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Construction of Sobolev spaces of fractional order with sub-riemannian vector fields

CONSTRUCTION OF SOBOLEV SPACES
OF FRACTIONAL ORDER
WITH SUB-RIEMANNIAN VECTOR FIELDS

by Sami MUSTAPHA & François VIGNERON

ABSTRACT. — Given a smooth family of vector fields satisfying Chow-Hörmander's condition of step 2 and a regularity assumption, we prove that the Sobolev spaces of fractional order constructed by the standard functional analysis can actually be "computed" with a simple formula involving the sub-riemannian distance.

Our approach relies on a microlocal analysis of translation operators in an anisotropic context. It also involves classical estimates of the heat-kernel associated to the sub-elliptic Laplacian.

RÉSUMÉ. — Étant donné une famille de champs de vecteurs $\mathscr{Z}$ vérifiant la condition de Chow-Hörmander de rang 2 et une hypothèse de régularité, on montre que les espaces de Sobolev d'ordre fractionnaire construits par les procédés standards de l'analyse fonctionnelle admettent une caractérisation géométrique à l'aide de la distance sous-riemannienne induite par la famille $\mathscr{Z}$. L'approche proposée est basée sur une analyse microlocale des opérateurs de translation dans un contexte anisotrope. Elle utilise aussi des estimations classiques du noyau de la chaleur associé à un Laplacien sous-elliptique.

1. Introduction

Let $\mathscr{Z} = (\mathscr{Z}_1, \ldots, \mathscr{Z}_m)$ be a family of smooth vector fields on an open subset Ω of $\mathbb{R}^q$, having constant rank r. Each field acts on functions as a Lie derivative:

$$(\mathscr{Z}_j f)(x) = df(x) \cdot \mathscr{Z}_j(x).$$

We are interested in better understanding the regularity properties that can be expressed through the fields $\mathscr{Z}_j$. The case we focus on is that of classical – second order – sub-riemannian geometry. Roughly speaking, we

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assume that the $\mathscr{Z}_j$ and their commutators $[\mathscr{Z}_k, \mathscr{Z}_l]$ span the tangent space, with locally bounded coefficients with respect to the base-point.

The main goal is to give an easily "computable" description of the Sobolev spaces of fractional order between the space $L^2(K)$ of square summable functions supported in a compact subset K of Ω, and the space

(1.1)
$$H^1_K(\mathscr{Z}) = \{\, u \in L^2(\Omega) \text{ s.t. } \mathscr{Z}_j u \in L^2(\Omega)\ (j=1,\dots,m), \text{ with supp}\, u \subset K \,\}.$$

For $0 < s < 1$, the space $H^s_K(\mathscr{Z}) = [L^2(K); H^1_K(\mathscr{Z})]_s$ and its norm are defined by complex interpolation. We will not concern ourselves with problems that might occur near the boundary, nor at infinity. This choice is legitimated by the fact that regularity is a local property of functions.

The simplest example is when the family $\mathscr{Z}$ is the usual gradient. In this case, the invariance of the Euclidean distance with respect to the translations T_h, and Parseval's identity, imply:

$$\iint_{\mathbb{R}^{2q}} \frac{|u(x) - u(y)|^2}{|x-y|^{q+2s}}\, dx dy = \int_{\mathbb{R}^q} \|(T_h^* - \mathrm{Id})u\|^2\, |h|^{-q-2s}\, dh$$

$$= \int_{\mathbb{R}^q} m_0(\xi)\, |\hat{u}(\xi)|^2\, d\xi$$

with $m_0(\xi) = \int |e^{ih\cdot\xi} - 1|^2\, |h|^{-q-2s}\, dh$. The homogeneity of the distance ensures that $m_0(\xi) = |\xi|^{2s} m_0(\xi/|\xi|)$; then the isotropy guarantees that m_0 is constant on the sphere $\mathbb{S}^{q-1}$. Finally, there exist some constant κ_s $(0 < s < 1)$ such that

(1.2)
$$\int_{\mathbb{R}^q} |\xi|^{2s}\, |\hat{u}(\xi)|^2\, d\xi = \kappa_s \iint_{\mathbb{R}^{2q}} \frac{|u(x) - u(y)|^2}{|x-y|^{q+2s}}\, dx dy.$$

This simple computation shows how crucial are the properties of the metric structure associated with the vector fields $\mathscr{Z}_j$. Let us recall briefly how it can be defined.

For $v \in \mathrm{T}_x\Omega$, one may define

$$\|v\|_x = \inf\left\{\, \sqrt{a_1^2 + \dots + a_m^2} \text{ s.t. } v = a_1 \mathscr{Z}_1(x) + \dots + a_m \mathscr{Z}_m(x) \,\right\},$$

with the convention that $\|v\|_x = \infty$ if $v \notin \mathscr{Z}(x) = \mathrm{Span}(\mathscr{Z}_1(x), \dots, \mathscr{Z}_m(x))$. For any absolutely continuous path $\gamma \in W^{1,1}([0,T];\Omega)$, its "length" is defined as

$$\mathscr{L}(\gamma) = \int_0^T \|\dot{\gamma}(t)\|_{\gamma(t)} dt.$$

The Carnot-Carathéodory (or sub-riemannian) distance is then given by

$$(1.3) \qquad d(x,y) = \inf \{ \, \mathscr{L}(\gamma) \mid \gamma \text{ abs. continuous path from } x \text{ to } y \, \}.$$

The paths of finite length joining x to y will be called $\mathscr{Z}$-paths, as $\dot{\gamma}(t) \in \mathscr{Z}(\gamma(t))$ for a.e. $t \in [0,T]$. Chow's theorem [2, thm. 2.4] asserts that, in the case that we are dealing with, the distance $d(x,y)$ is finite for all x and y in the same connex component of Ω.

The shape of the balls $\mathscr{B}_\rho^x = \{ y \in \Omega \text{ s.t. } d(x,y) < \rho \}$ may be very complicated. However, the Ball-Box theorem [14, thm. 2.10] implies the following (local) estimate:

$$(1.4a) \qquad \exists \beta > 1, \qquad \beta^{-1}|x-y| \leqslant d(x,y) \leqslant \beta|x-y|^{1/2}$$

for $|x-y| \leqslant 1$. The exponant $\frac{1}{2}$ reflects the fact that one only needs to compute the first commutators of the vector fields in order to span the whole tangent space.

The lower bound remains true, even if x and y are wide apart:

$$(1.4b) \qquad \forall x, y \in K, \qquad d(x,y) \geqslant \beta_K^{-1}|x-y|$$

for all compact subset $K \subset\subset \Omega$. In the following, one will assume that the coefficients of the vector fields $\mathscr{Z}_j$ (in the standard basis of $\mathbb{R}^q$) are bounded on $\bar{\Omega}$; then β_K may be chosen independently of K.

The main result of the article is the following:

THEOREM 1.1. — *Let $\mathscr{Z} = (\mathscr{Z}_1, \ldots, \mathscr{Z}_m)$ a family of smooth vector fields on an open connected subset Ω of $\mathbb{R}^q$, having constant rank r. Assume that any smooth vector field $\mathscr{X}$ may be decomposed (not necessarily in a unique way) as*

$$(1.5) \qquad \mathscr{X} = \sum \alpha_j \mathscr{Z}_j + \sum \beta_{k,l}[\mathscr{Z}_k, \mathscr{Z}_l]$$

with locally bounded functions α_j, $\beta_{k,l}$ on Ω. The associated Carnot-Carathéodory distance in Ω, defined by (1.3), will be denoted by d. Then, given $0 < s < 1$ and a compact subset K of Ω, there is a constant $C_{s,K} > 0$ such that

$$(1.6) \qquad C_{s,K}^{-1} \|u\|_{H_K^s(\mathscr{Z})}^2 \leqslant \|u\|_{L^2(\Omega)}^2 + \iint\limits_{\substack{\Omega \times \Omega \\ d(x,y)<1}} \frac{|u(x)-u(y)|^2}{d(x,y)^{2s}} \, \frac{dx\,dy}{\mathrm{Vol}(\mathscr{B}_{d(x,y)}^x)}$$

$$\leqslant C_{s,K} \|u\|_{H_K^s(\mathscr{Z})}^2$$

for all measurable function u having compact support in K, provided one of the expressions is finite.

This theorem complements the results of [8], which express an inclusion mapping from $H^s_{\mathrm{loc}}(\mathscr{Z})$ into Hölder-type spaces, constructed with the Carnot-Carathéodory distance, when $s > q - r/2$.

Assumption (1.5) is called Hörmander's condition. One says that the degree of nonholonomy of the system of vector fields is uniformly equal to 2 :
$$\forall x \in \Omega, \quad \mathrm{rank}\, \{\mathscr{Z}_j(x)\,; [\mathscr{Z}_k, \mathscr{Z}_l](x)\}_{j,k,l} = q,$$
and uniformity means that it is always possible, at least locally, to choose a basis of the sections of the tangent space among $\mathscr{Z}_j$ and $[\mathscr{Z}_k, \mathscr{Z}_l]$.

The volume $\mathrm{Vol}(\mathscr{B}^x_{d(x,y)})$ may be easily estimated. Indeed, sub-riemannian balls of small radius have the doubling property [12, eq. 3.1]:

(1.7a) $\exists\, c \geqslant 1, \quad \forall \rho \in [0,1], \qquad \mathrm{Vol}(\mathscr{B}^x_{2\rho}) \leqslant c\, \mathrm{Vol}(\mathscr{B}^x_\rho),$

so $\mathrm{Vol}\, \mathscr{B}^x_\rho \geqslant \rho^{\frac{\ln c}{\ln 2}}\, \mathrm{Vol}\, \mathscr{B}^x_1$.

Actually, we have supposed that rank $\mathscr{Z}(x) = r$ is constant and that the system is nonholonomic of order 2 ; therefore, one may define a homogeneous dimension
$$Q = r + 2(q - r)$$
such that:

(1.7b) $\exists\, \bar{C} \geqslant \underline{C} > 0, \quad \forall \rho \in [0,1], \qquad \underline{C}\, \rho^Q \leqslant \mathrm{Vol}(\mathscr{B}^x_\rho) \leqslant \bar{C}\, \rho^Q.$

See [14, thm. 2.17] for a more general statement (Mitchell's measure theorem).

Example

Before going further, let us analyse a concrete example: the family of left-invariant vector fields on the Heisenberg group $\mathbb{H}_n$.

