ANNALES

DE

L'INSTITUT FOURIER

Franziska HEINLOTH

A note on functional equations for zeta functions with values in Chow motives

Tome 57, nº 6 (2007), p. 1927-1945.

<http://aif.cedram.org/item?id=AIF_2007__57_6_1927_0>

Ann. Inst. Fourier, Grenoble
57, 6 (2007) 1927-1945

A NOTE ON FUNCTIONAL EQUATIONS FOR ZETA FUNCTIONS WITH VALUES IN CHOW MOTIVES

by **Franziska HEINLOTH**

Abstract. — We consider zeta functions with values in the Grothendieck ring of Chow motives. Investigating the λ–structure of this ring, we deduce a functional equation for the zeta function of abelian varieties. Furthermore, we show that the property of having a rational zeta function satisfying a functional equation is preserved under products.

Résumé. — Nous considérons les fonctions zêta à valeurs dans l'anneau de Grothendieck des motifs de Chow. L'étude de la λ-structure de cet anneau, nous permet d'obtenir une équation fonctionnelle pour la fonction zêta des variétés abéliennes. En outre nous montrons que l'existence d'une telle équation fonctionnelle est une propriété stable par produit.

1. Introduction

Let C be a geometrically irreducible smooth projective curve of genus g over a field k. Kapranov in [13] considers the *zeta function*

$$\zeta_\mu(C,T) = \sum_{i=0}^{\infty} \mu(\mathrm{Sym}^i(C))T^i,$$

where μ is a multiplicative Euler characteristic with compact support (*i.e.*, an invariant of k-varieties with values in a ring A satisfying $\mu(X) = \mu(X - Y) + \mu(Y)$ for $Y \subset X$ closed and $\mu(X \times Y) = \mu(X) \cdot \mu(Y)$), and $\mathrm{Sym}^i(C)$ denotes the i-th symmetric power of C. For example, if k is a finite field, the number of k-valued points is such an invariant, and the associated zeta function is the Hasse–Weil zeta function. Kapranov shows that if A is a

Keywords: zeta functions, Chow motives, functional equation.
Math. classification: 14G10, 14F42.

field and $\mathbb{L}_\mu = \mu(\mathbb{A}^1) \neq 0$, the zeta function of C with respect to μ is rational and satisfies the functional equation

$$\zeta_\mu(C, \frac{1}{\mathbb{L}_\mu T}) = \mathbb{L}_\mu^{1-g} T^{2-2g} \zeta_\mu(C, T).$$

Kapranov suggests that also zeta functions of higher dimensional smooth projective varieties should be rational and satisfy a functional equation.

Larsen and Lunts in [17] and [18] for $k = \mathbb{C}$ construct a multiplicative Euler characteristic with compact support μ such that the zeta function with respect to μ of smooth projective surfaces of nonnegative Kodaira dimension is not rational. (In their example, $\mathbb{L}_\mu = 0$.)

On the other hand, as they point out in [18], if A carries a λ–structure σ^i such that A (with its opposite structure) is *special* (compare Section 2), and if $\mu(\operatorname{Sym}^i X) = \sigma^i(\mu(X))$, the property of having a rational zeta function is *e.g.*, preserved under products.

In this note, we consider the value ring $\mathrm{K}_0(CM_k)$, the Grothendieck ring of Chow motives over k with rational coefficients. It is the free abelian group on isomorphism classes $[M]$ of Chow motives M modulo the relations $[M \oplus N] = [M] + [N]$ and carries a commutative ring structure induced by the tensor product of Chow motives. There is also the notion of the i-th symmetric power $\operatorname{Sym}^i M$ of a Chow motive M, which is defined as the image of the projector $\frac{1}{i!}\sum_{\sigma \in S_i} \sigma$ on $M^{\otimes i}$. The symmetric powers Sym^i endow $\mathrm{K}_0(CM_k)$ with the structure of a λ-ring. The opposite structure $(\operatorname{Alt}^i)_i$ is induced by the projectors $\frac{1}{i!}\sum_{\sigma \in S_i} (-1)^\sigma \sigma$ and turns out to be special (see Section 4 for details).

In characteristic zero, Gillet and Soulé as a corollary from [10] and Guillen and Navarro Aznar as a corollary from [12] get a multiplicative Euler characteristic with compact support μ with values in $\mathrm{K}_0(CM_k)$, such that $\mu(X) = [h(X)]$ for a smooth projective variety X. Here $h(X)$ is the Chow motive of X . Note that $\mu(\mathbb{A}^1)$ is the class of the Lefschetz motive $\mathbb{L}$. It follows from a result of Del Baño and Navarro Aznar in [5] that $\mu(\operatorname{Sym}^i X) = [\operatorname{Sym}^i h(X)]$ for a smooth projective variety X. Hence the zeta function of X associated to μ equals

$$Z_X(T) = \sum_{i=0}^{\infty} [\operatorname{Sym}^i h(X)] T^i.$$

This zeta function with values in $\mathrm{K}_0(CM_k)$ makes sense for any ground field k. Note that for k finite one can still read off the Hasse–Weil zeta function from it.

As pointed out by André in Section 4.3 of [2] and Chapter 13 of [1], varieties with a finite dimensional Chow motive in the sense of Kimura [14] and O'Sullivan (*i.e.*, whose Chow motive is the sum of two Chow motives X^+ and X^- such that $\mathrm{Alt}^i(X^+) = 0$ for $i \gg 0$ and $\mathrm{Sym}^i(X^-) = 0$ for $i \gg 0$) have a rational zeta function with coefficients in $\mathrm{K}_0(CM_k)$. More precisely, as Alt^i is the opposite structure to Sym^i (compare Section 2),

$$Z_X(T) = \frac{P(T)}{Q(-T)} \text{ in } \mathrm{K}_0(CM_k)[[T]], \tag{1.1}$$

where $P(T) = \sum_{i\geqslant 0}[\mathrm{Sym}^i(X^-)]T^i$ and $Q(T) = \sum_{i\geqslant 0}[\mathrm{Alt}^i(X^+)]T^i$ are polynomials and moreover $Q(T)$ is invertible in $\mathrm{K}_0(CM_k)[[T]]$. For example, this holds for an abelian variety over k.

In Chapter 13 of [1], André writes: «Nous laissons au lecteur le plaisir de spéculer sur d'éventuelles équations fonctionnelles...»

Kimura shows in [14] that if a smooth projective variety X over an algebraically closed field has a finite dimensional Chow motive $X^+ \oplus X^-$, the minimal e such that $\mathrm{Alt}^i(X^+) = 0$ for $i > e$ equals the dimension of the sum of the even cohomology groups for any Weil cohomology, and analogously for the minimal f such that $\mathrm{Sym}^i(X^-) = 0$ for $i > f$. Hence it seems natural to expect the functional equations

$$P\Big(\frac{1}{\mathbb{L}^n T}\Big) = T^{-f}\mathbb{L}^{-\frac{nf}{2}}P(T) \text{ and } Q\Big(\frac{1}{\mathbb{L}^n T}\Big) = T^{-e}\mathbb{L}^{-\frac{ne}{2}}Q(T)$$

in $\mathrm{K}_0(CM_k)[T,T^{-1}]$, where n is the dimension of X.

