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Gueorgui Tomov TODOROV

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PLURICANONICAL MAPS FOR THREEFOLDS OF GENERAL TYPE

by Gueorgui Tomov TODOROV (*)

ABSTRACT. — In this paper we will prove that for a threefold of general type and large volume the second plurigenus is positive and the fifth canonical map is birational.

RÉSUMÉ. — Nous prouvons que pour une variété de dimension 3 de type général et de grand volume le second plurigenre est positif et la cinquième application canonique est birationnelle

1. Introduction

One of the main problems in algebraic geometry is to understand the structure of pluricanonical maps. If X is a smooth complex projective variety of general type and dimension n , then by definition the plurigenus $P_r = h^0(X, \mathcal{O}_X(rK_X))$ grow like r^n and the pluricanonical maps $\phi_{rK_X} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(rK_X)))$ are birational for sufficiently large r . It is natural to look for minimal values r_n and r'_n such that $P_r \neq 0$ for $r \geq r_n$ and ϕ_{rK_X} is birational for $r \geq r'_n$. On a curve of general type the picture is clear, $P_r \neq 0$ for $r \geq 1$ and ϕ_{rK_X} is birational for $r \geq 3$. For surfaces we have that $P_r \neq 0$ for $r \geq 2$ and ϕ_{rK_X} is birational for $r \geq 5$, by a result of Bombieri (cf. [4]).

On threefolds the problem is much harder. The question has been extensively studied, for example by Kollár [11] and Benviste [3]. Chen and Hacon in [7] have a proof that if the irregularity is positive then ϕ_{7K_X}

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is birational. M. Chen [6] and S. Lee [15] prove that for X minimal and smooth ϕ_{6K_X} is birational. By a recent result of J. Chen, M. Chen and D. Zhang in [1] if X is a minimal Gorenstein threefold ϕ_{rK_X} is birational for $r \geq 5$. However Tsuji [17], and later Hacon-McKernan [5], Takayama [16] following ideas of Tsuji, have shown that for a smooth projective variety of general type and dimension n there exists an integer r_n , which depends only on n , such that ϕ_{rK_X} is birational for $r \geq r_n$. This theorem has also a nice consequence, namely if we fix a positive integer M the family of smooth projective varieties of general type, dimension n and volume smaller than M is birationally bounded.

Keeping this in mind it is natural to ask if we can find explicit values r_n and r'_n such that $P_r \neq 0$ for $r \geq r_n$ and ϕ_{rK_X} is birational for $r \geq r'_n$ for a class of varieties of dimension n and volume larger than a fixed positive number M . We will prove the following theorems:

THEOREM 1.1. — *Let X be a projective threefold of general type and assume that*

$$\text{vol}(X) > 2304^3.$$

Then $P_2 = h^0(X, \mathcal{O}_X(2K_X)) > 0$.

THEOREM 1.2. — *Let X be a projective threefold of general type and assume that*

$$\text{vol}(X) > 4355^3.$$

Then ϕ_{5K_X} is birational.

Here the volume of X is defined as

$$\text{vol}(X) = \limsup_{m \rightarrow \infty} \frac{3!h^0(X, \mathcal{O}_X(mK_X))}{m^3}.$$

Note that the above result is optimal since we can construct infinitely many families of threefolds of general type with increasing volume for which the ϕ_{rK_X} is not birational for $r \leq 4$. This can be done just by considering products of curves and surfaces of general type.

We will sketch the strategy of the proof for the non-vanishing of the second plurigenera. The proof of the second statement is similar. We would like to produce a divisor $D \sim \lambda K_X$ such that D has an isolated log canonical centre at a general point $x \in X$. If $\lambda < 1$, then we get a surjective homomorphism $H^0(X, \mathcal{O}_X(2K_X)) \rightarrow H^0(X, \mathbb{C}_x) \cong \mathbb{C}$ and so $P_2 \geq 1$. Since the volume of K_X is large, by an easy dimension count, one finds a small rational number $\lambda > 0$ and a divisor $D \sim \lambda K_X$ which has a non-trivial log canonical centre at a general point $x \in X$. However it could happen that the dimension of the log canonical centre is positive. This is not a

problem if the dimension of the log canonical centre at x is one since then we can apply a standard technique to cut down the dimension of the log canonical centre and have an isolated log canonical centre for $D' \sim \lambda' K_X$ for some small positive rational number λ' . If the dimension at x of the log canonical centre is two we could still try to cut down the dimension and indeed at the end we would produce a divisor $D \sim \lambda K_X$ with an isolated log canonical centre but λ will not be small enough to give us a bicanonical section. So in this case a different approach is needed. In fact using ideas of McKernan[10] and Tsuji we will produce a morphism to a curve and we will use this morphism to produce a section.

