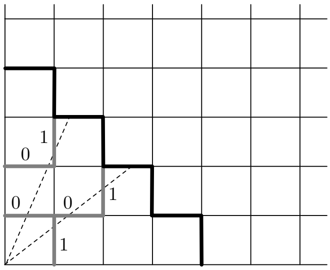

Figure 3.2. Words of billiard

In Figure 3.2 we show two billiard words in the square. The path represents the faces at length n of the initial square. In the figure we have $n = 3$. The two words are coded by 001 and 101, if we code the horizontal lines by 0, and the vertical lines by 1.

Lemma 3.8. — *Let v be a word in $\mathcal{BL}(n,d)^+$, then there exists only one positive diagonal associated to this word.*

Proof. — If we study the unfolding of a trajectory associated to v, the fact that we consider only words in $\mathcal{BL}(n,d)^+$ (and not in $\mathcal{BL}(n,d)$) implies that there are at most d choices for the suffix of v in the octant $\mathbb{R}_+^d$ (a suffix is a letter ℓ such that vl is a word), and the same result for the prefix.

We consider the faces related to the suffix letter. We claim that these faces have a non-empty intersection: by Lemma 3.7 these faces are in a same hypercube. They correspond to different letters of the coding, thus these faces intersect (by definition of the coding). The claim is proved.

Those faces have a non-empty intersection, if we consider the same intersection with the prefix, we have built a diagonal associated to this word, and by construction it is unique. □

DEFINITION 3.9. — *We call* discontinuity *a set of points*

$$X = \{(m,\omega),\, m \in A\}$$

in the phase space such that A is a face, and such that their orbits intersect another face of dimension $d-2$.

Let us remark that a discontinuity is of dimension at most $2(d-1)$, and that a diagonal is in the intersection of two discontinuities.

LEMMA 3.10. — *Consider a diagonal γ between two faces A, B of dimensions i, j. Then for all word $v \in \gamma$ (see Notations) we have*

$$d^2 \geqslant i(v) \geqslant 1.$$

Proof. — Consider a bispecial billiard word v, the cell of v in the phase space is an open set. It means that if a trajectory has v for coding, a small perturbation of v has still v for coding.

A face of dimension $d-2$ is at the intersection of two faces of dimension $d-1$, thus the face of dimension j is at the intersection of at least $d-j$ faces, and we deduce $m_r(v) \geqslant d-j$ by perturbation. The same method shows that $m_\ell(v) \geqslant d-i$. Now by definition of diagonal, see Definition 3.1, the diagonal is in the interior of the cell of v in the phase space. Moreover the cell of v is an open set. Now consider a segment $[a;b]$ inside the diagonal with $a \in A$. There exists an open set near a such that for all a' inside the segment $[a';b]$ is still coded by v. Now there exists a neighborhood of b such that for all b' the segment $[a';b']$ has v for coding. This implies $m_b(v) = m_r(v)m_\ell(v)$. Finally we obtain $i(v) = (m_\ell(v)-1)(m_r(v)-1) \geqslant 1$. The other inequality is obvious. □

3.1. Proof of Proposition 3.6

First we remark that the symmetries of $\mathbb{R}^d$ implies that $\sum_{v\in\mathcal{BL}(n,d)} i(v)$ is the same for each octant.

Now we are interested in the bispecial words which are neither in $\mathcal{BL}(n,d)^+$ nor in one of the symmetric sets. Their unfolding is in $[0,1]^{d-1}\times\mathbb{R}^+$. Thus for each coordinates plane their number is equal to the number of bispecial words of the cube of dimension $d-1$. Lemma 2.5 implies

$$\sum_{v\notin\mathcal{BL}(n,d)^+} i(v) \leqslant C\big|s(n+1,d-1)-s(n,d-1)\big| \qquad (C\in\mathbb{R}).$$

We consider the map

$$f : \mathcal{BL}(n,d)^+ \longrightarrow \mathrm{Diag}^+(n), \quad f : v \longmapsto \gamma.$$

Lemma 3.8 implies that f is well defined and onto, thus

$$\mathrm{card}\left(\mathcal{BL}(n,d)^+\right) = \sum_{\gamma \in \mathrm{Diag}(n)} \mathrm{card}\left(f^{-1}(\gamma)\right).$$

Then we obtain

$$\sum_{\mathcal{BL}(n,d)^+} i(v') = \sum_{\gamma \in \mathrm{Diag}(n)} \sum_{v \in \gamma} i(v).$$

Now we must bound $i(v)$ for each $v \in \gamma$. This is a consequence of Lemma 3.10.