In this note, we consider functional equations for zeta functions with coefficients in $\mathrm{K}_0(CM_k)$, where k is an arbitrary field. Using the well known decomposition of the Chow motive of an abelian variety, we prove

Proposition 1.1 (Proposition 5.1). — *Let A be an abelian variety of dimension g over k, and denote by $Z_A(T) = \sum_{i=0}^{\infty}[\mathrm{Sym}^i h(X)]T^i \in \mathrm{K}_0(CM_k)[[T]]$ its zeta function with values in $\mathrm{K}_0(CM_k)$. Then*

$$Z_A\Big(\frac{1}{\mathbb{L}^g T}\Big) = Z_A(T).$$

More precisely, $Z_A(T)$ can be written as $Z_A(T) = \frac{P^A(T)}{Q^A(-T)}$ as in Equation 1.1 in such a way that $P^A(T), Q^A(T) \in 1 + T\,\mathrm{K}_0(CM_k)[T]$ satisfy the expected functional equations

$$P^A\Big(\frac{1}{\mathbb{L}^g T}\Big) = T^{-f}\mathbb{L}^{-\frac{gf}{2}}P^A(T) \text{ and } Q^A\Big(\frac{1}{\mathbb{L}^g T}\Big) = T^{-e}\mathbb{L}^{-\frac{ge}{2}}Q^A(T)$$

in $\mathrm{K}_0(CM_k)[T,T^{-1}]$, where $e = f = 2^{2g-1}$.

Furthermore, in Proposition 6.1, we show that having a rational zeta function satisfying a functional equation is preserved by taking products.

To this end, in Section 4, we investigate the λ–structure on $\mathrm{K}_0(CM_k)$.

Acknowledgments. I am indebted to Hélène Esnault for her invaluable suggestions and comments and constant encouragement. I thank Niko Naumann for interesting discussions. I am grateful for the support of the DFG–Schwerpunkt “Globale Methoden in der Komplexen Geometrie”. I thank the referee for helpful comments and corrections.

2. λ–Rings

We recall the notion of a λ–ring. For more details, see for example Chapter I of Atiyah and Tall, [3].

A ring A endowed with operations λ^r for $r \in \mathbb{N}$ such that $\lambda^0(a) = 1$, $\lambda^1(a) = a$ and $\lambda^r(a+b) = \sum_{i+j=r} \lambda^i(a)\lambda^j(b)$ is called a *λ–ring*. This is equivalent to the datum of a group homomorphism $\lambda_t : (A, +) \longrightarrow (1 + tA[[t]], \cdot), a \mapsto 1 + \sum_{r\geqslant 1} \lambda^r(a)t^r$ such that $\lambda^1(a) = a$.

The *opposite* λ–structure on A is given by $\sigma_t(a) = (1+\sum_{r\geqslant 1}\lambda^r(a)(-t)^r)^{-1}$. Explicitely, σ^r is given recursively by

$$\sigma^r(a) - \sigma^{r-1}(a)\lambda(a) + \cdots + (-1)^r\lambda^r(a) = 0 \text{ for } r \geqslant 1.$$

$B = 1 + tA[[t]]$ itself carries the structure of a λ–ring:

Denote by σ_i^N the elementary symmetric polynomials in $\xi_1, \ldots, \xi_N$ and by s_i^N the elementary symmetric polynomials in $x_1, \ldots, x_N$. Let $P_n(\sigma_1^N, \ldots, \sigma_n^N, s_1^N, \ldots, s_n^N)$ be the coefficient of t^n in $\prod_{1\leqslant i,j\leqslant N}(1 + \xi_i x_j t)$, where $N \geqslant n$, and $P_{n,r}(\sigma_1^N, \ldots, \sigma_{rn}^N)$ the coefficient of t^n in $\prod_{1\leqslant i_1<\cdots<i_r\leqslant N}(1 + \xi_{i_1}\cdots\xi_{i_r}t)$, where $N \geqslant rn$.

Addition on B is given by multiplication, multiplication $\circ$ is given by

$$\Big(1 + \sum_{k\geqslant 1} a_k t^k\Big) \circ \Big(1 + \sum_{l\geqslant 1} b_l t^l\Big) = 1 + \sum_{n\geqslant 1} P_n(a_1, \ldots, a_n; b_1, \ldots, b_n)t^n$$

with neutral element $1 + t$, and the λ–structure is given by

$$\Lambda^r\Big(1 + \sum_{k\geqslant 1} a_k t^k\Big) = 1 + \sum_{n\geqslant 1} P_{n,r}(a_1, \ldots, a_{rn})t^n.$$

The λ–ring A is called *special*, if λ_t is a homomorphism of λ–rings.

Remark 2.1. — The λ–structure on B may be given in a more sophisticated manner without writing down the universal polynomials P_n and $P_{n,r}$ explicitly, compare Section I.1 of [3]. But we will need the precise shape of P_n and $P_{n,r}$ in Sections 5 and 6.

Remark 2.2. — A group homomorphism $\varphi : A \longrightarrow B$ between λ–rings is a homomorphism of λ–rings if there is a set of group generators $S \subseteq A$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in S$ and for all $r \in \mathbb{N}$ and $a \in S$ we have $\varphi(\lambda^r(a)) = \lambda^r(\varphi(a))$. Compare [18], Lemma 4.4.

3. Curves

As a motivation, let us briefly review the situation for curves.

First, we consider the zeta function associated to the universal Euler characteristic with compact support. Let C be a geometrically irreducible smooth projective curve of genus g over a field k. Denote its i-th symmetric power by $\mathrm{Sym}^i(C)$. The zeta function of C is defined as

$$\zeta_C(T) = \sum_{i=0}^{\infty} [\mathrm{Sym}^i(C)] T^i \text{ in } \mathrm{K}_0(\mathit{Var}_k)[[T]].$$

Here $\mathrm{K}_0(\mathit{Var}_k)$ is the value group of the universal Euler characteristic with compact support, *i.e.*, the free abelian group on isomorphism classes of varieties over k modulo the relations $[X] = [X - Y] + [Y]$, where $Y \subset X$ closed. It carries a commutative ring structure induced by the product of varieties. By abuse of notation, we denote the class of the affine line by $\mathbb{L}$. A stratification argument shows that we get the same Grothendieck ring if we take classes of *quasi-projective* varieties.

If there is a line bundle of degree 1 on C, Kapranov shows that $(1 - T)(1 - \mathbb{L}T)\zeta_C(T)$ is a polynomial of degree $2g$, and that the zeta function satisfies the functional equation

$$\zeta_C\Big(\frac{1}{\mathbb{L}T}\Big) = \mathbb{L}^{1-g} T^{2-2g} \zeta_C(T)$$

in $\mathcal{M}_k((T))$, where $\mathcal{M}_k := \mathrm{K}_0(\mathit{Var}_k)[\mathbb{L}^{-1}]$.

Let us give a slight reformulation of Kapranov's argument.

As pointed out by Larsen and Lunts in [18], the symmetric powers Sym^i of quasi-projective varieties induce the structure of a λ–ring on $\mathrm{K}_0(\mathit{Var}_k)$ and, since $\mathrm{Sym}^i(\mathbb{L}[X]) = \mathbb{L}^i \,\mathrm{Sym}^i([X])$ (see Göttsche, [11]), also on $\mathcal{M}_k$ via $\mathrm{Sym}^i(\mathbb{L}^k[X]) := \mathbb{L}^{ik}\,\mathrm{Sym}^i([X])$. In these terms, $\zeta_X(T) = \lambda_t([X])$. Since λ_t is a group homomorphism, we get

$$\zeta_C(T) = \Big(\sum_{i=0}^{\infty} \mathrm{Sym}^i([C] - [\mathbb{P}^1]) T^i\Big) \zeta_{\mathbb{P}^1}(T).$$

As $\zeta_{\mathbb{P}^1}(T) = \zeta_1(T)\zeta_{\mathbb{L}}(T) = \frac{1}{1-T}\frac{1}{1-\mathbb{L}T}$, multiplying this equation by $(1-T)(1-\mathbb{L}T)$ yields $\mathrm{Sym}^i([C]-[\mathbb{P}^1]) = [\mathrm{Sym}^i(C)]-[\mathrm{Sym}^{i-1}(C)][\mathbb{P}^1]+[\mathrm{Sym}^{i-2}(C)]\mathbb{L}$ for $i \geqslant 2$.