2. Preliminaries

2.1. Notation

We will work over the field of complex numbers \mathbb{C} . A \mathbb{Q} -Cartier divisor D is nef if $D \cdot C \geq 0$ for any curve C on X . We call two \mathbb{Q} -divisors D_1, D_2 \mathbb{Q} -linearly equivalent $D_1 \sim D_2$ if there exists an integer $m > 0$ such that mD_i are Cartier and linearly equivalent. We call two \mathbb{Q} -Cartier divisors D_1, D_2 numerically equivalent $D_1 \equiv D_2$ if $(D_1 - D_2) \cdot C = 0$ for any curve C on X . A log pair (X, Δ) is a normal variety X and an effective \mathbb{Q} -Weil divisor Δ such that $K_X + \Delta$ is \mathbb{Q} -Cartier. A projective morphism $\mu : Y \rightarrow X$ is a log resolution of the pair (X, Δ) if Y is smooth and $\mu^{-1}(\Delta) \cup \{\text{exceptional set of } \mu\}$ is a divisor with simple normal crossing support. For such μ we write

$$\mu^*(K_X + \Delta) = K_Y + \Gamma,$$

and $\Gamma = \sum a_i \Gamma_i$ where Γ_i are distinct integral divisors. A pair is called klt (resp. lc) if there is a log resolution $\mu : Y \rightarrow X$ such that in the above notation we have $a_i < 1$ (resp. $a_i \leq 1$). The number $1 - a_i$ is called log discrepancy of Γ_i with respect to the pair (X, Δ) . We say that a subvariety $V \subset X$ is a log canonical centre if it is the image of a divisor of log discrepancy at most zero. A log canonical place is a valuation corresponding to a divisor of log discrepancy at most zero. A log canonical centre is pure if $K_X + \Delta$ is log canonical at the generic point of V . If moreover there is a unique log canonical place lying over the generic point of V , then we say that V is exceptional. $\text{LCS}(X, \Delta, x)$ is the union of all log canonical centres of (X, Δ) through the point x . We will denote by $\text{LLC}(X, \Delta, x)$ the set of all log canonical centres containing a point $x \in X$.

2.2. Volumes

DEFINITION 2.1. — *Let X be irreducible projective variety of dimension n and let D be a \mathbb{Q} -Cartier divisor. The volume of D is*

$$\text{vol}(D) = \limsup_{m \rightarrow \infty} \frac{n! h^0(X, \mathcal{O}_X(mD))}{m^n}.$$

When D is nef we have that $\text{vol}(D) = D^n$, so that for a very ample divisor D the volume of D is just the degree of the image of X given by the linear system $|D|$ (cf. [14], Section 2.2C). The volume depends only on the numerical class of D and it can be extended to a continuous function $\text{vol}: N_{\mathbb{R}}^1 \rightarrow \mathbb{R}$, where $N_{\mathbb{R}}^1 = N^1(X) \otimes \mathbb{R}$ and $N^1(X)$ is the Neron-Severi group of X (cf. [14], Theorem 2.2.43). The volume is invariant under birational transformations (cf. [14], Proposition 2.2.42).

The following lemma is a standard tool to produce log canonical centres at a general point of a variety.

LEMMA 2.2. — (cf. [14], Lemma 10.4.11) *Let X be an irreducible projective variety of dimension d , L a \mathbb{Q} -Cartier divisor and $x \in X$ a smooth point. If for some positive rational number α we have that*

$$\text{vol}(L) > \alpha^d,$$

then for any sufficiently divisible integer $k \gg 0$ there exists a divisor

$$A = A_x \in |kL| \quad \text{with} \quad \text{mult}_x(A) > k\alpha.$$

Moreover we can take k to be independent of the smooth point.

We will also use the following variant of Lemma 2.2.

LEMMA 2.3. — *Let X be an irreducible projective variety of dimension d , L a \mathbb{Q} -Cartier divisor and $x, y \in X$ smooth points. If for some positive rational number β we have that*

$$\text{vol}(L) > 2\beta^d,$$

then for any sufficiently divisible integer $k \gg 0$, there exists a divisor

$$A = A_{x,y} \in |kL| \quad \text{with} \quad \text{mult}_x(A) > k\beta \text{ and } \text{mult}_y(A) > k\beta.$$

Proof. — The prove of this fact is well known. For a similar argument see for example [14], Proposition 1.1.31. \square

2.3. Multiplier Ideals

DEFINITION 2.4. — Let (X, D) be a log pair with X smooth and let $\mu : Y \rightarrow X$ be a log resolution. Then define the multiplier ideal sheaf of D to be

$$\mathcal{J}(X, D) = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor) \subset \mathcal{O}_X.$$

When no confusion is possible we will write simply $\mathcal{J}(D)$.

We will recall some facts, which will be needed in what follows. We will repeatedly make use of the following generalization of the Kodaira vanishing theorem, which will be referred to as Nadel vanishing. For a proof of this theorem we refer to section 9.4.B of [14].

THEOREM 2.5. — Let X be a smooth projective variety, D a \mathbb{Q} -divisor and L be an integral divisor such that $L - D$ is nef and big. Then

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) = 0 \quad \text{for } i > 0.$$

Define the log canonical threshold of D at x to be

$$c(D, x) = \inf\{c > 0 \mid \mathcal{J}(c \cdot D) \text{ is non-trivial at } x\}.$$

We will also define

$$\text{Nklt}(X, D) = \text{Supp}(\mathcal{O}_X / \mathcal{J}(X, D)) \subset X$$

with the reduced structure and we will call it the *non-klt locus* for (X, D)

Also we will say that (X, D) is *klt* (resp. *lc*) at $x \in X$ if $(U, D|_U)$ is *klt* (resp. *lc*) for some Zariski open neighborhood U of x . It is often useful to assume that $\text{LCS}(X, D, x)$ is irreducible at x . The next lemma asserts that one can achieve this after an arbitrarily small perturbation of D .

