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# INVARIANT MEASURES FOR THE STABLE FOLIATION ON NEGATIVELY CURVED PERIODIC MANIFOLDS

**by François LEDRAPPIER (*)**

ABSTRACT. — We classify reversible measures for the stable foliation on manifolds which are infinite covers of compact negatively curved manifolds. We extend the known results from hyperbolic surfaces to varying curvature and to all dimensions.

RÉSUMÉ. — Nous décrivons les mesures réversibles associées au feuilletage stable du flot géodésique sur une variété périodique de courbure négative. Nous étendons ainsi ce qui était connu pour les surfaces hyperboliques aux cas de courbure variable et de dimension supérieure.

## 1. Introduction

Let $M$ be a Riemannian manifold of dimension $n > 1$ and $T^1M$ its unit tangent bundle. The *geodesic flow* is the flow $g^s : T^1M \to T^1M$ which moves a line element at unit speed along the geodesic it determines. We write $X$ for the *geodesic spray*, the vector field defining the geodesic flow. At the line element $v \in T^1M$, the vector $X(v)$ is the horizontal lift of $v$. The manifold is called *Anosov* [(1)] if the sectional curvature is pinched between two negative constants and the first derivative of the curvature is bounded. Then, ([3], Appendix III) there is a uniformly continuous decomposition of $T^1M$ as the Whitney sum of three bundles $E^{ss}, E^{su}$ and $\mathbb{R}X$ and there

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(1) Contrarily to the usual definition, we do not assume here that $M$ is compact.

exist $C, \lambda$ positive such that, for $s > 0$:

$$||dg^s v||_{g^s\omega} \leqslant Ce^{-\lambda s}||v||_\omega \text{ for } v \in E^{ss}(\omega),$$
$$||dg^{-s} v||_{g^{-s}\omega} \leqslant Ce^{-\lambda s}||v||_\omega \text{ for } v \in E^{su}(\omega).$$

The bundles $E^{ss}, E^s := E^{ss} \oplus X, E^{su}, E^u := E^{su} \oplus X$ are all integrable and the integral manifolds at a point $\omega$ are respectively the *strong stable manifold* $W^{ss}(\omega)$, the *stable manifold* $W^s(\omega)$, the *strong unstable manifold* $W^{su}(\omega)$, the *unstable manifold* $W^u(\omega)$. The *strong stable manifold* at a line element $\omega$ is the geometric location of all $\omega' \in T^1(M)$ for which $d(g^s\omega, g^s\omega') \xrightarrow[s\to\infty]{} 0.$[2] The strong stable manifolds are smoothly embedded $(n-1)$ submanifolds of $T^1M$. The Sasake Riemannian metric on $T^1M$ defines by restriction to each $W^{ss}(\omega)$ a metric $g^{ss}$ and the associated distance $d^{ss}$ and Laplacian $\Delta^{ss}$ on $C^2$ functions. As $\omega$ varies the $W^{ss}(\omega)$ form a continuous lamination $\mathcal{W}^{ss}$, and, in local charts, the distance $d^{ss}$ and the Laplacian $\Delta^{ss}$ vary continuously. The stable lamination $\mathcal{W}^s$, the strong unstable lamination $\mathcal{W}^{su}$ and the unstable lamination $\mathcal{W}^u$ are defined analogously and have the same properties. In particular, denote by $\Delta^s$ the Laplacian along the stable manifolds $\mathcal{W}^s$ defined by the restriction of the Sasake metric. The manifold $M$ is called *periodic* if it is a regular cover of a finite volume manifold $M_0$. The *period* of $M$ is $M_0$, and the *symmetry group* of $M$ (relative to $M_0$) is the group of deck transformations. A periodic manifold is called *cocompact* if $M_0$ is compact.

In the particular case when $M$ is a cocompact periodic hyperbolic surface, the stable manifolds are one-dimensional and have a natural parametrization as the orbits of the horocycle flow. For an invariant ergodic Radon measure $m$ for the horocycle flow, there is a real $\alpha$ such that $m \circ g^s = e^{\alpha s}m$([15], Theorem 1). In this paper we extend this property to a general Anosov cocompact periodic manifold. We describe now a property of a Radon measure which generalizes invariance to higher dimensions and variable negative curvature. The following definitions are in the spirit of [6], [12] and [9]. Consider a second order differential operator $\mathcal{L}$ on a manifold $V$. We assume that $\mathcal{L}1 = 0$. A positive Radon measure on $V$ is said to be $\mathcal{L}$ harmonic (or harmonic with respect to $\mathcal{L}$) if for any $f$ $C^2$ function with compact support, we have

$$\int (\mathcal{L}f)dm = 0. \tag{1.1}$$

[2] Uniform continuity and regularity of the invariant manifolds are not explicit in [3], and are usually established in the compact case. But the proofs use only local coordinates around the trajectory with bounded geometry, see *e.g.*, [14].