If $i > 2g$, this expression vanishes, because for $j > 2g-2$ the morphism $\mathrm{Sym}^j(C) \longrightarrow \mathrm{Pic}^j(C) \cong \mathrm{Pic}^0(C)$ is a Zariski fibration with fiber $\mathbb{P}^{j-g}$ (we still assume that there is a line bundle of degree 1 on C). Therefore, $(1-T)(1-\mathbb{L}T)\zeta_C(T)$ is a polynomial of degree at most $2g$.

For the functional equation we need to show for $g \leqslant i \leqslant 2g$ that $\mathrm{Sym}^i([C]-[\mathbb{P}^1]) = \mathbb{L}^{i-g}\,\mathrm{Sym}^{2g-i}([C]-[\mathbb{P}^1])$: Consider the morphism $\mathrm{Sym}^j(C) \longrightarrow \mathrm{Pic}^j(C)$. It is a piecewise Zariski fibration with fiber $\mathbb{P}\,H^0(L)$ over L. There is an isomorphism $\mathrm{Pic}^i(C) \cong \mathrm{Pic}^{2g-2-i}(C)$, $L \mapsto \omega_C \otimes L^\vee$. By Riemann–Roch, $h^0(L) - h^0(\omega_C \otimes L^\vee) = \deg L + 1 - g$. Therefore,

$$[\mathrm{Sym}^i(C)] - \mathbb{L}^{i-g+1}[\mathrm{Sym}^{2g-2-i}(C)] = [\mathbb{P}^{i-g}][\mathrm{Pic}^i(C)] \text{ for } g \leqslant i \leqslant 2g-2.$$

Using $\mathrm{Pic}^j(C) \cong \mathrm{Pic}^0(C)$ again and adding up we conclude $\mathrm{Sym}^i([C]-[\mathbb{P}^1]) = \mathbb{L}^{i-g}\,\mathrm{Sym}^{2g-i}([C]-[\mathbb{P}^1])$.

Actually, the equation $\zeta_{\mathbb{P}^1}(T) = \frac{1}{1-T}\frac{1}{1-\mathbb{L}T}$ can be rephrased by saying that $\mathrm{Alt}^i(1) = \mathrm{Alt}^i(\mathbb{L}) = 0$ for $i \geqslant 2$, where Alt^i is the opposite λ–structure on $\mathrm{K}_0(\mathit{Var}_k)$.

Hence, $[C]$ can be written as the sum of two terms $x^+ + x^-$, where $\mathrm{Alt}^i(x^+) = 0$ for $i \gg 0$ and $\mathrm{Sym}^i(x^-) = 0$ for $i \gg 0$ and furthermore, $P^{x^-}(T) = \sum_i \mathrm{Sym}^i(x^-)T^i$ and $Q^{x^+}(T) = \sum_i \mathrm{Alt}^i(x^+)T^i$ satisfy the expected functional equations.

Remark 3.1. — If C carries a line bundle of degree d, a similar calculation shows that $(1-T^d)(1-\mathbb{L}^dT^d)\zeta_C(T)$ is a polynomial and that the functional equation still holds, as pointed out by Kapranov.

Now let us consider the zeta function of C with values in $\mathrm{K}_0(CM_k)$,

$$Z_C(T) = \sum_{i=0}^{\infty}[\mathrm{Sym}^i\, h(C)]T^i.$$

The Chow motive of C has a decomposition

$$h(C) = \mathbb{1} \oplus h^1(C) \oplus \mathbb{L},$$

where $h^1(C) = h^1(\mathrm{Pic}^0(C))$ (compare Section 5 and Sections 1.2.3 and 3.3 in [4]). Therefore $\mathrm{Sym}^i\, h^1(C) = 0$ for $i > 2g$ and $\mathrm{Sym}^i\, h^1(C) \cong \mathbb{L}^{i-g} \otimes \mathrm{Sym}^{2g-i}\, h^1(C)$. Hence the zeta function of C with values in $\mathrm{K}_0(CM_k)$ is also rational and satisfies the expected functional equation. In characteristic zero, this follows from the properties of the zeta function with values in $\mathcal{M}_k$ (compare Section 4.3).

4. λ–Structures on the Grothendieck ring of Chow motives

For the rest of the paper, we will restrict ourselves to the study of zeta functions with values in $\mathrm{K}_0(CM_k)$. In fact, the properties of the λ–structure on $\mathrm{K}_0(CM_k)$ which we need hold for the Grothendieck ring of any (pseudo-abelian) $\mathbb{Q}$-linear tensor category.

4.1. Schur functors

Let us recall some facts from Deligne, [7], Section 1.

Let κ be a field of characteristic zero, let $\mathcal{A}$ be a *κ-linear tensor category*, *i.e.*, a symmetric monoidal category, which is additive, pseudo-abelian and κ-linear such that $\otimes$ is κ-bilinear.

If V is a finite dimensional κ-vector space and X is an object of $\mathcal{A}$, there are objects $V \otimes X$ and $\mathcal{H}om(V, X)$ of $\mathcal{A}$ natural in V and X such that

$$\mathrm{Hom}(V \otimes X, Y) = \mathrm{Hom}(V, \mathrm{Hom}(X, Y))$$

and

$$\mathrm{Hom}(Y, \mathcal{H}om(V, X)) = \mathrm{Hom}(V \otimes Y, X).$$

There is a natural isomorphism $\mathcal{H}om(V, X) \cong V^\vee \otimes X$. The choice of a basis of V yields (non-canonical) isomorphisms $\mathcal{H}om(V, X) \cong X^{\oplus \dim V} \cong V \otimes X$.

If a finite group G acts on X, we define X^G as the image of the projector $\frac{1}{|G|} \sum_{g \in G} g \in \mathrm{End}(X)$.

If G acts on V and on X, it acts on $\mathcal{H}om(V, X)$ and we define $\mathcal{H}om_G(V,X)$ as $\mathcal{H}om(V,X)^G$. Note that $\mathrm{Hom}(Y, \mathcal{H}om_G(V, X)) = \mathrm{Hom}_G(V, \mathrm{Hom}(Y, X))$.

If all irreducible representations of G over $\overline{\kappa}$ are already defined over κ we have $\kappa[G] \cong \prod \mathrm{End}_\kappa(V_\lambda)$, where V_λ runs through a system of representatives for the isomorphism classes of irreducible representations. Therefore,

$$X \cong \mathcal{H}om_G(\kappa[G], X) \cong \bigoplus V_\lambda \otimes \mathcal{H}om_G(V_\lambda, X), \tag{4.1}$$

where the G-action on X corresponds to the G-action on the outer V_λ on the right hand side.

There is a natural isomorphism

$$\mathcal{H}om_{G \times H}(V \otimes W, X \otimes Y) \cong \mathcal{H}om_G(V, X) \otimes \mathcal{H}om_H(W, Y).$$

Under this isomorphism, the S_n-action on $\mathcal{H}om_G(V, X)^{\otimes n}$ corresponds to the S_n-action on $\mathcal{H}om_{G^n}(V^{\otimes n}, X^{\otimes n})$ induced by the actions on $V^{\otimes n}$ and $X^{\otimes n}$.

