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MAXIMAL INEQUALITIES AND RIESZ TRANSFORM ESTIMATES ON L^p SPACES FOR SCHRÖDINGER OPERATORS WITH NONNEGATIVE POTENTIALS

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ABSTRACT. — We show various L^p estimates for Schrödinger operators $-\Delta + V$ on $\mathbb{R}^n$ and their square roots. We assume reverse Hölder estimates on the potential, and improve some results of Shen. Our main tools are improved Fefferman-Phong inequalities and reverse Hölder estimates for weak solutions of $-\Delta + V$ and their gradients.

RÉSUMÉ. — On montre des estimations L^p pour des opérateurs de Schrödinger $-\Delta + V$ sur $\mathbb{R}^n$ et leurs racines carrées. Le potentiel est dans une classe Hölder inverse améliorant les résultats de Shen. On s'appuie sur une inégalité de type Fefferman-Phong améliorée et des inégalités Hölder inverse pour des solutions faibles de $-\Delta + V$ et leurs gradients.

1. Introduction and main results

Let $n \geqslant 1$ and V be a locally integrable nonnegative function on $\mathbb{R}^n$, not identically zero. The validity of the *a priori* $L^p(\mathbb{R}^n)$ inequality for $u \in C_0^\infty(\mathbb{R}^n)$

$$(1.1) \qquad \|\Delta u\|_p + \|Vu\|_p \leqslant C\| - \Delta u + Vu\|_p$$

with C independent of u has attracted many authors. Here, $\| \ \|_p$ denotes the norm in $L^p(\mathbb{R}^n)$. First, it allows to find the domain of the maximal realization of $-\Delta + V$ on $L^p(\mathbb{R}^n)$. Second, it provides estimates on the

fundamental solution (to be defined appropriately). It could well yield estimates for semi-linear equations but we have not seen any such applications in the literature and are not aware of any.

In this work, we search conditions, minimal in some sense, for this inequality to hold. Let us mention that to find the domain, it is enough to obtain this inequality with V replaced by $V + \lambda$ in the right hand side for some large constant $\lambda > 0$ with C depending on λ. We are interested in the possibility of having "homogeneous" inequalities ($\lambda = 0$) or, equivalently, on inequalities for $-\Delta + V + \lambda$ for $\lambda > 0$ with constant independent of λ.

The case $p = 1$ is well-known. For $u \in C_0^\infty(\mathbb{R}^n)$, real-valued, one has

$$(1.2) \qquad\qquad \|\Delta u\|_1 + \|Vu\|_1 \leqslant 3\| - \Delta u + Vu\|_1$$

as a consequence of the contractive inequality $\|Vu\|_1 \leqslant \| - \Delta u + Vu\|_1$ following either from work of Kato [25], or from work of Gallouët and Morel in semi-linear equations [17] inspired by the seminal paper of Brezis and Strauss [5] on semi-linear elliptic equations with L^1 data. Nevertheless, we shall give a simple account of this. This allows to define $-\Delta + V$ as an operator on $L^1(\mathbb{R}^n)$ with domain $\mathcal{D}_1(\Delta) \cap \mathcal{D}_1(V)$, a fact that was known before ([39]).

We turn to the L^p theory for $1 < p < \infty$. Assume that $V \in L^p_{loc}(\mathbb{R}^n)$. Then it is known that $-\Delta + V$ *a priori* defined on $C_0^\infty(\mathbb{R}^n)$ is essentially m-accretive in $L^p(\mathbb{R}^n)$ ([24, 25, 30]) and the domain of the m-accretive extension contains $\mathcal{D}_p(\Delta) \cap \mathcal{D}_p(V) = W^{2,p}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n, V^p)$ as a dense subspace. There are conditions to insure equality in [28, 39, 8, 34]. But this is still not enough to assert the validity of (1.1). A remark is that, by standard Calderón-Zygmund theory, one can replace $\|\Delta u\|_p$ by the equivalent quantity $\|\nabla^2 u\|_p$ as $1 < p < \infty$.

A natural question is which condition on V insures (1.1). An answer is the following.

THEOREM 1.1. — *Let* $1 < q \leqslant \infty$. *If* $V \in B_q$ *then for some* $\varepsilon > 0$ *depending only on* V, *(1.1) holds for* $1 < p < q + \varepsilon$.

Here, B_q, $1 < q \leqslant \infty$, is the class of the reverse Hölder weights with exponent q: $w \in B_q$ if $w \in L^q_{loc}(\mathbb{R}^n)$, $w > 0$ almost everywhere and there exists a constant C such that for all cube Q of $\mathbb{R}^n$,

$$(1.3) \qquad\qquad \left(\frac{1}{|Q|} \int_Q w^q(x)\,dx \right)^{1/q} \leqslant \frac{C}{|Q|} \int_Q w(x)\,dx.$$

If $q = \infty$, then the left hand side is the essential supremum on Q. The smallest C is called the B_q constant of w. Examples of B_q weights are

the power weights $|x|^{-\alpha}$ for $-\infty < \alpha < n/q$ and positive polynomials for $q = \infty$. Note that $B_q \subset B_p$ if $p < q$ and $w \in B_q$ implies $w \in B_{q+\varepsilon}$ for some $\varepsilon > 0$ depending on the B_q constant of w. Recall also that $w \in B_q$ implies that $d\nu = w\, dx$ is a doubling measure, namely the existence of a constant C such that $\nu(2Q) \leqslant C\nu(Q)$ for all cubes Q of $\mathbb{R}^n$, where λQ is the cube that is concentric with Q and dilated by a factor $\lambda > 0$. Properties of B_q weights presented here and used in the text are well-known facts among the harmonic analysis communauty. Good references to which the reader is referred to are [18, Chapter 9] or [38, Chapter 9] in order to keep the length reasonnable. There is one known property that we do prove in Section 11 as it is not standard.

Our result extends the one of Shen obtained under the restriction that $n/2 \leqslant q$ and $n \geqslant 3$ [31]. Prior to Shen's work, this was proved for positive polynomials when $p = 2$ in [27] and then when $1 < p < \infty$ in [20, 19, 40].[1] It is somewhat surprising (as it was to us) that we can go below the exponent $n/2$ which is critical for the regularity theory of the elliptic operator $-\Delta + V$.

Note that our result can be reformulated as : For $1 < p < \infty$, if $V \in B_p$ then (1.1) holds. Given the necessity of local L^p integrability, it is best possible within the class of all reverse Hölder weigths, which is the same as A_∞, the class of all Muckenhoupt weights.

A second family of inequalities concerns the square root (see below for a definition). We recall at this point the identity

$$\|\nabla u\|_2^2 + \|V^{1/2}u\|_2^2 = \|(-\Delta + V)^{1/2}u\|_2^2, \quad u \in C_0^\infty(\mathbb{R}^n).$$

The *a priori* inequalities

$$(1.4) \qquad \|\nabla u\|_{1,\infty} + \|V^{1/2}u\|_{1,\infty} \lesssim \|(-\Delta + V)^{1/2}u\|_1$$

and

$$(1.5) \qquad \|\nabla u\|_p + \|V^{1/2}u\|_p \lesssim \|(-\Delta + V)^{1/2}u\|_p$$

when $1 < p < 2$ hold for $u \in C_0^\infty(\mathbb{R}^n)$. Here, $\| \ \|_{p,\infty}$ is the "norm" in the Lorentz space $L^{p,\infty}(\mathbb{R}^n)$. Here, $\lesssim$ is the comparison in the sense of norms. Actually, the first inequality is attributed to Ouhabaz (unpublished) and the second one follows by interpolation. The proof of (1.4) uses the fact that the heat kernel of $-\Delta + V$ is controlled pointwise by the one of $-\Delta$ and a theorem in [11]. See [12] where the needed estimates are proved and [7] where a similar argument is done for Riesz transforms on manifolds. See

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[1] After this work was completed, we learned of a new recent proof using representations via Lie groups in [14], which also covers all positive fractional powers.

also [33] for a different proof using finite speed of propagation for the wave equation.

We are interested in pushing the range of p in (1.5) beyond 2 and also in studying the converse inequalities, that is *a priori* validity for smooth u of

$$\|(-\Delta + V)^{1/2}u\|_{1,\infty} \lesssim \|\nabla u\|_1 + \|V^{1/2}u\|_1 \tag{1.6}$$

and of

$$\|(-\Delta + V)^{1/2}u\|_p \lesssim \|\nabla u\|_p + \|V^{1/2}u\|_p. \tag{1.7}$$

Note that (1.5) for p implies (1.7) for the conjugate exponent p'. Hence, (1.7) already holds in the range $p > 2$ and necessary for (1.5) to holds with exponent p'. The statement summarizing our results is the following.

THEOREM 1.2. —

(1) Let $V \in B_q$ for some $q > 1$. Then (1.5) holds for $1 < p < 2(q + \varepsilon)$.
(2) If $V \in A_\infty = \cup_{q>1} B_q$, then (1.6) and (1.7) for $1 < p < 2$ hold.
(3) Let $V \in B_q$ for some $q > 1$ and $q \geqslant n/2$. Then $\|\nabla u\|_p \lesssim \|(-\Delta + V)^{1/2}u\|_p$ holds for $1 < p < q^* + \varepsilon$ if $q < n$, and for $1 < p < \infty$ if $q \geqslant n$.

Here, $q^* = qn/(n - q)$ is the Sobolev exponent of q if $q < n$. Note that $q^* \geqslant 2q$ exactly when $q \geqslant n/2$, hence item 3 improves over item 1 for the gradient part. We note that Shen proved item 3 when $n \geqslant 3$ and item 1 when $q \geqslant n/2$ and $n \geqslant 3$ [31]. We shall fully prove this theorem, even item 3 with an argument of a different nature that is interesting in its own right. The argument for (3) requires the proof of (2) first.

Note that one can also prove inequalities similar to (1.5) for fractional powers $(-\Delta + V)^s$, $0 < s < 1$, with range $1 < p < (q + \varepsilon)/s$. We shall not pursue this here.

Our results are satisfactory for reverse Hölder potentials as they make a bridge with the known results for L^1_{loc} nonnegative potentials. Let us list some other consequences to illustrate this.

COROLLARY 1.3. — Let $n \geqslant 1$, $1 < p < \infty$ and $V \in B_p$. Then the m-accretive extension on $L^p(\mathbb{R}^n)$ of $-\Delta + V$ defined on $C_0^\infty(\mathbb{R}^n)$ has domain equal to $\mathcal{D}_p(\Delta) \cap \mathcal{D}_p(V)$. In particular, for $p = 2$, $-\Delta + V$ defined on $H^2(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, V^2)$ is self-adjoint in $L^2(\mathbb{R}^n)$.

This applies to power weights $c|x|^{-\alpha}$ although this particular application is known by other methods [28].

COROLLARY 1.4. — *Let $n \geqslant 1$. Assume $V \in A_\infty$ and $1 < p < 2$ or $V \in B_{p/2}$ and $2 < p < \infty$, then $(-\Delta + V)^{1/2}$ has L^p-domain equal to $\mathcal{D}_p((-\Delta)^{1/2}) \cap \mathcal{D}_p(V^{1/2}) = W^{1,p}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n, V^{p/2})$.*

Further easy consequences are the following estimates. Set $\widehat{p} = \sup(2p, p^*)$ for $1 < p < \infty$ with $p^* = \infty$ if $p \geqslant n$ and $H = -\Delta + V$ with appropriate definition.

COROLLARY 1.5. — *Assume that $V \in B_q$ for some $q > 1$. Then for $\varepsilon > 0$ depending only on V,*
  (1) *$V^{1/2}H^{-1}V^{1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for $(2(q+\varepsilon))' < p < 2(q+\varepsilon)$.*
  (2) *$V^{1/2}H^{-1}$ div is bounded on $L^p(\mathbb{R}^n)$ for $(\widehat{q+\varepsilon})' < p < 2(q+\varepsilon)$.*
  (3) *$\nabla H^{-1}V^{1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for $(2(q+\varepsilon))' < p < \widehat{q+\varepsilon}$.*
  (4) *∇H^{-1} div is bounded on $L^p(\mathbb{R}^n)$ for $(\widehat{q+\varepsilon})' < p < \widehat{q+\varepsilon}$.*

Again, this result extends the ones of Shen in [31] obtained with the restriction $q \geqslant n/2$ and $n \geqslant 3$. He also proved bounds for $V^{1/2}\nabla H^{-1}$, which we can recover by our methods under the same hypotheses (and for $n \geqslant 1$ instead of $n \geqslant 3$). We therefore do not include such results.

We mention without proof that our results admit local versions, replacing $V \in B_q$ by $V \in B_{q,loc}$ which is defined by the same conditions on cubes with sides less than 1. Then we get the corresponding results and estimates for $H + 1$ instead of H. The results on operator domains are valid under local assumptions.

Our arguments are based on local estimates and this is fortunate because there is no auxiliary global weight as in [31] (see Section 2). Our main tools are

  1) An improved Fefferman-Phong inequality for A_∞ potentials.
  2) Criteria for proving L^p boundedness of operators in absence of kernels.
  3) Mean value inequalities for nonnegative subharmonic functions against A_∞ weights.
  4) Complex interpolation, together with L^p boundedness of imaginary powers of $-\Delta + V$ for $1 < p < \infty$.
  5) A Calderón-Zygmund decomposition adapted to level sets of the maximal function of $|\nabla f| + |V^{1/2}f|$.
  6) Reverse Hölder inequalities involving ∇u and $V^{1/2}u$ for weak solutions of $-\Delta u + Vu = 0$.

The latter estimates are of independent interest and we give a rather complete picture. This is more than necessary for applications to the inequality (1.5).

2. An improved Fefferman-Phong inequality

Usual Fefferman-Phong inequalities take the form

$$\int_{\mathbb{R}^n} m(x)^2 |u(x)|^2 \, dx \leqslant C \int_{\mathbb{R}^n} |\nabla u(x)|^2 + w(x)|u(x)|^2 \, dx$$

for $u \in C_0^\infty(\mathbb{R}^n)$ where m is a positive weight function depending on the potential w. If $w \in B_q$ and $q \geqslant n/2$, there is such a function m [31]. If $q < n/2$, it is not clear how to define m in function of w. Nevertheless, local inequalities on cubes Q still hold and depend on the scaling defined by the quantity $R^2 \operatorname{av}_Q w$ (The notation $\operatorname{av}_E v$ means $\frac{1}{|E|} \int_E v$).