In brief, $\mathbb{H}_n$ may be identified with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ through coordinates $(p, q\,; t)$. It has a non-commutative group structure, with left-invariant fields $\mathscr{L} = (\mathscr{X}_1, \ldots, \mathscr{X}_n; \mathscr{Y}_1, \ldots, \mathscr{Y}_n)$ given by
$$\mathscr{X}_j = \partial_{p_j} + 2q_j \partial_t \qquad \text{and} \qquad \mathscr{Y}_j = \partial_{q_j} - 2p_j \partial_t.$$
One has $[\mathscr{X}_j, \mathscr{Y}_k] = -4\delta_{j,k}\partial_t$; so $\mathbb{H}_n$ appears as a template of sub-riemannian geometry.

Thanks to the group structure, the Carnot-Carathéodory distance is given by:
$$d\left((p, q\,; t), (p', q'\,; t')\right) \simeq \left(|p - p'|^4 + |q - q'|^4 + |t - t' + 2(q \cdot p' - p \cdot q')|^2\right)^{1/4}.$$

More precisely, if $d'((p, q; t), (p', q'; t'))$ denotes the right-hand side, one can easily show that

$$C^{-1} d'((p, q; t), (p', q'; t')) \leqslant d((p, q; t), (p', q'; t')) \leqslant C d'((p, q; t), (p', q'; t')).$$

Explicit computations in the case $n = 1$ may be found in [7, §4].

Hörmander's theorem applied to the hypoelliptic operator $\Delta_{\mathbb{H}_n} = \sum \mathscr{X}_j^2 + \mathscr{Y}_j^2$ implies (for $s > 0$) a continuous embedding of $H_{\text{loc}}^s(\mathbb{H}_n, \mathscr{L})$ into the standard isotropic Sobolev space $H_{\text{loc}}^{s/2}(\mathbb{R}^{2n+1})$.

So, the left-hand side of (1.6) implies that for any compactly supported function u:

(1.8)
$$C_s^{-1} \iint\limits_{\mathbb{H}_n \times \mathbb{H}_n} \frac{|u(x) - u(y)|^2}{|x - y|^{2n+1+s}} \, dx dy \leqslant \int_{\mathbb{H}_n} |u(x)|^2 \, dx + \iint\limits_{\mathbb{H}_n \times \mathbb{H}_n} \frac{|u(x) - u(y)|^2}{d(x, y)^{2n+2+2s}} \, dx dy.$$

A direct proof of (1.8) might not be simple because it express the fact that the system $\mathscr{Z}(x)$ still behaves as an elliptic family in $2n$ variable directions, and that (1.4a) should only be used in the last direction.

A deeper reason for studying fractional Sobolev spaces is following. The equivalence stated in (1.6) express the invariance of the spaces $H^s(\mathscr{Z})$ under a large class of diffeomorphism. This property can be used to describe the regularity of the restriction to hypersurfaces of functions in $H^s(\mathscr{Z})$ when $s > 1/2$ (see [1] and [15]). This question arises naturaly in the study of the Dirichlet problem for the sub-elliptic Laplace operator $\Delta_{\mathbb{H}_n}$.

Structure of the article

The article is organised as follows. Section 2 contains the proof of the left-hand side of (1.6), using a bound of the heat-kernel $e^{t\Delta_{\mathscr{Z}}}$. In Section 3, we recall the standard notation and some basics facts of Weyl-Hörmander calculus. Section 4 is devoted to the proof of the right-hand side of (1.6), using microlocal analysis. Some final remarks about assumption (1.5) have been added in Section 5.

Let us now explain briefly, but with more details, the ideas involved in the proof of the main theorem, which rests on two very different approaches.

Roughly speaking, the left-hand side of (1.6) means that one may be able to control the fractional powers of $-\Delta_{\mathscr{Z}} = \sum \mathscr{L}_j^* \mathscr{L}_j$ by an expression involving negative powers of the sub-riemannian distance. The appropriate tool for doing this is functional calculus with the heat semi-group $e^{t\Delta_{\mathscr{Z}}}$ associated to the $\mathscr{L}_j$ and, more precisely, exponential estimates of its kernel.

In order to prove the right-hand side of (1.6), we will have to forget all the ideas related to trajectories, $\mathscr{L}$-paths, or geodesics. An efficient approach is in the spirit of the classical proof of (1.2), which we have recalled above. It consists of a frequential analysis of the the difference $u(y) - u(x)$, viewed as a translation operator $(T^*_{y-x} - \mathrm{Id})u$.

The main difficulty is the radical anisotropy of the ambient space. It is due to the fact that the subspace $\mathscr{L}(x)$ spanned by the fields $\mathscr{L}_j$ does not fill $T_x\Omega$. Darboux's theorem joined to assumption (1.5) implies that the variation of $\mathscr{L}(x)$ with the point x may not be "flattened" ; this means that it is not the tangent space to any embedded surface. In spite of technical problems that will be discussed in Section 5, the right tool to face this problem is a microlocal phase-space analysis.

Preliminary reduction of the problem

For the sake of simplicity, and also because Weyl-Hörmander's calculus requires us finally to work on $\mathbb{R}^q$ with the canonical identification $T^*\mathbb{R}^q \simeq \mathbb{R}^q \times \mathbb{R}^q$, one may notice that it suffices to prove the theorem in the case $\Omega = \mathbb{R}^q$.

Indeed, given $K \subset\subset \Omega$, one may chose a smooth cut-off function χ, compactly supported in Ω, such that $\chi \equiv 1$ in a neighborhoud of K. The family of vector fields that we will really deal with is

$$\widetilde{\mathscr{L}} = (\chi\mathscr{L}_1, \ldots, \chi\mathscr{L}_m, (1-\chi)\nabla) \ \text{ on } \mathbb{R}^q.$$

It has constant rank r on K and satisfies Chow-Hörmander's condition (1.5) on $\mathbb{R}^q$. We will not use the constant rank hypothesis outside K, so one should not worry about the increase of the rank of $\widetilde{\mathscr{L}}$ on $\mathrm{supp}(1-\chi)$. From now on, we will still denote the family $\widetilde{\mathscr{L}}$ by $\mathscr{L}$, in order to keep the notation compact.

Notice also that one may reduce the size of the support of u as needed. Indeed, if the result is proved on K and K', then it is obviously also true on $K \cup K'$.

Thanks

2. Approach based on the heat-kernel theory

In this section, we prove the left-hand side of (1.6), using an approach based on the heat-kernel theory.

The sub-elliptic Laplace operator defined by

$$\Delta_{\mathscr{X}} = -\sum \mathscr{X}_j^* \mathscr{X}_j = \sum \mathscr{X}_j^2 + (\operatorname{div} \mathscr{X}_j) \mathscr{X}_j$$

is symmetric hence a closable operator on L^2. As $-\Delta_{\mathscr{X}}$ is non-negative, one may consider its fractional powers.

For $0 < s < 1$, the complex interpolation space

$$H_K^s(\mathscr{X}) = [L^2(K); H_K^1(\mathscr{X})]_s$$

coincides with the completion of $\mathscr{D}(K)$ with respect to the norm

(2.1)
$$\|u\|_{s,\mathscr{X}}^2 = \|u\|_{L^2}^2 + \left\| (-\Delta_{\mathscr{X}})^{s/2} u \right\|_{L^2}^2,$$

and the norms $\|\cdot\|_{s,\mathscr{X}}$ and $\|\cdot\|_{H^s(\mathscr{X})}$ are actually equivalent.

Let us sum up the main idea. The semi-group $\{e^{t\Delta_{\mathscr{X}}}\}_{t \geqslant 0}$ generated by $(-\Delta_{\mathscr{X}})$ is holomorphic and contracting on L^2. For any $s \in]0,1[$ one has $(-\Delta_{\mathscr{X}})^s = -\Delta_{\mathscr{X}} \circ (-\Delta_{\mathscr{X}})^{s-1}$, with $s-1 < 0$. Functional calculus ensures that:

(2.2)
$$\forall \sigma > 0, \quad \exists c_\sigma > 0, \qquad (-\Delta_{\mathscr{X}})^{-\sigma} = c_\sigma \int_0^\infty t^\sigma e^{t\Delta_{\mathscr{X}}} \frac{dt}{t}$$

so $(-\Delta_{\mathscr{X}})^s = -c_{1-s} \int_0^\infty t^{1-s} \Delta_{\mathscr{X}} \left(e^{t\Delta_{\mathscr{X}}} \right) \frac{dt}{t}$. Its kernel is therefore given by the integral:

$$(-\Delta_{\mathscr{X}})^s u(x) = -c_{1-s} \int_{\mathbb{R}^q} \int_0^\infty t^{1-s} \frac{\partial p_t}{\partial t}(x,y) \, u(y) \, \frac{dt}{t} \, dy$$

where $p_t(x,y)$ denotes the heat-kernel:

$$\frac{\partial p_t}{\partial t} = \Delta_{\mathscr{X}} \, p_t \quad (y \text{ fixed}), \qquad \text{and} \qquad p_0(x,y) = \delta_0(x-y).$$

Classical bounds of $p_t(x,y)$ involving the Carnot-Carathéodory distance are well known and may be found e.g. in [13]. Let us now move on to a detailed proof of the left-hand side of (1.6).

2.1. An exact formula for $\left\|(-\Delta_{\mathscr{X}})^{s/2}u\right\|_{L^2}$

One may compute $\|\cdot\|_{s,\mathscr{X}}$ in the following way:

$$
\left\|(-\Delta_{\mathscr{X}})^{s/2}u\right\|_{L^2}^2 = \left((-\Delta_{\mathscr{X}})^2 \circ (-\Delta_{\mathscr{X}})^{s-2}u \mid u\right)_{L^2}
$$

$$
= c_s' \left((-\Delta_{\mathscr{X}})^2 \left(\int_0^\infty t^{2-s}\, e^{2t\Delta_{\mathscr{X}}}\, \frac{dt}{t}\right) u \,\Big|\, u\right)_{L^2}
$$

with $c_s' = 2^{2-s}c_{2-s}$. As $\Delta_{\mathscr{X}}$ commutes to the heat-operator $e^{t\Delta_{\mathscr{X}}}$, one has:

$$
\text{(2.3a)} \qquad \left\|(-\Delta_{\mathscr{X}})^{s/2}u\right\|_{L^2}^2 = c_s' \int_0^\infty t^{2-s} \left\|\Delta_{\mathscr{X}}(e^{t\Delta_{\mathscr{X}}}u)\right\|_{L^2}^2 \frac{dt}{t}.
$$

The right manner to bound this integral depends on the size of t ; so we will use instead :

$$
\text{(2.3b)} \qquad \frac{1}{c_s'}\left\|(-\Delta_{\mathscr{X}})^{s/2}u\right\|_{L^2}^2 = \int_{\mathbb{R}^q}\int_0^1 t^{2-s}\left|\frac{\partial}{\partial t}(e^{t\Delta_{\mathscr{X}}}u)(x)\right|^2 \frac{dt}{t}\, dx
$$
$$
+ \int_1^\infty t^{2-s}\left\|\Delta_{\mathscr{X}}(e^{t\Delta_{\mathscr{X}}}u)\right\|_{L^2}^2 \frac{dt}{t}.
$$

2.2. Asymptotical decay due to the diffusion process

As $e^{t\Delta_{\mathscr{X}}}$ is a holomorphic semi-group, one has the following decay [10, lem. 2.38]:

$$
\text{(2.4)} \qquad\qquad \left\|\Delta_{\mathscr{X}}(e^{t\Delta_{\mathscr{X}}})\right\|_{\mathscr{L}(L^2)} \leqslant \frac{C}{t}
$$

so the integral corresponding to $t \geqslant 1$ in (2.3b) is simply controled by $\frac{C}{s}\|u\|_{L^2}^2$.