LEMMA 2.6. — (cf. [5] Lemma 2.5, [9], [2]) Let X be a smooth projective variety and Δ an effective \mathbb{Q} -divisor and assume that (X, Δ) is *lc* at some point $x \in X$. If $W_1, W_2 \in \text{LLC}(X, \Delta, x)$ and W is an irreducible component of $W_1 \cap W_2$ containing x , then $W \in \text{LLC}(X, \Delta, x)$. Therefore, if (X, Δ) is not *klt* at x , then $\text{LLC}(X, \Delta, x)$ has a unique minimal irreducible element, say V . Moreover, there exists an effective \mathbb{Q} divisor E such that

$$\text{LLC}(X, (1 - \epsilon)\Delta + \epsilon E, x) = \{V\}$$

for all $0 < \epsilon \ll 1$. We may also assume that there is a unique place laying above V and if $x \in X$ is general and L is a big divisor, then one can take $E = aL$ for some positive number a .

The next theorem shows how multiplier ideal sheaves behave with respect to restrictions to smooth divisors.

THEOREM 2.7. — (cf. [14], Theorem 9.5.1) *Let X be a smooth projective variety, let D be an effective \mathbb{Q} -divisor on X and let $H \subset X$ be a smooth irreducible hypersurface which is not contained in the support of D . Then there is an inclusion*

$$\mathcal{J}(H, D_H) \subset \mathcal{J}(X, D)_H.$$

Here D_H is the restriction of D to H , which is by assumption an effective \mathbb{Q} -divisor on H so that the multiplier ideal sheaf in question is defined. $\mathcal{J}(X, D)_H$ indicates the image of $\mathcal{J}(X, D)$ under the natural maps

$$\mathcal{J}(X, D) \hookrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H.$$

Finally we will use the following Lemma from Hacon-McKernan ([5], Lemma 2.6).

LEMMA 2.8. — *Let X be a smooth projective variety and D a big divisor on X . Let x be a general point, λ a rational number with $0 < \lambda < 1$ and $\Delta \sim \lambda D$ a \mathbb{Q} -divisor such that x is a component of $\text{LCS}(X, \Delta, x)$. Then $h^0(X, \mathcal{O}_X(K_X + D)) > 0$.*

3. Proof of the Theorem 1.1

Let us start by replacing X by a desingularization and assume that

$$\text{vol}(X) > \alpha^3.$$

The value of α as in Theorem 1.1 will be determined in the course of the proof. We are going to proceed in the following way. The first step will be to produce a log canonical centre at some point $x \in X$ and then to cut down the dimension of the log canonical centre until it is an isolated point. This can be done as long as the log canonical centre is not a surface of small volume. In this last case we are able to produce a morphism to a curve with special properties which will help us to produce sections of multiples of the canonical bundle.

Our main tool to cut down the dimension of the log canonical centres is the following result of Hacon-McKernan [5]. Fix a log pair (X, Δ) where X is \mathbb{Q} -factorial and Δ is an effective \mathbb{Q} -divisor. Let V be an exceptional log canonical centre of $K_X + \Delta$. Let $f : W \rightarrow V$ be a resolution of V and let Θ be a \mathbb{Q} -divisor on W . Suppose that there are positive rational numbers λ and μ such that $\Delta \sim \lambda K_X$ and $\Theta \sim \mu K_W$. Let $\nu = (\lambda + 1)(\mu + 1) - 1$. We suppose that V is not contained in the augmented base locus of K_X and W is of general type. Here the augmented base locus is defined as follows. For

a divisor L by the stable base locus $\mathbf{B}(L)$ we mean the algebraic set that is $\cap \text{Bs}(|mL|)$, where the intersection is taken over all positive integers m . For a big divisor L the sets $\mathbf{B}(L - \epsilon A)$ are the same for any ample divisor A and any $0 < \epsilon \ll 1$ (cf. [14], Lemma 10.3.1) and this is the augmented base locus of L . In this setting we have the following result.

THEOREM 3.1. — (*Hacon-McKernan*) *There is a very general subset U of V with the following property. Suppose that $W' \subset W$ is a pure log canonical centre of $K_W + \Theta$, whose image $V' \subset V$ intersects U . Then for every positive rational number δ we may find a divisor Δ' on X such that V' is an exceptional log canonical centre of $K_X + \Delta'$ where $\Delta' \sim (\nu + \delta)K_X$.*

Proof of Theorem 1.1. — Denote by X_0 the complement of all subvarieties contained in X that are not of general type (since X is of general type, this is a countable union of closed subsets) and of the augmented base locus of K_X . Choose a general point $x \in X_0$.