LEMMA 2.1. — Let $w \in A_\infty$ and $1 \leqslant p < \infty$. Then there are constants $C > 0$ and $\beta \in (0,1)$ depending only on the A_∞ constant of w, p and n such that for all cubes Q (with sidelength R) and $u \in C^1(\mathbb{R}^n)$, one has

$$\int_Q |\nabla u|^p + w|u|^p \geqslant \frac{C m_\beta(R^p \operatorname{av}_Q w)}{R^p} \int_Q |u|^p$$

where $m_\beta(x) = x$ for $x \leqslant 1$ and $m_\beta(x) = x^\beta$ for $x \geqslant 1$.

We recall that A_∞ is the class of all Muckenhoupt weights and is also the class of all reverse Hölder weights ([18, Chapter 9])

This lemma with $\beta = 0$ is already in [31] when $p = 2$. The improvement occurs when $R^p \operatorname{av}_Q w \geqslant 1$ and is crucial for us in Section 8. Such an improvement has also applications to criteria for compactness of resolvents for magnetic Schrödinger operators (personal communication of B. Helffer)[2].

Proof. — We begin as in Fefferman-Phong argument (see [15] and also [31]) we have

$$\int_Q |\nabla u|^p \geqslant \frac{C}{R^{n+p}} \int_Q \int_Q |u(x) - u(y)|^p \, dx dy$$

and

$$\int_Q w|u|^p = \frac{1}{R^n} \int_Q \int_Q w(x)|u(x)|^p \, dx dy.$$

Hence,

$$(2.1) \qquad \int_Q |\nabla u|^p + w|u|^p \geqslant \operatorname{av}_Q \big[\min(CR^{-p}, w)\big] \int_Q |u(y)|^p \, dy.$$

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[2] Added in proof: compactness of the solution operator to $\overline{\partial}$ in weighted L^2-spaces, preprint 2007, F Haslinger and B. Helffer.

Now, we use that $w \in A_\infty$. There exists $\varepsilon > 0$, independent of Q, such that $E = \{x \in Q \,;\, w(x) > \varepsilon \operatorname{av}_Q w\}$ satisfies $|E| \geqslant \frac{1}{2}|Q|$. Hence

$$\operatorname{av}_Q \left[\min(CR^{-p}, w)\right] \geqslant \frac{1}{2} \min(CR^{-p}, \varepsilon \operatorname{av}_Q w).$$

This proves the desired inequality when $R^p \operatorname{av}_Q w \leqslant 1$.

Assume now that $R^p \operatorname{av}_Q w \geqslant 1$. Subdivide Q in a dyadic manner and stop the first time that $R(Q')^p \operatorname{av}_{Q'} w < 1$. One obtains a collection $\{Q_i\}$ of strict dyadic subcubes of Q which are maximal for the property $R_i^p \operatorname{av}_{Q_i} w < 1$. Furthermore, since $w(x)\,dx$ is a doubling measure and as the ancestor $\widehat{Q_i}$ of Q_i satisfies $(2R_i)^p \operatorname{av}_{\widehat{Q_i}} w \geqslant 1$, there exists $A > 0$ such that $R_i^p \operatorname{av}_{Q_i} w \geqslant A$. The last observation is that the Q_i form a disjoint covering of Q up to a set of null measure. Indeed, for almost all $x \in Q$, $\operatorname{av}_{Q'} w$ converges to $w(x)$, and therefore $R(Q')^2 \operatorname{av}_{Q'} w$ to 0, whenever Q' describes the sequence of dyadic subcubes of Q that contain x and shrink to 0. Hence,

$$\int_Q |\nabla u|^p + w|u|^p = \sum_i \int_{Q_i} |\nabla u|^p + w|u|^p$$

$$\geqslant C' \sum_i \min(R_i^{-p}, \operatorname{av}_{Q_i} w) \int_{Q_i} |u|^p$$

$$\geqslant AC' \sum_i R_i^{-p} \int_{Q_i} |u|^p$$

$$\geqslant AC' \min_i \left(\frac{R}{R_i}\right)^p R^{-p} \int_Q |u|^p.$$

It remains to estimate $\min_i \left(\frac{R}{R_i}\right)^p$ from below. As $w \in A_\infty$, there exists $1 \leqslant \alpha < \infty$ such that $w \in A_\alpha$. This implies that for any cube Q and measurable subset E of Q, we have

$$\left(\frac{\operatorname{av}_E w}{\operatorname{av}_Q w}\right) \geqslant C \left(\frac{|E|}{|Q|}\right)^{\alpha-1}.$$

Applying this to $E = Q_i$ and Q, we obtain,

$$\left(\frac{R}{R_i}\right)^p = \left(\frac{R^p \operatorname{av}_Q w}{R_i^p \operatorname{av}_{Q_i} w}\right)\left(\frac{\operatorname{av}_{Q_i} w}{\operatorname{av}_Q w}\right)$$

$$\geqslant R^p \operatorname{av}_Q w \left(\frac{\operatorname{av}_{Q_i} w}{\operatorname{av}_Q w}\right)$$

$$\geqslant CR^p \operatorname{av}_Q w \left(\frac{R_i}{R}\right)^{n(\alpha-1)}.$$

This yields $\min_i \left(\frac{R}{R_i}\right)^p \geqslant C(R^p \operatorname{av}_Q w)^\beta$ with $\beta = \frac{p}{p+n(\alpha-1)}$ and the lemma is proved. $\square$

Remark 2.2. — If $w \in B_q$ for $q > n/p$ (as in [31] with $p = 2$), then R/R_i is also bounded by $C(R^p \operatorname{av}_Q w)^{p-n/q}$, that is R/R_i is logarithmically comparable to $R^p \operatorname{av}_Q w$. No such thing is true if $q < n/p$. For example, if $w(x) = |x|^{-\alpha}$ with $p < \alpha < n$ (hence $w \in B_q$ for $q < n/\alpha < n/p$) then it is easy to show that $\max R/R_i$ can be unbounded. Furthermore, for all x then $R^p \operatorname{av}_{Q(x,R)} w$ tends to 0 as $R \to +\infty$, which is not the case when $0 < \alpha < p$. The case $\alpha = p$ is different in the sense that $R^p \operatorname{av}_{Q(x,R)} w$ tends to a non zero constant as $R \to +\infty$.

3. Definitions of the Schrödinger operator

Recall that V is a nonnegative locally integrable function on $\mathbb{R}^n$. The definition of the Schrödinger operator associated to $-\Delta + V$ is via the quadratic form method (See [9, Sections 1.3 & 1.8]. Let

$$\mathcal{V} = \{f \in L^2(\mathbb{R}^n)\,;\, \nabla f \ \& \ V^{1/2}f \in L^2(\mathbb{R}^n)\}.$$

Equipped with the norm

$$\|f\|_{\mathcal{V}} = \left(\|f\|_2^2 + \|\nabla f\|_2^2 + \|V^{1/2}f\|_2^2\right)^{1/2}$$

it is a Hilbert space and $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{V}$. The sesquilinear form

$$Q(u,v) = \int_{\mathbb{R}^n} \nabla u \cdot \overline{\nabla v} + Vu\,\overline{v}$$

on $\mathcal{V} \times \mathcal{V}$ is bounded below and non-negative and, therefore, there exists a unique positive self-adjoint operator H such that

$$\langle Hu, v\rangle = Q(u,v) \quad \forall\, u \in \mathcal{D}(H)\ \forall\, v \in \mathcal{V}.$$

The Beurling-Deny theory implies that $\varepsilon(H + \varepsilon)^{-1}$ is a positivity-preserving contraction on $L^p(\mathbb{R}^n)$ for all $1 \leqslant p \leqslant \infty$ and $\varepsilon > 0$. Moreover, if $V' \in L^1_{loc}(\mathbb{R}^n)$ is another potential with $0 \leqslant V' \leqslant V$ and H' is the corresponding operator then one has for any $\varepsilon > 0$ and for any $f \in L^p$, $1 \leqslant p \leqslant \infty$, $f \geqslant 0$

$$0 \leqslant (H + \varepsilon)^{-1}f \leqslant (H' + \varepsilon)^{-1}f.$$

This fact can be seen from [29, Theorem 2.4.7] on L^2 for the semigroups generated by $-H$ and $-H'$ by applying the Laplace transform or from the Trotter-product formula [9, p. 49]. Passing from L^2 to L^p is standard. Taking $V' = 0$ yields the pointwise domination of the kernel of resolvent

of H by the kernel of the resolvent of the negative Laplacian $(-\Delta + \varepsilon)^{-1}$ which is a convolution operator with $\varepsilon^{n-1}G(\varepsilon x)$ for some $G \in L^1(\mathbb{R}^n)$. Note that if V is bounded below by some positive constant $\varepsilon > 0$, then H^{-1} is bounded on $L^p(\mathbb{R}^n)$ for all $1 \leqslant p \leqslant \infty$ and is dominated by $(-\Delta + \varepsilon)^{-1}$.

Since $\mathcal{D}(H)$ is dense in $\mathcal{V}$, there is a natural extension $\widetilde{H}$ of H as a bounded operator from $\mathcal{V}$ to $\mathcal{V}'$ (not identified with $\mathcal{V}$). Further, for any $\varepsilon > 0$, $\widetilde{H} + \varepsilon$ is invertible but this ceases at $\varepsilon = 0$ so it is useful to introduce an "homogeneous" version of H as follows: Let $\dot{\mathcal{V}}$ be the closure of $C_0^\infty(\mathbb{R}^n)$ under the semi-norm

$$\|f\|_{\dot{\mathcal{V}}} = \left(\|\nabla f\|_2^2 + \|V^{1/2}f\|_2^2\right)^{1/2}.$$

By (2.1), there is a continuous inclusion $\dot{\mathcal{V}} \subset L^2_{loc}(\mathbb{R}^n)$ if V is not identically 0, which is assumed from now on, hence, this is a norm. The form Q is the inner product on $\dot{\mathcal{V}}$ associated to this norm so that $\dot{\mathcal{V}}$ is a Hilbert space. But if we choose not to identify $\dot{\mathcal{V}}$ and its dual, then there is a unique bounded and invertible operator $\dot{H} : \dot{\mathcal{V}} \to \dot{\mathcal{V}}'$ such that for all $u, v \in \dot{\mathcal{V}}$, $\langle \dot{H}u, v \rangle = Q(u, v)$. Here, $\langle \, , \, \rangle$ is the duality (sesquilinear) form between $\dot{\mathcal{V}}'$ and $\dot{\mathcal{V}}$. Note that since $C_0^\infty(\mathbb{R}^n)$ is densely contained in $\dot{\mathcal{V}}$, this coincides with the usual duality between distributions and test functions when $v \in C_0^\infty(\mathbb{R}^n)$. By abuse, we do not distinguish the two notations, which we write as an integral when the integrand is integrable.

In concrete terms, if $f \in \dot{\mathcal{V}}'$ there exists a unique $u \in \dot{\mathcal{V}}$ such that

$$(3.1) \qquad \int_{\mathbb{R}^n} \nabla u \cdot \nabla \overline{v} + V u \overline{v} = \langle f, v \rangle \quad \forall v \in C_0^\infty(\mathbb{R}^n).$$

In particular, $-\Delta u + Vu = f$ holds in the sense of distributions. There is a classical approximation procedure to obtain u for nice f.

LEMMA 3.1. — Assume that $f \in \dot{\mathcal{V}}' \cap L^2(\mathbb{R}^n)$. For $\varepsilon > 0$, let $u_\varepsilon = (H + \varepsilon)^{-1}f \in \mathcal{D}(H)$. Then (u_ε) is a bounded sequence in $\dot{\mathcal{V}}$ which converges strongly to $\dot{H}^{-1}f$.

Proof. — By definition,

$$\int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla \overline{v} + (V + \varepsilon)u_\varepsilon \, \overline{v} = \int_{\mathbb{R}^n} f \, \overline{v} \quad \forall v \in \mathcal{V}$$

and in particular

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 + (V + \varepsilon)|u_\varepsilon|^2 = \int_{\mathbb{R}^n} f \, \overline{u_\varepsilon}.$$

The boundedness of (u_ε) in $\dot{\mathcal{V}}$ follows readily using that $|\int_{\mathbb{R}^n} f \, \overline{u_\varepsilon}| \leqslant \|f\|_{\dot{\mathcal{V}}'}\|u_\varepsilon\|_{\dot{\mathcal{V}}}$ and $f \in \dot{\mathcal{V}}'$.

Let us see first the weak convergence. Let $u \in \dot{\mathcal{V}}$ be a weak limit of a subsequence (u_{ε_j}). One can take limits in the first equation when $v \in C_0^\infty(\mathbb{R}^n)$ and we see that u satisfies (3.1). By uniqueness, $u = \dot{H}^{-1}f$ and (u_ε) converges weakly to u. Since $f \in \dot{\mathcal{V}}'$, we have

$$\int_{\mathbb{R}^n} |\nabla u|^2 + V|u|^2 = \langle f, u \rangle.$$

Weak convergence implies

$$\int_{\mathbb{R}^n} |\nabla u|^2 + V|u|^2 \leqslant \liminf \int_{\mathbb{R}^n} |\nabla u_\varepsilon| + V|u_\varepsilon|^2$$

$$\leqslant \limsup \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 + V|u_\varepsilon|^2$$

$$\leqslant \limsup \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 + (V + \varepsilon)|u_\varepsilon|^2$$

$$= \limsup \int_{\mathbb{R}^n} f\,\overline{u_\varepsilon} = \langle f, u \rangle.$$

Thus $\|u_\varepsilon\|_{\dot{\mathcal{V}}} \to \|u\|_{\dot{\mathcal{V}}}$ and together with weak convergence, this yields strong convergence. $\square$

Remark 3.2. — The continuity of the inclusion $\dot{\mathcal{V}} \subset L^2_{loc}(\mathbb{R}^n)$ has two further consequences: first, we have that $L^2_{comp}(\mathbb{R}^n)$, the space of compactly supported L^2 functions on $\mathbb{R}^n$, is continuously contained in $\dot{\mathcal{V}}' \cap L^2(\mathbb{R}^n)$. Second, (u_ε) has a subsequence converging to u almost everywhere.