2.3. Short-time dynamic

Let us now focus on the integral corresponding to $t \in [0,1]$.
The first valuable remark is that:

$$
\forall t > 0, \quad \forall x \in \mathbb{R}^q, \qquad \int_{\mathbb{R}^q} p_t(x,y)\, dy = 1,
$$

so $\int_{\mathbb{R}^q} \partial_t p_t(x,y)\, dy = 0$. This allows us to exhibit the difference $u(y) - u(x)$:

$$
\frac{\partial}{\partial t}(e^{t\Delta_{\mathscr{X}}}u)(x) = \int_{\mathbb{R}^q} \frac{\partial p_t(x,y)}{\partial t}\, u(y)\, dy = \int_{\mathbb{R}^q} \frac{\partial p_t(x,y)}{\partial t}\, [u(y) - u(x)]\, dy.
$$

LEMMA 2.1. — *The kernel $p_t(x,y)$ of the heat-operator $e^{t\Delta_{\mathscr{X}}}$ satisfies the following pointwise estimate:*

$$(2.5) \qquad \forall t \in \,]0,1[, \qquad \left|\frac{\partial p_t(x,y)}{\partial t}\right| \leqslant \frac{\kappa}{t}\, p_{\mu t}(x,y) \leqslant \frac{\bar{\kappa}}{t}\, \frac{e^{-\bar{\mu}d^2(x,y)/t}}{\mathrm{Vol}(\mathscr{B}^x_{\sqrt{t}})}$$

for some constants $\kappa, \bar{\kappa}, \mu, \bar{\mu} > 0$.

Proof. — Let us recall the classical estimates given in [13, thm. 3 and 4]:

$$\left|\partial_t^k\, p_t(x,y)\right| \leqslant \frac{C_k}{t^k}\, \frac{e^{-\nu\, d(x,y)^2/t}}{\mathrm{Vol}(\mathscr{B}^x_{\sqrt{t}})} \quad (k=0,1) \text{ and } p_t(x,y) \geqslant C_0'\, \frac{e^{-\nu'd(x,y)^2/t}}{\mathrm{Vol}(\mathscr{B}^x_{\sqrt{t}})}.$$

Let $\mu = \nu'/\nu \geqslant 1$, $\quad \bar{\mu} = \nu^2/\nu'$ $\quad$ and $\quad \bar{\kappa} = C_0 \kappa$ where

$$\kappa = \frac{C_1}{C_0'}\, \sup_{\substack{x \in \mathbb{R}^q \\ 0<t<1}}\, \frac{\mathrm{Vol}(\mathscr{B}^x_{\sqrt{\mu t}})}{\mathrm{Vol}(\mathscr{B}^x_{\sqrt{t}})}.$$

As the sub-riemannian balls of small radius have the doubling property (1.7b), κ is bounded by $C_1 C_0'^{-1}\bar{C}\underline{C}^{-1}\mu^{Q/2}$ and (2.5) follows immediately. $\qquad\square$

Using the first part of (2.5), Hölder's inequality applied to the probability measure $p_{\mu t}(x,y)\,dy$ leads to the following pointwise estimate:

$$(2.6) \qquad \left|\frac{\partial}{\partial t}(e^{t\Delta_{\mathscr{X}}}u)(x)\right|^2 \leqslant \frac{\kappa^2}{t^2}\int_{\mathbb{R}^q} p_{\mu t}(x,y)\,|u(y)-u(x)|^2\,dy.$$

Substitution in (2.3b) gives:

$$c_s'^{-1}\left\|(-\Delta_{\mathscr{X}})^{s/2}u\right\|_{L^2}^2 \leqslant \frac{C}{s}\,\|u\|_{L^2}^2 + \kappa^2\iiint_{[0,1]\times\mathbb{R}^{2q}} p_{\mu t}(x,y)\,|u(x)-u(y)|^2\frac{dt\,dx\,dy}{t^{1+s}}.$$

Using the second part of (2.5), and then the change of variable $t = \bar{\mu}d^2(x,y)\tau$ gives:

$$(2.7) \quad \kappa^2\int_0^1 p_{\mu t}(x,y)\,\frac{dt}{t^{1+s}} \leqslant \frac{\kappa\,\bar{\kappa}}{\bar{\mu}^s\,d(x,y)^{2s}}\int_0^{\frac{1}{\bar{\mu}d(x,y)^2}} \frac{\tau^{-s}\,e^{-1/\tau}}{\mathrm{Vol}(\mathscr{B}^x_{d(x,y)\sqrt{\bar{\mu}\tau}})}\,\frac{d\tau}{\tau}.$$

The radius of the ball involved in the above formula is always smaller than 1.

If $d(x,y) < 1$, one uses the doubling property (1.7b) again:

$$\int_0^{\frac{1}{\bar{\mu}d(x,y)^2}} \frac{\tau^{-s}\,e^{-1/\tau}}{\mathrm{Vol}(\mathscr{B}^x_{d(x,y)\sqrt{\bar{\mu}\tau}})}\,\frac{d\tau}{\tau} \leqslant \frac{\mathrm{Cte}}{\mathrm{Vol}(\mathscr{B}^x_{d(x,y)})}\int_0^\infty \tau^{-s-\frac{Q}{2}}\,e^{-1/\tau}\,\frac{d\tau}{\tau}$$

$$\leqslant \frac{\mathrm{Cte}'}{\mathrm{Vol}(\mathscr{B}^x_{d(x,y)})}.$$

If $d(x,y) \geqslant 1$, one uses version (1.7b) of the doubling property

$$\mathrm{Vol}(\mathscr{B}^x_{d(x,y)\sqrt{\bar\mu\tau}}) \geqslant \underline{C}\bar\mu^{Q/2}\,\tau^{Q/2}\,d(x,y)^Q$$

whence, thanks to the remark (1.4b):

$$\int_0^{\frac{1}{\bar\mu d(x,y)^2}} \frac{\tau^{-s}e^{-1/\tau}}{\mathrm{Vol}(\mathscr{B}^x_{d(x,y)\sqrt{\bar\mu\tau}})} \frac{d\tau}{\tau} \leqslant \frac{\mathrm{Cte}}{|x-y|^Q} \int_0^{\frac{1}{\bar\mu}} \tau^{-s-\frac{Q}{2}}e^{-1/\tau}\frac{d\tau}{\tau} \leqslant \frac{\mathrm{Cte}'}{|x-y|^Q}.$$

In this case, one has $|x-y| \geqslant \beta^{-2}$ (if not, one would have $d(x,y) \leqslant \beta|x-y|^{1/2} < 1$, which is contradictory).

2.4. End of the proof of the left-hand side of (1.6)

Finally putting those estimates together leads to:

$$\left\|(-\Delta_{\mathscr{X}})^{s/2}u\right\|_{L^2}^2 \leqslant C_1\left(\|u\|_{L^2}^2 + \iint\limits_{|x-y|\geqslant\beta^{-2}} \frac{|u(x)-u(y)|^2}{|x-y|^{Q+2s}}\,dxdy\right)$$

$$+ C_2 \iint\limits_{d(x,y)<1} \frac{|u(x)-u(y)|^2}{d(x,y)^{2s}} \frac{dx\,dy}{\mathrm{Vol}(\mathscr{B}^x_{d(x,y)})}$$

$$\leqslant C_1'\,\|u\|_{L^2}^2 + C_2 \iint\limits_{d(x,y)<1} \frac{|u(x)-u(y)|^2}{d(x,y)^{2s}} \frac{dx\,dy}{\mathrm{Vol}(\mathscr{B}^x_{d(x,y)})}.$$

Remark. — The real numbers $C_1, C_1', C_2 > 0$ depend on $s \in \,]0,1[$ and on $\mathscr{X}$ but only through the constants appearing in (1.4b), (1.7b) and (2.5).

This concludes the proof of the left-hand side of (1.6) in the main theorem, and ends the first stage of the article.

3. Quick review of Weyl-Hörmander's calculus

The rest of the article is devoted to the proof of the right-hand side of (1.6). As it requires a microlocal analysis of the translation operators $T_h - \mathrm{Id}$, we will briefly recall the results that we are going to use and fix the notation for the subsequent proof. For more details, we refer to [9] and the references therein.

3.1. Confinement

Confinement is defined through a family of semi-norms on the phase-space (the cotangent bundle $\mathrm{T}^*\Omega$, equipped with its natural symplectic structure).

Let g be a metric on a symplectic space $(\mathbb{R}^{2q}, \sigma)$; the dual metric is

$$g^\sigma(S) = \sup_{T \neq 0} \frac{\sigma(S, T)^2}{g(T)}.$$

From a matricial point of view, one has simply $G^\sigma = -\Sigma G^{-1} \Sigma$ with obvious notation. Given a Borel-measurable set $\mathcal{U} \subset \mathbb{R}^{2q}$ and ϕ a smooth function, one may define the *confinement semi-norms* :

$$(3.1) \quad \|\phi\|_{N;\mathrm{Conf}_\mathcal{U}(g)} = \sup_{\substack{X \in \mathrm{T}^*\mathbb{R}^q \\ k \leqslant N\, ;\, T_i \neq 0}} \left(1 + g^\sigma(X - \mathcal{U})\right)^N \frac{|\partial_{T_1} \dots \partial_{T_k} \phi(X)|}{g(T_1)^{1/2} \dots g(T_k)^{1/2}}$$

for $N \in \mathbb{N}$. Obviously, $\|\phi\|_{N;\mathrm{Conf}_\mathcal{U}(g)} \leqslant \|\phi\|_{N+1;\mathrm{Conf}_\mathcal{U}(g)}$.