4. Equations of diagonals

In these section we give in Lemma 4.1 the equations of a diagonal, we deduce in Proposition 4.2 that several diagonals can not overlay, and we finish the section by a description of the diagonals of fixed combinatorial length. Remark that these equations are homogeneous in ω.

Lemma 4.1. — *Let A, B two faces of dimension $d-2$, we consider*

$$\gamma_{A,B} = \left\{(m,\omega) \in \mathbb{R}^d \times \mathbb{R}^{*d};\ m \in A,\ m + \mathbb{R}\omega \cap B \neq \emptyset\right\}.$$

Then $\gamma_{A,B}$ has one of the following equations:

1) $n\omega_i = p\omega_j$, *with* $n, p \in \mathbb{N}$;
2) $m_i + n\omega_i/\omega_j = p$ *with* $n, p \in \mathbb{N}$;
3) $\omega_j m_i - \omega_i m_j = n\omega_i + p\omega_j$ *with* $n, p \in \mathbb{N}$.

Proof. — First we can assume that the point $m \in A$ have coordinates of the following form:

$${}^t(m_1, \dots, m_{d-2}, 0, 0).$$

Then each point of B have two coordinates equal to integers n, p. Thus its coordinates are of the form:

$$B : {}^t(b_1, \dots, n, \dots, p, \dots, b_{d-2}).$$

If the line $m + \mathbb{R}\omega$ intersects B it means that there exists λ such that $m + \lambda\omega \in B$. Then there are three choices, depending on the position of n, p in the coordinates.

- If n, p are at positions $d-1, d$ we obtain a system of the form

$$\lambda\omega_{d-1} = n, \quad \lambda\omega_d = p.$$

This gives equation 1).

- If n is at a position i less or equal than $d-2$, and p is at position $d-1$ or d, we obtain

$$\lambda\omega_{d-1} = p, \quad m_i + \lambda\omega_i = n.$$

This gives the second equation.

- If n and p are at position less than $d-2$, we are in case 3). □

Proposition 4.2. — *Let $A, B, C_i, i = 1, \dots, \ell$ be $\ell + 2$ faces of dimension $d-2$. We deduce the equivalence*

$$\gamma_{A,B} = \bigcup_i \gamma_{A,C_i} \iff A, B, C_i \text{ are contained in a hyperplane of } \mathbb{R}^d.$$

Proof. — We consider the three functions which appear in Lemma 4.1:

$$\begin{aligned} f(\omega) &= n\omega_i - p\omega_j, \\ g(m,\omega) &= m_i + \frac{n\omega_i}{\omega_j} = p, \\ h(m,\omega) &= \omega_j m_i - \omega_i m_j - (n\omega_i + p\omega_j). \end{aligned}$$

The diagonals $\gamma_{A,B}, \gamma_{A,C_i}$ have equations of the type f, g, h by preceding lemma (with different n, p, i, j). Remark that these equations are quadratic in the variables m, ω. Thus these maps are analytic.