We continue with square roots. As H is self-adjoint, it has a unique square root $H^{1/2}$, which is self-adjoint with domain $\mathcal{V}$ and for all $u \in C_0^\infty(\mathbb{R}^n)$, $\|H^{1/2}u\|_2^2 = \|\nabla u\|_2^2 + \|V^{1/2}u\|_2^2$. This allows us to extend $H^{1/2}$ from $\dot{\mathcal{V}}$ into L^2. If S denotes this extension, then we have $S^*S = \dot{H}$ where $S^* \colon L^2 \to \dot{\mathcal{V}}'$ is the adjoint of S.

Our results are all about $\dot{H}$ or alternately about $H + \varepsilon$ with a uniform control of constants with respect to $\varepsilon > 0$. By abuse, we write H for $\dot{H}$ and $H^{1/2}$ for its extension S or its adjoint S^*. The context will make clear which object is the right one.

4. An L^1 maximal inequality

The following result is essentially a consequence of a result of Gallouët and Morel [17] in the semi-linear setting or can be seen from [25]. We present a simple and complete proof in this situation. We assume that V is not identically 0

LEMMA 4.1. — *Let $f \in L^\infty_{comp}(\mathbb{R}^n)$, $f \geqslant 0$ and $u = H^{-1}f$. Then $u \geqslant 0$, Vu and Δu are in $L^1(\mathbb{R}^n)$ with*

$$\int_{\mathbb{R}^n} Vu \leqslant \int_{\mathbb{R}^n} f,$$

$$\int_{\mathbb{R}^n} |\Delta u| \leqslant 2 \int_{\mathbb{R}^n} f.$$

Furthermore, $u \in W^{1,1}_{loc}(\mathbb{R}^n)$ and for any measurable set E with bounded measure,

$$\int_E |\nabla u| \leqslant C(n)|E|^{1/n} \int_{\mathbb{R}^n} f,$$

and for all compact set K in $\mathbb{R}^n$,

$$\int_K |u| \leqslant C(K,n,V) \int_{\mathbb{R}^n} f.$$

Remark 4.2. — In fact, more is true. If $n = 1$, the estimate on u'' tells us that u' is bounded. If $n \geqslant 2$, then $u \in W^{1,q}_{loc}(\mathbb{R}^n)$ for $1 \leqslant q < \frac{n}{n-1}$.

Proof. — For $N \geqslant \varepsilon > 0$, set $V_{\varepsilon,N} = \inf(V + \varepsilon, N)$. Let $f \in L^\infty_{comp}(\mathbb{R}^n)$, $f \geqslant 0$ and set $u = H^{-1}f$, $u_\varepsilon = (H+\varepsilon)^{-1}f$ and $u_{\varepsilon,N} = H^{-1}_{\varepsilon,N}f$ where $H_{\varepsilon,N}$ is associated to the potential $V_{\varepsilon,N}$. By Lemma 3.1 we know that $u \in L^1_{loc}(\mathbb{R}^n)$ (with norm controlled by $\|f\|_\infty$ which is not enough). From the preceding section $u, u_\varepsilon, u_{\varepsilon,N} \geqslant 0$. Further, $V_{\varepsilon,N} \leqslant V + \varepsilon$ implies $u_\varepsilon \leqslant u_{\varepsilon,N}$ and $(V+\varepsilon)_{\varepsilon>0}$ being non-decreasing implies that $(u_\varepsilon)_{\varepsilon>0}$ is non-increasing with $u_\varepsilon \leqslant u$. In addition, it follows from the remark after Lemma 3.1 that u_ε converges almost everywhere to u as $\varepsilon \to 0$. Indeed, a subsequence already converges almost everywhere to u, hence the family itself by monotonicity. As a consequence, (u_ε) converges to u in $L^1_{loc}(\mathbb{R}^n)$ by the monotone convergence theorem. $\qquad\square$

Let us see the estimate for Vu. Since $\varepsilon \leqslant V_{\varepsilon,N}$, the operator $H^{-1}_{\varepsilon,N}$ is a bounded operator on $L^1(\mathbb{R}^n)$ as it is dominated by $(-\Delta + \varepsilon)^{-1}$ (see the preceding section). As $V_{\varepsilon,N} \leqslant N$ is bounded, $V_{\varepsilon,N}H^{-1}_{\varepsilon,N}$ is a bounded operator on $L^1(\mathbb{R}^n)$ and also $\Delta H^{-1}_{\varepsilon,N}$ by difference. As $u_{\varepsilon,N} \in L^1(\mathbb{R}^n)$ and $\Delta u_{\varepsilon,N} \in L^1(\mathbb{R}^n)$, an easy argument via the Fourier transform implies that $\int_{\mathbb{R}^n} \Delta u_{\varepsilon,N} = 0$, and so

$$\int_{\mathbb{R}^n} V_{\varepsilon,N} u_{\varepsilon,N} = \int_{\mathbb{R}^n} f.$$

Next, we have $0 \leqslant V_{\varepsilon,N}u_\varepsilon \leqslant V_{\varepsilon,N}u_{\varepsilon,N}$, so $V_{\varepsilon,N}u_\varepsilon$ is integrable and by monotone convergence as $N \to \infty$, $(V + \varepsilon)u_\varepsilon$ is integrable and

$$\int_{\mathbb{R}^n} (V + \varepsilon)u_\varepsilon \leqslant \int_{\mathbb{R}^n} f.$$

Finally, we have

$$\int_{\mathbb{R}^n} V u_\varepsilon \leqslant \int_{\mathbb{R}^n} (V + \varepsilon) u_\varepsilon \leqslant \int_{\mathbb{R}^n} f$$

and monotone convergence as $\varepsilon \to 0$ yields

$$\int_{\mathbb{R}^n} V u \leqslant \int_{\mathbb{R}^n} f.$$

We turn to the term with Δu. As (u_ε) converges to u in $L^1_{loc}(\mathbb{R}^n)$, Δu is the limit of Δu_ε in $\mathcal{D}'(\mathbb{R}^n)$. If $h \in C_0^\infty(\mathbb{R}^n)$, then

$$-\int_{\mathbb{R}^n} \Delta u_\varepsilon\, h = \int_{\mathbb{R}^n} \nabla u_\varepsilon \cdot \nabla h = \int_{\mathbb{R}^n} fh - \int_{\mathbb{R}^n} V u_\varepsilon h - \int_{\mathbb{R}^n} \varepsilon u_\varepsilon h.$$

As h has compact support, the last integral converges to 0, hence $-\Delta u$ is equal to $f - Vu \in \mathcal{D}'(\mathbb{R}^n)$ and its L^1 control follows.

We turn to the gradient estimate. As $u_\varepsilon \in L^1(\mathbb{R}^n)$ and $\Delta u_\varepsilon \in L^1(\mathbb{R}^n)$, it can be shown (see [4, Appendix]) that if E is a measurable subset of $\mathbb{R}^n$ with bounded measure and $1 \leqslant q < \frac{n}{n-1}$,

$$\int_E |\nabla u_\varepsilon|^q \leqslant C(n,q)|E|^{1-\frac{(n-1)q}{n}} \|\Delta u_\varepsilon\|_1^q$$

hence

(4.1) $$\int_E |\nabla u_\varepsilon|^q \leqslant C(n,q)|E|^{1-\frac{(n-1)q}{n}} \|f\|_1^q.$$

Next, recall that if Q is a cube, by (2.1) we have,

$$\mathrm{av}_Q\left[\min(CR^{-1}, V)\right] \int_Q u_\varepsilon \leqslant \int_Q |\nabla u_\varepsilon| + V u_\varepsilon$$

hence

(4.2) $$\int_Q u_\varepsilon \leqslant C(Q, n, V)\|f\|_1.$$

It follows easily from these two estimates and Poincaré inequality that $u_\varepsilon \in W^{1,q}_{loc}(\mathbb{R}^n)$ for $1 \leqslant q < \frac{n}{n-1}$ and is bounded in that space. Thus, for any $1 < q < \frac{n}{n-1}$, $u \in W^{1,q}_{loc}(\mathbb{R}^n)$ by taking weak limits. The estimate (4.1) passes to the liminf and by Hölder becomes true for $q = 1$. The estimate (4.2) also passes to the limit by convergence in $L^1_{loc}(\mathbb{R}^n)$. This finishes the proof.

Let $B = \{u \in L^1_{loc}(\mathbb{R}^n)\,;\, \Delta u \in L^1(\mathbb{R}^n), Vu \in L^1(\mathbb{R}^n)\}$ equipped with the topology defined by the semi-norms for $L^1_{loc}(\mathbb{R}^n)$, $\|\Delta u\|_1$ and $\|Vu\|_1$. We have obtained

THEOREM 4.3. — *The operator H^{-1} a priori defined on $L_{comp}^{\infty}(\mathbb{R}^n)$ extends to a bounded operator from $L^1(\mathbb{R}^n)$ into B. Denoting again H^{-1} this extension, VH^{-1} is a positivity-preserving contraction on $L^1(\mathbb{R}^n)$ and $\frac{1}{2}\Delta H^{-1}$ a contraction on $L^1(\mathbb{R}^n)$.*

PROPOSITION 4.4. — *Let $f \in L^1(\mathbb{R}^n)$. There is uniqueness of solutions for the equation $-\Delta u + Vu = f$ in the class $L^1(\mathbb{R}^n) \cap B$. In particular, if $u \in C_0^{\infty}(\mathbb{R}^n)$ and $f = -\Delta u + Vu$, then $u = H^{-1}f$.*

Proof. — Since $-\Delta u + Vu = 0$, then for $\varepsilon > 0$ we have $-\Delta u + Vu + \varepsilon u = \varepsilon u$. As $u \in L^1(\mathbb{R}^n)$, we can write $|u| \leqslant (-\Delta + \varepsilon)^{-1}(\varepsilon|u|) = (-\varepsilon^{-1}\Delta + 1)^{-1}|u|$. Using the explicit expression of the kernel of $(-\varepsilon^{-1}\Delta + 1)^{-1}$ and taking limits as $\varepsilon \to 0$ proves that $u = 0$. □

COROLLARY 4.5. — *Equation (1.2) holds.*

Proof. — If $u \in C_0^{\infty}(\mathbb{R}^n)$ and $f = -\Delta u + Vu$, then $Vu = VH^{-1}f$ and $\Delta u = \Delta H^{-1}f$ by the proposition above. Applying Theorem 4.3 proves that $\|Vu\|_1 \leqslant \| -\Delta u + Vu\|_1$ and $\|\Delta u\|_1 \leqslant 2\| -\Delta u + Vu\|_1$. □

Remark 4.6. — We have obtained existence in B and uniqueness in $B \cap L^1(\mathbb{R}^n)$, which is enough for our needs. Uniqueness can even be shown in a larger space if $n \geqslant 3$ for any V and under some conditions on V if $n \leqslant 2$. See [17].

5. L^p maximal inequalities

The main sledge hammer is the following criterion for L^p boundedness ([3]). A slightly weaker version appears in Shen [32].

THEOREM 5.1. — *Let $1 \leqslant p_0 < q_0 \leqslant \infty$. Suppose that T is a bounded sublinear operator on $L^{p_0}(\mathbb{R}^n)$. Assume that there exist constants $\alpha_2 > \alpha_1 > 1$, $C > 0$ such that*

$$(5.1) \qquad \left(\mathrm{av}_Q \, |Tf|^{q_0} \right)^{\frac{1}{q_0}} \leqslant C \left\{ \left(\mathrm{av}_{\alpha_1 Q} \, |Tf|^{p_0} \right)^{\frac{1}{p_0}} + (S|f|)(x) \right\},$$

for all cube Q, $x \in Q$ and all $f \in L_{comp}^{\infty}(\mathbb{R}^n)$ with support in $\mathbb{R}^n \setminus \alpha_2 Q$, where S is a positive operator. Let $p_0 < p < q_0$. If S is bounded on $L^p(\mathbb{R}^n)$, then, there is a constant C such that

$$\|Tf\|_p \leqslant C \|f\|_p$$

for all $f \in L_{comp}^{\infty}(\mathbb{R}^n)$.

Note that in this statement, f can be valued in a Banach space and $|f|$ denotes its norm. Also the space $L^\infty_{comp}(\mathbb{R}^n)$ can be replaced by $C_0^\infty(\mathbb{R}^n)$.

Fix an open set Ω. By a weak solution of $-\Delta u + Vu = 0$ in Ω, we mean $u \in L^1_{loc}(\Omega)$ with $V^{1/2}u, \nabla u \in L^2_{loc}(\Omega)$ and the equation holds in the distribution sense on Ω. Remark that by Poincaré's inequality if u is a weak solution, then $u \in L^2_{loc}(\Omega)$. A subharmonic function on Ω is a function $v \in L^1_{loc}(\Omega)$ such that $\Delta v \geqslant 0$ in $\mathcal{D}'(\Omega)$. It should be observed that if u is a weak solution in Ω of $-\Delta u + Vu = 0$ then

$$(5.2) \qquad \Delta|u|^2 = 2V|u|^2 + 2|\nabla u|^2$$

and in particular, $|u|^2$ is a nonnegative subharmonic function in Ω.

