ANNALES

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Tome 57, nº 6 (2007), p. 2031-2040.

<http://aif.cedram.org/item?id=AIF_2007__57_6_2031_0>

Ann. Inst. Fourier, Grenoble
57, 6 (2007) 2031-2040

BERNSTEIN-SATO POLYNOMIALS AND SPECTRAL NUMBERS

by Andréa G. GUIMARÃES & Abramo HEFEZ (*)

ABSTRACT. — In this paper we will describe a set of roots of the Bernstein-Sato polynomial associated to a germ of complex analytic function in several variables, with an isolated critical point at the origin, that may be obtained by only knowing the spectral numbers of the germ. This will also give us a set of common roots of the Bernstein-Sato polynomials associated to the members of a μ-constant family of germs of functions. An example will show that this set may sometimes consist of all common roots.

RÉSUMÉ. — Dans cet article nous décrivons un ensemble de racines du polynôme de Bernstein-Sato associées à un germe de fonction analytique à plusieurs variables complexes, avec un point critique isolé à l'origine, qui peuvent être obtenues en connaissant seulement les nombres spectraux du germe. Ceci nous donnera aussi un ensemble de racines communes aux polynômes de Bernstein-Sato associées aux membres d'une famille à μ-constant de germes de fonctions. Un exemple nous montrera que cet ensemble peut parfois donner toutes les racines communes.

1. Introduction

Let $\mathcal{O}_n$ be the ring $\mathbb{C}\{x_1,\dots,x_n\}$ of germs of holomorphic functions at the origin of $\mathbb{C}^n$. Given $f \in \mathcal{O}_n$, it is well known (cf. [1] and [2]) that there exist a non-zero polynomial $b(s) \in \mathbb{C}[s]$ and a differential operator $P(s)$, holomorphic in $x_1,\dots,x_n$ and polynomial in s and in $\partial/\partial x_1,\dots,\partial/\partial x_n$, such that

$$P(s)f(x)^{s+1} = b(s)f(x)^s.$$

The monic generator $b_f(s)$ of the ideal of such polynomials $b(s)$ is the Bernstein-Sato polynomial (or, simply, the Bernstein polynomial) of the

Keywords: Bernstein polynomial, Spectral numbers, Gauss-Manin connection and Brieskorn lattice.
Math. classification: 32S40, 58K15, 58K65.
(*) Partially supported by a CAPES fellowship and by a CNPq grant, respectively.

germ f. Since $b_f(s)$ is always divisible by $s+1$, we will consider the reduced Bernstein polynomial $\widetilde{b}_f(s) = b_f(s)/(s+1)$.

Malgrange, in [11], proved that, when f has an isolated critical point at the origin and $f(0) = 0$, the roots of $b_f(s)$ are rational numbers. In fact, he reinterpreted the reduced Bernstein-Sato polynomial as the minimal polynomial of a linear operator on some quotient of Brieskorn lattices, proving, as a consequence, that if α is a root of $\widetilde{b}_f(s)$, then $\exp(-2\pi\sqrt{-1}\alpha)$ is an eigenvalue of the monodromy of the Gauss-Manin connection associated to f, which, by Brieskorn's Monodromy Theorem (cf. [4]), is a root of unity. Afterwards, Kashiwara in [8], using resolutions of singularities, proved the rationality of the roots of $b_f(s)$ for any f.

Associated to a given f, vanishing and with an isolated critical point at the origin, there are μ rational numbers, called the spectral numbers of f, which are known to be topological invariants; that is, they are constant along any μ-constant deformation of f (cf. [16]). It is also known that for each root of $\widetilde{b}_f(s)$ there is a spectral number such that the sum of the two numbers is an integer. In general, it is only possible to give bounds on these integers (see [15]). In this paper we will show that, for such f, a large set of spectral numbers already determines a corresponding set of roots of the Bernstein-Sato polynomial of f.