Furthermore, if we have a short exact sequence of finite groups

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1,$$

and V is a representation of H while W is a representation of G, which also acts on $X \in \mathcal{A}$, we have a natural isomorphism

$$\mathcal{H}om_G(V \otimes W, X) \cong \mathcal{H}om_H(V, \mathcal{H}om_K(W, X)).$$

Finally, if G acts on X and V is a representation of a subgroup $H < G$, we get the Nakayama relation (or Frobenius reciprocity)

$$\mathcal{H}om_H(V, X) \cong \mathcal{H}om_G(\operatorname{Ind}_H^G V, X).$$

If G is the symmetric group S_n and V_λ is an irreducible representation of S_n, indexed by a partition λ of $n = |\lambda|$,

$$S_\lambda(X) := \mathcal{H}om_{S_n}(V_\lambda, X^{\otimes n})$$

is called *Schur functor*. For the trivial representation $\operatorname{Triv}(S_n)$ we get

$$\operatorname{Sym}^n(X) := S_{(n)}(X) = \operatorname{im}\Big(\frac{1}{n!}\sum_{\sigma \in S_n} \sigma\Big) \subseteq X^{\otimes n},$$

for the alternating representation $\operatorname{Sign}(S_n)$ we obtain

$$\operatorname{Alt}^n(X) := S_{(1^n)}(X) = \operatorname{im}\Big(\frac{1}{n!}\sum_{\sigma \in S_n} (-1)^\sigma \sigma\Big) \subseteq X^{\otimes n}.$$

By 4.1, there is a canonical isomorphism

$$X^{\otimes n} \cong \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda(X),$$

where the S_n–action on $X^{\otimes n}$ corresponds to the action on V_λ on the right hand side. Note that in particular

$$S_\lambda(X) = 0 \text{ for } \lambda \neq (n), \tag{4.2}$$

if $\operatorname{Sym}^n(X) = X^{\otimes n}$.

If V_{μ_i}, $i = 1, \dots, r$ are irreducible representations of S_{n_i} and V_λ is an irreducible representation of S_n, where $n = \sum n_i$, we denote by

$$[\lambda : \mu_1, \dots, \mu_r]$$

the multiplicity of $\bigotimes V_{\lambda_i}$ in $\operatorname{Res}^{S_n}_{\prod S_{n_i}} V_\lambda$ (which equals the multiplicity of V_λ in $\operatorname{Ind}^{S_n}_{\prod S_{n_i}} \bigotimes V_{\lambda_i}$ by Frobenius reciprocity).

With this notation, we get

$$S_\mu(X) \otimes S_\nu(X) \cong \mathcal{H}om_{S_m \times S_n}(V_\mu \otimes V_\nu, X^{\otimes m+n})$$

$$\cong \mathcal{H}om_{S_{m+n}}(\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} V_\mu \otimes V_\nu, X^{\otimes m+n}) \tag{4.3}$$

$$\cong \bigoplus_{|\lambda|=|\mu|+|\nu|} [\lambda : \mu, \nu] S_\lambda(X) \tag{4.4}$$

$$S_\lambda(X \oplus Y) \cong \bigoplus_{|\mu|+|\nu|=|\lambda|} [\lambda : \mu, \nu] S_\mu(X) \otimes S_\nu(Y) \tag{4.5}$$

$$S_\lambda(X \otimes Y) \cong \bigoplus_{|\mu|=|\nu|=|\lambda|} [V_\mu \otimes V_\nu : V_\lambda] S_\mu(X) \otimes S_\nu(Y) \tag{4.6}$$

Furthermore, for a S_m–representation V and a G–representation W,

$$\mathcal{H}om_{S_m}(V, \mathcal{H}om_{G^m}(W^{\otimes m}, X^{\otimes m})) \cong \mathcal{H}om_{S_m \ltimes G^m}(V \otimes W^{\otimes m}, X^{\otimes m}). \tag{4.7}$$

In particular, for $|\mu| = m$ and $|\nu| = n$,

$$S_\mu(S_\nu(X)) = \mathcal{H}om_{S_m \ltimes S_n{}^m}(V_\mu \otimes V_\nu{}^{\otimes m}, X^{\otimes nm})$$

$$\cong \mathcal{H}om_{S_{nm}}(\mathrm{Ind}_{S_m \ltimes S_n{}^m}^{S_{nm}} V_\mu \otimes V_\nu{}^{\otimes m}, X^{\otimes nm}).$$

4.2. The λ–structure

Denote by $\mathrm{K}_0(\mathcal{A})$ the free abelian group on isomorphism classes $[X]$ of objects of $\mathcal{A}$ modulo the relations $[X \oplus Y] = [X] + [Y]$. It is the Grothendieck group associated to the abelian monoid of isomorphism classes of objects in $\mathcal{A}$ with direct sum. The tensor product of $\mathcal{A}$ induces a commutative ring structure on $\mathrm{K}_0(\mathcal{A})$. We call $\mathrm{K}_0(\mathcal{A})$ the *Grothendieck ring of $\mathcal{A}$.*

Note that for any $X \in \mathcal{A}$ with G–action we obtain a group homomorphism from the Grothendieck group of G–representations to $\mathrm{K}_0(\mathcal{A})$ sending a representation V to $\mathcal{H}om_G(V, X)$.

Lemma 4.1. — *The exterior powers* Alt^n *induce a special λ–ring structure on* $\mathrm{K}_0(\mathcal{A})$ *with opposite λ–structure given by the symmetric powers* Sym^n.

Proof. — Due to Equation 4.5 and the Littlewood–Richardson rule (see *e.g.*, [9], Appendix A),

$$[X] \mapsto 1 + \sum_{n \geqslant 1} [\mathrm{Alt}^n(X)] t^n$$

induces a λ–ring structure on $\mathrm{K}_0(\mathcal{A})$. The fact that the opposite structure is given by Sym^n follows from Equation 4.4 and the Littlewood–Richardson rule or more precisely from the fact that for $i, j \geqslant 1$ we have

$$\mathrm{Sym}^i(X) \otimes \mathrm{Alt}^j(X) \cong S_{(i+1,1^{j-1})}(X) \oplus S_{(i,1^j)}(X).$$

To show that the λ–structure given by Alt^i is special, we use an argument by Larsen and Lunts from [18], Theorem 5.1, in a slightly more general setting.

Recall one possible description of the free special λ-ring R on one generator: $R = \bigoplus_{n \geqslant 0} R_n$, where R_n is the representation ring over κ of the symmetric group S_n (with the convention that S_0 is the trivial group and R_0 therefore is $\mathbb{Z}$). It has a $\mathbb{Z}$-basis consisting of the elements (n, V_ν), where V_ν is an irreducible S_n-representation. The product is given by

$$(m, V_\mu)(n, V_\nu) = \big(m+n, \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} V_\mu \otimes V_\nu\big),$$

while the λ-structure is is given by

$$\lambda^r(n, V_\nu) = \big(rn, \mathrm{Ind}_{S_r \ltimes {S_n}^r}^{S_{rn}} \mathrm{Sign}(S_r) \otimes {V_\nu}^{\otimes r}\big).$$

Its generator as a λ-ring is $(1, \kappa)$.