Consider now the phase-space $\mathrm{T}^*\mathbb{R}^q$ (canonically identified with $\mathbb{R}^q \times \mathbb{R}^q$) and the family of vector fields $\mathscr{Z}$, suitably extended to $\mathbb{R}^q$, as explained § 1. To each point $X = (x, \xi)$ one associates the following metric:
$$(3.2)$$
$$g_X = \mathfrak{m}^{-2}(X) \left\{ \langle \xi \rangle^2 \, dx^2 + d\xi^2 \right\} \quad \text{with} \quad \mathfrak{m}(X)^2 = 1 + |\xi| + \sum |\langle \mathscr{Z}_j(x) | \xi \rangle|^2.$$

The family $(g_Y)_{Y \in \mathrm{T}^*\mathbb{R}^q}$ satisfies the classical assumptions of Weyl-Hörmander's calculus [6, lem. 1.2.1], *i.e.*, the uncertainty principle $g \leqslant g^\sigma$, and assumptions called slowness and temperance:

$$\forall T \neq 0, \qquad \left(\frac{g_X(T)}{g_Y(T)}\right)^{\pm 1} \leqslant C_0 \left(1 + g_Y^\sigma(X - \mathcal{U}_Y^{\delta_0})\right)^{N_0}$$

where $\mathcal{U}_Y^\delta = \{X \in \mathrm{T}^*\mathbb{R}^q \, ;\, g_Y(X - Y) \leqslant \delta^2\}$.

Let $\delta \in]0, \delta_0[$. The confinement near Y is defined by (3.1), with respect to the metric g_Y and the balls $\mathcal{U}_Y^\delta$. In the following, $\mathrm{Conf}_{\mathcal{U}_Y^\delta}(g_Y)$ will be shortened to Conf_Y.

There is a partition of the unity $(\vartheta_Y)_{Y \in \mathrm{T}^*\mathbb{R}^q}$ involving only confined symbols:

$$(3.3) \quad \int_{\mathrm{T}^*\mathbb{R}^q} \vartheta_Y \, d_g Y = 1 \quad \text{and} \quad \forall N, \quad \sup_{Y \in \mathrm{T}^*\mathbb{R}^q} \|\vartheta_Y\|_{N;\mathrm{Conf}_Y} < \infty,$$

with the standard symplectic renormalisation of the measure $d_g Y = |\det G_Y|^{1/2} dY$ on the phase-space. Here, G_Y denotes the matrix of g_Y in the local coordinates in which dY is computed.

The metric g is dominated by the metric $\langle\xi\rangle\, dx^2 + \langle\xi\rangle^{-1}\, d\xi^2$, which is strongly temperate [4, déf. 7.1]. One may assume, without loss of generality, that ϑ_Y splits as $\vartheta_Y = \psi_Y \# \varphi_Y$ with a partition $(\varphi_Y)_{Y\in\mathrm{T}^*\mathbb{R}^q}$ and functions $(\psi_Y)_{Y\in\mathrm{T}^*\mathbb{R}^q}$ that are also uniformly confined [4, §7]. This technical property may be substituted for the useful idea that a compactly supported function f may be written as a product $f = f\widetilde{f}$ with $\widetilde{f}$ being a smooth function equal to 1 in a small neighbourhood of $\mathrm{supp}\, f$.

The law of composition of symbols is defined by the formula:

$$(\phi\#\psi)(X) = \pi^{-2q} \iint e^{-2i[X-S,X-T]}\,\phi(S)\,\psi(T)\, dS\, dT.$$

where $[X, Y] = y\cdot\xi - x\cdot\eta$ denotes the canonical symplectic structure on $\mathbb{R}^{2q}$.

Multi-confinement of a composed symbol (see [5, §3.2]). *There is a symmetric function Δ such that, for all N, one may find an integer M and a non-negative constant C_N verifying*

(3.4a)
$$\|\phi\#\psi\|_{N;\mathrm{Conf}_X} + \|\phi\#\psi\|_{N;\mathrm{Conf}_Y} \leqslant C_N\,\Delta(X,Y)^{-N}\,\|\phi\|_{M;\mathrm{Conf}_X}\,\|\psi\|_{M;\mathrm{Conf}_Y}$$

for all symbols ϕ, ψ. Moreover, $\Delta(X,Y) \geqslant 1$ and there is an integer N_0 such that

(3.4b)
$$\sup_X \int_{\mathrm{T}^*\mathbb{R}^q} \Delta(X,Y)^{-N_0}\, d_gY < \infty.$$

3.2. Weyl's quantization

Weyl's quantization of a symbol ϕ is defined (at least formally) by:

$$\phi^{\mathrm{w}} u(x) = \iint e^{i\langle x-z|\zeta\rangle}\,\phi\Big(\frac{x+z}{2};\zeta\Big)\,u(z)\,\frac{dz d\zeta}{(2\pi)^q}.$$

The composition of quantified symbols is quite simple: $\phi^{\mathrm{w}} \circ \psi^{\mathrm{w}} = (\phi\#\psi)^{\mathrm{w}}$.

Confined symbols define bounded operators on L^2 ; see [5, 2.4.1]:

(3.5)
$$\|\phi^{\mathrm{w}}\|_{\mathscr{L}(L^2)} \leqslant C_1 \inf_X \|\phi\|_{N_1;\mathrm{Conf}_X}.$$

As g_X may be written $a(X)\, dx^2 + b(X)\, d\xi^2$, one obtains the following deeper result.

Microlocal version of Cotlar's lemma (see [8, 2.16]). *Let $(\psi_Y)_{Y\in T^*\mathbb{R}^q}$ be a uniformly confined family of symbols and $f : \mathbb{R}^q \times T^*\mathbb{R}^q \to \mathbb{C}$ a*

mesurable function. Define also

$$(3.6\text{a}) \qquad u(x) = \int_{\mathrm{T}^*\mathbb{R}^q} \left(\psi_Y^{\mathrm{w}} f(\cdot, Y) \right)(x) \ d_g Y.$$

Then, there is a constant $C_\psi > 0$ (depending only on the ψ_Y) such that:

$$(3.6\text{b}) \qquad \|u\|_{L^2}^2 \leqslant C_\psi \int_{\mathrm{T}^*\mathbb{R}^q} \|f(\cdot, Y)\|_{L^2}^2 \ d_g Y.$$

3.3. Sobolev spaces and symbol classes

For $s \geqslant 0$, the space $H(\mathfrak{m}^s)$ consists of all functions $u \in L^2(\mathbb{R}^q)$ such that:

$$(3.7) \qquad \|u\|_{H(\mathfrak{m}^s)}^2 = \int_{\mathrm{T}^*\mathbb{R}^q} \mathfrak{m}(Y)^{2s} \|\vartheta_Y^{\mathrm{w}} u\|_{L^2}^2 \ d_g Y < \infty$$

for a partition of unity $(\vartheta_Y)_{Y \in \mathrm{T}^*\mathbb{R}^q}$ into confined symbols. The space $H(\mathfrak{m}^s)$ and its natural Hilbertian structure do not depend on the choice of the partition of unity [4, prop. 4.3].

Given a *weight* function $\mathfrak{M}$, *i.e.*, a smooth function such that

$$\left(\frac{\mathfrak{M}(X)}{\mathfrak{M}(Y)} \right)^{\pm 1} \leqslant \bar{C} \left(1 + g_Y^\sigma(X - \mathcal{U}_Y^{\delta_0}) \right)^{\bar{N}},$$

the corresponding symbol classes $S(\mathfrak{M})$ are defined through their semi-norms by:

$$(3.8) \qquad \|\phi\|_{N;S(\mathfrak{M})} = \sup_{\substack{X \in \mathrm{T}^*\mathbb{R}^q \\ k \leqslant N \, ; \, T_i \neq 0}} \mathfrak{M}(X)^{-1} \frac{|\partial_{T_1} \ldots \partial_{T_k} \phi(X)|}{g_X(T_1)^{1/2} \ldots g_X(T_k)^{1/2}}.$$

Actually, when dealing with symbol classes (3.8), one uses the semi-norms

$$(3.9) \quad \|\phi\|_{N;\mathrm{Conf}_Y'} = \sup_{\substack{X \in \mathrm{T}^*\mathbb{R}^q \\ k \leqslant N \, ; \, T_i \neq 0}} \left(1 + g_Y^\sigma(X - \mathcal{U}_Y^\delta) \right)^N \frac{|\partial_{T_1} \ldots \partial_{T_k} \phi(X)|}{g_X(T_1)^{1/2} \ldots g_X(T_k)^{1/2}}$$

instead of $\|\cdot\|_{N;\mathrm{Conf}_Y}$. As

$$C_0^{-N/2} \|\phi\|_{N;\mathrm{Conf}_Y'} \leqslant \|\phi\|_{(1+N_0)N;\mathrm{Conf}_Y} \leqslant C_0^N \|\phi\|_{(1+N_0)^2 N;\mathrm{Conf}_Y'},$$

both families of semi-norms define the same Frechet space.

Examples.

1. The space $H(1)$ is exactly $L^2(\mathbb{R}^q)$ and both norms are equivalent.

2. The space $H(\langle\xi\rangle^s)$ is the standard Sobolev space $H^s(\mathbb{R}^q)$ for all $s \geqslant 0$.

3. The weight $\mathfrak{m}$ in (3.2) is an admissible weight ; $H(\mathfrak{m}^k)$ is the space of square integrable functions u such that

$$\forall \ell \in \{0,\ldots,k\}, \quad \forall(j_i)_{1\leqslant i\leqslant\ell} \in \{1,\ldots,m\}^\ell, \quad \mathscr{Z}_{j_1}\ldots\mathscr{Z}_{j_\ell}u \in L^2.$$

In other words, it is the Hilbert-space $H^k(\mathscr{Z})$. As the family $\{H(\mathfrak{m}^s)\}_{s\geqslant 0}$ is stable by complex interpolation [4, rmk. 4.2], one has for all $s \geqslant 0$:

$$H(\mathfrak{m}^s) \simeq H^s(\mathbb{R}^q; \mathscr{Z}).$$

Moreover, $\mathfrak{m}$ is a regular weight [6, lem. 1.2.1], i.e., $\mathfrak{m}^s \in S(\mathfrak{m}^s)$ for all $s \geqslant 0$.