Several other authors contributed to this subject (eg. [3], [14], [13], [18], [5], [6], [10], [7]). In particular, some of them have already considered the connection among the roots of $\widetilde{b}_f(s)$ and spectral numbers. For example, M. Saito, in [14], exhibits a well determined set of spectral numbers such that the symmetric of each element of this set, subtracted by 1, is a root of $\widetilde{b}_f(s)$. In [7], a set of spectral numbers containing Saito's one was found with the same property as above, but in the particular case of a two variable function f with isolated critical point at the origin and finite monodromy.

In the present work, more precisely in Theorem 3.3, we extend the above quoted result of [7], without any restriction on the monodromy or on the number of variables of f, with the only assumption of isolated critical point. As an application, in Corollary 3.5, we get a set of common roots of the Bernstein-Sato polynomials of all members in a μ-constant family of germs of functions vanishing at the origin and with an isolated critical point there.

2. Brieskorn Lattices and Bernstein Polynomials

In what follows f will always be in the maximal ideal of $\mathcal{O}_n$ and with an isolated critical point at the origin. We denote by

$$J(f) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$$

the Jacobian ideal of f in $\mathcal{O}_n$, whose $\mathbb{C}$-codimension is Milnor's number μ.

Let $\Omega^m = \Omega^m_{\mathbb{C}^n,0}$ be the $\mathcal{O}_n$-module of germs of holomorphic m-forms at the origin of $\mathbb{C}^n$. Consider (cf. [4] or [10])

$$H'' = \Omega^n / df \wedge d\Omega^{n-2},$$

$$H' = df \wedge \Omega^{n-1} / df \wedge d\Omega^{n-2}.$$

It follows immediately from the above definitions that, as $\mathbb{C}$-vector spaces,

$$H''/H' \simeq \Omega^n / \left(df \wedge \Omega^{n-1}\right) \simeq \mathcal{O}_n / J(f).$$

The $\mathcal{O}_n$-modules H'' and H' are endowed with a structure of $\mathbb{C}\{t\}$-module, defining

$$t[w] = [fw], \quad \forall [w] \in H''.$$

The $\mathbb{C}$-linear isomorphism

$$\begin{array}{cccc} \nabla : & H' & \longrightarrow & H'' \\ & [df \wedge w] & \mapsto & [dw] \end{array}$$

satisfies Leibniz rule, with respect to the $\mathbb{C}\{t\}$-module structure, so it extends uniquely to a meromorphic connection

$$\partial_t : H'' \otimes_{\mathbb{C}\{t\}} \mathcal{K} \longrightarrow H'' \otimes_{\mathbb{C}\{t\}} \mathcal{K},$$

where $\mathcal{K}$ is the field of fraction of $\mathbb{C}\{t\}$ (cf. [12]). The set $H = H'' \otimes_{\mathbb{C}\{t\}} \mathcal{K}$ is called the Gauss-Manin system and the operator ∂_t is called the Gauss-Manin connection, associated to f.

The connection ∂_t is regular and $\dim_{\mathcal{K}} H = \mu$ (cf. [4]). Notice that H' and H'' are lattices in H (i.e., free $\mathbb{C}\{t\}$-submodules of H which generate H over $\mathcal{K}$).

Since the connection is regular, the lattices H' and H'' may be saturated with respect to $\partial_t t$ as follows:

$$\widetilde{H''} = \sum_{k \geqslant 0} (\partial_t t)^k H'' \quad \text{and} \quad \widetilde{H'} = \sum_{k \geqslant 0} (\partial_t t)^k H'.$$

One can prove that $t\widetilde{H''} = \widetilde{H'}$ (cf. [11]).

It was shown in [11] that the minimal polynomial of the $\mathbb{C}$-endomorphism $-\overline{\partial_t t} : \widetilde{H''}/\widetilde{H'} \to \widetilde{H''}/\widetilde{H'}$, induced by $-\partial_t t : \widetilde{H''} \to \widetilde{H''}$, is the reduced Bernstein polynomial of f.