The main technical lemma is interesting on its own right. It states that a form of the mean value inequality for subharmonic functions still holds if the Lebesgue measure is replaced by a weighted measure of Muckenhoupt type. More precisely,

LEMMA 5.2. — *Assume $w \in A_\infty$ and f is a nonnegative subharmonic function in Ω, Q is a cube in $\mathbb{R}^n$ with $\overline{2Q} \subset \Omega$, $1 < \mu \leqslant 2$ and $0 < s < \infty$. Then for some C depending on the A_∞ constant of w, s, μ (and independent of f and Q) and almost all $x \in Q$, we have*

$$f(x) \leqslant \left(\frac{C}{w(\mu Q)} \int_{\mu Q} wf^s \right)^{1/s}.$$

Here $w(E) = \int_E w$. As A_∞ weights have the doubling property we have $\mathrm{av}_{\mu Q}\, w \sim \mathrm{av}_Q\, w$ and the inequality above rewrites (the notation sup meaning essential supremum)

$$(5.3) \qquad \left(\mathrm{av}_Q\, w \right)\left(\sup_Q f^s \right) \leqslant C \,\mathrm{av}_{\mu Q}(wf^s).$$

Proof. — This is a consequence of a result of S. Buckley [6]. We give the proof for the convenience of the reader. Since $w \in A_\infty$, there is $t < \infty$ such that $w \in A_t$. Hence for any nonnegative measurable function g, we have

$$\mathrm{av}_{\mu Q}\, g \leqslant C \Big(\frac{1}{w(\mu Q)} \int_{\mu Q} wg^t \Big)^{1/t} = C \big(\mathrm{av}_{\mu Q}(wg^t) \big)^{1/t} \big(\mathrm{av}_{\mu Q}\, w \big)^{-1/t}.$$

But subharmonicity of f in Ω implies for almost all $x \in Q$ and all $0 < r < \infty$

$$(5.4) \qquad f(x) \leqslant C_{r,\mu} \left(\mathrm{av}_{\mu Q}\, f^r \right)^{1/r}$$

(See [16]. It can also be obtained from classical facts on weak reverse Hölder weights [22, Theorem 2]). Applying this with $r = s/t$ yields

$$f(x) \leqslant C \big(\mathrm{av}_{\mu Q}\, f^{s/t} \big)^{t/s} \leqslant C \Big(\mathrm{av}_{\mu Q}(wf^s) \Big)^{1/s} \big(\mathrm{av}_{\mu Q}\, w \big)^{-1/s}.$$

$\square$

COROLLARY 5.3. — *Let $w \in B_r$ for some $1 < r \leqslant \infty$ and let $0 < s < \infty$. Then there is $C \geqslant 0$ depending only on the B_r constant of w, s, μ such that for any cube Q and any nonnegative subharmonic function f in a neighborhood of $\overline{2Q}$ we have for all $1 < \mu \leqslant 2$*

$$\big(\mathrm{av}_Q(wf^s)^r \big)^{1/r} \leqslant C\, \mathrm{av}_{\mu Q}(wf^s).$$

Proof. — We have

$$\big(\mathrm{av}_Q(wf^s)^r \big)^{1/r} \leqslant \big(\mathrm{av}_Q\, w^r \big)^{1/r} \sup_Q f^s$$

$$\leqslant C \big(\mathrm{av}_Q\, w \big) \sup_Q f^s$$

$$\leqslant C\, \mathrm{av}_{\mu Q}(wf^s).$$

The second inequality uses the B_r condition on w and the last inequality is (5.3). $\square$

Let us come back to the Schrödinger operator.

THEOREM 5.4. — *Let $V \in B_q$ for some $1 < q \leqslant \infty$. Then there is $r > q$ (or $r = \infty$ if $q = \infty$) such that VH^{-1} and ΔH^{-1}, defined on $L^1(\mathbb{R}^n)$ from Theorem 4.3, extend to bounded operators on $L^p(\mathbb{R}^n)$ for $1 < p < r$.*

Proof. — By difference, it suffices to prove the theorem for VH^{-1}. We know that this is a bounded operator on $L^1(\mathbb{R}^n)$. Let $r > q$ be given by self-improvement of the reverse Hölder inequalities of V. Fix a cube Q and let $f \in L^\infty(\mathbb{R}^n)$ with compact support contained in $\mathbb{R}^n \setminus 4Q$. Then $u = H^{-1}f$ is well-defined in $\dot{\mathcal{V}}$ and is a weak solution of $-\Delta u + Vu = 0$ in $4Q$. Since $|u|^2$ is subharmonic, the above corollary applies with $w = V$, $f = |u|^2$ and $s = 1/2$. It yields (5.1) with $T = VH^{-1}$, $p_0 = 1$, $q_0 = r$, $S = 0$, $\alpha_1 = 2$ and $\alpha_2 = 4$. Hence, T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < r$ by Theorem 5.1. $\square$

PROOF OF THEOREM 1.1. — *Let $u \in C_0^\infty(\mathbb{R}^n)$ and $f = -\Delta u + Vu$. We know that $u = H^{-1}f$ by Proposition 4.4. Now, using the hypothesis $V \in B_q$, we have bounded extensions on $L^p(\mathbb{R}^n)$ of VH^{-1} and ΔH^{-1} for $1 < p < q + \varepsilon$ for some $\varepsilon > 0$ depending on V. We conclude that $\|Vu\|_p + \|\Delta u\|_p \lesssim \|f\|_p$.*

6. Complex interpolation

We shall use complex interpolation to obtain item 1 of Theorem 1.2, relying on the following result due to Hebisch [21].

PROPOSITION 6.1. — *Let V be a nonnegative locally integrable function on $\mathbb{R}^n$. Then for all $y \in \mathbb{R}$, H^{iy} has a bounded extension on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and for fixed p its operator norm does not exceed $C(\delta, p)e^{\delta|y|}$ for all $\delta > 0$.*

Here, H^{iy} is defined as a bounded operator on $L^2(\mathbb{R}^n)$ by functional calculus. For $V = 0$, this is standard result for the singular integral operator $(-\Delta)^{iy}$. Actually, the operator norm can be improved but we do not need sharp estimates.

LEMMA 6.2. — *The space $\mathcal{D} = \mathcal{R}(H) \cap L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

Proof. — It suffices to show that $\mathcal{R}(H) \cap L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for the $L^p(\mathbb{R}^n)$ norm. Let $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Since $f \in L^2(\mathbb{R}^n)$, for $\varepsilon > 0$, $f_\varepsilon = H(H + \varepsilon)^{-1}f \in \mathcal{R}(H)$. Also $f_\varepsilon = f - \varepsilon(H + \varepsilon)^{-1}f$. Thus $f_\varepsilon \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ as $|(H + \varepsilon)^{-1}f| \leqslant (-\Delta + \varepsilon)^{-1}|f|$ and the kernel of $(-\Delta + \varepsilon)^{-1}$ is integrable. It remains to see that f_ε converges to f in $L^p(\mathbb{R}^n)$. But again $|f - f_\varepsilon| \leqslant \varepsilon(-\Delta + \varepsilon)^{-1}|f|$ and the latter expression is easily seen to converge to 0 in L^p as ε tends to 0.

We now prove the boundedness of $\nabla H^{-1/2}$ and $V^{1/2}H^{-1/2}$ on $L^p(\mathbb{R}^n)$ for $1 < p < 2(q + \varepsilon)$, which is half of item 3 of Theorem 1.2. Let $f \in \mathcal{D}$, $g \in C_0^\infty(\mathbb{R}^n)$. We define for $z \in S = \{x + iy \in \mathbb{C} \, ; \, 0 \leqslant x \leqslant 1, y \in \mathbb{R}\}$,

$$A(z) = \langle (-\Delta)^z H^{-z}f, g \rangle$$

We shall use the Stein interpolation theorem for families of operators (see [36, Chapter 5]).

Observe that for all $z \in S$, $(-\Delta)^{\bar{z}}g \in L^2$ with $\|(-\Delta)^{\bar{z}}g\|_2 \leqslant C\|g\|_{H^2}$ (the Sobolev space of order 2). Since $f \in \mathcal{R}(H)$, $f = H\tilde{f}$ with $M = \|\tilde{f}\|_2 + \|H\tilde{f}\|_2 < \infty$. Hence,

$$\|H^{-z}f\|_2 = \|H^{1-z}\tilde{f}\|_2 \leqslant \|H^{-iy}\|_{2,2}\|H^{1-x}\tilde{f}\|_2 \leqslant C(\delta)e^{\delta|y|}M.$$

Thus $|A(z)| \leqslant C_\delta e^{\delta|y|}M\|g\|_{H^2}$. It follows that A satisfies the admissible growth condition. It is not difficult to establish continuity on S and analyticity on $\operatorname{Int} S$ of A. Then, for $z = iy$ and $1 < p < \infty$, we have

$$|A(iy)| \leqslant \|H^{-iy}f\|_p \|(-\Delta)^{-iy}g\|_{p'} \leqslant C(\delta, p)e^{\delta|y|}\|f\|_p\|g\|_{p'}.$$

And for $z = 1 + iy$ and $1 < p < q + \varepsilon$,

$$|A(1 + iy)| \leqslant \|\Delta H^{-1} H^{-iy} f\|_p \|(-\Delta)^{-iy} g\|_{p'}$$
$$\leqslant \|\Delta H^{-1}\|_{p,p} C(\delta, p) e^{\delta|y|} \|f\|_p \|g\|_{p'}.$$

Thus, for $z = 1/2$ and $1 < p < 2(q + \varepsilon)$, we obtain

$$|A(1/2)| \leqslant C(p) \|f\|_p \|g\|_{p'}.$$

We conclude by a density argument that $(-\Delta)^{1/2} H^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < 2(q + \varepsilon)$.

Similarly, for $f \in \mathcal{D}$, $g \in C_0^\infty(\mathbb{R}^n)$, we define for $z \in S = \{x + iy \in \mathbb{C} \,;\, 0 \leqslant x \leqslant 1\}$ and fixed $N > 0$,

$$B(z) = \langle V_N^z H^{-z} f, g \rangle,$$

with $V_N = \inf(V, N)$. Then,

$$|B(z)| \leqslant C(\delta, q') e^{\delta|y|} M(\|g\|_2 + \|V_N g\|_2),$$

hence B has the admissible growth condition. It is also clearly continuous on S and analytic on Int S. Then, for $z = iy$ and $1 < p < \infty$.

$$|B(iy)| \leqslant \|H^{-iy} f\|_p \|V_N^{-iy} g\|_{p'} \leqslant C(\delta, p) e^{\delta|y|} \|f\|_p \|g\|_{p'}.$$

And for $z = 1 + iy$ and $1 < p < q + \varepsilon$,

$$|A(1 + iy)| \leqslant \|V_N H^{-1} H^{-iy} f\|_p \|V_N^{-iy} g\|_{p'}$$
$$\leqslant \|V H^{-1}\|_{p,p} C(\delta, p) e^{\delta|y|} \|f\|_p \|g\|_{p'}.$$

where we used that $\|V_N H^{-1}\|_{p,p} \leqslant \|V H^{-1}\|_{p,p}$ as $0 \leqslant V_N \leqslant V$ almost everywhere. Thus, for $z = 1/2$ and $1 < p < 2(q + \varepsilon)$, we obtain

$$|B(1/2)| \leqslant C(p) \|f\|_p \|g\|_{p'}.$$

We conclude by a density argument that $V_N^{1/2} H^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < 2(q + \varepsilon)$ with a bound that is uniform with respect to N. By monotone convergence, this yields the $L^p(\mathbb{R}^n)$ boundedness of $V^{1/2} H^{-1/2}$ in the same range.

To finish the proof, fix $1 < p < 2(q + \varepsilon)$. Let $u \in C_0^\infty(\mathbb{R}^n)$. The only thing to establish is $\|\nabla u\|_p + \|V^{1/2} u\|_p \leqslant C(p) \|H^{1/2} u\|_p$. Since $u \in \mathcal{V}$, $f = H^{1/2} u$ is well-defined. We assume that $f \in L^p(\mathbb{R}^n)$, otherwise there is nothing to prove. Then, by Calderón-Zygmund theory and the above,

$$\|\nabla u\|_p + \|V^{1/2} u\|_p \leqslant C(p) \|(-\Delta)^{1/2} H^{-1/2} f\|_p + \|V^{1/2} H^{-1/2} f\|_p$$
$$\leqslant C'(p) \|f\|_p$$

and the proof is finished. □

Remark 6.3. — This interpolation argument also gives us a proof of the L^p boundedness of ∇H^{-1} and $V^{1/2} H^{-1/2}$ for $1 < p < 2$ for all non zero $V \in L^1_{loc}(\mathbb{R}^n)$.

7. Reverse Riesz transforms

This section is concerned with the proof of Theorem 1.2, item 2. We first want to show that there exists $C > 0$ depending only on the A_∞ constant of V such that for all $\alpha > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$ then

(7.1) $\qquad |\{x \in \mathbb{R}^n \,;\, |H^{1/2}f(x)| > \alpha\}| \leqslant \dfrac{C}{\alpha} \displaystyle\int |\nabla f| + V^{1/2}|f|.$

First, it is not too hard to show that if $\varepsilon \leqslant V \leqslant N$ for some $N > \varepsilon > 0$ then this inequality holds with C depending on ε, N (in fact, the next argument gives this also). Let C_1 be the best constant in this inequality with V replaced by $V_{\varepsilon,N} = \min(V + \varepsilon, N)$. We want to show that C_1 is bounded independently of ε and N. Assume it is the case, then for $\varepsilon, N > 0$, all $\alpha > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$

$$|\{x \in \mathbb{R}^n \,;\, |(-\Delta + V_{\varepsilon,N})^{1/2}f(x)| > \alpha\}| \leqslant \frac{C_1}{\alpha} \int |\nabla f| + V_{\varepsilon,N}^{1/2}|f|$$
$$\leqslant \frac{C_1}{\alpha} \int |\nabla f| + (V + \varepsilon)^{1/2}|f|.$$

Now, it is easy to show that $(-\Delta + V_{\varepsilon,N})^{1/2}f$ converges in L^2 to $(H+\varepsilon)^{1/2}f$ hence up to extraction of a subsequence, the above inequality passes to the limit as $N \to +\infty$. Then, as $f \in C_0^\infty(\mathbb{R}^n) \subset \mathcal{V} = \mathcal{D}(H^{1/2})$, $(H + \varepsilon)^{1/2}f$ converges to $H^{1/2}f$ in $L^2(\mathbb{R}^n)$ by functional calculus as ε tends to 0 and we obtain (7.1) with $C = C_1$.

Remark that if $V \in A_\infty$, then for all $N > \varepsilon > 0$, $V_{\varepsilon,N}$ is also in A_∞ with constants that are uniform with respect to ε and N. So as long as we only use the A_∞ information, we are safe. Therefore, we assume that $\varepsilon \leqslant V \leqslant N$ but we do not use this information quantitatively.

We also define C_p as the best constant C such that for $1 < p < 2$ and $f \in C_0^\infty(\mathbb{R}^n)$

$$\|H^{1/2}f\|_p \leqslant C_p(\|\nabla f\|_p + \|V^{1/2}f\|_p).$$

By extension, we can take f to be in the closure of $C_0^\infty(\mathbb{R}^n)$ for the norm defined by the right hand side. Since V is bounded below and above, this is the usual Sobolev space $W^{1,p}(\mathbb{R}^n)$.