The link between Sobolev spaces and symbol classes is fully described by the following theorem.

THEOREM (see [4, cor. 6.6 & 6.7].). — *A function u belongs to $H(\mathfrak{m}^s)$ if and only if $a^w u \in L^2$ for all symbols $a \in S(\mathfrak{m}^s)$. Moreover, there are symbols $b \in S(\mathfrak{m}^s)$ and $h \in S(\mathfrak{m}^{-s})$ such that*

$$(3.10) \qquad\qquad h\#b = b\#h = 1.$$

The mapping $h^w : L^2 \to H(\mathfrak{m}^s)$ is a Hilbert-space isomorphism, whose inverse is b^w.

4. Approach based on a frequential analysis of $u(y) - u(x)$

In this section, we are going to prove the right-hand side of (1.6), using an approach based on phase-space analysis. The key point is to define a frequency cut-off corresponding to the "sub-riemannian amplitude" of the translation $T^*_{y-x} - \mathrm{Id}$; this amplitude cannot be understood outside an adapted microlocal framework. This idea is often used in nonlinear analysis (see, e.g. [3]). We will follow [8], where it was succesfully applied to prove Sobolev embeddings.

4.1. Anisotropic projectors and scaling

The preliminary stage of the proof is the introduction of a decomposition of each tangent space $\mathrm{T}_x\Omega$, adapted to the local properties of the sub-riemannian family $\mathscr{Z}$. As the tangent space is identified with $\mathbb{R}^q$, this decomposition will induce a local anisotropic scaling.

Thanks to the constant rank hypothesis near the support of u (and narrowing it if needed with a partition of unity), one may chose a smooth basis $(\mathscr{X}_j)_{1\leqslant j\leqslant q}$ and integers n_j, n'_j such that

$$\begin{cases} \mathscr{X}_j = \mathscr{L}_{n_j} & \text{for} \quad 1 \leqslant j \leqslant r, \\ \mathscr{X}_k = [\mathscr{L}_{n_k}, \mathscr{L}_{n'_k}] & \text{for} \quad r+1 \leqslant k \leqslant q \end{cases}$$

on $\operatorname{supp} u$.

Given $\underline{x} \in \operatorname{supp} u$, the flows $e^{x_j \mathscr{X}_j} \underline{x}$ define a local coordinate system $(x_j)_{1 \leqslant j \leqslant q}$ centered at the point $\underline{x}$. This system is a privileged one in the terminology of Bellaiche. It means that the Carnot-Carathéodory distance to $\underline{x}$ can be estimated precisely:

$$C^{-1}\Big(\sum_{1 \leqslant j \leqslant r} |x_j| + \sum_{r < k \leqslant q} |x_k|^{1/2} \Big) \leqslant d(x, \underline{x}) \leqslant C \Big(\sum_{1 \leqslant j \leqslant r} |x_j| + \sum_{r < k \leqslant q} |x_k|^{1/2} \Big).$$

This property is the heart of the proof of the Ball-Box theorem given in [14, prop. 2.14].

The hidden constants may be chosen uniformly with respect to $\underline{x}$ because it is a regular point ; see [11, lem. 1] and the references therein.

The tangent space may now be decomposed in an obvious way:
(4.1)

$$\mathrm{T}_x \Omega = \mathscr{N}_1(x) \oplus \mathscr{N}_2(x) \quad \text{where} \quad \begin{cases} \mathscr{N}_1(x) = \operatorname{Span}\{\mathscr{X}_1, \ldots, \mathscr{X}_r\}, \\ \mathscr{N}_2(x) = \operatorname{Span}\{\mathscr{X}_{r+1}, \ldots, \mathscr{X}_n\}. \end{cases}$$

The corresponding projector on $\mathscr{N}_i$ is denoted by π_i^x $(i = 1, 2)$. The subspaces $\mathscr{N}_i(x)$ vary smoothly with x (i.e., $\mathscr{N}_i$ is a sub-bundle of $\mathrm{T}\Omega$).

In this notation, the index i gives the length of the commutators of the initial fields that are needed to span $\mathscr{N}_i$: the higher the index, the more difficult to access is the direction, in regard to the sub-riemannian metric.

The first projector π_1 has a deep link with the microlocal weight $\mathfrak{m}$ defined above in (3.2).

PROPOSITION 4.1. — Let $\nu \in \mathbb{S}^{q-1}$. The function $(x, \xi) \mapsto \langle \pi_1^x \nu | \xi \rangle$ is a symbol of class $S(\mathfrak{m})$. The constants may be chosen uniformly for $\nu \in \mathbb{S}^{q-1}$. One has also $\langle \pi_2^x \nu | \xi \rangle \in S(\mathfrak{m}^2)$.

The proof relies essentialy on the following lemma. Recall that a function $a(x, \xi)$ is a classical symbol of class $S^1_{1,0}$ when

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leqslant C_{\alpha,\beta} \, \langle \xi \rangle^{1-|\alpha|}$$

for all multi-indices α, β.

LEMMA 4.2. — *There is an inclusion mapping of Frechet spaces*

(4.2a) $$S_{1,0}^1 \hookrightarrow S(\mathfrak{m}^2).$$

Moreover, a symbol $a \in S_{1,0}^1$ *such that*

(4.2b) $$|a(X)| \leqslant C \, \mathfrak{m}(X)$$

for some finite constant $C > 0$ *actually belongs to* $S(\mathfrak{m})$.

Proof. — First, let us notice that

$$|a(x, \xi)| \leqslant C_0 \, \langle \xi \rangle \leqslant C_0 \, \mathfrak{m}^2(X).$$

It is sufficient to deal with $D_x^{\mathfrak{m}} = \frac{\mathfrak{m}(X)}{\langle \xi \rangle} \partial_x$ and $D_\xi^{\mathfrak{m}} = \mathfrak{m}(X) \partial_\xi$ because

$$\forall T \in \mathrm{T}^* \mathbb{R}^q, \qquad g_X(T)^{-1/2} \, |\partial_T \phi(X)| \leqslant |D_x^{\mathfrak{m}} \phi(X)| + |D_\xi^{\mathfrak{m}} \phi(X)|.$$

The first derivatives are given by:

$$D_x^{\mathfrak{m}}\big(a(x,\xi)\big) = b(X) \, \mathfrak{m}(X) \quad \text{and} \quad D_\xi^{\mathfrak{m}}\big(a(x,\xi)\big) = c(X) \, \mathfrak{m}(X)$$

with $b(X) = \partial_x a / \langle \xi \rangle$ and $c(X) = \partial_\xi a$. One may check immediately that $b, c \in S_{1,0}^0$. According to [6, lem. 1.2.1] one also has $\mathfrak{m} \in S(\mathfrak{m})$.

For the derivatives of higher order, one uses the fact that for $f \in S_{1,0}^0$ and $h \in S(\mathfrak{m})$:

$$D_x^{\mathfrak{m}}(f(X) \cdot h(X)) = \partial_x f(X) \cdot \frac{\mathfrak{m}(X)}{\langle \xi \rangle} \, h(X) + f(X) \cdot D_x^{\mathfrak{m}} h(X)$$

and

$$D_\xi^{\mathfrak{m}}(f(X) \cdot h(X)) = \langle \xi \rangle \, \partial_\xi f(X) \cdot \frac{\mathfrak{m}(X)}{\langle \xi \rangle} \, h(X) + f(X) \cdot D_\xi^{\mathfrak{m}} h(X).$$

Both derivatives are therefore also the sums of terms of the form $\widetilde{f}(X) \, \widetilde{h}(X)$ with $\widetilde{f} \in S_{1,0}^0$ and $\widetilde{h} \in S(\mathfrak{m})$. $\qquad\square$

It is now possible to conclude the proof of Proposition 4.1.

Proof (of Prop. 4.1). — Given $\nu \in \mathbb{S}^{q-1}$, there are smooth functions α_j such that

$$\pi_1^x \nu = \sum_{1 \leqslant j \leqslant r} \alpha_j(x) \mathscr{X}_j(x) \quad \text{with} \quad \mathscr{X}_j(x) = \sum_{1 \leqslant k \leqslant q} X_j^k(x) \partial_k.$$

For all $\xi = (\xi_k)_{1 \leqslant k \leqslant q} \in \mathbb{R}^q$, the Euclidean scalar product with ξ is:

(4.3) $$\langle \pi_1^x \nu | \xi \rangle = \sum_{j,k} \alpha_j(x) X_j^k(x) \xi_k = \sum_{1 \leqslant j \leqslant r} \alpha_j(x) \, \langle \mathscr{X}_j(x) | \xi \rangle,$$

so $|\langle \pi_1^x \nu | \xi \rangle| \leqslant \mathfrak{m}(X) \left(\sum \|\alpha_j\|_{L^\infty}^2 \right)^{1/2}$. Proposition 4.1 follows now from the preceding lemma. $\qquad\square$

As $\Omega = \mathbb{R}^q$, the tangent spaces are all naturally identified with $\mathbb{R}^q$ and anisotropic translations may be defined by

(4.4) $\forall \nu \in \mathbb{S}^{q-1}, \quad \forall \rho > 0, \qquad \mathrm{D}_{\nu,\rho}(x) = x + \rho\,\pi_1^x(\nu) + \rho^2\,\pi_2^x(\nu).$

Let $\mathrm{D}_{\nu,\rho}^* u = u \circ \mathrm{D}_{\nu,\rho}$, the associated scaling operator.

PROPOSITION 4.3. — *There is a constant $C > 1$ such that*

(4.5) $\forall \nu \in \mathbb{S}^{q-1}, \quad \forall \rho \in [0,1], \qquad C^{-1}\,\rho \leqslant d(x, \mathrm{D}_{\nu,\rho}(x)) \leqslant C\,\rho,$

uniformly for x near $\mathrm{supp}\, u$.

Proof. — Obvious, from the remarks preceding (4.1) in the current section. $\qquad\square$

4.2. Anisotropic polar coordinates

Thanks to the doubling property (1.7b), one has to estimate:

(4.6) $\displaystyle \Im_s u = \iint_{d(x,y)<1} |u(x) - u(y)|^2\, d(x,y)^{-Q-2s}\, dx\, dy.$

In order to compute this integral, one uses polar coordinates, centered at x.