For $\alpha \in \mathbb{Q}$, set

$$\mathcal{V}^{\alpha}H = \bigoplus_{\alpha \leqslant \beta < \alpha+1} \mathbb{C}\{t\}C_{\beta}, \qquad \mathcal{V}^{>\alpha}H = \bigoplus_{\alpha < \beta \leqslant \alpha+1} \mathbb{C}\{t\}C_{\beta},$$

where $C_{\beta} = \{v \in H;\ \exists l \in \mathbb{N}, (-\partial_t t + \beta + 1)^{l} v = 0\}$, and

$$\mathrm{Gr}_{\mathcal{V}}^{\alpha}H = \mathcal{V}^{\alpha}H/\mathcal{V}^{>\alpha}H.$$

Then $\mathcal{V}$ defines a decreasing filtration on H, called the $\mathcal{V}$-filtration, that has the following properties (cf. [13] and [14]):

(2.1) $H = \bigcup_{\alpha} \mathcal{V}^{\alpha}H$ and $\mathcal{V}^{\alpha}H$ is a lattice of H.

(2.2) $t(\mathcal{V}^{\alpha}H) \subset \mathcal{V}^{\alpha+1}H$ and $\partial_t(\mathcal{V}^{\alpha}H) \subset \mathcal{V}^{\alpha-1}H$.

(2.3) If $v \in \mathcal{V}^{\alpha}H$, then $hv \in \mathcal{V}^{\alpha}H$, for all $h \in \mathcal{O}_n$.

(2.4) The operator $(-\partial_t t + \alpha + 1)$ is nilpotent on $\mathrm{Gr}_{\mathcal{V}}^{\alpha}H \cong C_{\alpha}$.

If $N \subset M$ are lattices of H, then the $\mathcal{V}$-filtration on H induces a $\mathcal{V}$-filtration on M by intersection, i.e.,

$$\mathcal{V}^{\alpha}M = (\mathcal{V}^{\alpha}H) \cap M, \qquad \mathcal{V}^{>\alpha}M = \left(\mathcal{V}^{>\alpha}H\right) \cap M;$$

and on the quotient M/N by the expressions

$$\mathcal{V}^{\alpha}\left(M/N\right) = \left(\mathcal{V}^{\alpha}M + N\right)/N, \qquad \mathcal{V}^{>\alpha}\left(M/N\right) = \left(\mathcal{V}^{>\alpha}M + N\right)/N.$$

It is easy to check that

$$\mathrm{Gr}_{\mathcal{V}}^{\alpha}M = \frac{\mathcal{V}^{\alpha}M}{\mathcal{V}^{>\alpha}M} \simeq \frac{\mathcal{V}^{\alpha}M + \mathcal{V}^{>\alpha}H}{\mathcal{V}^{>\alpha}H}, \qquad \mathrm{Gr}_{\mathcal{V}}^{\alpha}\left(M/N\right) \simeq \frac{\mathrm{Gr}_{\mathcal{V}}^{\alpha}M}{\mathrm{Gr}_{\mathcal{V}}^{\alpha}N}.$$

A number $\alpha \in \mathbb{Q}$, is called a spectral number of f of multiplicity $\mathrm{d}(\alpha)$, if

$$\mathrm{d}(\alpha) := \dim_{\mathbb{C}}\mathrm{Gr}_{\mathcal{V}}^{\alpha}\left(H''/H'\right) > 0.$$

Since $\dim_{\mathbb{C}}\left(H''/H'\right) = \dim_{\mathbb{C}}\mathcal{O}_n/J(f)) = \mu$, it follows that there are exactly μ spectral numbers $\alpha_1, \ldots, \alpha_{\mu}$, where each α_i is counted with its multiplicity $d(\alpha_i)$. In [16], Varchenko proved that the spectral numbers are invariant in any μ-constant deformation of f.