Since $\text{vol}(K_X) > \alpha^3$ by Lemma 2.2 for any integer $k \gg 0$ there exists a divisor

$$A = A_x \in |kK_X| \quad \text{with} \quad \text{mult}_x(A) > k\alpha.$$

So, if we consider the divisor

$$\Delta' = \lambda' \frac{A}{k}$$

for $\lambda' < \frac{3}{\alpha}$ but close enough to $\frac{3}{\alpha}$, we have that

$$\text{mult}_x \Delta' > 3.$$

But then (X, Δ') is not klt at x and so by Lemma 2.6, if V is the unique minimal irreducible element of $\text{LCC}(X, \Delta', x)$, we have that

$$\text{LLC}(X, (1 - \epsilon)\Delta' + \epsilon E, x) = \{V\}$$

for all $0 < \epsilon \ll 1$. Here we can take $E \sim aK_X$ for some positive integer a . Now by taking $\epsilon < \frac{3\alpha^{-1} - \lambda'}{a - \lambda'}$ we obtain a divisor D such that $D \sim \lambda K_X$ with $\lambda < \frac{3}{\alpha}$ and

$$\text{LCC}(X, D, x) = \{V\}.$$

We will analyze different cases based on the dimension of this log canonical centre.

Case I - $\dim V = 0$.

To produce a section of $H^0(X, \mathcal{O}_X(2K_X))$ we want to apply Lemma 2.8. So what we need is that $\lambda < 1$. This can be achieved for $\alpha = 3$.

It is now clear that $P_2(X) > 0$.

Case II - $\dim V = 1$.

We want to cut down the dimension of the log canonical centre to reduce to the previous case. We wish to apply Theorem 3.1, so we need to produce a divisor of high multiplicity at a very general smooth point of V . Let $f : W \rightarrow V$ be the normalization of V . By our choice of x we know that W is of general type, so $\text{vol}(K_W) \geq 2 > \frac{15}{8}$.

Lemma 2.2 implies that for a smooth point $w \in W$ there exists a \mathbb{Q} -divisor $\Theta' \sim K_W$ with multiplicity greater than $\frac{15}{8}$ at w . Then $\Theta = \mu\Theta' \sim \mu K_W$ has multiplicity one at w for some $\mu < \frac{8}{15}$. By Theorem 3.1 there is a divisor D' on X such that

$$D' \sim (\lambda + \mu + \lambda\mu + \delta)K_X,$$

where δ is any small positive rational number, and the pair (X, D') has some very general point of V as an exceptional log canonical centre of singularities. Choose $\delta = 10^{-6}$. As before we just have to make sure that $\lambda + \mu + \lambda\mu + \delta < 1$. Now $\lambda + \mu + \lambda\mu + \delta < (\frac{8}{15} + \frac{23}{15}\lambda + \delta)$ so it is enough that $\frac{8}{15} + \frac{23}{15}\lambda + \delta < 1$, and since $\lambda < \frac{3}{\alpha}$ this is possible as long as $\alpha \geq 10$.

Case III - $\dim V = 2$.

Let $f : W \rightarrow V$ be a resolution of singularities of V . As in the previous case W is of general type. We will divide our argument in two cases based on whether the volume of K_W is large or small. In the first case when the volume of K_W is large we may proceed as before, that is produce a log canonical centre of tK_X for some $t < 1$ and of smaller dimension. In the small volume case we will show that there exists a fibration onto a curve and using this fibration we will be able to prove the existence of the required sections.

Suppose now that $\text{vol}(K_W) > 64$. By Theorem 3.1 and using that $\text{vol}(K_W) > 64$ we can guarantee there is a divisor $D' \sim (\frac{1}{4} + \frac{5}{4}\lambda + \delta)K_X$ such that V' is an exceptional log canonical centre of (X, D') of dimension not greater than one. If $\dim V' = 0$ then we are done by the discussion in Case I. If instead $\dim V' = 1$ replacing V by V' and using the discussion in Case II we see that there is a divisor D'' such that

$$D'' \sim \left(\frac{8}{15} + \frac{23}{15} \left(\frac{1}{4} + \frac{4}{5}\lambda + \delta \right) + \delta \right) K_X \sim \left(\frac{55}{60} + \frac{92}{75}\lambda + \frac{38}{15}\delta \right) K_X$$

for which for $\text{LLC}(X, D'', x') = \{x'\}$ for a point $x' \in X$. As we have seen already, once we have an isolated centre we are done as long as $\frac{55}{60} + \frac{92}{75}\lambda + \frac{38}{15}\delta < 1$, and this can be achieved if $\alpha > 45$.

To proceed we need a simplified version of a Lemma 3.2 from McKernan [10]. For the convenience of the reader we include a proof of the result.

Recall that a subset P of X is called countably dense if it is not contained in the union of countably many closed subsets of X .

LEMMA 3.2. — *Let Y be a smooth projective variety and suppose that for every point $y \in P$, where P is a countably dense subset of Y , we may find a pair (Δ_y, W_y) such that W_y is a pure log canonical centre for $K_Y + \Delta_y$ at y and $\Delta_y \sim \Delta/w_y$ for some big Cartier divisor Δ on Y . Then there exists a diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{\pi} & Y \\ \downarrow f & & \\ B & & \end{array}$$

such that f is a dominant morphism of normal projective varieties with connected fibres and for a general fibre Y'_b of f there exists $y \in f(Y'_b)$ so that $f(Y'_b)$ is a pure log canonical centre for $K_Y + \Delta_y$ with $\Delta_y \sim \Delta/w$ at y , for some w . Also π is a generically finite and dominant morphism of normal varieties.