LEMMA 4.4. — *Given $x \in \Omega$, the mapping $(\nu, \rho) \mapsto \mathrm{D}_{\nu,\rho}(x)$ is a diffeomorphism*

$$\mathbb{S}^{q-1} \times\,]0, +\infty[\xrightarrow{\simeq} \mathbb{R}^q$$

whose jacobian determinant is uniformly equivalent to ρ^{Q-1}.

Proof. — In the coordinates system centered at $\underline{x}$ defined at the begining of § 4.1, the point $\mathrm{D}_{\nu,\rho}(\underline{x})$ is represented by $(\rho\nu_1, \ldots, \rho\nu_r,\, \rho^2\nu_{r+1}, \ldots, \rho^2\nu_n)$ where (ν_i) are the coordinates of ν. The jacobian determinant grow like ρ^{Q-1}, with $Q = r + 2(n-r)$. As the fields $\mathscr{X}_i$ are smooth, so are the transfer matrices from those special coordinates to a fixed one. $\qquad\square$

Using the preceding lemma and Fubini's theorem:

(4.7) $\displaystyle \Im_s u \leqslant C \sup_{\nu \in \mathbb{S}^{q-1}} \int_0^R \left\| (\mathrm{D}_{\nu,\rho}^* - \mathrm{Id})\, u \right\|_{L^2}^2\, \rho^{-1-2s} d\rho$

for some finite $R > 0$.

Remark. — One may ask whether (4.7) is related to the belonging of the map $\rho \mapsto \mathrm{D}_{\nu,\rho}^* u$ to the space $H^s\left([0,1]; L^2\right)$. This is generally not the case because $\mathrm{D}_{\nu,\rho+\rho'} \neq \mathrm{D}_{\nu,\rho} \circ \mathrm{D}_{\nu,\rho'}$ as it would be in the Euclidean case.

4.3. Microlocal decomposition

For each value of ρ, one decomposes $u = \underline{u}_\rho + \bar{u}_\rho$ in order to separate low and high frequencies of u. Using the invariance of the L^2-norm with respect to translations, one gets:

$$(4.8)$$
$$C^{-1}\Im_s u \leqslant \sup_{\nu \in \mathbb{S}^{q-1}} \int_0^R \left\| (\mathrm{D}_{\nu,\rho}^* - \mathrm{Id})\underline{u}_\rho \right\|_{L^2}^2 \rho^{-1-2s} d\rho + 2 \int_0^R \|\bar{u}_\rho\|_{L^2}^2 \rho^{-1-2s} d\rho$$

The appropriate notion of frequency is the microlocal weight $\mathfrak{m}(X)$ defined by (3.2).

Precisely, with the notation of § 3.1, let $(\vartheta_Y)_{Y \in \mathrm{T}^*\mathbb{R}^q}$ be a partition of unity on the phase space. The real symbols ϑ_Y are supposed to be uniformly confined in g_Y-balls $\mathcal{U}_Y^\delta$ of small radius $\delta \in]0, \delta_0[$, and split $\vartheta_Y = \psi_Y \# \varphi_Y$.

For $\rho > 0$, one defines the low frequency part of u as

$$(4.9) \qquad \underline{u}_\rho = \int_{\mathfrak{m}(Y) \leqslant \frac{c_0}{\rho}} \vartheta_Y^{\mathrm{w}} u(x)\ d_g Y.$$

The length ρ may be thought as the inverse of the cut-off frequency. The parameter c_0 is arbitrary but it should be chosen in $]0, 1[$.

4.4. Estimate of the high frequencies

High-frequencies are dealt with by a standard computation, using the splitting of ϑ_Y.

Precisely, the L^2-norm being equivalent to that of $H(1)$, one has:

$$\|\bar{u}_\rho\|_{L^2}^2 = \iint_{\substack{\rho\, \mathfrak{m}(Y) > c_0 \\ \rho\, \mathfrak{m}(Z) > c_0}} \left((\psi_Z \# \psi_Y)^{\mathrm{w}} \circ \varphi_Y^{\mathrm{w}} u \,\big|\, \varphi_Z^{\mathrm{w}} u \right)_{L^2} d_g Y\, d_g Z.$$

The estimates (3.4), (3.5) and Cauchy-Schwarz inequality imply:

$$\|\bar{u}_\rho\|_{L^2}^2 \leqslant C_N \iint_{\substack{\rho\, \mathfrak{m}(Y) > c_0 \\ \rho\, \mathfrak{m}(Z) > c_0}} \|\varphi_Y^{\mathrm{w}} u\|_{L^2} \|\varphi_Z^{\mathrm{w}} u\|_{L^2} \frac{d_g Y\, d_g Z}{\Delta(Y, Z)^N}$$

$$\leqslant C' \int_{\mathfrak{m}(Y) > \frac{c_0}{\rho}} \|\varphi_Y^{\mathrm{w}} u\|_{L^2}^2\ d_g Y.$$

Computing first the integral in ρ, the formula $\int_A^R \rho^{-1-2s}\, d\rho \leqslant \frac{1}{2s} A^{-2s}$ for $s > 0$ and $A \leqslant R$ gives

$$(4.10) \qquad \int_0^R \|\bar{u}_\rho\|_{L^2}^2\, \rho^{-1-2s} d\rho \leqslant C_s \, \|u\|_{H(\mathfrak{m}^s)}^2 \,.$$

4.5. Estimate of the low frequencies

It remains now to bound the integral involving $\underline{u}_\rho$ in (4.8).

Taking advantage of the fact that (ϑ_X) is a partition of unity on $\mathrm{T}^*\mathbb{R}^q$, one has:

$$(\mathrm{D}_{\nu,\rho}^* - \mathrm{Id})\, \underline{u}_\rho = \iint\limits_{\mathfrak{m}(Y) \leqslant \frac{c_0}{\rho}} \vartheta_Z^{\mathrm{w}} \circ (\mathrm{D}_{\nu,\rho}^* - \mathrm{Id}) \circ \vartheta_Y^{\mathrm{w}} u \; d_g Y \, d_g Z.$$

As $\vartheta_X = \psi_X \# \varphi_X$, the microlocal version of Cotlar's lemma recalled above implies :

$$\left\|(\mathrm{D}_{\nu,\rho}^* - \mathrm{Id})\, \underline{u}_\rho\right\|_{L^2}^2 \leqslant \iint\limits_{\mathfrak{m}(Y) \leqslant \frac{c_0}{\rho}} \left\|\varphi_Z^{\mathrm{w}} \circ (\mathrm{D}_{\nu,\rho}^* - \mathrm{Id}) \circ \psi_Y^{\mathrm{w}} \circ \varphi_Y^{\mathrm{w}} u\right\|_{L^2}^2 \; d_g Y d_g Z.$$

The main result is contained in the following lemma.

LEMMA 4.5. — *For all integer N, there is a $C_N' > 0$ such that $\mathfrak{m}(Y) \leqslant \frac{c_0}{\rho}$ imply:*

$$(4.11) \qquad \|\Theta_{\nu,\rho}(Y, Z)\|_{\mathscr{L}(L^2)} \leqslant C_N' \, \frac{\rho\,\mathfrak{m}(Y)}{\Delta(Y, Z)^N}$$

where $\Theta_{\nu,\rho}(Y, Z) = \varphi_Z^{\mathrm{w}} \circ (\mathrm{D}_{\nu,\rho}^* - \mathrm{Id}) \circ \psi_Y^{\mathrm{w}}$ and Δ is the symmetric function of (3.4).

Remark. —   This lemma is actually the microlocal version of the inequality

$$\|(T_h^* - \mathrm{Id})\Delta_q u\|_{L^2} \leqslant C\, 2^q |h| \, \|\Delta_q u\|_{L^2}$$

of the Littlewood-Paley theory with $\Delta_q u = \mathscr{F}^{-1}[\varphi(2^{-q}\xi)\widehat{u}(\xi)]$ and φ a smooth function supported in a ring. The frequency is replaced by the weight $\mathfrak{m}(Y)$. The restrictions on the amplitude of the translation and the technical "sandwich" of ψ_Y^{w} and φ_Z^{w} is due to the lack of commutativity between translations and confinement operators of Weyl's calculus.

Once this lemma is obtained, the conclusion is straightforward. Choosing $2N \geqslant N_0$ in (3.4):

$$
\left\| (\mathrm{D}_{\nu,\rho}^* - \mathrm{Id}) \underline{u}_\rho \right\|_{L^2}^2 \leqslant C_N'^2 \iint\limits_{\mathfrak{m}(Y) \leqslant \frac{c_0}{\rho}} \rho^2 m(Y)^2 \left\| \varphi_Y^{\mathrm{w}} u \right\|_{L^2}^2 \frac{d_g Y \, d_g Z}{\Delta(Y,Z)^{2N}}
$$

$$
\leqslant C' \int\limits_{\mathfrak{m}(Y) \leqslant \frac{c_0}{\rho}} \rho^2 m(Y)^2 \left\| \varphi_Y^{\mathrm{w}} u \right\|_{L^2}^2 \, d_g Y.
$$

As $s < 1$, one has $\int_0^B \rho^{1-2s} d\rho = \dfrac{1}{2(1-s)} B^{2(1-s)}$. Therefore, for all $\nu \in \mathbb{S}^{q-1}$:

$$
(4.12) \quad
\begin{aligned}
& \int_0^R \left\| (\mathrm{D}_{\nu,\rho}^* - \mathrm{Id}) \underline{u}_\rho \right\|_{L^2}^2 \rho^{-1-2s} d\rho \\
& \leqslant C' \iint\limits_{\rho \leqslant c_0 \mathfrak{m}(Y)^{-1}} m(Y)^2 \left\| \varphi_Y^{\mathrm{w}} u \right\|_{L^2}^2 \rho^{1-2s} \, d\rho \, d_g Y \leqslant C_s \left\| u \right\|_{H(\mathfrak{m}^s)}^2.
\end{aligned}
$$

Finally, it follows from (4.8), (4.10) and (4.12) that

$$
(4.13) \quad \iint\limits_{d(x,y) < 1} |u(x) - u(y)|^2 \, d(x,y)^{-Q-2s} \, dx \, dy \leqslant C_s \left\| u \right\|_{H(\mathfrak{m}^s)}^2.
$$

This concludes the proof of the right-hand side of (1.6) up to that of lemma 4.5.