Finally, a rational number $-(\alpha+1)$ is a root of $\widetilde{b}_f$ if, and only if, $\mathrm{Gr}_{\mathcal{V}}^{\alpha}\left(\widetilde{H''}/\widetilde{H'}\right) \neq (0)$ and the multiplicity of this root is the nilpotency degree of the action of $(-\partial_t t + \alpha + 1)$ over $\mathrm{Gr}_{\mathcal{V}}^{\alpha}\left(\widetilde{H''}/\widetilde{H'}\right)$ (cf. [11]).

The next proposition will be fundamental for the proof of our result.

Proposition 2.1. — *If $h' \in \mathcal{O}_n$ is such that $f - h' \in J(f)$, then*

$$\widetilde{H'} \subset H' + \sum_{k \geqslant 0} (\partial_t t)^k \mathcal{O}_n[h'dx] \subset \widetilde{H''}.$$

Proof. — Initially, observe that from the definition of H' it follows that

$$H' = [J(f)dx]. \tag{2.6}$$

□

Let $z = [df \wedge \omega] \in H'$, then by Leibniz' rule we have

$$(\partial_t t)z = \partial_t(tz) = z + t\partial_t z = z + t[d\omega] = z + [fd\omega] \in H' + \mathcal{O}_n[fdx].$$

This, in view of the definition of h' and of (2.6), yields

$$(\partial_t t)H' \subset H' + \mathcal{O}_n[fdx] = H' + \mathcal{O}_n[h'dx].$$

Now, suppose inductively that

$$(\partial_t t)^i H' \subset H' + \sum_{k=0}^{i-1} (\partial_t t)^k \mathcal{O}_n[h'dx].$$

Then one has

$$(\partial_t t)^{i+1} H' \subset (\partial_t t)^i (H' + \mathcal{O}_n[h'dx]) = (\partial_t t)^i H' + (\partial_t t)^i \mathcal{O}_n[h'dx] \subset$$

$$H' + \sum_{k=0}^{i-1} (\partial_t t)^k \mathcal{O}_n[h'dx] + (\partial_t t)^i \mathcal{O}_n[h'dx] = H' + \sum_{k=0}^{i} (\partial_t t)^k \mathcal{O}_n[h'dx].$$

This proves that, for all i,

$$\sum_{k=0}^{i} (\partial_t t)^k H' \subset H' + \sum_{k=0}^{i-1} (\partial_t t)^k \mathcal{O}_n[h'dx],$$

hence establishing the result. □

3. Roots of Bernstein-Sato Polynomials and Spectral Numbers

In this section we will prove our main result, Theorem 3.3, in which we describe a set of roots of $\tilde{b}_f(s)$ which may be determined by only knowing the spectral numbers of f. As a consequence, we get a set of roots of $\tilde{b}_f(s)$ which are invariant in a μ-constant deformation of f. Example 3.6 will show that, in general, our result is optimal, in the sense that in some cases it may give all common roots for the members of the family.

Proposition 3.1. — *Let $h' \in \mathcal{O}_n$ be such that $f - h' \in J(f)$. If $[h'dx] \in \mathcal{V}^{\alpha'}\widetilde{H''}$, for some rational number α', then*

$$\widetilde{H'} \subset H' + \mathcal{V}^{\alpha'}\widetilde{H''}.$$

Proof. — For any $h' \in \mathcal{O}_n$ such that $[h'dx] \in \mathcal{V}^{\alpha'}\widetilde{H''}$, we get from (2.3) and (2.2) that

$$\sum_{k\geqslant 0}(\partial_t t)^k \mathcal{O}_n[h'dx] \subset \mathcal{V}^{\alpha'}\widetilde{H''}.$$

□

On the other hand, if $f - h' \in J(f)$, it follows from Proposition 2.1 that

$$\widetilde{H'} \subset H' + \mathcal{V}^{\alpha'}\widetilde{H''}.$$

□

Proposition 3.2. — *If for some rational number α' one has*

$$\widetilde{H'} \subset H' + \mathcal{V}^{\alpha'}\widetilde{H''},$$

then for every spectral number α associated to f such that $\alpha < \alpha'$, the number $-(\alpha+1)$ is a root of b_f.