We know that $C_2 = 1$. We shall prove that for some numbers C, M under control, we have

(7.2) $$C_1 \leqslant CC_2^2 + M = C + M.$$

This will require the use of a specific Calderón-Zygmund decomposition on f adapted to level sets of $|\nabla f| + V^{1/2}|f|$.

The Marcinkiewicz interpolation theorem would give us

(7.3) $$C_p \lesssim C_1^{\frac{2}{p}-1}$$

provided it applies. But it is not known whether the spaces defined by the seminorms $\|\nabla f\|_q + \|V^{1/2}f\|_q$, $1 \leqslant q \leqslant 2$, interpolate by the real method[3]. If we use the assumption $\varepsilon \leqslant V \leqslant N$, then we may interpolate but the constants would depend on ε, N. Instead, we prove (7.3) by adapting Marcinkiewicz theorem argument using again our Calderón-Zygmund decomposition.

LEMMA 7.1. — Let $n \geqslant 1$, $1 \leqslant p < 2$, $V \in A_\infty$ and $f \in C_0^\infty(\mathbb{R}^n)$, hence $\|\nabla f\|_p + \|V^{1/2}f\|_p < \infty$. Let $\alpha > 0$. Then, one can find a collection of cubes (Q_i), functions g and b_i such that

(7.4) $$f = g + \sum_i b_i$$

and the following properties hold:

(7.5) $$\|\nabla g\|_2 + \|V^{1/2}g\|_2 \leqslant C\alpha^{1-p/2}(\|\nabla f\|_p + \|V^{1/2}f\|_p)^{p/2},$$

(7.6) $$\operatorname{supp} b_i \in Q_i \text{ and } \int_{Q_i} |\nabla b_i|^p + R_i^{-p}|b_i|^p \leqslant C\alpha^p|Q_i|,$$

(7.7) $$\sum_i |Q_i| \leqslant C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p + |V^{1/2}f|^p,$$

(7.8) $$\sum_i \mathbf{1}_{Q_i} \leqslant N,$$

where N depends only on dimension and C on dimension, p and the A_∞ constant of V. Here, R_i denotes the sidelength of Q_i and gradients are taken in the sense of distributions in $\mathbb{R}^n$.

We remark that the decomposition is on f while the control is on $|\nabla f|^p + |V^{1/2}f|^p$.

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[3] Added in proof: this has been shown recently by N. Badr: Real interpolation of Sobolev spaces associated to a weight, preprint 2007, Orsay.

Proof. — Let Ω be the open set $\{x \in \mathbb{R}^n; M(|\nabla f|^p + |V^{1/2}f|^p)(x) > \alpha^p\}$ where M is the uncentered maximal operator over cubes of $\mathbb{R}^n$. If Ω is empty, then set $g = f$ and $b_i = 0$. Otherwise, the maximal theorem gives us

$$|\Omega| \leqslant C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p + |V^{1/2}f|^p.$$

Let (Q_i) be a Whitney decomposition of Ω by dyadic cubes: Ω is the disjoint union of the Q_i's, the cubes $2Q_i$ are contained in Ω and have the bounded overlap property, but the cubes $4Q_i$ intersect $F = \mathbb{R}^n \setminus \Omega$.[4] As usual, λQ is the cube co-centered with Q with sidelength λ times that of Q. Hence (7.7) and (7.8) are satisfied by the cubes $2Q_i$. We remark that since $V \in A_\infty$, we have $V^{p/2} \in A_\infty$ when $1 \leqslant p \leqslant 2$ (see Section 11). Hence we have by Lemma 2.1

$$\int_{2Q_i} |\nabla f|^p + |V^{1/2}f|^p \geqslant C \min(\mathrm{av}_{2Q_i} V^{p/2}, R_i^{-p}) \int_{2Q_i} |f|^p.$$

We declare Q_i of type 1 if $\mathrm{av}_{2Q_i} V^{p/2} \geqslant R_i^{-p}$ and of type 2 if $\mathrm{av}_{2Q_i} V^{p/2} < R_i^{-p}$.

Let us now define the functions b_i. Let $(\mathcal{X}_i)$ be a partition of unity on Ω associated to the covering (Q_i) so that for each i, $\mathcal{X}_i$ is a C^1 function supported in $2Q_i$ with $\|\mathcal{X}_i\|_\infty + R_i\|\nabla \mathcal{X}_i\|_\infty \leqslant c(n)$. Set

$$b_i = \begin{cases} f\mathcal{X}_i, & \text{if } Q_i \text{ is of type 1}, \\ (f - \mathrm{av}_{2Q_i} f)\mathcal{X}_i, & \text{if } Q_i \text{ is of type 2}. \end{cases}$$

If Q_i is of type 2, then it is a direct consequence of the L^p-Poincaré inequality that

$$\int_{2Q_i} |\nabla b_i|^p + R_i^{-p}|b_i|^p \leqslant C \int_{2Q_i} |\nabla f|^p.$$

As $\int_{4Q_i} |\nabla f|^p \leqslant \alpha^p |4Q_i|$ we get the desired inequality in (7.6).

If Q_i is of type 1,

$$\int_{2Q_i} R_i^{-p}|b_i|^p \leqslant \int_{2Q_i} R_i^{-p}|f|^p \leqslant C \int_{2Q_i} |\nabla f|^p + |V^{1/2}f|^p.$$

As the same integral but on $4Q_i$ is controlled by $\alpha^p|4Q_i|$ we get $\int_{2Q_i} R_i^{-p}|b_i|^p \leqslant C\alpha^p|Q_i|$. Since $\nabla b_i = \mathcal{X}_i \nabla f + f\nabla \mathcal{X}_i$ we obtain the same bound for $\int_{2Q_i} |\nabla b_i|^p$.

Set $g = f - \sum b_i$ where the sum is over both types of cubes and is locally finite by (7.8). It is clear that $g = f$ on $F = \mathbb{R}^n \setminus \Omega$ and $g = \sum^2 (\mathrm{av}_{2Q_i} f)\, \mathcal{X}_i$

---

[4] In fact, the factor 2 should be some $c = c(n) > 1$ explicitly given in [35, Chapter 6]. We use this convention to avoid too many irrelevant constants.

on Ω, where $\sum^j$ means that we are summing over cubes of type j. Let us prove (7.4).

First, by the differentiation theorem, $V^{1/2}|f| \leqslant \alpha$ almost everywhere on F. Next, since $V \in A_\infty$ and $p < 2$ implies $V^{p/2} \in B_{2/p}$ (see Section 11) and $\mathrm{av}_{2Q_i} V \leqslant C(\mathrm{av}_{2Q_i} V^{p/2})^{2/p}$. Hence

$$\int_\Omega V|g|^2 \leqslant \sum{}^2 \int_{2Q_i} V|\,\mathrm{av}_{2Q_i} f|^2$$

$$\leqslant C \sum{}^2 \left((\mathrm{av}_{2Q_i} V^{p/2})|\,\mathrm{av}_{2Q_i} f|^p\right)^{2/p} |Q_i|.$$

Now, by construction of the type 2 cubes and the L^p version of Fefferman-Phong inequality,

$$(\mathrm{av}_{2Q_i} V^{p/2})|\,\mathrm{av}_{2Q_i} f|^p \leqslant C \,\mathrm{av}_{2Q_i}(|\nabla f|^p + |V^{1/2}f|^p) \leqslant C\alpha^p.$$

Hence,

$$\int_\Omega V|g|^2 \leqslant C \sum{}^2 \alpha^2 \,|Q_i| \leqslant C\alpha^{2-p} \int_{\mathbb{R}^n} |\nabla f|^p + |V^{1/2}f|^p.$$

Combining the estimates on F and Ω, we obtain the desired bound for $\int_{\mathbb{R}^n} V|g|^2$. We finish the proof by estimating $\|\nabla g\|_\infty$ and $\|\nabla g\|_p$. First, it is easy to see that the inequality $\|b_i\|_p^p \leqslant C\alpha^p R_i^p |Q_i|$ together with the fact that Whitney cubes have sidelength comparable to their distance to the boundary, imply that $\sum b_i$ converges in the sense of distributions in $\mathbb{R}^n$ (not just in Ω, which is a trivial fact!), hence $\nabla g = \nabla f - \sum \nabla b_i$. It follows from the L^p estimates on ∇b_i and the bounded overlap property that

$$\left\|\sum \nabla b_i\right\|_p \leqslant C(\|\nabla f\|_p + \|V^{1/2}f\|_p),$$

therefore the same estimate holds for $\|\nabla g\|_p$. Next, a computation of the sum $\sum \nabla b_i$ leads us to

$$\nabla g = \mathbf{1}_F(\nabla f) + \sum{}^2 (\mathrm{av}_{2Q_i} f) \nabla \mathcal{X}_i.$$

By definition of F and the differentiation theorem, $|\nabla g|$ is bounded by α almost everywhere on F. It remains to control $\|h_2\|_\infty$ where $h_2 = \sum^2 (\mathrm{av}_{2Q_i} f) \nabla \mathcal{X}_i$. Set $h_1 = \sum^1 (\mathrm{av}_{2Q_i} f) \nabla \mathcal{X}_i$. By already seen arguments for type 1 cubes, $|\mathrm{av}_{2Q_i} f| \leqslant C\alpha R_i$. Hence, $|h_1| \leqslant C \sum^1 \mathbf{1}_{2Q_i}\alpha \leqslant CN\alpha$ and it suffices to show that $h = h_1 + h_2$ is bounded by $C\alpha$. To see this, observe that $\sum_i \mathcal{X}_i(x) = 1$ on Ω and 0 on F. Since it is a locally finite sum we have $\sum_i \nabla \mathcal{X}_i(x) = 0$ for $x \in \Omega$. Fix $x \in \Omega$. Let Q_j be the Whitney cube containing x and let I_x be the set of indices i such that $x \in 2Q_i$. We know

that $\sharp I_x \leqslant N$. Also for $i \in I_x$ we have that $C^{-1}R_i \leqslant R_j \leqslant CR_i$ (see [35]). Therefore, we may write

$$|h(x)| = \left| \sum_{i \in I_x} (\mathrm{av}_{2Q_i} f - \mathrm{av}_{2Q_j} f) \nabla \mathcal{X}_i(x) \right| \leqslant C \sum_{i \in I_x} |\mathrm{av}_{2Q_i} f - \mathrm{av}_{2Q_j} f| R_i^{-1}.$$

But $2Q_i$ and $2Q_j$ are contained in CQ_j for some $C > 4$ independent of j. Hence, the Poincaré inequality and the definition of Q_j yields

$$|\mathrm{av}_{2Q_i} f - \mathrm{av}_{2Q_j} f| \leqslant CR_j(\mathrm{av}_{CQ_j} |\nabla f|^p)^{1/p} \leqslant CR_j \alpha.$$

We have finished the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of item 3 in Theorem 1.2. — First, we prove (7.2). Let $f \in C_0^\infty(\mathbb{R}^n)$. We use the following resolution of $H^{1/2}$:

$$H^{1/2}f = c \int_0^\infty He^{-t^2 H} f \, dt$$

where $c = 2\pi^{-1/2}$ is forgotten from now on. It suffices to obtain the result for the truncated integrals $\int_\varepsilon^R \ldots$ with bounds independent of ε, R, and then to let $\varepsilon \downarrow 0$ and $R \uparrow \infty$. For the truncated integrals, all the calculations are justified. We thus consider that $H^{1/2}$ is one of the truncated integrals but we still write the limits as 0 and $+\infty$ to simplify the exposition.

Apply the Calderón-Zygmund decomposition of Lemma 7.1 with $p = 1$ to f at height α and write $f = g + \sum_i b_i$.

Concerning g, we have

$$\left| \left\{ x \in \mathbb{R}^n; |H^{1/2}g(x)| > \frac{\alpha}{3} \right\} \right| \leqslant \frac{9}{\alpha^2} \int |H^{1/2}g|^2$$
$$\leqslant \frac{9}{\alpha^2} \int |\nabla g|^2 + V|g|^2$$
$$\leqslant \frac{C}{\alpha} \int |\nabla f| + |V^{1/2}f|$$

where we used (7.5).

The argument to estimate $H^{1/2}b_i$ will use the Gaussian upper bounds of the kernels of e^{-tH} which are valid for all potentials $V \geqslant 0$. Let $r_i = 2^k$ if $2^k \leqslant R_i < 2^{k+1}$ (R_i is the sidelength of Q_i) and set $T_i = \int_0^{r_i} He^{-t^2 H} dt$ and $U_i = \int_{r_i}^\infty He^{-t^2 H} dt$. It is enough to estimate

$$A = |\{x \in \mathbb{R}^n; |\sum_i T_i b_i(x)| > \alpha/3\}|$$

and

$$B = |\{x \in \mathbb{R}^n; |\sum_i U_i b_i(x)| > \alpha/3\}|.$$

First,

$$A \leqslant |\cup_i 4Q_i| + \left| \left\{ x \in \mathbb{R}^n \setminus \cup_i 4Q_i; \left| \sum_i T_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right|,$$

and by (7.7), $|\cup_i 4Q_i| \leqslant \frac{C}{\alpha} \int |\nabla f| + |V^{1/2} f|$.