Proof. — As Δ is big by Lemma 2.6, possibly passing to a subset of P , we may assume that W_y is an exceptional log canonical centre. If we decompose a countably dense set as a countable union at least one of the sets in the union is countably dense. The field of rational numbers is countable so on a countably dense subset w_y does not depend on y . We obtain a countably dense subset Q of P such that $w_y = w$, for some fixed rational number w , for every $y \in Q$. Then there is an integer m' such that $m'\Delta_y$ and $m'\Delta/w$ are integral and linearly equivalent, so $m'\Delta_y \in |m'\Delta/w|$. In the linear system $|m'\Delta/w|$ take the closure B of the points that correspond to divisors Δ_y . Replace B by an irreducible component which contains a set corresponding to every point of a countably dense subset of Y and let $f : H \rightarrow B$ be the universal family over B , $H \subset B \times Y$. Pick a log resolution of the generic fiber and extend this to an embedded resolution of $Y \times U$ for some open subset U of B . By our assumptions there is a unique exceptional divisor of log discrepancy zero over the generic point of the base corresponding to the log canonical centres W_y . Thus by taking a finite cover of B and passing to an open subset of U we may assume that there is a morphism $f : Y' \rightarrow B$ whose fibre over $b \in U$ is a log canonical centre for $K_Y + \Delta_b$.

By passing to an open subset of U we may assume that f is flat, so U maps to the Hilbert scheme. Replace B by the normalization of the closure of the image of U in the corresponding Hilbert scheme and Y' by the normalization of the pull back of the universal family. Cutting B

by hyperplanes we may assume that the map from Y' to Y is generically finite. \square

Going back to the proof of Theorem 1.1 we have left to discuss the case in which for every very general $x \in X$ we have a pair of (D_x, V_x) , such that

- (1): $D_x \sim \lambda K_X$.
- (2): V_x is a pure log canonical center of D_x .
- (3): $\dim V_x = 2$.

Observe that here we can take the same λ for every point in a countably dense subset of X by the argument of the first part of the proof of the Lemma above. Now by the previous lemma we have a diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \downarrow f & & \\ B & & \end{array}$$

where π is dominant and generically finite of normal varieties, and the image of the general fiber of f is V_x for some $x \in X$.

We claim that we can also assume that the map π is birational. There is a general proof of this fact in McKernan's paper [10]. In our case, because of the low dimension, we present a simpler argument. We will use an elementary lemma of calculation of centres of log canonical singularities.

LEMMA 3.3. — *Let Y be a smooth projective variety and let (Δ_i, W_i) for $i = 1, 2$ be a pair such that at some point $y \in Y$ we have that W_i is an exceptional centre of log canonical singularities of codimension 1 at y for $K_Y + \Delta_i$ with Δ_i smooth at y . Then there exists $Z \subset W_1 \cap W_2$ a minimal pure centre of log canonical singularities at y for a pair (Y, Δ) with $\Delta = k(\Delta_1 + \Delta_2)$ for some rational number $0 < k \leq 1$.*

Proof. — First of all notice that if we set $c := c(Y, \Delta_1 + \Delta_2, y)$ we have that $c \leq 1$. Also $\text{LCS}(X, c(\Delta_1 + \Delta_2), y) \subset W_1 \cap W_2$.

The lemma now follows from the first part of Lemma 2.6. \square

Let X_b be the fibre of f over $b \in B$. We claim that there exists a closed subset X_1 of X such that for any pair (D_x, V_x) as above, $x \notin X_1$ and $\pi(X_b) = V_x$ for some $b \in B$ we have that D_x is smooth at x . In fact there is an open subset U of B that is parametrizing a universal family Y of divisors D_b so that for $x \in \pi(f^{-1}(U))$ and $f(X'_b) = V_x$ we have $D_x = D_b$. But after shrinking U the singular locus of Y , which is at most two dimensional, dominates the union of singularities of divisors D_x with $x \in \pi(f^{-1}(U))$. Since $B \setminus U$ is a finite set the claim follows.

Suppose now that the inverse image of a general $x \in X \setminus X_1$ under π is contained in two different fibres of f . Then through some general point $x \in X$ there are two log canonical centres, say V_1 and V_2 , of the pairs (X, D_1) and (X, D_2) , with $D_i \sim \lambda K_X$ for $i = 1, 2$. But since $x \notin X_1$ it follows that D_i are smooth at x for $i = 1, 2$ and we can apply the previous lemma. So there is C a minimal pure log canonical centre of (X, D) with $D = k(D_1 + D_2)$ and $0 < k \leq 1$ contained in the intersection of V_1 and V_2 through x . We have $D \sim k(2\lambda)K_X$. Since $\dim C \leq 1$ by the analysis in Case I and II if $\frac{8}{15} + \frac{23}{15}(2\lambda) < 1$ we obtain a section of $2K_X$. The inequality can be arranged for $\alpha > 20$.