$R \otimes_{\mathbb{Z}} R$ is the free special λ-ring on two generators. It has a $\mathbb{Z}$-basis consisting of the elements $(n_1, n_2, V_{\nu_1} \otimes V_{\nu_2})$, where V_{ν_i} is an irreducible S_{n_i}-representation. The product is given by

$$\begin{aligned}&(m_1, m_2, V_{\mu_1} \otimes V_{\mu_2})(n_1, n_2, V_{\nu_1} \otimes V_{\nu_2}) \\ &= (m_1+n_1, m_2+n_2, \mathrm{Ind}_{(S_{m_1} \times S_{n_1}) \times (S_{m_2} \times S_{n_2})}^{S_{m_1+n_1} \times S_{m_2+n_2}} (V_{\mu_1} \otimes V_{\nu_1}) \otimes (V_{\mu_2} \otimes V_{\nu_2})),\end{aligned}$$

while the λ-structure is is given by

$$\lambda^r(n_1, n_2, V_{\nu_1} \otimes V_{\nu_2}) = (rn_1, rn_2, \mathrm{Ind}_{S_r \ltimes ({S_{n_1}}^r \times {S_{n_2}}^r)}^{S_{rn_1} \times S_{rn_2}} \mathrm{Sign}(S_r) \otimes {V_{\nu_1}}^{\otimes r} \otimes {V_{\nu_2}}^{\otimes r}).$$

Now let X_1, X_2 be two objects of $\mathcal{A}$. Then, by 4.3 and 4.7,

$$(n_1, n_2, V_{\nu_1} \otimes V_{\nu_2}) \mapsto S_{\nu_1}(X_1) \otimes S_{\nu_2}(X_2)$$

defines a λ-ring homomorphism $R \otimes_{\mathbb{Z}} R \longrightarrow \mathrm{K}_0(\mathcal{A})$, hence every pair $[X_1], [X_2]$ is contained in a special λ-subring of $\mathrm{K}_0(\mathcal{A})$.

Therefore, $\lambda_t : \mathrm{K}_0(\mathcal{A}) \longrightarrow 1 + t\,\mathrm{K}_0(\mathcal{A})[[t]]$ satisfies

$$\lambda_t(xy) = \lambda_t(x) \circ \lambda_t(y)$$

and

$$\lambda_t(\lambda^r x) = \Lambda^r(\lambda_t(x))$$

for elements $x = [X]$ and $y = [Y]$ and due to Remark 2.2 therefore for all $x, y \in \mathrm{K}_0(\mathcal{A})$. □

We will need some more identities relating symmetric and exterior powers.

For a representation V of S_n, let $V' := \mathrm{Sign}(S_n) \otimes V$. Note that

$$\mathrm{Sign}(S_{mn}) \otimes \mathrm{Ind}_{S_m \ltimes S_n{}^m}^{S_{mn}} V \otimes W^{\otimes m} \cong \begin{cases} \mathrm{Ind}_{S_m \ltimes S_n{}^m}^{S_{mn}} V' \otimes W'^{\otimes m} & \text{if } n \text{ is odd} \\ \mathrm{Ind}_{S_m \ltimes S_n{}^m}^{S_{mn}} V \otimes W'^{\otimes m} & \text{if } n \text{ is even,} \end{cases}$$

because

$$(-1)^{\iota(\sigma)} = \begin{cases} (-1)^{\sigma} & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even,} \end{cases}$$

where $\iota : S_m \hookrightarrow S_{mn}$. Furthermore,

$$\mathrm{Sign}(S_{m+n}) \otimes \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} V \otimes W \cong \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} V' \otimes W'.$$

Therefore, for n odd, we obtain the following equation in R:

$$\begin{aligned} &\mathrm{Ind}_{S_m \ltimes S_n{}^m}^{S_{mn}} \mathrm{Triv}(S_m) \otimes \mathrm{Triv}(S_n)^{\otimes m} \\ &\quad = \mathrm{Ind}_{S_m \ltimes S_n{}^m}^{S_{mn}} \mathrm{Sign}(S_m)' \otimes \mathrm{Sign}(S_n)'^{\otimes m} \\ &\quad = \mathrm{Sign}(S_{mn}) \otimes P_{m,n}(\mathrm{Sign}(S_1), \dots, \mathrm{Sign}(S_{mn})) \\ &\quad = P_{m,n}(\mathrm{Triv}(S_1), \dots, \mathrm{Triv}(S_{mn})). \end{aligned}$$

Similarly, for n even, we get

$$\mathrm{Ind}_{S_m \ltimes S_n{}^m}^{S_{mn}} \mathrm{Sign}(S_m) \otimes \mathrm{Triv}(S_n)^{\otimes m} = P_{m,n}(\mathrm{Triv}(S_1), \dots, \mathrm{Triv}(S_{mn})).$$

Hence for every $x = [X] \in \mathrm{K}_0(\mathcal{A})$ we get

$$\mathrm{Sym}^m(\mathrm{Sym}^n(x)) = P_{m,n}(\mathrm{Sym}^1(x), \dots, \mathrm{Sym}^{mn}(x)) \text{ if } n \text{ is odd,} \tag{4.8}$$

$$\mathrm{Alt}^m(\mathrm{Sym}^n(x)) = P_{m,n}(\mathrm{Sym}^1(x), \dots, \mathrm{Sym}^{mn}(x)) \text{ if } n \text{ is even.} \tag{4.9}$$

Now consider the two generators $e_1 = (1, 0, \kappa \otimes \kappa)$ and $e_2 = (0, 1, \kappa \otimes \kappa)$ of $R \otimes_{\mathbb{Z}} R$.

We know that

$$\lambda^n(e_1 e_2) = P_n(\lambda^1(e_1), \dots, \lambda^n(e_1), \lambda^1(e_2), \dots, \lambda^n(e_2)),$$

where $\lambda^i(e_1) = (i, 0, \mathrm{Sign}(S_i) \otimes \kappa)$ and $\lambda^i(e_2) = (0, i, \kappa \otimes \mathrm{Sign}(S_i))$. On the other hand,

$$\lambda^n(e_1 e_2) = \sum_{|\mu| = |\nu| = n} [V_\mu \otimes V_\nu : \mathrm{Sign}(S_n)](n, n, V_\mu \otimes V_\nu).$$

As $[V_\mu \otimes V_\nu : \mathrm{Sign}(S_n)] = [V'_\mu \otimes V'_\nu : \mathrm{Sign}(S_n)]$, applying Equations 4.4 and 4.3, for $x = [X], y = [Y] \in \mathrm{K}_0(\mathcal{A})$ we get

$$\mathrm{Alt}^n(xy) = P_n(\mathrm{Sym}^1(x), \dots, \mathrm{Sym}^n(x), \mathrm{Sym}^1(y), \dots, \mathrm{Sym}^n(y)). \tag{4.10}$$

Similarly, as $[V_\mu \otimes V_\nu : \mathrm{Triv}(S_n)] = [V'_\mu \otimes V_\nu : \mathrm{Sign}(S_n)] = [V_\mu \otimes V'_\nu : \mathrm{Sign}(S_n)]$, we have

$$\mathrm{Sym}^n(xy) = P_n(\mathrm{Sym}^1(x), \dots, \mathrm{Sym}^n(x), \mathrm{Alt}^1(y), \dots, \mathrm{Alt}^n(y)) \tag{4.11}$$

$$= P_n(\mathrm{Alt}^1(x), \dots, \mathrm{Alt}^n(x), \mathrm{Sym}^1(y), \dots, \mathrm{Sym}^n(y)). \tag{4.12}$$

4.3. The Grothendieck ring of Chow motives

Everything in this section applies to the $\mathbb{Q}$–linear tensor category CM_k of Chow motives over a field k with rational coefficients as in Manin, [19] or Scholl, [20], where the equivalence relation on cycles is rational equivalence. Note that $\mathrm{Sym}^n \mathbb{L} = \mathbb{L}^{\otimes n}$ and therefore $S_\lambda(\mathbb{L}) = 0$ for $\lambda \neq (n)$ due to Equation 4.2. Hence it follows from Equation 4.6 that

$$S_\lambda(\mathbb{L} \otimes M) \cong \mathbb{L}^{\otimes|\lambda|} \otimes S_\lambda(X). \tag{4.13}$$

We denote the Grothendieck ring of CM_k by $\mathrm{K}_0(CM_k)$. If k is of characteristic zero, Gillet and Soulé as a corollary from [10] get a ring homomorphism from $\mathcal{M}_k$ to $\mathrm{K}_0(CM_k)$ such that for a smooth projective variety X the class $[X]$ of X is sent to $[h(X)]$, where $h(X)$ is the Chow motive of X. Del Baño and Navarro Aznar have shown in [5] that for a finite group G acting on X, the class $[X/G]$ of the quotient is sent to $[h(X)^G]$, where $h(X)^G$ is the image of the projector $\frac{1}{|G|}\sum_{g\in G} g$ in $h(X)$. In particular, the ring homomorphism $\mathcal{M}_k \longrightarrow \mathrm{K}_0(CM_k)$ is actually a homomorphism of λ–rings.