We compute the jacobian of these maps:

$$\begin{aligned} \mathrm{d}f &= \big(\dots 0 \dots || \dots n \dots - p \dots\big), \\ \mathrm{d}g &= \big(0 \dots \omega_j \dots || \dots n \dots m_i - p \dots\big), \\ \mathrm{d}h &= \big(0 \dots \omega_j \dots - \omega_i \dots || \dots - m_i - n \dots m_i - p \dots\big). \end{aligned}$$

Now without loss of generality we treat the case $\ell = 1$. The sets $\gamma_{A,B}$, $\gamma_{A,C}$ are equal if and only if two of the preceding functions are equal on a set of positive measure. It implies that the linear forms are proportional. Assume that two different forms are proportional (for example $\mathrm{d}f$ and $\mathrm{d}g$). It implies that $m_i = 0$, thus the equality is true on an hyperplane, and they are not equal on a set of positive measure. Thus the only possibility is that the two equations are of the same type (i.e. two equations $\mathrm{d}f$ or two equations $\mathrm{d}g$). Then the same argument shows that the equality of two equations of the type $\mathrm{d}g$ or $\mathrm{d}h$ implies that (m, ω) lives on a set of zero measure. Thus the only possibility is the equality of two vectors $\mathrm{d}f$. And it is equivalent to the fact that A, B, C belong to the same hyperplane. □

Lemma 4.3. — *Let A, B be two faces of dimension less or equal than $d-2$. Assume A, B are at combinatorial length n, and that the elements of A are* *of the form*

$$^t(m_1, \dots, m_{d-2}, 0, 0).$$

Then we have:

- *either* A, B *are in a subspace of dimension* $d-2$ *then points of* B *are of the form*

$$^t(b_1, \dots, b_{d-2}, n_{d-1}, n_d),$$

with $n_{d-1}, n_d \in \mathbb{N}$, $\gcd(n_{d-1}, n_d) = 1$ *and* $\sum_{i=1}^{d-2} E(b_i) + n_{d-1} + n_d = n$*;*

- *or the points of* B *have the following coordinates:*

$$^t(b_1, \dots, n_i, \dots, n_j, \dots, b_{d-2}), \quad (i, j) \neq (d-1, d),$$

with $n_i, n_j \in \mathbb{N}$ *and* $\sum_{\ell=1}^{d-2} E(b_\ell) + n_i + n_j = n$*.*

Proof. — First of all we consider the faces of dimension $d-1$ which are at combinatorial length n of A. We claim that the points $(b_i)_{i \leqslant d}$ of these faces verify $\sum_{i=1}^{d} E(b_i) = n$.

The proof is made by induction on n. It is clear for $n = 1$, now consider a billiard trajectory of length n, it means that just before the last face we intersect another face of the same cube. These face is at combinatorial length $n-1$, and we can apply the induction process. Now consider a point of these faces, denote by $(c_i)_{i \leqslant d}$ its coordinates. We verify easily that

$$\sum_{i=1}^{d-2} E(b_i) - \sum_{i=1}^{d} E(c_i) = 1$$

for all point b, c. This finishes the proof of the claim.

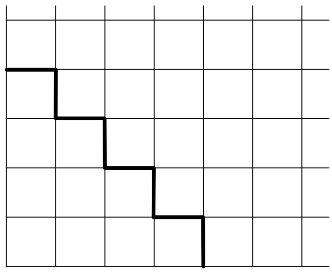

Figure 4.1. Length of billiard words

In Figure 4.1 the path represents the faces at length n of the initial square. In the figure we have $n = 3$.

A trajectory between A and B is a diagonal if the trajectory does not intersect another face. It means that $\gamma_{A,B}$ must not be the union of γ_{A,C_i}, where C_i are at length less than n from A. We use Proposition 4.2 which implies that the only bad case is when A, B are on a same hyperplane. Thus the second point of the Lemma is proved.

Now assume A, B are contained in a hyperplane. The fixed coordinates of all points in A and B are at the same places. Then we project on the plane generated by these coordinates. The diagonal projects on a line. This line does not contain integer points. Thus we obtain the primality condition. □

Corollary 4.4. — *We deduce that there exists $C > 0$ such that*

$$\operatorname{card} \operatorname{Diag}(n) \leqslant Cn^{d-1}.$$

Proof. — A diagonal $\gamma_{A,B}$ can be of several forms among the dimension of the faces. Since a face of dimension $d-1$ has a bounded number of faces of dimension less than $d-1$ in its boundary, we can reduce to count the diagonals between faces of dimension $d-1$. Then the number of diagonals is bounded by a constant $C(d)$ times the number of diagonals between faces of dimension $d-2$. The preceding lemma shows that we have the inequality

$$\operatorname{card} \operatorname{Diag}(n) \leqslant C \operatorname{card} \Big\{(n_i)_{1\leqslant i\leqslant d};\, n_i \in \mathbb{N},\, \sum_{i=1}^{d} n_i = n\Big\}.$$