So we will replace our X with an appropriate smooth birational model such that we have a morphism to a curve $f : X \rightarrow B$ and for a general fiber $X_{b'}$ over a point $b' \in B$ there is a divisor $D_{b'} \sim \lambda K_X$ for which we have

$$\mathcal{J}(D_{b'}) \subset \mathcal{O}_X(-X_{b'}).$$

Now run the relative minimal model program for the morphism $f : X \rightarrow B$. Denote the resulting morphism by $f_0 : X_0 \rightarrow B$ and here X_0 is \mathbb{Q} -factorial with terminal singularities. Since the singularities of X_0 are isolated points (cf. [13]), we can replace X_0 by a resolution which does not change the general fibre of f_0 . Thus by replacing f by the appropriate resolution we can assume that the general fibre of f is a minimal and smooth surface.

We need the following lemma.

LEMMA 3.4. — *Let $g : Y \rightarrow Z$ be a surjective morphism from a smooth projective variety Y to a normal variety Z . Let L be a divisor on Y such that $L \equiv g^*M + \Delta$, where M is nef and big \mathbb{Q} -Cartier divisor on Z and Δ is an effective \mathbb{Q} -Cartier divisor on Y . Then $H^i(Z, R^j g_* (\mathcal{O}_Y(K_Y + L) \otimes \mathcal{J}(\Delta))) = 0$ for $i > 0, j \geq 0$.*

Proof. — Let $g' : Y' \rightarrow Y$ be a log resolution of the pair (Y, Δ) and let $h = g \circ g'$. By Corollary 10.15 of [12] we have that $H^i(Z, R^j h_* (K_{Y'} - \llcorner g'^* \Delta \lrcorner + g'^* L)) = 0$ for $i > 0, j \geq 0$. Notice that we have $R^j g'_* \mathcal{O}_{Y'}(K_{Y'}/Y - \llcorner g'^* \Delta \lrcorner) = 0$ for $j > 0$ (cf. [14], Theorem 9.4.1) and so the Lemma follows by the projection formula. □

For an integer m sufficiently large and divisible consider a general divisor $G \in |m(K_X + X_b)|$ for some general $b \in B$. For a general $b' \in B$ set $D = D'_{b'}$ and let $F = 3D + \frac{1-3\lambda}{m}G + (3\lambda - 2)X_{b'}$. We have that $\mathcal{J}(F) \subset \mathcal{O}_X(-X_{b'})$. The divisor F is effective since $X_{b'}$ is a pure log canonical centre for (X, D) .

Consider

$$E := f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(F)).$$

This is a vector bundle, since it is a torsion free sheaf on a curve.

CLAIM 1. — *Let r be the rank of the bundle E . If $\alpha > 2304$ then we have that $r > 0$.*

Proof of the Claim. The general fiber X_b is a smooth surface and it is enough to prove that

$$H^0(X_b, \mathcal{O}_{X_b}(2K_{X_b}) \otimes \mathcal{J}(F)|_{X_b}) \neq 0.$$

The hypothesis of the restriction Theorem 2.7 for multiplier ideal sheaves are satisfied, so we have that $\mathcal{J}(X_b, (3D + \frac{1-3\lambda}{m}G)|_{X_b}) \subset \mathcal{J}(X, 3D + \frac{1-3\lambda}{m}G)|_{X_b}$. Also since K_X is big we can apply Kawamata’s Theorem A from [8], so we obtain a surjection

$$H^0(X, \mathcal{O}_X(m(K_X + X_b))) \rightarrow H^0(X_b, \mathcal{O}_X(mK_{X_b})).$$

But X_b is minimal, so $|mK_{X_b}|$ is base point free. This implies that

$$\mathcal{J}(X_b, (3D + \frac{1-3\lambda}{m}G)|_{X_b}) = \mathcal{J}(X_b, 3D|_{X_b}).$$

Define $\Delta = D|_{X_b}$ and observe that since $(K_X)|_{X_b} \sim K_{X_b}$ we have that $\Delta \sim \lambda K_{X_b}$. So it is enough to show that

$$H^0(X_b, \mathcal{O}_{X_b}(2K_{X_b}) \otimes \mathcal{J}(3\Delta)) \neq 0.$$

By a result of Bombieri (cf. [4]) we have that the linear system $|4K_{X_b}|$ is base point free. If we arrange that $3\Delta \cdot C < 1$, for a curve C that is a component of a divisor in $|4K_{X_b}|$ and not a component of Δ , we will have that $\text{mult}_x 3\Delta < 1$ for $x \in \Delta$, and this implies that $\mathcal{J}(3\Delta)$ is trivial (cf. [14], Proposition 9.5.11). The inequality could be arranged for $\lambda < \frac{1}{768}$ and thus for $\alpha > 2304$. But the second plurigenera on a surface of general type is always non-zero, and the claim follows. □