Proof. — For every rational number $\alpha < \alpha'$, the hypothesis implies, that

$$\mathcal{V}^{\alpha}\widetilde{H'} \subset \mathcal{V}^{\alpha}H' + \mathcal{V}^{\alpha'}\widetilde{H''},$$

hence

$$\mathrm{Gr}^{\alpha}_{\mathcal{V}}\left(\frac{\widetilde{H'}}{H'}\right) = (0).$$

So,

$$\mathrm{Gr}^{\alpha}_{\mathcal{V}}\left(\frac{\widetilde{H''}}{\widetilde{H'}}\right) \simeq \mathrm{Gr}^{\alpha}_{\mathcal{V}}\left(\frac{\widetilde{H''}/H'}{\widetilde{H'}/H'}\right) \simeq \mathrm{Gr}^{\alpha}_{\mathcal{V}}\left(\frac{\widetilde{H''}}{H'}\right). \tag{3.1}$$

But, if α is a spectral number, then

$$0 \neq \mathrm{Gr}^{\alpha}_{\mathcal{V}}\left(\frac{H''}{H'}\right) \subset \mathrm{Gr}^{\alpha}_{\mathcal{V}}\left(\frac{\widetilde{H''}}{H'}\right).$$

This, together with (3.1), imply that

$$\mathrm{Gr}^{\alpha}_{\mathcal{V}}\left(\frac{\widetilde{H''}}{\widetilde{H'}}\right) \neq 0.$$

Therefore, as mentioned just before Proposition 2.1, we have that $-(\alpha+1)$ is a root of b_f, for all spectral number $\alpha < \alpha'$. □

Now, put

$$\eta = \sum_{i=1}^{n}(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

then $dx = \dfrac{1}{n} d\eta$, and consequently, from the definition of ∂_t, we have that

$$\partial_t{}^{-1}[dx] = \left[\frac{1}{n} df \wedge \eta\right] = \left[\frac{1}{n}\left(x_1 f_{x_1} + x_2 f_{x_2} + \cdots + x_n f_{x_n}\right) dx\right]. \tag{3.2}$$

Theorem 3.3. — *Let $f \in \mathcal{O}_n$ with an isolated critical point at the origin and vanishing there. If α_1 is the smallest spectral number of f, then for every spectral number α such that $\alpha < \alpha_1 + 1$, the number $-(\alpha+1)$ is a root of b_f.*

Proof. — It is well known that $H'' \subset \mathcal{V}^{\alpha_1} H''$ (cf. [9], Lemma 3.2.7). Hence we have $[dx] \in \mathcal{V}^{\alpha_1} H''$. So, for $h' = f - (1/n)(x_1 f_{x_1} + \cdots + x_n f_{x_n})$, we have by (3.2) and (2.2) that

$$[h' dx] = (t - \partial_t^{-1})[dx] \in \mathcal{V}^{\alpha_1+1} H'' \subset \mathcal{V}^{\alpha_1+1} \widetilde{H''}.$$

Now, the result follows from Propositions 3.1 and 3.2, where we put $\alpha' = \alpha_1 + 1$. □

Using other methods, Saito, in [14] (Theorem 0.7), has shown that if α is a non-positive spectral number, then $-(\alpha+1)$ is a root of b_f. Since one always has $\alpha_1 + 1 > 0$ (cf. [9], Chap. II, (8.3.4)), then Theorem 3.3, above, contains Saito's result and the example below shows that in fact it may improve it.