Therefore $\operatorname{card} \operatorname{Diag}(n) \leqslant Cn^{d-1}$. □

5. Upper bound

In this section we show

Theorem 5.1. — *There exists $C > 0$ such that*

$$s(n,d) \leqslant Cn^{3d-4}.$$

Lemma 5.2. — *Let A, B two faces of dimension less than $d-2$, then the set $\gamma_{A,B}$ is of dimension less than $2(d-2)$. For all k, for all subset I of $\mathbb{N}$ of cardinality k, there exists $C(k) > 0$ such that for all $(A_i)_{i\in I}$ faces of dimension less than $d-2$, $\bigcap_{i,j\in I} \gamma_{A_i,A_j}$ has at most $C(k,d)$ connected components.*

Proof. — The first part is a consequence of Lemma 4.1. Indeed A is of dimension less than $d-2$, the directions lives in $\mathbb{R}P^{d-1}$ which is of dimension $d-1$ and the manifold has one equation.

For the second part we use again Lemma 4.1. The equation of these sets are polynomial equation of bounded degree (2), and a theorem of [9, Exercise 8.4.5] finishes the proof. □

Proposition 5.3. — *There exists $C>0$ such that for all A,B faces of dimension less than $d-2$, at combinatorial length n, we have*

$$\sum_{v\in\gamma_{A,B}} 1 \leqslant Cn^{2d-4}.$$

Proof. — We consider the cell related to $\gamma_{A,B}$. This space is partitioned with several discontinuities. The number of sets of the partition is equal to the number of words v in $\gamma_{A,B}$. First if a discontinuity does not partition, we prolong it. It gives an upper bound for the number of words. Then we consider the partition made by two discontinuities, Lemma 5.2 implies that the number of connected components is bounded by C. Then we apply Corollary 2.10 with Cn hyperplanes (in fact algebraic varieties of degree at most 2, see Remark 2.11), and $x=2(d-2)$ due to the first part of Lemma 5.2. □

5.1. Proof of Theorem 5.1

We make an induction on d.

If $d=2$ it is a consequence of [15] or [6], or [8].

By Proposition 3.6 we deduce

$$\sum_{v\in\mathcal{BL}(n,d)} i(v) \leqslant 2^d\Big[\sum_{\gamma\in\mathrm{Diag}(n)}\sum_{v\in\gamma} 1 + s(n+1,d-1)\Big].$$

Then the preceding proposition shows

$$\sum_{v\in\mathcal{BL}(n,d)} i(v) \leqslant 2^d\Big[\sum_{\gamma\in\mathrm{Diag}(n)} Cn^{2d-4} + s(n+1,d-1)\Big].$$

Corollary 4.4 implies that $\mathrm{card}(\mathrm{Diag}(n)) \leqslant n^{d-1}$, we deduce

$$\sum_{v\in\mathcal{BL}(n,d)} i(v) \leqslant 2^d\big[Cn^{3d-5} + s(n+1,d-1)\big].$$

By induction we deduce

$$s(n+1,d)-s(n,d) \leqslant C[n^{3d-5}+n^{3d-7}], \quad s(n,d)\leqslant Cn^{3d-4}.$$

The induction is proved.

6. Lower bound

We prove

THEOREM 6.1. — *There exists $C > 0$ such that for all n*

$$s(n+1,d) - s(n,d) \geqslant Cn^{3d-5}.$$

The proof is made by induction on d. It is clear for $d = 2$ due to [15] or [6] or [8], assume it is true for $i \leqslant d-1$.