4.6. Proof of lemma 4.5

The following discussion, joined to the characterisation of symbol classes given by [4, thm. 5.5] implies that the symbol of the operator $(\mathrm{D}_{\nu,\rho}^* - \mathrm{Id}) \circ \psi_Y^{\mathrm{w}}$ belongs to $S(\rho \, \mathfrak{m}(Y))$, uniformly when the parameters ρ and Y vary in the domain $\rho \leqslant c_0 \, \mathfrak{m}(Y)^{-1}$. Roughly speaking, this domain corresponds to translations $\mathrm{D}_{\nu,\rho}^*$ in the physical space, which do not destroy the microlocalisation of $\psi_Y^{\mathrm{w}} u$.

The translation $\mathrm{D}_{\nu,\rho}^*$ may be written as a sum of small translations, using Taylor's formula:

$$
\Theta_{\nu,\rho}(Y,Z) = \varphi_Z^{\mathrm{w}} \circ \left(\int_0^\rho \frac{\partial}{\partial t} \mathrm{D}_{\nu,t}^* \, dt \right) \circ \psi_Y^{\mathrm{w}}.
$$

Coming back to the definition (4.4), the derivative of $\mathrm{D}_{\nu,\rho}^*$ is given by:

$$
\frac{\partial}{\partial \rho} \mathrm{D}_{\nu,\rho}^* = \mathrm{D}_{\nu,\rho}^* \circ \left(\partial_{\pi_1^x \nu} + 2\rho \, \partial_{\pi_2^x \nu} \right).
$$

Derivatives and translations of Weyl's quantization can be computed easily:

$$\partial_{x_k} \circ \phi^{\mathrm{w}} = \left[\left(i\xi_k + \frac{1}{2}\partial_{x_k} \right)\phi \right]^{\mathrm{w}} \qquad \text{and} \qquad T_h^* \circ \phi^{\mathrm{w}} = \phi(x+h,\xi)^{\mathrm{w}} \circ T_h^*.$$

Here, one denotes $T_h^* v(x) = v(x+h)$. Putting those formulas together, one gets:

$$(4.14) \qquad \Theta_{\nu,\rho}(Y,Z) = \int_0^\rho \left(\varphi_Z \,\#\, \phi_Y^{\nu,t}(\mathrm{D}_{\nu,t}(x),\xi) \right)^{\mathrm{w}} \circ \mathrm{D}_{\nu,t}^* \, dt$$

with

$$\phi_Y^{\nu,\rho}(x,\xi) = \left(i\,\langle \pi_1^x \nu | \xi \rangle + 2i\rho\,\langle \pi_2^x \nu | \xi \rangle + \frac{1}{2}(\pi_1^x \nu + 2\rho\,\pi_2^x \nu)\cdot \nabla_x \right) \psi_Y(x,\xi).$$

The first trouble is that $\phi_Y^{\nu,\rho}$ is not computed at $X = (x,\xi)$, but at the point $(\mathrm{D}_{\nu,\rho}(x),\xi)$. The response is given by the following lemma.

LEMMA 4.6. — *For all $T \in T^*\mathbb{R}^q$ such that $g_Y^\sigma(T) \leqslant c^2$ and any symbol ϕ, one has:*

$$(4.15) \qquad \|\phi(X+T)\|_{N;\mathrm{Conf}_Y} \leqslant C^N \, \|\phi(X)\|_{N;\mathrm{Conf}_Y}$$

with $C = \max(2; 1+2c^2)$ and the same confinement radius δ.

Proof. — The proof is simpler with the semi-norms $\|\cdot\|_{\mathrm{Conf}_Y}$ because one may change X into $X - T$:

$$\|\phi(X+T)\|_{N;\mathrm{Conf}_Y} = \sup_{\substack{X \in \mathrm{T}^*\mathbb{R}^q \\ k \leqslant N\,;\, T_i \neq 0}} \left(1 + g_Y^\sigma((X-T) - \mathcal{U}_Y^\delta) \right)^N$$

$$\frac{|\partial_{T_1} \ldots \partial_{T_k} \phi(X)|}{g_Y(T_1)^{1/2} \ldots g_Y(T_k)^{1/2}}.$$

As $g_Y^\sigma(T) \leqslant c^2$, one has:

$$g_Y^\sigma((X-T) - \mathcal{U}_Y^\delta) \leqslant 2g_Y^\sigma(X - \mathcal{U}_Y^\delta) + 2g_Y^\sigma(T) \leqslant 2c^2 + 2g_Y^\sigma(X - \mathcal{U}_Y^\delta),$$

whence (4.15). $\qquad\qquad\square$

The rest of the proof will use many times a simple estimate of symbolic calculus that we recall first for the reader's convenience.

LEMMA 4.7. — *Given smooth symbols A and B, a weight function $\mathfrak{M}$, and an integer N, there are a constant C and integers K, M such that:*

$$(4.16) \qquad \|AB\|_{N;\mathrm{Conf}_Y} \leqslant C\,\mathfrak{M}(Y)\,\|A\|_{K;S(\mathfrak{M})}\,\|B\|_{M;\mathrm{Conf}_Y}$$

uniformly for $Y \in T^*\mathbb{R}^q$, and with the same confinement radius δ.

Proof. — One uses the equivalent semi-norms (3.9). Leibniz' formula implies

$$\frac{\partial_{T_1}\ldots\partial_{T_k}\big(A(X)B(X)\big)}{g_X(T_1)^{1/2}\ldots g_X(T_k)^{1/2}} = \sum \frac{\partial_{T_{\sigma(1)}}\ldots\partial_{T_{\sigma(\ell)}}A(X)}{g_X(T_{\sigma(1)})^{1/2}\ldots g_X(T_{\sigma(\ell)})^{1/2}}$$
$$\cdot \frac{\partial_{T_{\sigma(\ell+1)}}\ldots\partial_{T_{\sigma(k)}}B(X)}{g_X(T_{\sigma(\ell+1)})^{1/2}\ldots g_X(T_{\sigma(k)})^{1/2}}$$

where the sum is over the permutations $\sigma \in \mathfrak{S}_k$ such that $\sigma(1) < \ldots < \sigma(\ell)$ and $\sigma(\ell+1) < \ldots < \sigma(k)$. It follows:

$$\frac{\partial_{T_1}\ldots\partial_{T_k}\big(A(X)B(X)\big)}{g_X(T_1)^{1/2}\ldots g_X(T_k)^{1/2}} \leqslant C\mathfrak{M}(X)\frac{\|A\|_{k;S(\mathfrak{M})}\,\|B\|_{N+\bar{N};\mathrm{Conf}'_Y}}{\big(1+g_Y^\sigma(X-\mathcal{U}_Y^\delta)\big)^{N+\bar{N}}}$$
$$\leqslant \frac{C\bar{C}\mathfrak{M}(Y)}{\big(1+g_Y^\sigma(X-\mathcal{U}_Y^\delta)\big)^N}\,\|A\|_{k;S(\mathfrak{M})}\|B\|_{N+\bar{N};\mathrm{Conf}'_Y}$$

whence the lemma.   □

The last step is the study of the confinement of $\phi_Y^{\nu;\rho}$.

LEMMA 4.8. — *For all integer N, there are a constant C_N and an integer M such that:*

$$(4.17) \qquad \sup_{\rho\,\mathfrak{m}(Y)\leqslant c_0} \mathfrak{m}(Y)^{-1}\,\|\phi_Y^{\nu;\rho}\|_{N;\mathrm{Conf}_Y} \leqslant C_N\,\|\psi_Y\|_{M;\mathrm{Conf}_Y}\,.$$

Proof. — Let us study each term appearing in $\phi_Y^{\nu;\rho}$ separately.

The "elliptic" term is $\langle\pi_1^x\nu|\xi\rangle\,\psi_Y(X)$. Proposition 4.1 asserts that $\langle\pi_1^x\nu|\xi\rangle \in S(\mathfrak{m})$ so
(4.18a)
$$\|\langle\pi_1^x\nu|\xi\rangle\,\psi_Y(X)\|_{N;\mathrm{Conf}_Y} \leqslant C\,\mathfrak{m}(Y)\,\|\langle\pi_1^x\nu|\xi\rangle\|_{K;S(\mathfrak{m})}\,\|\psi_Y\|_{M;\mathrm{Conf}_Y}$$
by Lemma 4.7.

Focus now on the "sub-elliptic" term: $\rho\,\langle\pi_2^x\nu|\xi\rangle\,\psi_Y(X)$. As $\langle\pi_2^x\nu|\xi\rangle \in S(\mathfrak{m}^2)$, one has:

$$\|\rho\,\langle\pi_2^x\nu|\xi\rangle\,\psi_Y(x,\xi)\|_{N;\mathrm{Conf}_Y} \leqslant C\,\rho\,\mathfrak{m}(Y)^2\,\|\langle\pi_2^x\nu|\xi\rangle\|_{K;S(\mathfrak{m}^2)}\,\|\psi_Y\|_{M;\mathrm{Conf}_Y}\,,$$

again by Lemma 4.7. The assumption $\rho\,\mathfrak{m}(Y) \leqslant c_0$ implies
(4.18b)
$$\|\rho\,\langle\pi_2^x\nu|\xi\rangle\,\psi_Y(x,\xi)\|_{N;\mathrm{Conf}_Y} \leqslant C\,c_0\,\mathfrak{m}(Y)\,\|\langle\pi_2^x\nu|\xi\rangle\|_{K;S(\mathfrak{m}^2)}\,\|\psi_Y\|_{M;\mathrm{Conf}_Y}\,.$$

For the last term, an immediate application of the definition (3.1) gives

$$\|(\pi_1^x\nu + 2\rho\,\pi_2^x\nu)\cdot\nabla_x\,\psi_Y\|_{N;\mathrm{Conf}_Y} \leqslant g_Y(T)^{1/2}\,\|\psi_Y\|_{N+1;\mathrm{Conf}_Y}$$
$$(4.18c) \qquad\qquad\qquad\qquad\qquad \leqslant (1+2\rho)\,\mathfrak{m}(Y)\,\|\psi_Y\|_{N+1;\mathrm{Conf}_Y}$$

with $T = (\pi_1^x \nu + 2\rho \, \pi_2^x \nu, 0)$ and $g_Y(\partial_x)^{1/2} = \frac{\langle \eta \rangle}{\mathfrak{m}(Y)} \leqslant \mathfrak{m}(Y)$. $\qquad\square$

Let us now conclude the proof of Lemma 4.5.