By Lemma 3.4 since $K_X \equiv X_b + F$ we have the vanishing

$$H^1(B, f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(F))) = 0.$$

Tensoring the exact sequence

$$0 \rightarrow \mathcal{O}_B \rightarrow \mathcal{O}_B(b') \rightarrow \mathcal{O}_{b'} \rightarrow 0$$

by E we get a section of $f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(F)) \otimes \mathcal{O}_B(b')$. But we have the inclusion

$$f_*(\mathcal{O}_X(2K_X) \otimes \mathcal{J}(F)) \otimes \mathcal{O}_B(b') \hookrightarrow f_*(\mathcal{O}_X(2K_X))$$

and we obtain a section of $\mathcal{O}_X(2K_X)$. □

4. Proof of Theorem 1.2

We want to prove that the map given by the linear system $|5K_X|$ is birational. So we would like to produce a divisor $D \sim tK_X$ with $x, y \in \text{LLC}(X, D)$, where t is a positive rational number. Then apply Kawamata-Viehweg vanishing to lift sections from $\mathbb{C}_x \oplus \mathbb{C}_y$. Unfortunately if we proceed as above we are not able to do so for $t < 4$. We will have to use a slightly different approach.

As before we will divide the proof in two parts. First we will study the case when all the surfaces that appear as log canonical centres of some divisor linearly equivalent to a multiple of the canonical class are of large volume. In this case we deduce the result from Takayama's results [16] by carefully choose the necessary constants.

Proof of Theorem 1.2. Start by replacing X by a desingularization. Recall that we are assuming

$$\text{vol}(K_X) > \beta^3.$$

By Theorem 3.1 and Proposition 5.3 (see also Notation 5.2) of Takayama [16] for $\epsilon > 0$ there is a birational morphism $\mu : X' \rightarrow X$ and a decomposition

$$\mu^* K_X \sim A + M,$$

where A is an ample \mathbb{Q} -divisor and M is an effective divisor whose support contains all the exceptional locus of μ , such that for two general points x_1 and x_2 there is a divisor D ,

$$D \sim \lambda A$$

with

$$\lambda < \lambda'_1 + \frac{4\sqrt{2}}{(1-\epsilon)\alpha_1} + 2 \frac{2 + \lambda'_1 + \frac{4\sqrt{2}}{(1-\epsilon)\alpha_1} + 2\epsilon}{(1-\epsilon)\alpha_2} + 2\epsilon,$$

such that $x_i \in \text{Nklt}(X', D)$ for $i = 1, 2$ and $\text{codim Nklt}(X', D) = 3$ around at least one of the x_i , $i = 1, 2$. Then by Nadel vanishing we see that $(\lceil \lambda \rceil + 1)K_X$ gives a birational map. Here α_1 is the square root of a lower bound for the volume of the surfaces of general type contained in X , α_2 is a lower bound for the volume of the curves of general type contained in X , and

$$\lambda'_1 < 2^{1/3} \frac{3}{(1-\epsilon)\beta}.$$

In any case we would like to ensure that $\lambda < 4$. First note that for any curve C of general type we have $\text{vol}(C) \geq 2$. So in the inequality above we may take $\alpha_2 = 2$. We also may choose $\epsilon = 0.01$. Since we will argue as

before we will take $\alpha_1 = 6$. To achieve that $\lambda < 4$ it suffices to be able to choose $\lambda'_1 < \frac{1}{20}$, and this can be done as long as $\beta > 76$.

Finally arguing as in the proof of Theorem 1.1 and the proof of Lemma 3.2 we can assume that through every point $x \in X$ in a countably dense subset of X we have a surface V_x that is a centre of log canonical singularities for (X, D_x) , $D_x \sim \lambda'_1 K_X$ and so we get a diagram

$$\begin{array}{ccc} X'' & \xrightarrow{\pi} & X \\ \downarrow f & & \\ B & & \end{array}$$

As before we argue that we can assume that π is birational. In fact let x and y be two general points of X as above. We would like to produce a divisor D that is lc, but not klt at one of the points with centre a curve and that is not klt at the other point and apply the inductive steps of Takayama. Consider the divisor $D_x + D_y$. By rescaling we can assume that we have a divisor that is lc, but not klt at one of the points, say x , and it is not klt at y . That is, we have $D_1 \sim \lambda' K_X$ with $\lambda' \leq 2\lambda'_1$. Also we may assume that the centre of log canonical singularities at x is a surface. If π is not birational then there is another divisor D'_x with centre at x another surface. Consider $D' = D_1 + kD'_x$, where $k = \max\{c, \text{such that } (X, D_1 + cD'_x) \text{ is lc at } x\}$. Since x and y are general points we observe that $0 < k \leq 1$. By possibly applying Lemma 3.3, we obtain a divisor $D \sim \lambda'' K_X$ with the desired properties and $\lambda'' \leq 3\lambda'_1$. Then Proposition 5.3 of [16] implies that by taking

$$\lambda < 3\lambda'_1 + \frac{2 + 3\lambda'_1 + \epsilon}{1 - \epsilon} + \epsilon$$

we have that $(\lceil \lambda \rceil + 1)K_X$ gives a birational map. We can take $\lambda < 4$ for $\beta \geq 16$.