Example 3.4. — Let

$$f = x_1^6 + x_2^5 + x_1^4 x_2^3 + x_1^4 x_2^2 + x_1^3 x_2^3.$$

Then by means of the Singular software(1) one may easily determine the following numbers:

Spectral Numbers:

$$-\tfrac{19}{30}, -\tfrac{7}{15}, -\tfrac{13}{30}, -\tfrac{3}{10}, -\tfrac{4}{15}, -\tfrac{7}{30}, -\tfrac{2}{15}, -\tfrac{1}{10}, -\tfrac{1}{15}, -\tfrac{1}{30}, \tfrac{1}{30}, \tfrac{1}{15}, \tfrac{1}{10}, \tfrac{2}{15}, \tfrac{7}{30}, \tfrac{4}{15}, \tfrac{3}{10}, \tfrac{13}{30}, \tfrac{7}{15}, \tfrac{19}{30}.$$

Roots of b_f of the form $-(\alpha+1)$, for α a spectral number smaller than $\alpha_1 + 1 = \frac{11}{30}$ (Theorem 3.3):

$$-\tfrac{11}{30}, -\tfrac{8}{15}, -\tfrac{17}{30}, -\tfrac{7}{10}, -\tfrac{11}{15}, -\tfrac{23}{30}, -\tfrac{13}{15}, -\tfrac{9}{10}, -\tfrac{14}{15}, -\tfrac{29}{30}, -\tfrac{31}{30}, -\tfrac{16}{15}, -\tfrac{11}{10}, -\tfrac{17}{15}, -\tfrac{37}{30}, -\tfrac{19}{15}, -\tfrac{13}{10}.$$

Roots of b_f of the form $-(\alpha+1)$, for α a non-positive spectral number ([14], Theorem 0.7):

$$-\tfrac{11}{30}, -\tfrac{8}{15}, -\tfrac{17}{30}, -\tfrac{7}{10}, -\tfrac{11}{15}, -\tfrac{23}{30}, -\tfrac{13}{15}, -\tfrac{9}{10}, -\tfrac{14}{15}, -\tfrac{29}{30}.$$

(1) www.singular.uni-kl.de

We also point out that Hertling and Stahlke [7], Theorem 5.1, got the same result as our Theorem 3.3, but when f belongs to $\mathbb{C}\{x_1, x_2\}$ and has finite monodromy. Also, Briançon et al. [3], Proposition B.3.1.2.1.b, showed that the same holds for $f \in \mathbb{C}\{x_1, x_2\}$ and non-degenerate with respect to its Newton polygon.

Now, from the invariance of the the spectral numbers in a μ-constant family (cf. [16], Theorem 2), we get the following result.

Corollary 3.5. — *Let a μ-constant family of germs of functions in $\mathcal{O}_n$ vanishing at the origin, with an isolated critical point there, be given. Let α_1 be the smallest spectral number of the members of the family. Then for any spectral number α such that $\alpha < \alpha_1 + 1$, we have that $-(\alpha + 1)$ is a common root of the Bernstein-Sato polynomials of all member of the family.*

The next example will show that this is the best result one can get in general, without knowing any additional information about the family of germs of functions.

Example 3.6. — Consider the μ-constant ($\mu = 24$) family of germs:

$$f_t = x_1^4 x_2 + x_1 x_2^6 + t x_1^3 x_2^3.$$

Using the Singular software, we get:

Spectral Numbers of f_0 and f_1 less than $\alpha_1 + 1 = \frac{8}{23}$:

$$-\tfrac{15}{23}, -\tfrac{12}{23}, -\tfrac{10}{23}, -\tfrac{9}{23}, -\tfrac{7}{23}, -\tfrac{6}{23}, -\tfrac{5}{23}, -\tfrac{4}{23}, -\tfrac{3}{23}, -\tfrac{2}{23}, -\tfrac{1}{23}, 0, \tfrac{1}{23}, \tfrac{2}{23}, \tfrac{3}{23}, \tfrac{4}{23}, \tfrac{5}{23}, \tfrac{6}{23}, \tfrac{7}{23};$$

Common roots of b_{f_0} and b_{f_1}:

$$-\tfrac{8}{23}, -\tfrac{11}{23}, -\tfrac{13}{23}, -\tfrac{14}{23}, -\tfrac{16}{23}, -\tfrac{17}{23}, -\tfrac{18}{23}, -\tfrac{19}{23}, -\tfrac{20}{23}, -\tfrac{21}{23}, -\tfrac{22}{23}, -1, -\tfrac{24}{23}, -\tfrac{25}{23}, -\tfrac{26}{23}, -\tfrac{27}{23}, -\tfrac{28}{23}, -\tfrac{29}{23}, -\tfrac{30}{23};$$