5. Abelian varieties

Let us recall some facts about the Chow motive of an abelian variety. For more details and references, see for example [20].

Shermenev has established a decomposition of the Chow motive of an abelian variety of dimension g as a sum

$$h(A) \cong \bigoplus_{0\leqslant i\leqslant 2g} h^i(A), \tag{5.1}$$

where

$$h^i(A) \cong \mathrm{Sym}^i(h^1(A)) \tag{5.2}$$

(in particular, $\mathrm{Sym}^i(h^1(A)) = 0$ for $i > 2g$) and furthermore, $h^0(A) = \mathbb{1}$ and $h^{2g}(A) = \mathbb{L}^g$.

Using the Fourier transformation developed by Beauville in [6], Deninger and Murre showed in [8] that there is a canonical decomposition of the form (5.1) such that multiplication by n acts on $h^i(A)$ as n^i. Künnemann proved in [16] that this decomposition satisfies Relations (5.2), and in [15] he obtained a hard Lefschetz theorem.

We use the canonical decomposition to prove the following

Proposition 5.1. — *Let A be an abelian variety of dimension g over k, and denote by $Z_A(T) = \sum_{i=0}^{\infty}[\mathrm{Sym}^i\, h(X)]T^i \in \mathrm{K}_0(CM_k)[[T]]$ its zeta function with values in $\mathrm{K}_0(CM_k)$. Then $Z_A(\frac{1}{\mathbb{L}^g T}) = Z_A(T)$. More precisely, $Z_A(T)$ can be written as $Z_A(T) = \frac{P^A(T)}{Q^A(-T)}$, such that $P^A(T), Q^A(T) \in 1 + T\,\mathrm{K}_0(CM_k)[T]$ satisfy the expected functional equations*

$$P^A\Big(\frac{1}{\mathbb{L}^g T}\Big) = T^{-f}\mathbb{L}^{-\frac{gf}{2}}P^A(T) \quad \text{and} \quad Q^A\Big(\frac{1}{\mathbb{L}^g T}\Big) = T^{-e}\mathbb{L}^{-\frac{ge}{2}}Q^A(T)$$

in $\mathrm{K}_0(CM_k)[T, T^{-1}]$, where $e = f = 2^{2g-1}$.

Proof. — Due to Relations (5.1) and (5.2), the zeta function of A with values in $\mathrm{K}_0(CM_k)$ equals

$$Z_A(T) = \frac{\prod_{\substack{0\leqslant n\leqslant 2g\\ n \text{ odd}}} P_n^A(T)}{\prod_{\substack{0\leqslant n\leqslant 2g\\ n \text{ even}}} Q_n^A(-T)},$$

where

$$P_n^A(T) := \sum_{m\geqslant 0}[\mathrm{Sym}^m(\mathrm{Sym}^n(h^1(A)))]T^m,$$
$$Q_n^A(T) := \sum_{m\geqslant 0}[\mathrm{Alt}^m(\mathrm{Sym}^n(h^1(A)))]T^m.$$

Denote by $\sigma_1^N, \ldots, \sigma_N^N$ the elementary symmetric functions in $\xi_1, \ldots, \xi_N$. From the commutativity of the diagram

$$\begin{tikzcd} \mathbb{Z}[\sigma_1^K, \ldots, \sigma_K^K] \arrow[r] \arrow[d, hook] & \mathbb{Z}[\sigma_1^k, \ldots, \sigma_k^k] \arrow[d, hook] \\ \mathbb{Z}[\xi_1, \ldots, \xi_K] \arrow[r] & \mathbb{Z}[\xi_1, \ldots, \xi_k] \end{tikzcd}$$

where $k \leqslant K$, $\sigma_l^K \mapsto 0$ for $k < l \leqslant K$ and to σ_l^k for $l \leqslant k$ and $\xi_l \mapsto 0$ for $k < l \leqslant K$ under the horizontal maps, it follows that

$$q_n^g(t) := \sum_{m\geqslant 0} P_{m,n}(\sigma_1^{2g}, \ldots, \sigma_{2g}^{2g}, 0, \ldots, 0)t^m = \prod_{1\leqslant i_1<\cdots<i_n\leqslant 2g}(1 + \xi_{i_1}\cdots\xi_{i_n}t)$$

is a polynomial of degree $b_n^g = \binom{2g}{n}$. For convenience, let us denote σ_{2g}^{2g} by σ. The polynomial $q_n^g(t)$ satisfies

$$\begin{aligned} q_n^g\Big(\frac{1}{\sigma t}\Big) &= \prod_{1\leqslant i_1<\cdots<i_n\leqslant 2g} \Big(1+\xi_{i_1}\cdots\xi_{i_n}\frac{1}{\sigma t}\Big) \\ &= \Big(\frac{1}{\sigma t}\Big)^{b_n^g} \prod_{1\leqslant i_1<\cdots<i_n\leqslant 2g} \xi_{i_1}\cdots\xi_{i_n} \prod_{1\leqslant i_1<\cdots<i_n\leqslant 2g} \Big(1+\frac{\sigma}{\xi_{i_1}\cdots\xi_{i_n}}t\Big) \\ &= \Big(\frac{1}{\sigma t}\Big)^{b_n^g} \sigma^{\frac{b_n^g n}{2g}} q_{2g-n}^g(t). \end{aligned}$$

(For the last step, note that $\prod_{1\leqslant i_1<\cdots<i_n\leqslant 2g}\xi_{i_1}\cdots\xi_{i_n}$ is a symmetric monomial of degree $b_n^g n$, therefore it must equal $\sigma^{\frac{b_n^g n}{2g}}$.) Hence, for n odd, it follows from 4.8 that $P_n^A(T)$ is a polynomial of degree b_n^g and

$$P_n^A\Big(\frac{1}{\mathbb{L}^g T}\Big) = \Big(\frac{1}{\mathbb{L}^g T}\Big)^{b_n^g} \mathbb{L}^{\frac{b_n^g n}{2}} P_{2g-n}^A(T).$$

In particular, $P^A(T) := \prod_{\substack{0\leqslant n\leqslant 2g\\ n \text{ odd}}} P_n^A(T)$ satisfies $P^A(\frac{1}{\mathbb{L}^g T}) = T^{-f}\mathbb{L}^{-\frac{gf}{2}} P^A(T)$.

On the other hand, for n even, we deduce from 4.9 that $Q_n^A(T)$ is a polynomial of degree $b_n^g = \binom{2g}{n}$ and satisfies

$$Q_n^A\Big(\frac{1}{\mathbb{L}^g T}\Big) = \Big(\frac{1}{\mathbb{L}^g T}\Big)^{b_n^g} \mathbb{L}^{\frac{b_n^g n}{2}} Q_{2g-n}^A(T),$$

hence $Q^A(T) := \prod_{\substack{0\leqslant n\leqslant 2g\\ n \text{ even}}} Q_n^A(T)$ satisfies $Q^A(\frac{1}{\mathbb{L}^g T}) = T^{-e}\mathbb{L}^{-\frac{ge}{2}} Q^A(T)$. □

Remark 5.2. — If A is the Jacobian of a curve C, we have $Z_C(T) = \frac{P_1^A(T)}{(1-T)(1-\mathbb{L}T)}$ (compare Section 3).