DEFINITION 6.2. — *Let A, B two faces of dimension less than $d-2$.*

- *We denote by $C_{A,B}$ the vector space generated by $\vec{A}, \vec{B}$.*
- *Let $\pi : \mathbb{R}^d \to C_{A,B}$ be the orthogonal projection.*

Consider a trajectory of a fixed diagonal, it is coded by a word v, the image of the trajectory by π is a billiard trajectory, due to Lemma 2.2. Thus the map π can be extended to words, we denote it again by π:

$$\begin{array}{ccc} \gamma & \xrightarrow{\pi} & \pi(\gamma) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ v & \xrightarrow{\pi} & \pi(v) \end{array}$$

The map π consists to erase some letters, due to Lemma 2.2.

6.1. Projection and language

The aim of this section is to prove

PROPOSITION 6.3. — *We have*

$$\bigcup_{\gamma_{A,B} \in \mathrm{Diag}(n)} \left\{\pi(v), v \in \gamma_{A,B}\right\} = \bigcup_{i \leqslant n-1} \mathcal{L}(i, d-1).$$

Proof. — The inclusion

$$\bigcup_{\gamma_{A,B} \in \mathrm{Diag}(n)} \left\{\pi(v), v \in \gamma_{A,B}\right\} \subset \bigcup_{i \leqslant n-1} \mathcal{L}(i, d-1)$$

is a consequence of Lemma 2.2.

To prove the second inclusion we need:

LEMMA 6.4. — *Let $i \leqslant n-1$, and let $v \in \mathcal{L}(i, d-1)$ be a billiard word between two faces A, B' of dimension $d-2$. There exists a face B of dimension $d-2$ such that*

$$d(A,B) = n, \quad \gamma_{A,B} \text{ is a diagonal}, \quad \pi_{A,B}(B) = B'.$$

Proof. — By Lemma 4.3, we can always lift the face B' in a face B with $d(A,B)=n$. We just have to translate B' to the coordinate $x_d = n-i$. The only point to prove is that the trajectories between A, B form a diagonal. We make a proof by contradiction. Then each trajectory between A, B intersects another face C_i. It implies that $\gamma_{A,B}$ is cover by some γ_{A,C_i}. Contradiction with Proposition 4.2. □

Now the proof of the proposition is a simple consequence of this lemma and of Lemma 2.2. □

Corollary 6.5. — *For any diagonal γ we have*

$$\sum_{v\in\gamma} 1 \geqslant \sum_{v\in\pi(\gamma)} 1.$$

Proof. — By preceding Lemma, a word of $\pi(\gamma)$ can be lift in a word of γ. In other word the map π is surjective on the billiard words. □

6.2. Proof of Theorem 6.1

We fix the face A as in Lemma 4.1. By Lemma 4.3 the coordinates $(n_1, \ldots, n_d)$ of B can be of two types:

$$n_1 + \cdots + n_d = n,\ \gcd(n_{d-1}, n_d) = 1 \quad \text{or} \quad n_1 + \cdots + n_d = n.$$

Definition 6.6. — *We denote these sets of diagonals by* $\mathrm{Diag}_1(n)$ *and* $\mathrm{Diag}_2(n)$. *Let γ be a diagonal, the number $\sum_{v\in\gamma} 1$ is denoted by $f(n_1, \ldots, n_d)$ or $g(n_1, \ldots, n_d)$ if γ is in* $\mathrm{Diag}_1(n)$ *or not.*

Due to Proposition 3.6 we must compute

$$X = \sum_{\gamma\in\mathrm{Diag}(n)} \sum_{v\in\gamma} 1.$$

By Corollary 6.5 we can write this sum as

$$\begin{aligned} X &= \sum_{\gamma\in\mathrm{Diag}_1(n)} \sum_{v\in\gamma} 1 + \sum_{\gamma\in\mathrm{Diag}_2(n)} \sum_{v\in\gamma} 1. \\ &= \sum_{\gamma\in\mathrm{Diag}_1(n)} f(n_1, \ldots, n_d) + \sum_{\gamma\in\mathrm{Diag}_2(n)} g(n_1, \ldots, n_d). \end{aligned}$$