Thus we may assume that we have a morphism to a curve $f : X \rightarrow B$, such that the general fiber X_b of f is minimal and smooth and it is log canonical centre for (X, D_x) with $D_x \sim \lambda K_X$ and $\lambda < 2^{\frac{1}{3}} \frac{3}{(1-\epsilon)\beta}$. Moreover we have that the $\text{vol}(X_b) < 36$.

The goal is to find a section of $\mathcal{O}_X(5K_X)$ that separates two general points. In the case when the two general points x and y in X do not lie in the same fiber, say $x \in X_{b_1}$ and $y \in X_{b_2}$, we will create a section of $f_*\mathcal{O}_X(5K_X)$ that separates $f(x)$ and $f(y)$. For b and b' general points in B consider a general divisor $G \in |m(K_X + X_{b'})|$ for some integer m sufficiently large and divisible and define $F = D_{b_1} + D_{b_2} + 6D_b + \frac{4-8\lambda}{m}G + (8\lambda - 5)X_b$. Then arguing as in Claim 1 we can guarantee that $E = f_*(\mathcal{O}_X(5K_X) \otimes \mathcal{J}(F))$

is a vector bundle of positive rank for $\lambda < \frac{1}{1152}$ so that $\beta \geq 4355$. Since $4K_X \equiv X_b + F \sim f^*b + F$ we can apply Lemma 3.4. It follows that $H^1(B, f_*(\mathcal{O}_X(5K_X) \otimes \mathcal{J}(F))) = 0$. Then tensoring the short exact sequence

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \mathcal{O}_B(b_1 + b_2) \longrightarrow \mathcal{O}_{b_1} \oplus \mathcal{O}_{b_2} \longrightarrow 0,$$

with E we see that there is a section of $f_*(\mathcal{O}_X(5K_X) \otimes \mathcal{J}(F)) \otimes \mathcal{O}_B(b_1 + b_2)$ that separates b_1 and b_2 . But then the inclusion

$$f_*(\mathcal{O}_X(5K_X) \otimes \mathcal{J}(F)) \otimes \mathcal{O}_B(b_1 + b_2) \hookrightarrow f_*\mathcal{O}_X(5K_X)$$

implies that the linear system $|5K_X|$ separates points that are not on the same general fibre of f .

Suppose now that x and y lie in the same fiber, say X_b over $b \in B$. Let b' be another general point and let $(D_{b'}, X_{b'})$ be the corresponding pair such that $\mathcal{J}(D_{b'}) \subset \mathcal{O}_X(-X_{b'})$. Write $K_X \sim A' + M'$ as a sum of an ample and an effective divisor. Consider a general divisor $G \in |m(K_X + X_b)|$ for some integer m sufficiently large and divisible and define $F = 6D_{b'} + (\frac{4-6\lambda}{m} - \epsilon)G + (4\lambda - 5 + \epsilon)X_{b'} + \epsilon M'$ with ϵ a small rational number $0 < \epsilon \ll 1$.

Let $\Delta = D|_{X_{b'}}$. Observe that $\mathcal{J}(X_b, F|_{X_b}) = \mathcal{J}(X_b, 6\Delta)$ is trivial for $\lambda < \frac{1}{864}$ (and this follows as in the proof of Claim 1 and Corollary 2.35 of [13]). We can choose such lambda for $\beta \geq 3266$. It follows that

$$H^0(X_b, \mathcal{O}_{X_b}(5K_{X_b}) \otimes \mathcal{J}(X_b, 6\Delta)) \cong H^0(X_b, \mathcal{O}_{X_b}(5K_{X_b})).$$

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(5K_X - X_b) \otimes \mathcal{J}(F) \longrightarrow \mathcal{O}_X(5K_X) \otimes \mathcal{J}(F) \longrightarrow \mathcal{O}_{X_b}(5K_{X_b}) \otimes \mathcal{J}(F)|_{X_b} \longrightarrow 0.$$

Since $5K_X - X_b - F \equiv \epsilon A'$ by Nadel vanishing (Theorem 2.5) we have a surjection

$$H^0(X, \mathcal{O}_X(5K_X) \otimes \mathcal{J}(F)) \longrightarrow H^0(X_b, \mathcal{O}_{X_b}(5K_{X_b}) \otimes \mathcal{J}(F)|_{X_b}) \longrightarrow 0.$$

Observing that

$$\begin{aligned} H^0(X_b, \mathcal{O}_{X_b}(5K_{X_b})) &\cong H^0(X_b, \mathcal{O}_{X_b}(5K_{X_b}) \otimes \mathcal{J}(X_b, F|_{X_b})) \\ &\subseteq H^0(X_b, f_*\mathcal{O}_{X_b}(5K_{X_b}) \otimes \mathcal{J}(F)|_{X_b}), \end{aligned}$$

and since we have sections in $\mathcal{O}_{X_b}(5K_{X_b})$ that separate x and y , we get such a section in $\mathcal{O}_X(5K_X)$. □

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Gueorgui Tomov TODOROV
University of Utah
Dep. of mathematics, JWB 107
155 S 1400 E RM 233
Salt Lake City TU 84112-0090 (USA)
todorov@math.utah.edu