Lemma 4.6 can actually be applied, because $\rho \leqslant \frac{c_0}{\mathfrak{m}(Y)} \leqslant c_0 < 1$; therefore $\rho^2 \leqslant \rho$ and the Euclidean length of the translation is

$$|x - \mathrm{D}_{\nu,\rho}(x)| = |\rho \, \pi_1^x \nu + \rho^2 \pi_2^x \nu| \leqslant 2\rho \leqslant \frac{2c_0}{\mathfrak{m}(Y)}.$$

so $g_Y^\sigma(\rho \, \pi_1^x \nu + \rho^2 \pi_2^x \nu, 0) \leqslant 4c_0^2$.

Applying now Lemma 4.8, one obtains that the family $\left(\mathfrak{m}(Y)^{-1} \, \phi_Y^{\nu,\rho}(\mathrm{D}_{\nu,t}(x), \xi) \right)_{Y \in \mathrm{T}^* \mathbb{R}^q}$ is uniformly confined in the domain $\rho \, \mathfrak{m}(Y) \leqslant c_0$.

Finally, one may apply (3.4) and (3.5) to the formula (4.14):

$$\|\Theta_{\nu,\rho}(Y, Z)\|_{\mathscr{L}(L^2)} \leqslant C_1 \int_0^\rho \left\| \varphi_Z \# \phi_Y^{\nu,t}(\mathrm{D}_{\nu,t}(x), \xi) \right\|_{N_1; \mathrm{Conf}_Y} dt$$
$$\leqslant C_N' \frac{\rho \, \mathfrak{m}(Y)}{\Delta(Y, Z)^N},$$

whence (4.11).

5. Final remarks

This section contains some additional remarks about the necessity of assumption (1.5). A weaker one, known as the *finite rank bracket generating* condition, requires the existence of an integer $n_0 \geqslant 2$ such that any smooth vector field $\mathscr{X}$ may be decomposed (not necessarily in an unique way) as

$$(5.1) \qquad \begin{aligned} \mathscr{X} &= \sum \alpha_j^1 \mathscr{L}_j + \sum \alpha_{j_1,j_2}^2 [\mathscr{L}_{j_1}, \mathscr{L}_{j_2}] \\ &+ \ldots + \sum \alpha_{j_1,\ldots,j_{n_0}}^{n_0} [\mathscr{L}_{j_1}, [\mathscr{L}_{j_2}, [\ldots [\mathscr{L}_{j_{n_0-1}}, \mathscr{L}_{j_{n_0}}]]]] \end{aligned}$$

with locally bounded functions $\alpha_{j_1,\ldots,j_k}^k$ on Ω. On says that n_0 is the degree of nonholonomy of the family $\mathscr{L}$. The Carnot-Carathéodory distance (1.3) may still be defined. In this case, the constant rank hypothesis should be replaced by a *regularity* assumption: the dimension of the subspace

$$(5.2) \qquad \mathscr{V}_k = \mathrm{Span} \left\{ \mathscr{L}_j(x), \ldots, [\mathscr{L}_{j_1}, [\ldots [\mathscr{L}_{j_{k-1}}, \mathscr{L}_{j_k}]]] \right\}$$

of $\mathrm{T}_x \Omega$ will be assumed to be constant for all $k \leqslant n_0$.

The proof of the left-hand side of (1.6) does not require assumption (1.5) but only (5.1) because the main estimate (2.5) of the heat-kernel remains true in this case. Therefore, the proof given above holds without changes.

Our proof of the right-hand side of (1.6) uses Weyl-Hörmander calculus. This technique was chosen to overcome the following difficulty: the geometry is not "flat" in the sense that $\mathscr{L}(x)$ cannot be realized as the tangent space of any r-dimensional submanifold.

However, Weyl's calculus behaves badly when the microlocal regularity jumps from $H_{x,\xi}^{s_1}$ to $H_{x,\xi}^{s_2}$ when $s_2 > 2s_1$ or $s_2 < s_1/2$. Recall that a function u is said to be in H_{x_0,ξ_0}^{s} if

$$\langle \xi \rangle^s |\widehat{\phi u}(\xi)| \in L^2 \left(\left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \varepsilon \right)$$

for some (small) $\varepsilon > 0$ and all cut-off functions $\phi \in \mathcal{D}(\Omega)$ such that $\phi(x_0) = 1$.

To illustrate this fact, one may consider the following example. In $\mathbb{R}^4$, the Goursat family is

$$U = \partial_{x_1} + x_3 \partial_{x_2} + x_4 \partial_{x_3} \qquad \text{and} \qquad V = \partial_{x_4}.$$

The degree of noholonomy of this system is 3 because:

$$[V, U] = \partial_{x_3} \qquad \text{and} \qquad [[V, U], U] = \partial_{x_2}.$$

The natural microlocal weight associated with it is:

$$m(X) = \left((\xi_1 + x_3 \xi_2 + x_4 \xi_3)^{12} + \xi_4^{12} + \xi_3^6 + \langle \xi \rangle^4 \right)^{1/12}.$$

The main problem is the following negative result.

PROPOSITION 5.1. — *There is no metric g on the phase-space $T^*\mathbb{R}^4$ having the form*

(5.3)
$$g_X = \sum_j \frac{dx_j^2}{a_j(X)^2} + \frac{d\xi_j^2}{b_j(X)^2},$$

satisfying the uncertainty principle $g_X \leqslant g_X^\sigma$ and such that $m \in S(m, g)$.

Proof. — The uncertainty principle has a simple expression:

$$a_j(X) b_j(X) \geqslant 1 \qquad (j = 1, \ldots, 4).$$

If the weight m belongs to the symbol class $S(m, g)$, the following estimates will hold:

$$\begin{cases} a_j(X) |\partial_{x_j} m(X)| \leqslant C\, m(X), \\ b_j(X) |\partial_{\xi_j} m(X)| \leqslant C\, m(X). \end{cases}$$

Therefore, one has for all $j \in \{1, \ldots, 4\}$:

(5.4)
$$|\partial_{x_j} m(X)|\, |\partial_{\xi_j} m(X)| \leqslant C^2\, m(X)^2.$$

An easy computation gives

$$\partial_{x_3} m(X) = \left(\frac{\xi_1 + x_3 \xi_2 + x_4 \xi_3}{m(X)} \right)^{11} \xi_2$$

and

$$\partial_{\xi_3} m(X) = \left(\frac{\xi_1 + x_3 \xi_2 + x_4 \xi_3}{m(X)} \right)^{11} x_4 + \frac{2 \langle \xi \rangle^2 \xi_3 + 3 \xi_3^5}{6 \, m(X)^{11}}.$$

On the domain defined by

$$x_3 = 0, \quad \xi_3 = 0, \quad \xi_2 = \xi_1^3 \quad \text{and} \quad |\xi_4| \leqslant |\xi_1|$$

one has $\left(\xi_1^{12} + \xi_1^4 + 1 \right)^{1/12} \leqslant m(X) \leqslant C \left(\xi_1^{12} + \xi_1^4 + 1 \right)^{1/12}$ for $C > 1$ sufficiently large. For $|\xi_1| \geqslant 1$, it implies:

$$|x_4| \, |\xi_1|^3 \leqslant C' \, |\partial_{x_j} m(X)| \, |\partial_{\xi_j} m(X)| \qquad \text{and} \qquad m(X)^2 \leqslant C'' \, |\xi_1|^2,$$

which contradicts (5.4) when $|x_4| \, |\xi_1| \to \infty$. $\qquad\qquad\qquad\square$

Nonetheless, if one "flattens" artificially the geometry, the microlocal regularity may jump without restrictions. Actually, it is just a simple computation with the Fourier transform.

PROPOSITION 5.2. — *Let* $1 \leqslant \omega_1 \leqslant \omega_2 \leqslant \ldots \leqslant \omega_q$ *be real numbers and*

(5.5a)
$$\delta_\omega(x, y) = \sum_{j=1}^{q} |x_j - y_j|^{1/\omega_j}$$

the corresponding anisotropic distance on $\mathbb{R}^q$. *Denote by* $Q = \sum \omega_j$ *the homogeneous dimension. Then, for all function* u *and* $s \in]0, 1[$, *one has*

(5.5b)
$$\iint_{\mathbb{R}^{2q}} \frac{|u(x) - u(y)|^2}{\delta_\omega(x,y)^{Q+2s}} \, dx dy = \int_{\mathbb{R}^q} m_\omega(\xi)^{2s} \, |\hat{u}(\xi)|^2 \, d\xi$$

with $C^{-1} \sum |\xi_j|^{1/\omega_j} \leqslant m_\omega(\xi) \leqslant C \sum |\xi_j|^{1/\omega_j}$.

Proof. — Let $T_h^* v(x) = v(x+h)$. The change of variable $x = y+h$ gives:

$$\iint_{\mathbb{R}^{2q}} \frac{|u(x) - u(y)|^2}{\delta_\omega(x,y)^{Q+2s}} \, dx dy = \int_{\mathbb{R}^q} \frac{\|(T_h^* - \text{Id})u\|_{L^2}^2}{\left(\sum |h_j|^{1/\omega_j} \right)^{Q+2s}} \, dh.$$

Then Parseval's identity and Fubini's theorem imply:

$$\int_{\mathbb{R}^q} \frac{\|(T_h^* - \text{Id})u\|_{L^2}^2}{\left(\sum |h_j|^{1/\omega_j} \right)^{Q+2s}} \, dh = \int_{\mathbb{R}^q} \mu(\xi) \, |\hat{u}(\xi)|^2 \, d\xi$$

with $\mu(\xi) = \int |e^{ih\cdot\xi} - 1|^2 \left(\sum |h_j|^{1/\omega_j}\right)^{-Q-2s} dh$. As $s < 1 \leqslant \omega_j$, one has

$$0 < \mu(\xi) \leqslant 4\omega_1 \cdots \omega_q \left(|\xi|^2 \sum \int_0^1 |\vartheta_j|^{2\omega_j} \frac{d\vartheta}{|\vartheta|^{q+2s}} + \int_1^\infty \frac{d\vartheta}{|\vartheta|^{q+2s}}\right) \leqslant C \langle\xi\rangle^2.$$

The homogeneity of the distance δ_ω imply that for all $\lambda > 0$

$$\mu(\xi) = \lambda^{2s} \mu\left(\frac{\xi_1}{\lambda^{\omega_1}}, \ldots, \frac{\xi_q}{\lambda^{\omega_q}}\right).$$

Applying this identity with $\lambda = \sum |\xi_j|^{1/\omega_j}$ provides the result because the function μ is regular on the compact sphere

$$\sum |\xi_j|^{1/\omega_j} = 1$$

and therefore bounded from above and from below. $\qquad\qquad\square$

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