Now we use the projection π. By Lemma 4.3 and 6.5 we deduce

$$X \geqslant \sum_{n_d, n_{d-1}} \sum_{n_1, \ldots, n_{d-2}} \big[f(n_1, \ldots, n_{d-2})\chi(n_d, n_{d-1}) + g(n_1, \ldots, n_d)\big],$$

where $\chi(n_d, n_{d-1}) = 1$ if $\gcd(n_d, n_{d-1}) = 1$ and 0 either. Now Proposition 6.3 implies that

$$\sum_{n_1+\cdots+n_{d-1}=n} \big[f(n_1,\ldots,n_{d-1}) + g(n_1,\ldots,n_{d-1})\big] = p(n, d-1).$$

This can be written as

$$\sum_{n_{d-1}\leqslant n} \sum_{n_1+\cdots+n_{d-2}=n-n_{d-1}} \big[f(n_1,\ldots,n_{d-1}) + g(n_1,\ldots,n_{d-1})\big] = p(n, d-1),$$

$$\sum_{n_1+\cdots+n_{d-2}=n-n_{d-1}} \big[f(n_1,\ldots,n_{d-1}) + g(n_1,\ldots,n_{d-1})\big] = s(n_{d-1}, d-1).$$

Then we deduce

$$\begin{aligned} X &\geqslant \sum_{n_d,n_{d-1}} \sum_{n_1,\ldots,n_{d-2}} \big[f(n_1,\ldots,n_{d-2}) + g(n_1,\ldots,n_d)\big]\chi(n_{d-1}, n_d), \\ &\geqslant \sum_{n_d}\sum_{n_{d-1}} \sum_{n_1,\ldots,n_{d-2}} \big[f(n_1,\ldots,n_{d-2}) + g(n_1,\ldots,n_d)\big]\chi(n_{d-1}, n_d), \\ &\geqslant \sum_{n_d\leqslant n} \sum_{n_{d-1}\leqslant n-n_d} s(n_d-1, d-1)\chi(n_d, n_{d-1}). \end{aligned}$$

Then the induction hypothesis shows that

$$\big|s(n+1, d-1) - s(n, d-1)\big| \geqslant n^{3d-8}.$$

It implies a lower bound on $s(n, d-1)$:

$$\begin{aligned} X &\geqslant \sum_{n_d\leqslant n} \sum_{n_{d-1}\leqslant n-n_d} (n_{d-1})^{3d-7}\chi(n_d, n_{d-1}). \\ &\geqslant \sum_{n_d\leqslant n} \sum_{n_{d-1}\leqslant n_d} (n_{d-1})^{3d-7}\chi(n-n_d, n_{d-1}). \\ &\geqslant \sum_{n_d\leqslant n} \sum_{\substack{n_{d-1}\leqslant n_d \\ \gcd(n_{d-1},n_d)=1}} (n_{d-1})^{3d-7}. \end{aligned}$$

We apply Lemma 2.15 with $p = 3d - 7$ and obtain

$$X \geqslant Cn^{3d-5}.$$

Now by Proposition 3.6 we have

$$s(n+1, d) - s(n, d) \geqslant CX + O\big(s(n, d-1) - s(n, d-1)\big).$$

We apply again the induction to obtain a lower bound for the error term of Proposition 3.6. This term is bounded by n^{3d-8}. Thus we have

$$s(n+1, d) - s(n, d) \geqslant Cn^{3d-5} - n^{3d-8}, \quad s(n+1, d) - s(n) \geqslant Cn^{3d-5}.$$

The proof by induction is finished.

7. Proof of the main theorem

We just have to join Theorems 5.1 and 6.1. □

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Nicolas BEDARIDE

Pascal HUBERT
Fédération de recherches des unités de
mathématiques de Marseille UMR 6632
Laboratoire d'Analyse Topologie et Probabilités
Av. Escadrille Normandie-Niemen
13397 Marseille Cedex 20 (France)

nicolas.bedaride@univ-cezanne.fr

hubert@cmi.univ-mrs.fr