For the other term, we have

$$\left| \left\{ x \in \mathbb{R}^n \setminus \cup_i 4Q_i; \left| \sum_i T_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right| \leqslant \frac{C}{\alpha^2} \int \left| \sum_i h_i \right|^2$$

with $h_i = \mathbf{1}_{(4Q_i)^c} |T_i b_i|$. To estimate the L^2 norm, we dualize against $u \in L^2(\mathbb{R}^n)$ with $\|u\|_2 = 1$:

$$\int |u| \sum_i h_i = \sum_i \sum_{j=2}^{\infty} A_{ij}$$

where

$$A_{ij} = \int_{C_j(Q_i)} |T_i b_i| |u|, \quad C_j(Q_i) = 2^{j+1} Q_i \setminus 2^j Q_i.$$

Using the well-known Gaussian upper bounds for the kernels of tHe^{-tH}, $t > 0$, and $r_i \sim R_i$, we obtain

$$\|He^{-t^2 H} b_i\|_{L^2(C_j(Q_i))} \leqslant \frac{C}{t^{\gamma+2}} e^{-\frac{c4^j r_i^2}{t^2}} \|b_i\|_1$$

where $\gamma = \frac{n}{2}$. By (7.6), $\|b_i\|_1 \leqslant c \alpha R_i |Q_i|$, hence, by Minkowski integral inequality, for some appropriate positive constants C, c,

$$\|T_i b_i\|_{L^2(C_j(Q_i))} \leqslant \int_0^{r_i} \|He^{-t^2 H} b_i\|_{L^2(C_j(Q_i))} \, dt$$
$$\leqslant C \alpha e^{-c4^j} |Q_i|^{1/2}.$$

Now remark that for any $y \in Q_i$ and any $j \geqslant 2$,

$$\left(\int_{C_j(Q_i)} |u|^2 \right)^{1/2} \leqslant \left(\int_{2^{j+1} Q_i} |u|^2 \right)^{1/2} \leqslant (2^{n(j+1)} |Q_i|)^{1/2} (M(|u|^2)(y))^{1/2}.$$

Applying Hölder inequality, one obtains

$$A_{ij} \leqslant C \alpha 2^{nj/2} e^{-c4^j} |Q_i| (M(|u|^2)(y))^{1/2}.$$

Averaging over Q_i yields

$$A_{ij} \leqslant C \alpha 2^{nj/2} e^{-c4^j} \int_{Q_i} (M(|u|^2)(y))^{1/2} \, dy.$$

Summing over $j \geqslant 2$ and i, we have

$$\int |u| \sum_i h_i \leqslant C\alpha \int \sum_i \mathbf{1}_{Q_i}(y)\big(M(|u|^2)(y)\big)^{1/2} \, dy.$$

Using finite overlap (7.8) of the cubes Q_i and Kolmogorov's inequality, one obtains

$$\int |u| \sum_i h_i \leqslant C'N\alpha |\cup_i Q_i|^{1/2} \||u|^2\|_1^{1/2}.$$

Hence

$$\left|\left\{x \in \mathbb{R}^n \setminus \cup_i 4Q_i; \left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right| \leqslant C|\cup_i Q_i| \leqslant \frac{C}{\alpha^p} \int |\nabla f| + |V^{1/2} f|$$

by (7.8) and (7.7).

It remains to handling the term B. Using functional calculus for H one can compute U_i as $r_i^{-1}\psi(r_i^2 H)$ with ψ the holomorphic function on the sector $|\arg z| < \frac{\pi}{2}$ given by

$$\psi(z) = \int_1^\infty e^{-t^2 z} z \, dt.$$

It is easy to show that $|\psi(z)| \leqslant C|z|^{1/2} e^{-c|z|}$, uniformly on subsectors $|\arg z| \leqslant \mu < \frac{\pi}{2}$.

Let $q = 2$ if $n = 1$ and $q = 1^* = \frac{n}{n-1}$ for $n \geqslant 2$. By Poincaré-Sobolev inequality, $b_i \in L^q$ and

$$\|b_i\|_q \leqslant cR_i^{1-(n-\frac{n}{q})} \|\nabla b_i\|_1 \leqslant C\alpha R_i^{1+\frac{n}{q}}.$$

We invoke the estimate

$$(7.9) \qquad \left\|\sum_{k\in\mathbb{Z}} \psi(4^k H)\beta_k\right\|_q \lesssim \left\|\left(\sum_{k\in\mathbb{Z}} |\beta_k|^2\right)^{1/2}\right\|_q.$$

Indeed, by duality, this is equivalent to the Littlewood-Paley inequality

$$\left\|\left(\sum_{k\in\mathbb{Z}} |\psi(4^k H)\beta|^2\right)^{1/2}\right\|_{q'} \lesssim \|\beta\|_{q'}.$$

For $q = 2$, this is a simple estimate using Borel functional calculus on L^2 since H is self-adjoint. For $q \neq 2$, this is a consequence of the Gaussian estimates for the kernels of e^{-tH}, $t > 0$ (this was first proved in [2] using the vector-valued version of the work in [11]. See [1] for a more general argument in this spirit or [26] for an abstract proof relying on functional calculus).

To apply (7.9), observe that the definitions of r_i and U_i yield

$$\sum_i U_i b_i = \sum_{k \in \mathbb{Z}} \psi(4^k H) \beta_k$$

with

$$\beta_k = \sum_{i, r_i = 2^k} \frac{b_i}{r_i}.$$

Using the bounded overlap property (7.8), one has that

$$\left\| \left(\sum_{k \in \mathbb{Z}} |\beta_k|^2 \right)^{1/2} \right\|_q^q \leqslant C \int \sum_i \frac{|b_i|^q}{r_i^q}.$$

Using $R_i \sim r_i$,

$$\int \sum_i \frac{|b_i|^q}{r_i^q} \leqslant C \alpha^q \sum_i |Q_i|.$$

Hence, by (7.7)

$$\left| \left\{ x \in \mathbb{R}^n; \left| \sum_i U_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right| \leqslant C \sum_i |Q_i| \leqslant \frac{C}{\alpha} \int |\nabla f| + |V^{1/2} f|.$$

We turn to the proof of (7.3). Fix $1 < p < 2$ and $f \in C_0^\infty(\mathbb{R}^n)$. Choose $0 < \delta < 1$ so that $1 < p\delta$. Let $\alpha > 0$ and apply the Calderón-Zygmund decomposition of Lemma 7.1 to f with exponent $p\delta$ and threshold α. We may do this since $\|\nabla f\|_{p\delta} + \|V^{1/2} f\|_{p\delta} < \infty$. Of course we do not want to use its value in a quantitative way. We obtain that $f = g_\alpha + b_\alpha$ with $b_\alpha = \sum_i b_i$.

Write

$$\|H^{1/2} f\|_p^p = p 2^p \int_0^\infty \alpha^{p-1} |\{x \in \mathbb{R}^n; |H^{1/2} f(x)| > 2\alpha\}| \, d\alpha$$

$$\leqslant p 2^p \int_0^\infty \alpha^{p-1} |\{x \in \mathbb{R}^n; |H^{1/2} g_\alpha(x)| > \alpha\}| \, d\alpha$$

$$+ p 2^p \int_0^\infty \alpha^{p-1} |\{x \in \mathbb{R}^n; |H^{1/2} b_\alpha(x)| > \alpha\}| \, d\alpha$$

$$\leqslant I + II$$

with

$$I = C p 2^p \int_0^\infty \alpha^{p-1} \frac{\|\nabla g_\alpha\|_2^2 + \|V^{1/2} g_\alpha\|_2^2}{\alpha^2} \, d\alpha = I_g + I_v$$

and

$$II = C p 2^p \int_0^\infty \alpha^{p-1} \frac{\|\nabla b_\alpha\|_1 + \|V^{1/2} b_\alpha\|_1}{\alpha} \, d\alpha = II_g + II_v,$$

where I_g and II_g denote the gradient term in I and II respectively. To estimate these integrals, we need to come back to the construction of g_α and b_α.

Set $Tf = (|\nabla f|^{p\delta} + |V^{1/2}f|^{p\delta})^{1/p\delta}$. Write F_α as the complement of $\Omega_\alpha = \{M(Tf^{p\delta}) > \alpha^{p\delta}\}$. Then recall that $\nabla g_\alpha = \mathbf{1}_{F_\alpha}(\nabla f) + \mathbf{1}_{\Omega_\alpha}h$ where $|h| \leqslant C\alpha$ and $|\nabla f| \leqslant \alpha$ on F_α. Thus I_g splits into $I_{g1} + I_{g2}$ according to this decomposition. The treatment of I_{g1} is done using the definition of F_α, Fubini's theorem and $p < 2$ as follows:

$$
\begin{aligned}
I_{g1} &= \frac{Cp2^p}{2-p}\int |\nabla f|^2 \left(M(Tf^{p\delta})\right)^{\frac{p-2}{p\delta}} \\
&\leqslant \frac{Cp2^p}{2-p}\int |\nabla f|^p,
\end{aligned}
$$

where we used

$$
|\nabla f|^2 = |\nabla f|^p \, |\nabla f|^{2-p} \leqslant |\nabla f|^p \, (Tf^{p\delta})^{\frac{2-p}{p\delta}} \leqslant |\nabla f|^p \left(M(Tf^{p\delta})\right)^{\frac{2-p}{p\delta}}
$$

almost everywhere. For I_{g2}, we use the bound of h to obtain

$$
\begin{aligned}
I_{g2} &\leqslant Cp2^p \int_0^\infty \alpha^{p-1}|\Omega_\alpha|\, d\alpha \\
&= C2^p \int \left(M(Tf^{p\delta})\right)^{\frac{1}{\delta}} \\
&\leqslant C\int |\nabla f|^p + |V^{1/2}f|^p
\end{aligned}
$$

by the strong type $(\frac{1}{\delta}, \frac{1}{\delta})$ of the maximal operator.

Next, we turn to the term II_g. We have $\nabla b_\alpha = \mathbf{1}_{\Omega_\alpha}(\nabla f) - \mathbf{1}_{\Omega_\alpha}h$ so that $II_g \leqslant (II_{g1} + I_{g2})$ and I_{g2} is already controlled. For II_{g1} we have by using Hölder's inequality and the strong type $(\frac{1}{\delta}, \frac{1}{\delta})$ of the maximal operator

$$
\begin{aligned}
II_{g1} &= \frac{Cp2^p}{p-1}\int |\nabla f| \left(M(Tf^{p\delta})\right)^{\frac{p-1}{p\delta}} \\
&\leqslant \frac{Cp2^p}{p-1}\left(\int |\nabla f|^p\right)^{1/p}\left(\int \left(M(Tf^{p\delta})\right)^{(\frac{p-1}{p\delta})p'}\right)^{1/p'} \\
&\leqslant C\int |\nabla f|^p + |V^{1/2}f|^p.
\end{aligned}
$$

It remains to look at I_v and II_v. Recall that $g_\alpha = f$ on F_α and $g_\alpha = h_\alpha$ on Ω_α, and we have proved $\int V|h_\alpha|^2 \leqslant C\alpha^2|\Omega_\alpha|$. Hence, I_v splits as $I_{v1} + I_{v2}$.

First,

$$I_{v1} = \frac{Cp2^p}{2-p} \int |V^{1/2}f|^2 \left(M(Tf^{p\delta})\right)^{\frac{p-2}{p\delta}}$$

$$\leqslant \frac{Cp2^p}{2-p} \int |V^{1/2}f|^p.$$

with the similar argument as for I_{g1}. Next,

$$I_{v2} \leqslant Cp2^p \int_0^\infty \alpha^{p-1}|\Omega_\alpha| \, d\alpha$$

$$= C2^p \int \left(M(Tf^{p\delta})\right)^{\frac{1}{\delta}}$$

$$\leqslant C \int |\nabla f|^p + |V^{1/2}f|^p.$$

Now, $b_\alpha = f - g_\alpha = f - h_\alpha$ on Ω_α and $b_\alpha = 0$ on F_α. Hence, $II_v \leqslant II_{v1} + I_{v2}$ and

$$II_{v1} = \frac{Cp2^p}{p-1} \int |V^{1/2}f| \left(M(Tf^{p\delta})\right)^{\frac{p-1}{p\delta}}$$

$$\leqslant \frac{Cp2^p}{p-1} \left(\int |V^{1/2}f|^p\right)^{1/p} \left(\int \left(M(Tf^{p\delta})\right)^{\left(\frac{p-1}{p\delta}\right)p'}\right)^{1/p'}$$

$$\leqslant C \int |\nabla f|^p + |V^{1/2}f|^p.$$

This concludes the proof of item 3 of Theorem 1.2. $\qquad\square$

8. Estimates for weak solutions

In this section, Q denotes a cube, R its radius, and u a weak solution of $-\Delta u + Vu = 0$ in a neighborhood of $\overline{2Q}$. Recall that under the assumption $V \geqslant 0$, we have the mean value inequality

$$(8.1) \qquad \sup_Q |u| \leqslant C(r,n,\mu)\left(\text{av}_{\mu Q} |u|^r\right)^{1/r}$$

for any $0 < r < \infty$ and $1 < \mu \leqslant 2$. And we have also shown a mean value inequality against arbitrary A_∞ weights.

We state some further estimates that are interesting in their own right assuming $V \in A_\infty$. By splitting real and imaginary parts, we may suppose u real-valued. All constants are independent of Q and u but they may depend on V through the constants in the A_∞ condition or the B_q condition when assumed.