Remark 5.3. — An easy calculation using Equation 4.13 and the decomposition of the motive of a blow–up as *e.g.*, in [19], Section 9, shows that the property of having a rational zeta function satisfying a functional equation is closed under blow–ups along smooth centers satisfying a functional equation.

More precisely, suppose that X is an n–dimensional smooth projective variety such that $[h(X)] = [X^+] + [X^-]$, where $[\mathrm{Alt}^i(X^+)] = 0$ for $i > e(X^+)$ and $[\mathrm{Sym}^i(X^-)] = 0$ for $i > f(X^-)$. Let $Q^{X^+}(T) = \sum_{i\geqslant 0}[\mathrm{Alt}^i(X^+)]T^i$ and $P^{X-}(T) = \sum_{i\geqslant 0}[\mathrm{Alt}^i(X^-)]T^i$. Suppose furthermore that

$$Q^{X^+}\Big(\frac{1}{\mathbb{L}^n T}\Big) = T^{-e(X^+)}\mathbb{L}^{-\frac{ne(X^+)}{2}} Q^{X^+}(T) \text{ in } \mathrm{K}_0(CM_k)[T,T^{-1}]$$

and

$$P^{X^-}\left(\frac{1}{\mathbb{L}^n T}\right) = T^{-f(X^-)}\mathbb{L}^{-\frac{nf(X^-)}{2}} P^{X^-}(T) \text{ in } \mathrm{K}_0(CM_k)[T,T^{-1}],$$

and likewise for a smooth closed subvariety Y of X of pure codimension d. Then the same holds for the blow–up $\mathrm{Bl}_Y X$ of X along Y, where $(\mathrm{Bl}_Y X)^+ = X^+ \oplus \bigoplus_{i=1}^{d-1} \mathbb{L}^i \otimes Y^+$, $(\mathrm{Bl}_Y X)^- = X^- \oplus \bigoplus_{i=1}^{d-1} \mathbb{L}^i \otimes Y^-$, $e((\mathrm{Bl}_Y X)^+) = e(X^+) + (d-1)e(Y^+)$ and $f((\mathrm{Bl}_Y X)^-) = f(X^-) + (d-1)f(Y^-)$.

Remark 5.4. — For Kummer surfaces X, an explicit calculation of $[h(X)]$ (we know how multiplication by -1 acts on the Chow motive of an abelian variety) yields $[\mathrm{Alt}^i(h(X))] = 0$ for $i > 24$ and the expected functional equation $Q^{h(X)}(\frac{1}{\mathbb{L}^2 T}) = T^{-24}\mathbb{L}^{-24}Q^{h(X)}(T)$ in $\mathrm{K}_0(CM_k)[T,T^{-1}]$.

6. Products

In this section, we investigate zeta functions of products of varieties whose zeta functions satisfy a functional equation.

For the class of a Chow motive $x \in \mathrm{K}_0(CM_k)$, we define $Q^x(T) := \sum_{i\geqslant 0} \mathrm{Alt}^i(x)T^i$ and $P^x(T) := \sum_{i\geqslant 0} \mathrm{Sym}^i(x)T^i$.

Proposition 6.1. — *The property of having a rational zeta function with values in $\mathrm{K}_0(CM_k)$ satisfying a functional equation is closed under products. More precisely, suppose that X is an n–dimensional smooth projective variety such that $[h(X)] = [X^+] + [X^-]$, where $[\mathrm{Alt}^i(X^+)] = 0$ for $i > e(X^+)$ and $[\mathrm{Sym}^i(X^-)] = 0$ for $i > f(X^-)$. Suppose furthermore that*

$$Q^{X^+}\left(\frac{1}{\mathbb{L}^n T}\right) = T^{-e(X^+)}\mathbb{L}^{-\frac{ne(X^+)}{2}} Q^{X^+}(T) \text{ in } \mathrm{K}_0(CM_k)[T,T^{-1}]$$

and

$$P^{X^-}\left(\frac{1}{\mathbb{L}^n T}\right) = T^{-f(X^-)}\mathbb{L}^{-\frac{nf(X^-)}{2}} P^{X^-}(T) \text{ in } \mathrm{K}_0(CM_k)[T,T^{-1}],$$

and likewise for a smooth projective variety Y. Then the same holds for $X \times Y$, where $(X\times Y)^+ = X^+ \otimes Y^+ \oplus X^- \otimes Y^-$, $(X \times Y)^- = X^+ \otimes Y^- \oplus X^- \otimes Y^+$, $e((X\times Y)^+) = e(X^+)e(Y^+) + f(X^-)f(Y^-)$ and $f((X\times Y)^-) = e(X^+)f(Y^-) + f(X^-)e(Y^+)$.

We start with a special case.

Lemma 6.2. — *Suppose that $x \in \mathrm{K}_0(CM_k)$ is the class of a Chow motive satisfying*

$$\deg Q^x = e \quad \textit{and} \quad Q^x\Big(\frac{1}{\mathbb{L}^m T}\Big) = T^{-e}\mathbb{L}^{-\frac{me}{2}} Q^x(T)$$

and that $y \in \mathrm{K}_0(CM_k)$ is the class of a Chow motive satisfying

$$\deg Q^y = f \quad \textit{and} \quad Q^y\Big(\frac{1}{\mathbb{L}^n T}\Big) = T^{-f}\mathbb{L}^{-\frac{nf}{2}} Q^y(T).$$

Then the class $xy \in \mathrm{K}_0(CM_k)$ satisfies

$$\deg Q^{xy} = ef \quad \textit{and} \quad Q^{xy}\Big(\frac{1}{\mathbb{L}^{m+n} T}\Big) = T^{-ef}\mathbb{L}^{-\frac{(m+n)ef}{2}} Q^{xy}(T).$$

Proof of Lemma. — Denote the elementary symmetric functions in $\xi_1, \dots, \xi_e$ by σ_i and the elementary symmetric functions in $x_1, \dots, x_f$ by s_i. Consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}[\xi_1, \xi_1^{-1}, \dots, \xi_e, \xi_e^{-1}, s_1, \dots, s_f, L, L^{-1}, t, t^{-1}] \arrow[r, "\psi'"] & \mathrm{K}_0(CM_k)[\xi_1, \xi_1^{-1}, \dots, \xi_e, \xi_e^{-1}, T, T^{-1}] \\
& \mathrm{K}_0(CM_k)[\sigma_1, \dots, \sigma_e, T, T^{-1}] \arrow[u, hook] \arrow[d] \\
\mathbb{Z}[\sigma_1, \dots, \sigma_e, s_1, \dots, s_f, L, L^{-1}, t, t^{-1}] \arrow[uu, hook] \arrow[dd, hook] \arrow[ur] \arrow[r, "\varphi"] \arrow[dr] & \mathrm{K}_0(CM_k)[T, T^{-1}] \\
& \mathrm{K}_0(CM_k)[s_1, \dots, s_f, T, T^{-1}] \arrow[u] \arrow[d, hook] \\
\mathbb{Z}[\sigma_1, \dots, \sigma_e, x_1, x_1^{-1}, \dots, x_f, x_f^{-1}, L, L^{-1}, t, t^{-1}] \arrow[r, "\psi''"] & \mathrm{K}_0(CM_k)[x_1, x_1^{-1}, \dots, x_f, x_f^{-1}, T, T^{-1}]
\end{tikzcd}$$

where $\varphi(t) = T$, $\varphi(\sigma_i) = \mathrm{Alt}^i(x)$, $\varphi(s_j) = \mathrm{Alt}^j(y)$ and $\varphi(L) = \mathbb{L}$.