The other roots of b_{f_1}:

$$-\tfrac{9}{23}, -\tfrac{10}{23}, -\tfrac{12}{23}, -\tfrac{15}{23};$$

The other roots of b_{f_0}:

$$-\tfrac{32}{23}, -\tfrac{33}{23}, -\tfrac{35}{23}, -\tfrac{38}{23}.$$

With this, we see that the only common roots of b_{f_0} and b_{f_1} are the numbers given by Corollary 3.5.

The use of Propositions 3.1 and 3.2, could give more roots of Bernstein-Sato polynomials than those obtained by Theorem 3.3, if one could find an h', with $f - h' \in J(f)$, such that $[h'dx] \in \mathcal{V}^{\alpha'}\widetilde{H''}$, for some rational number α' larger than $\alpha_1 + 1$. This, in general, is very difficult to check. However, in some circumstances this is possible. For example, when f is non-degenerated with respect to its Newton polyhedron, Varchenko and Khovanskii in [17] have shown that the $\mathcal{V}$-filtration of $[hdX] \in H''$ corresponds to the Newton-order of the series $x_1 \dots x_n h$ minus 1. This is illustrated in the following example.

Example 3.7. — Consider $f = X_1^5 + X_2^7 + X_1^3 X_2^5$.

It is easy to verify that f is non-degenerated with respect to its Newton polygon.

The spectral numbers of f are:

$$-\tfrac{23}{35}, -\tfrac{18}{35}, -\tfrac{16}{35}, -\tfrac{13}{35}, -\tfrac{11}{35}, -\tfrac{9}{35}, -\tfrac{8}{35}, -\tfrac{6}{35}, -\tfrac{4}{35}, -\tfrac{3}{35}, -\tfrac{2}{35}, -\tfrac{1}{35},$$
$$\tfrac{1}{35}, \tfrac{2}{35}, \tfrac{3}{35}, \tfrac{4}{35}, \tfrac{6}{35}, \tfrac{8}{35}, \tfrac{9}{35}, \tfrac{11}{35}, \tfrac{13}{35}, \tfrac{16}{35}, \tfrac{18}{35}, \tfrac{23}{35}.$$

If

$$h' = f - \frac{1}{5}x_1 f_{x_1} - \frac{1}{7}x_2 f_{x_2} = -\frac{11}{35}x_1^3 x_2^5,$$

then the Newton-order of $[h'dX]$ is $\rho([h'dX]) = \frac{23}{35}$. Then, from Propositions 3.1 and 3.2, we have that the numbers of the form $-(\alpha+1)$, where α is a spectral number of f and $\alpha < \frac{23}{33}$, are roots of the Bernstein-Sato Polynomial of f. These are:

$$-\tfrac{12}{35}, -\tfrac{17}{35}, -\tfrac{19}{35}, -\tfrac{22}{35}, -\tfrac{24}{35}, -\tfrac{26}{35}, -\tfrac{27}{35}, -\tfrac{29}{35}, -\tfrac{31}{35}, -\tfrac{32}{35}, -\tfrac{33}{35}, -\tfrac{34}{35},$$
$$-\tfrac{36}{35}, -\tfrac{37}{35}, -\tfrac{38}{35}, -\tfrac{39}{35}, -\tfrac{41}{35}, -\tfrac{43}{35}, -\tfrac{44}{35}, -\tfrac{46}{35}, -\tfrac{48}{35}, -\tfrac{51}{35}, -\tfrac{53}{35}.$$

Giving the roots $-\frac{48}{35}, -\frac{51}{35}, -\frac{53}{35}$, in addition to those given by Theorem 3.3.

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Manuscrit reçu le 11 mai 2006,
accepté le 12 octobre 2006.

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