LEMMA 8.1. — *For all $1 \leqslant \mu < \mu' \leqslant 2$ and $k > 0$, there is a constant C such that*

$$\operatorname{av}_{\mu Q} |u|^2 \leqslant \frac{C}{(1 + R^2 \operatorname{av}_Q V)^k} \left(\operatorname{av}_{\mu' Q} |u|^2 \right).$$

and

$$\operatorname{av}_{\mu Q}(|\nabla u|^2 + V|u|^2) \leqslant \frac{C}{(1 + R^2 \operatorname{av}_Q V)^k} \left(\operatorname{av}_{\mu' Q}(|\nabla u|^2 + V|u|^2) \right).$$

LEMMA 8.2. — *For all $1 < \mu \leqslant 2$ and $k > 0$, there is a constant C such that*

$$(R \operatorname{av}_Q V)^2 \operatorname{av}_Q |u|^2 \leqslant \frac{C}{(1 + R^2 \operatorname{av}_Q V)^k} \left(\operatorname{av}_{\mu Q}(V|u|^2) \right).$$

LEMMA 8.3. — *For all $1 < \mu \leqslant 2$, $k > 0$ and $\sup(n, 2) < p < \infty$, there is a constant C such that*

$$(R \operatorname{av}_Q V)^2 \operatorname{av}_Q |u|^2 \leqslant \frac{C}{(1 + R^2 \operatorname{av}_Q V)^k} \left(\operatorname{av}_{\mu Q} |\nabla u|^p \right)^{2/p}.$$

LEMMA 8.4. — *Assume $V \in B_q$ and set $\tilde{q} = \inf(q^*, 2q)$. For all $1 < \mu \leqslant 2$ and $k > 0$ there is a constant C such that*

$$\left(\operatorname{av}_Q |\nabla u|^{\tilde{q}} \right)^{1/\tilde{q}} \leqslant \frac{C}{(1 + R^2 \operatorname{av}_Q V)^k} \left(\operatorname{av}_{\mu Q}(|\nabla u|^2 + V|u|^2) \right)^{1/2}.$$

LEMMA 8.5. — *Assume $V \in B_q$. For all $1 < \mu \leqslant 2$, if $n/2 \leqslant q < n$ then there is a constant C such that*

$$\left(\operatorname{av}_Q |\nabla u|^{q^*} \right)^{1/q^*} \leqslant C \left(\operatorname{av}_{\mu Q} |\nabla u|^2 \right)^{1/2},$$

and if $q \geqslant n$ then there is a constant C such that

$$\sup_Q |\nabla u| \leqslant C \left(\operatorname{av}_{\mu Q} |\nabla u|^2 \right)^{1/2}.$$

LEMMA 8.6. — *Assume $V \in B_q$. For all $1 < \mu \leqslant 2$ and $k > 0$, if $n/2 \leqslant q < n$ then there is a constant C such that*

$$\left(\operatorname{av}_Q |\nabla u|^{q^*} \right)^{1/q^*} \leqslant \frac{C}{R(1 + R^2 \operatorname{av}_Q V)^k} \left(\sup_{\mu Q} |u| \right).$$

and if $q \geqslant n$ then there is a constant C such that

$$\sup_Q |\nabla u| \leqslant \frac{C}{R(1 + R^2 \operatorname{av}_Q V)^k} \left(\sup_{\mu Q} |u| \right).$$

LEMMA 8.7. — *Assume $V \in B_q$ with $q > 1$ and $q \geqslant n/2$. For all $1 < \mu \leqslant 2$ and $k > 0$ there is a constant C such that*

$$(R \operatorname{av}_Q V)^2 \operatorname{av}_Q |u|^2 \leqslant \frac{C}{(1 + R^2 \operatorname{av}_Q V)^k} \left(\operatorname{av}_{\mu Q} |\nabla u|^2 \right).$$

LEMMA 8.8. — *In Lemma 8.5, the constant C can be replaced by $C(1+R^2 \operatorname{av}_Q V)^{-k}$ for any $k > 0$.*

Let us postpone the proofs and make some remarks concerning these inequalities.

Remark 8.9. —

1) Lemma 8.5 is a weak reverse Hölder inequality for the gradient of weak solutions. It improves over Lemma 8.4 in the fact that the right hand side does not have terms involving $V|u|^2$ but this is under the assumption $q \geqslant n/2$. Using self-improvement of weak reverse Hölder inequalities (see [22, Theorem 2]), we may replace the exponent 2 in the right hand sides by any $0 < p < 2$.
2) We do not know if Lemma 8.5 holds for $q < n/2$.
3) In Lemma 8.4, note that $\tilde{q} = q^* < 2q$ when $q < n/2$ and it would be natural the estimate holds for the larger exponent $2q$.
4) Lemma 8.7 is a Poincaré type inequality for weak solutions. As $\sup_Q |u|$ can be compared to $\left(\operatorname{av}_Q |u|^2\right)^{1/2}$, we see that it is a converse to the Caccioppoli inequality in the regime $R^2 \operatorname{av}_Q V \geqslant 1$.
5) Except for Lemma 8.1 and 8.6 which are closely related to Lemma 4.6 and Remark 4.9 in [31], these lemmata appear to be new.

Proof of Lemma 8.1. — There is nothing to prove if $R^2 \operatorname{av}_Q V \leqslant 1$ and we assume $R^2 \operatorname{av}_Q V \geqslant 1$. The well-known Caccioppoli type argument yields for $1 \leqslant \mu < \mu' \leqslant 2$

$$(8.2) \qquad \int_{\mu Q} |\nabla u|^2 + V|u|^2 \leqslant \frac{C}{R^2} \int_{\mu' Q} |u|^2.$$

The improved Fefferman-Phong inequality of Lemma 2.1 and the fact that the averages of V on μQ with $1 \leqslant \mu \leqslant 2$ are all uniformly comparable tell us for some $\beta > 0$,

$$\frac{1}{R^2} \int_{\mu Q} |u|^2 \leqslant \frac{C}{(R^2 \operatorname{av}_Q V)^\beta} \int_{\mu Q} |\nabla u|^2 + V|u|^2.$$

The desired estimates follow readily by iterating these two inequalities. $\qquad \square$

Proof of Lemma 8.2. — Using Lemma 8.1 with $k > 1$ and $1 < \mu' < \mu$ and then Lemma 5.2, we have,

$$
\begin{aligned}
(R \operatorname{av}_Q V)^2 \operatorname{av}_Q |u|^2 &\leqslant \frac{C \operatorname{av}_Q V \operatorname{av}_{\mu'Q} |u|^2}{(1 + R^2 \operatorname{av}_Q V)^{k-1}} \\
&\leqslant \frac{C \operatorname{av}_{\mu'Q} V \sup_{\mu'Q} |u|^2}{(1 + R^2 \operatorname{av}_Q V)^{k-1}} \\
&\leqslant \frac{C \operatorname{av}_{\mu Q}(V|u|^2)}{(1 + R^2 \operatorname{av}_Q V)^{k-1}}.
\end{aligned}
$$

$\square$

Proof of Lemma 8.3. — Of course, if $\operatorname{av}_{\mu Q} |\nabla u|^p = \infty$ there is nothing to prove. Assume, therefore, that $\operatorname{av}_{\mu Q} |\nabla u|^p < \infty$. Let $1 < \nu < \mu$ and η be a smooth non-negative function, bounded by 1, equal to 1 on νQ with support on μQ and whose gradient is bounded by C/R and Laplacian by C/R^2. Integrating the equation $-\Delta u + V u = 0$ against $u \eta^2$, we find

$$
\int |\nabla u|^2 \eta^2 + V|u|^2 \eta^2 = 2 \int \nabla u \cdot \nabla \eta \, u \eta \leqslant \frac{C}{R} \left(\int_{\mu Q} |\nabla u|^2 \right)^{1/2} \left(\int |u|^2 \eta^2 \right)^{1/2},
$$

hence

$$
X \leqslant C \, (R^2 \operatorname{av}_Q V)^{1/2} |\mu Q|^{1/2} \, Y^{1/2} \, Z^{1/2}
$$

where we have set $X = (R^2 \operatorname{av}_Q V) \int V|u|^2 \eta^2$, $Y = \left(\operatorname{av}_{\mu Q} |\nabla u|^p \right)^{2/p}$ and $Z = \operatorname{av}_Q V \int |u|^2 \eta^2$. By Morrey's embedding theorem, u is Hölder continuous with exponent $\alpha = 1 - n/p$ and for all $x, y \in \mu Q$,

$$
|u(x) - u(y)| \leqslant C \left(\frac{|x-y|}{R} \right)^{\alpha} R \left(\operatorname{av}_{\mu Q} |\nabla u|^p \right)^{1/p} = C \left(\frac{|x-y|}{R} \right)^{\alpha} R \, Y^{1/2}.
$$

We pick $y \in \overline{Q}$ such that $|u(y)| = \inf_Q |u|$. Then

$$
\begin{aligned}
Z &= \operatorname{av}_Q V \int |u|^2 \eta^2 \\
&\leqslant 2(\operatorname{av}_Q V) \inf_Q |u|^2 \int \eta^2 + 2(\operatorname{av}_Q V) \int |u(x) - u(y)|^2 \eta^2(x) \, dx \\
&\leqslant 2 \left(\operatorname{av}_Q (V|u|^2) \right) \int \eta^2 + C(\operatorname{av}_Q V) R^2 Y \int \left(\frac{|x-y|}{R} \right)^{2\alpha} \eta^2(x) \, dx \\
&\leqslant C \left(\operatorname{av}_Q (V|u|^2) \right) |Q| + C(\operatorname{av}_Q V) R^2 Y |\mu Q| \\
&\leqslant C \int V|u|^2 \eta^2 + C(\operatorname{av}_Q V) R^2 Y |\mu Q|,
\end{aligned}
$$

where, in the penultimate inequality, we used the support condition on η and $0 \leqslant \eta \leqslant 1$, and in the last, $\eta = 1$ on Q. Using the previous inequalities,

we obtain

$$X \leqslant C|\mu Q|^{1/2} Y^{1/2} \left(CX + C(R^2 \operatorname{av}_Q V)^2 |\mu Q| Y \right)^{1/2}$$

which, by $2ab \leqslant \varepsilon^{-1} a^2 + \varepsilon b^2$ for all $a, b \geqslant 0$ and $\varepsilon > 0$, implies

$$X \leqslant C(1 + R^2 \operatorname{av}_Q V)^2 |\mu Q| Y.$$

Next, let $1 < \nu' < \nu$. Using $\eta = 1$ on νQ, Lemma 5.2 and Lemma 8.1,

$$\int V|u|^2 \eta^2 \geqslant \int_{\nu Q} V|u|^2$$

$$\geqslant C \operatorname{av}_{\nu' Q} V \int_{\nu' Q} |u|^2$$

$$\geqslant C(\operatorname{av}_Q V)(1 + R^2 \operatorname{av}_Q V)^k \int_Q |u|^2,$$

hence

$$X \geqslant C(R \operatorname{av}_Q V)^2 (1 + R^2 \operatorname{av}_Q V)^k \int_Q |u|^2.$$

The upper and lower bounds for X yield the lemma. $\qquad\square$

Proof of Lemma 8.4. — First note that if $q \leqslant \frac{2n}{n+2}$ then $\tilde{q} \leqslant 2$ and the conclusion (useless for us) follows by a mere Hölder inequality. Henceforth, we assume $q > \frac{2n}{n+2}$. Also, by Lemma 8.1, it suffices to obtain the estimate with $k = 0$. Let us assume $\mu = 2$ for simplicity of the argument. Let v be the harmonic function on $2Q$ with $v = u$ on $\partial(2Q)$ and set $w = u - v$ on $2Q$. Since $w = 0$ on $\partial(2Q)$, we have

$$\left(\operatorname{av}_{2Q} |\nabla w|^2 \right)^{1/2} \leqslant \left(\operatorname{av}_{2Q} |\nabla u|^2 \right)^{1/2}.$$

By elliptic estimates for harmonic functions, we have for all $2 \leqslant p \leqslant \infty$, and in particular for $p = \tilde{q}$,

$$\left(\operatorname{av}_Q |\nabla v|^p \right)^{1/p} \leqslant C(\operatorname{av}_{2Q} |\nabla v|^2)^{1/2} \leqslant 2C(\operatorname{av}_{2Q} |\nabla u|^2)^{1/2}.$$

Let $1 < \mu < 2$ and η be a smooth non-negative function, bounded by 1, equal to 1 on Q with support contained in μQ and whose gradient is bounded by C/R and Laplacian by C/R^2. As $\Delta w = \Delta u = Vu$ on $2Q$, we have

$$\Delta(w\eta) = Vu\eta + 2\nabla w \cdot \nabla \eta + w \Delta \eta \quad \text{on } \mathbb{R}^n.$$

Hence, if $n \geqslant 2$ by Green's representation for the Laplace equation

$$\nabla(w\eta)(x) = \int_{\mathbb{R}^n} \nabla\Gamma(x - y) \left[(Vu\eta)(y) + 2\nabla w(y) \cdot \nabla\eta(y) + w(y)\Delta\eta(y) \right] dy$$

$$= I + II + III$$

where Γ is the fundamental solution of Δ so that $|\nabla\Gamma(x)| \leqslant C|x|^{1-n}$. Since $\tilde{q} \leqslant q^*$, we have

$$\left(\operatorname{av}_Q |\nabla w|^{\tilde{q}}\right)^{1/\tilde{q}} \leqslant \left(\operatorname{av}_Q |\nabla w|^{q^*}\right)^{1/q^*}$$

so that it suffices to bound the latter integral. Using support conditions on η, we obtain the pointwise bounds for $x \in Q$,

$$II \leqslant C \operatorname{av}_{2Q} |\nabla w| \leqslant C\left(\operatorname{av}_{2Q} |\nabla w|^2\right)^{1/2} \leqslant C\left(\operatorname{av}_{2Q} |\nabla u|^2\right)^{1/2}$$

and

$$III \leqslant \frac{C}{R} \operatorname{av}_{2Q} |w| \leqslant C\left(\operatorname{av}_{2Q} |\nabla w|^2\right)^{1/2} \leqslant C\left(\operatorname{av}_{2Q} |\nabla u|^2\right)^{1/2}$$

where we used Poincaré inequality for w on $2Q$ as $w = 0$ on the boundary. By the $L^q - L^{q^*}$ boundedness of the Riesz potential

$$\left(\int_{\mathbb{R}^n} I^{q^*}\right)^{1/q^*} \leqslant C\left(\int_{\mathbb{R}^n} |Vu\eta|^q\right)^{1/q} \leqslant C\left(\int_{\mu Q} |V|^q\right)^{1/q} \sup_{\mu Q} |u|.$$

Normalizing by taking averages and using the B_q condition on V yields

$$(8.3) \qquad\qquad \left(\operatorname{av}_Q I^{q^*}\right)^{1/q^*} \leqslant CR \operatorname{av}_{\mu Q} V \sup_{\mu Q} |u|.$$

Now, if $\mu < \mu' < 2$, subharmonicity of $|u|^2$ and Lemma 5.2 yield

$$R \operatorname{av}_{\mu Q} V \sup_{\mu Q} |u| \leqslant CR \operatorname{av}_{\mu' Q} V \left(\operatorname{av}_{\mu' Q} |u|^2\right)^{1/2}$$

which by Lemma 8.2 is bounded by $C\left(\operatorname{av}_{2Q}(V|u|^2)\right)^{1/2}$. Gathering the estimates obtained for ∇v and ∇w, the lemma is proved when $n \geqslant 2$.