We know that

$$q^x(t) := \prod_{1 \leqslant i \leqslant e} (1 + \xi_i t)$$

$$q^y(t) := \prod_{1 \leqslant j \leqslant f} (1 + x_j t)$$

$$q^{xy}(t) := \prod_{\substack{1 \leqslant i \leqslant e \\ 1 \leqslant j \leqslant f}} (1 + \xi_i x_j t)$$

are mapped by φ to $Q^x(T)$, $Q^y(T)$ and $Q^{xy}(T)$, and similarly, $q^x(\frac{1}{L^m t})$ is mapped to $Q^x(\frac{1}{\mathbb{L}^m T})$, and so on.

Now

$$q^{xy}\Big(\frac{1}{L^{m+n}t}\Big) = \prod_{1\leqslant i\leqslant e} q^y\Big(\frac{\xi_i}{L^{m+n}t}\Big) = \prod_{1\leqslant i\leqslant e} q^y\Big(\frac{1}{L^n t_i}\Big),$$

where $t_i = \frac{L^m t}{\xi_i}$.

We know that

$$\psi'\Big(q^y\Big(\frac{1}{L^n t_i}\Big)\Big) = \psi'\Big(t_i^{-f} L^{-\frac{nf}{2}} q^y(t_i)\Big)$$

and

$$q^y(t_i) = \prod_{1\leqslant j\leqslant f}(1+x_j t_i) = t_i^f \prod_{1\leqslant j\leqslant f} x_j \prod_{1\leqslant j\leqslant f}\Big(1+\frac{1}{x_j t_i}\Big).$$

Therefore,

$$\psi'(q^y\Big(\frac{1}{L^n t_i})\Big) = \psi'\Big(L^{-\frac{nf}{2}} \prod_{1\leqslant j\leqslant f} x_j \prod_{1\leqslant j\leqslant f}\Big(1+\frac{\xi_i}{L^m x_j t}\Big)\Big)$$

and hence

$$\begin{aligned}\varphi\Big(q^{xy}\Big(\frac{1}{L^{m+n}t}\Big)\Big) &= \varphi\Big(L^{-\frac{nef}{2}}\Big(\prod_{1\leqslant j\leqslant f} x_j\Big)^e \prod_{1\leqslant i\leqslant e}\prod_{1\leqslant j\leqslant f}\Big(1+\frac{\xi_i}{L^m x_j t}\Big)\Big)\\ &= \varphi\Big(L^{-\frac{nef}{2}}\Big(\prod_{1\leqslant j\leqslant f} x_j\Big)^e \prod_{1\leqslant j\leqslant f} q^x\Big(\frac{1}{L^m\theta_j}\Big)\Big),\end{aligned}$$

where $\theta_j = x_j t$. As

$$\psi''\Big(q^x\Big(\frac{1}{L^m\theta_j}\Big)\Big) = \psi''(\theta_j^{-e} L^{-\frac{me}{2}} q^x(\theta_j)),$$

we conclude that

$$\begin{aligned}\varphi\Big(q^{xy}\Big(\frac{1}{L^{m+n}t}\Big)\Big) &= \varphi\Big(L^{-\frac{nef}{2}} L^{-\frac{mef}{2}} t^{-ef}\Big(\prod_{1\leqslant j\leqslant f} x_j\Big)^e \prod_{1\leqslant j\leqslant f} x_j^{-e} q^x(\theta_j)\Big)\\ &= \varphi\Big(t^{-ef} L^{-\frac{(m+n)ef}{2}} q^{xy}(t)\Big).\end{aligned}$$

□

Now due to Equations 4.10, 4.11 and 4.12 we have the same behavior for $Q^{xy}(T)$ if $P^x(T)$ and $P^y(T)$ fulfill similar conditions, and likewise for $P^{xy}(T)$ given the conditions for $P^x(T)$ and $Q^y(T)$ or for $Q^x(T)$ and $P^y(T)$. This establishes the Proposition.

BIBLIOGRAPHY

[1] Y. André, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 17, Société Mathématique de France, Paris, 2004, xii+261 pages.

[2] ———, "Motifs de dimension finie d'après S.-I. Kimura, P. O'Sullivan,…", *Astérisque* **299** (2005), p. 115-145, Séminaire Bourbaki, Vol. 2003/2004.

[3] M. F. Atiyah & D. O. Tall, "Group representations, λ-rings and the J-homomorphism", *Topology* **8** (1969), p. 253-297.

[4] S. del Baño, "On motives and moduli spaces of stable bundles over a curve", PhD thesis, Universitat Politècnica de Catalunya, Barcelona, 1998, Available from http://www-mal.upc.es/recerca/1998.html.

[5] S. del Baño & V. N. Aznar, "On the motive of a quotient variety", *Collect. Math.* **49** (1998), no. 2-3, p. 203-226, Dedicated to the memory of Fernando Serrano.

[6] A. Beauville, "Sur l'anneau de Chow d'une variété abélienne", *Math. Ann.* **273** (1986), no. 4, p. 647-651.

[7] P. Deligne, "Catégories tensorielles", *Mosc. Math. J.* **2** (2002), no. 2, p. 227-248, Dedicated to Yuri I. Manin on the occasion of his 65th birthday.

[8] C. Deninger & J. Murre, "Motivic decomposition of abelian schemes and the Fourier transform", *J. Reine Angew. Math.* **422** (1991), p. 201-219.

[9] W. Fulton & J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, xvi+551 pages.

[10] H. Gillet & C. Soulé, "Descent, motives and K-theory", *J. Reine Angew. Math.* **478** (1996), p. 127-176.

[11] L. Göttsche, "On the motive of the Hilbert scheme of points on a surface", *Math. Res. Lett.* **8** (2001), no. 5-6, p. 613-627.

[12] F. Guillén & V. N. Aznar, "Un critère d'extension des foncteurs définis sur les schémas lisses", *Publ. Math. Inst. Hautes Études Sci.* **95** (2002), p. 1-91.

[13] M. Kapranov, "The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups", MSRI Preprint 2000-006, `arXiv:math.AG/0001005`.

[14] S.-I. Kimura, "Chow groups are finite dimensional, in some sense", *Math. Ann.* **331** (2005), no. 1, p. 173-201.

[15] K. Künnemann, "A Lefschetz decomposition for Chow motives of abelian schemes", *Invent. Math.* **113** (1993), no. 1, p. 85-102.

[16] ———, "On the Chow motive of an abelian scheme", in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, p. 189-205.

[17] M. Larsen & V. A. Lunts, "Motivic measures and stable birational geometry", *Mosc. Math. J.* **3** (2002), no. 1, p. 85-95, 259.

[18] ———, "Rationality criteria for motivic zeta functions", *Compos. Math.* **140** (2004), no. 6, p. 1537-1560.

[19] Y. I. Manin, "Correspondences, motifs and monoidal transformations", *Mat. Sb. (N.S.)* **77 (119)** (1968), p. 475-507.

[20] A. J. Scholl, "Classical motives", in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, p. 163-187.

Manuscrit reçu le 2 décembre 2005,
accepté le 23 mars 2006.

Franziska HEINLOTH
Universität Duisburg—Essen
Standort Essen
FB6, Mathematik
45117 Essen (German)
franziska.heinloth@uni-duisburg-essen.de