When $n = 1$, we have

$$(w\eta)'(x) = -\int_x^\infty Vu\eta + 2w'\eta' + w\eta''$$

and we obtain readily for $x \in Q$,

$$|w'(x)| \leqslant CR(\operatorname{av}_{\mu Q} V) \sup_{\mu Q} |u| + C\left(\operatorname{av}_{\mu Q} |w'|^2\right)^{1/2}.$$

The rest of the proof is as before. $\qquad\qquad\qquad\qquad\qquad\square$

Proof of Lemma 8.5. — Assume $n/2 < q < n$. The previous lemma shows that $\operatorname{av}_{\mu' Q} |\nabla u|^{\tilde{q}} < \infty$ for all $1 < \mu' \leqslant \mu$. As $\tilde{q} = 2q > n$, Lemma 8.3 applies and using it with $k = 0$ instead of Lemma 8.2 in the previous argument, we obtain,

$$\left(\operatorname{av}_Q |\nabla w|^{q^*}\right)^{1/q^*} \leqslant C\left(\operatorname{av}_{\mu Q} |\nabla u|^{2q}\right)^{1/2q}.$$

As the similar estimate holds for v in place of w, we obtain

$$\big(\operatorname{av}_Q |\nabla u|^{q^*} \big)^{1/q^*} \leqslant C \big(\operatorname{av}_{\mu Q} |\nabla u|^{2q} \big)^{1/2q}.$$

Note that this inequality holds not just for Q but for all cubes Q' with $\overline{2Q'}$ contained in the open set where u is a weak solution. As $q^* > 2q$, this set of inequalities self-improves with $2q$ replaced by any $0 < p < 2q$ (see [22]) and, in particular,

$$\big(\operatorname{av}_Q |\nabla u|^{q^*} \big)^{1/q^*} \leqslant C \big(\operatorname{av}_{\mu Q} |\nabla u|^2 \big)^{1/2}.$$

If $q \geqslant n$ and $n \geqslant 2$, then we may as well consider $q > n$. Then (8.3) becomes

$$\sup_Q I \leqslant CR \operatorname{av}_{\mu Q} V \sup_{\mu Q} |u|$$

so that the pointwise bound for ∇u follows by Lemma 8.3. If $n = 1$, we already obtained a pointwise bound for ∇u and again Lemma 8.3 applies. $\square$

Proof of Lemma 8.6. — It suffices to incorporate the Caccioppoli inequality (8.2) in the inequalities of Lemma 8.6. $\square$

Proof of Lemma 8.7. — It suffices to combine Lemma 8.3 and Lemma 8.5. $\square$

Proof of Lemma 8.8. — It suffices to see the case $R^2 \operatorname{av}_Q V \geqslant 1$. Then, combine Lemma 8.6, the mean value inequality (8.1) with $r = 2$ and Lemma 8.7. $\square$

9. Riesz transforms

This section is concerned with the proof of Theorem 1.2, item 3. We present an argument inspired by [32] which also gives us a second proof of part of item 1[5].

9.1. A reduction

We know that it suffices to establish the boundedness of $\nabla H^{-1/2}$ and of $V^{1/2} H^{-1/2}$ on L^p for the appropriate ranges of p. As already observed, the case $1 < p \leqslant 2$ is already taken care of with no assumption on V. We henceforth assume $p > 2$ and $V \in A_\infty$.

---

[5] In this section, L^p denotes either $L^p(\mathbb{R}^n, \mathbb{C})$ or $L^p(\mathbb{R}^n, \mathbb{C}^n)$.

By duality, we know that $H^{-1/2}\operatorname{div}$ and $H^{-1/2}V^{1/2}$ are bounded on L^p. Thus, if $\nabla H^{-1/2}$ is also bounded on L^p, we obtain that $\nabla H^{-1}\operatorname{div}$ and $\nabla H^{-1}V^{1/2}$ are bounded on L^p.

Reciprocally, if $\nabla H^{-1}\operatorname{div}$ and $\nabla H^{-1}V^{1/2}$ are bounded on L^p, then their adjoints are bounded on $L^{p'}$. Thus, if $F \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$,

$$\|H^{-1/2}\operatorname{div} F\|_{p'} = \|H^{1/2}H^{-1}\operatorname{div} F\|_{p'}$$
$$\leqslant C(\|\nabla H^{-1}\operatorname{div} F\|_{p'} + \|V^{1/2}H^{-1}\operatorname{div} F\|_{p'}) \leqslant C\|F\|_{p'}$$

where the first inequality follows from item 2 of Theorem 1.2. Hence, by duality, $\nabla H^{-1/2}$ is bounded on L^p.

The same treatment can be done on $V^{1/2}H^{-1/2}$. We have obtained

LEMMA 9.1. — *If $V \in A_\infty$ and $p > 2$, the L^p boundedness of $\nabla H^{-1/2}$ is equivalent to that of $\nabla H^{-1}\operatorname{div}$ and $\nabla H^{-1}V^{1/2}$, and the L^p boundedness of $V^{1/2}H^{-1/2}$ is equivalent to that of $V^{1/2}H^{-1}V^{1/2}$ and $V^{1/2}H^{-1}\operatorname{div}$.*

It suffices therefore to establish part of Corollary 1.5 namely,

PROPOSITION 9.2. — *Assume that $V \in B_q$ for some $q > 1$. Then for $2 < p \leqslant 2(q + \varepsilon)$, for some $\varepsilon > 0$ depending only on V, $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and $F \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$,*

$$\|V^{1/2}H^{-1}V^{1/2}f\|_p \leqslant C_p\|f\|_p, \quad \|V^{1/2}H^{-1}\operatorname{div} F\|_p \leqslant C_p\|F\|_p.$$

PROPOSITION 9.3. — *Assume that $V \in B_q$ for some $q > 1$. Then for $2 < p \leqslant q^* + \varepsilon$ for some $\varepsilon > 0$ depending only on V, $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and $F \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$,*

$$\|\nabla H^{-1}V^{1/2}f\|_p \leqslant C_p\|f\|_p, \quad \|\nabla H^{-1}\operatorname{div} F\|_p \leqslant C_p\|F\|_p.$$

The interest of such a reduction is that this allows us to use properties of weak solutions of H.

Note that Proposition 9.3 is void if $q \leqslant \frac{2n}{n+2}$ as $q^* \leqslant 2$. Note also that $q^* < 2q$ exactly when $q < n/2$. In this case, this statement yields a smaller range than the interpolation method in Section 6.

Proof of Proposition 9.2. — Fix a cube Q and and let $f \in C_0^\infty(\mathbb{R}^n)$ supported away from $4Q$. Then $u = H^{-1}V^{1/2}f$ is well defined on $\mathbb{R}^n$ with $\|V^{1/2}u\|_2 + \|\nabla u\|_2 \leqslant \|f\|_2$ by construction of H and

$$\int_{\mathbb{R}^n} Vu\varphi + \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^n} V^{1/2}f\varphi$$

for all $\varphi \in L^2$ with $\|V^{1/2}\varphi\|_2 + \|\nabla\varphi\|_2 < \infty$. In particular, the support condition on f implies that u is a weak solution of $-\Delta u + Vu = 0$ in $4Q$, hence $|u|^2$ is subharmonic on $4Q$. Let r such that $V \in B_r$ and note that

$V^{1/2} \in B_{2r}$ (see section 11). By Lemma 5.3 with $w = V^{1/2}$ $f = |u|^2$ and $s = 1/2$, we have

$$\left(\mathrm{av}_Q (V^{1/2} |u|)^{2r} \right)^{1/2r} \leqslant C \ \mathrm{av}_{\mu Q} (V^{1/2} |u|).$$

Thus, (5.1) holds with $T = V^{1/2} H^{-1} V^{1/2}$, $q_0 = 2r$, $p_0 = 2$ and $S = 0$. By Theorem 5.1, $V^{1/2} H^{-1} V^{1/2}$ is bounded on L^p for $2 < p < 2r$.

The argument is the same for $V^{1/2} H^{-1} \operatorname{div}$. This finishes the proof. □

Proof of Proposition 9.3. — We assume $q > \frac{2n}{n+2}$, that is $q^* > 2$. otherwise there is nothing to prove. We consider first the operator $\nabla H^{-1} V^{1/2}$.

Assume $q < n/2$. Fix a cube Q and and let $f \in C_0^\infty(\mathbb{R}^n)$ supported away from $4Q$. Let $u = H^{-1} V^{1/2} f$. As before, the support condition on f implies that u is a weak solution of $-\Delta u + V u = 0$ in $4Q$. Thanks to Lemma 8.4, (5.1) holds with $T = \nabla H^{-1} V^{1/2}$, $q_0 = q^*$ and $S = V^{1/2} H^{-1} V^{1/2}$. As S is bounded on L^{q^*} by Proposition 9.2 and $2 < q^* \leqslant 2q$, Theorem 5.1 implies that $\nabla H^{-1} V^{1/2}$ is bounded on L^p for $2 < p < q^*$. Finally, by the self-improvement of reverse Hölder estimates we can replace q by a slightly larger value and, therefore, L^p boundedness for $p < q^* + \varepsilon$ holds.

Assume next that $n/2 \leqslant q < n$. In this case, $q^* \geqslant 2q$. Again, we may as well assume $q > n/2$. Then, Lemma 8.5 yields, this time, (5.1) with $T = \nabla H^{-1} V^{1/2}$, $q_0 = q^*$ and $S = 0$. Hence, Theorem 5.1 implies that $\nabla H^{-1} V^{1/2}$ is bounded on L^p for $2 < p < q^*$. Again, by self-improvement of the B_q condition, it holds for $p < q^* + \varepsilon$.

Finally, if $q \geqslant n$, then, Lemma 8.5 yields (5.1) for any $2 < q_0 < \infty$ with $T = \nabla H^{-1} V^{1/2}$ and $S = 0$. Hence, Theorem 5.1 implies that $\nabla H^{-1} V^{1/2}$ is bounded on L^p for $2 < p < \infty$.

The argument is the same for $\nabla H^{-1} \operatorname{div}$ and this finishes the proof. □

10. L^p Domains of H and $H^{1/2}$

Proof of Corollary 1.3. — It is known that $-\Delta + V$ defined on $C_0^\infty(\mathbb{R}^n)$ is essentially m-accretive on $L^p(\mathbb{R}^n)$ if $V \in L_{loc}^p(\mathbb{R}^n)$. The domain of its extension is $\{u \in L^p(\mathbb{R}^n) \, ; \, -\Delta u + V u \in L^p(\mathbb{R}^n)\}$ with norm $\|u\|_p + \| - \Delta u + V u\|_p$. By (1.1) this norm is equivalent to $\|u\|_p + \|\Delta u\|_p + \|V u\|_p$ on $C_0^\infty(\mathbb{R}^n)$ when $V \in B_p$. The result follows. □

Proof of Corollary 1.4. — Let $E^p(\mathbb{R}^n) = \mathcal{D}_p(\nabla) \cap \mathcal{D}_p(V^{1/2})$ $= W^{1,p}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n, V^{p/2})$. Let us begin with the following lemma. □

LEMMA 10.1. — *If $1 < p < \infty$ and $V^{p/2} \in L_{loc}^1(\mathbb{R}^n)$, then $C_0^\infty(\mathbb{R}^n)$ is dense in $E^p(\mathbb{R}^n)$.*

Indeed, for $p = 2$ this is a well-known fact as $C_0^\infty(\mathbb{R}^n)$ is a core of the form domain of $-\Delta + V$. The proof of this fact (see, for instance, [10, pp. 157-158]) adapts to any p with $1 < p < \infty$.

We also remark that under the assumption $V \in L^1_{loc}$, $-\Delta + V$ has a bounded holomorphic functional calculus on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ ([13]), and in particular, $\|(-\Delta + V + 1)^{1/2}u\|_p \sim \|(-\Delta + V)^{1/2}u\|_p + \|u\|_p$ for all $u \in C_0^\infty(\mathbb{R}^n)$. Thus, it suffices to find the domain of $(-\Delta + V + 1)^{1/2}$.

Now, assume $V \in A_\infty$ and $1 < p < 2$ or $V \in B_{p/2}$ and $2 < p < \infty$. We have shown that $\|(-\Delta + V)^{1/2}u\|_p \sim \|\nabla u\|_p + \|V^{1/2}u\|_p$ for $u \in C_0^\infty(\mathbb{R}^n)$. Thus, using this and the lemma, $(-\Delta + V + 1)^{1/2}$ has a bounded extension from $E^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and this extension is invertible. This proves the result.

Remark 10.2. — It is not hard to show that the L^p-domain $(1 < p < \infty)$ of $(-\Delta + V)^{1/2}$ coincides with the domain of the square root of (minus) the infinitesimal generator of the semigroup $(e^{-tH})_{t>0}$ seen as an analytic and C_0-semigroup on L^p.

11. Some facts about A_∞ weights

That $V \in A_\infty$ implies $V^s \in B_{1/s}$ for $0 < s < 1$ was first observed implicitly in [37]. See also [23]. We give a direct proof for convenience.

PROPOSITION 11.1. — *Let V be a nonnegative measurable function. Then the followings are equivalent:*

(1) $V \in A_\infty$.
(2) *For all $s \in (0, 1)$, $V^s \in B_{1/s}$.*
(3) *There exists $s \in (0, 1)$, $V^s \in B_{1/s}$.*

Proof. — If $V^s \in B_{1/s}$ for some $s \in (0, 1)$, then by the self-improvement property of the B_q class, $V^s \in B_{\varepsilon+1/s}$ for some $\varepsilon > 0$. Hence, $V \in B_{1+s\varepsilon}$, which implies $V \in A_\infty$. Thus, (2) implies (3) implies (1).

Assume $V \in A_\infty$ and $s \in (0, 1)$. Since A_∞ weights satisfy a reverse Hölder inequality, there is $r > 1$ such that $V \in B_r$. Hence, for $A > 1$ and any cube Q, the set $E_Q = \{x \in Q \,;\, V^s(x) > A \operatorname{av}_Q V^s\}$ satisfies

$$\frac{\int_{E_Q} V}{\int_Q V} \leqslant C \left(\frac{|E_Q|}{|Q|} \right)^{1/r'}.$$

Since $|E_Q| \leqslant A^{-1}|Q|$ by Tchebytchev's inequality, we obtain $\int_{E_Q} V \leqslant CA^{-1/r'} \int_Q V$.

Choose A such that $CA^{-1/r'} \leqslant 1/2$. We have

$$\int_Q V = \int_{Q \setminus E_Q} V + \int_{E_Q} V \leqslant (A \operatorname{av}_Q V^s)^{1/s} |Q| + \frac{1}{2} \int_Q V$$

which yields

$$\operatorname{av}_Q V \leqslant 2(A \operatorname{av}_Q V^s)^{1/s}.$$

$\square$

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