A harmonic measure is called ergodic if it cannot be decomposed into a nontrivial convex combination of harmonic measures. A positive Radon measure on $V$ is said to be $\mathcal{L}$ reversible (or reversible with respect to $\mathcal{L}$) if for any $f, g$ $C^2$ functions with compact support, we have

$$\int (\mathcal{L}f) g dm = \int f(\mathcal{L}g) dm. \tag{1.2}$$

This property is called 'fully invariant' in [12] and 'self-adjoint' in [9]. Even if this not used here, 'reversible' alludes to a property of $m$ with respect to the associated leafwise diffusion. [9] contains a full discussion and examples of harmonic and reversible finite measures with respect to a leafwise diffusion operator along the stable foliation on a compact Anosov manifold.

By choosing $g$ in (1.2) constant on the support of $f$, we see that reversible measures are harmonic. A Radon measure $m$ which is reversible with respect to the Laplacian $\Delta^{ss}$ has conditional measures along $W^{ss}$ leaves which are proportional to the Riemannian volume. Indeed, we know ([6], Theorem 1,(c)) that a $\Delta^{ss}$ harmonic measure has conditional measures along strong stable manifolds that have a density with respect to the Riemannian volume $m^{ss}$. Moreover, this density $k$ is a positive function satisfying $\Delta^{ss} k = 0$. Write the reversibility condition in a decomposition into local strong stable manifolds: for all $f, g$ with compact support on $M$:

$$\int \left( \int_{W^{ss}_{loc}} (\Delta^{ss} f) g k dm^{ss} \right) dm = \int \left( \int_{W^{ss}_{loc}} f (\Delta^{ss} g) k dm^{ss} \right) dm.$$

On the other hand, since $\Delta^{ss}$ is symmetric on $W^{ss}_{loc}$, we can also write:

$$\int \left( \int_{W^{ss}_{loc}} (\Delta^{ss} f) g k dm^{ss} \right) dm = \int \left( \int_{W^{ss}_{loc}} f \Delta^{ss} (gk) dm^{ss} \right) dm.$$

Comparing the two formulas, we see that for all $f, g$ with compact support

$$\int \left( \int_{W^{ss}_{loc}} f < \nabla^{ss} g, \nabla^{ss} k > dm^{ss} \right) dm = 0.$$

This is possible only if $k$ is a constant function. Conversely, if the conditional measures of $m$ along strong stable manifolds are proportional to the Riemannian volume, then the same computation shows that the measure $m$ is $\Delta^{ss}$ reversible. Therefore, in higher dimension and variable curvature, we propose to replace invariance under the horocycle flow by $\Delta^{ss}$ reversibility. In this paper, we show how to extend the result of [15]. A reversible measure is called extremal reversible if it cannot be decomposed into a nontrivial convex combination of reversible measures; it is called ergodic reversible if it cannot be further decomposed into a nontrivial convex combination of harmonic measures. For a smooth function $f$ with compact support, we denote $Xf =< X, \nabla^s f >$ the derivative of $f$ in the direction of the flow.

Consider the universal cover $\widetilde{M}$ of $M$. Since $M$ is negatively curved, $\widetilde{M}$ is homeomorphic to an open ball. Denote by $\partial\widetilde{M}$ the geometric boundary of $\widetilde{M}$ and by $\Gamma$ the associated covering group of isometries of $\widetilde{M}$. The action of $\Gamma$ on $\widetilde{M}$ extends to a continuous action of $\Gamma$ on $\partial\widetilde{M}$. Define the Busemann function $B_\xi(x,y)$ for $\xi \in \partial\widetilde{M}, x, y \in \widetilde{M}$ by

$$B_\xi(x,y) = \lim_{z\to\xi}(d(y,z) - d(x,z)). \tag{1.3}$$

We fix once and for all a base point $o \in \widetilde{M}$. Let $\alpha$ be real. A measure $\nu$ on $\partial\widetilde{M}$ is called $\alpha$-conformal if, for all $\gamma \in \Gamma$, $\frac{d\nu\circ\gamma}{d\nu}(\xi) = e^{\alpha B_\xi(o,\gamma^{-1}o)}$. A measure $\nu$ on $\partial\widetilde{M}$ is called $\alpha$-conformal ergodic if it is $\alpha$-conformal and cannot be decomposed as a convex combination of $\alpha$-conformal measures.

Fix $\eta \in \partial\widetilde{M}$. The weak unstable manifold $W^u_\eta$ is made of all vectors $v \in T^1\widetilde{M}$ such that the geodesic $\sigma_v$ defined by $v$ goes to $\eta$ at time $-\infty$. We can take as coordinates on $W^u_\eta$ the pair $(\xi, s)$, where $\xi = \sigma_v(+\infty)$ and $s$ is given by $s = -B_\xi(o, \sigma_v(0))$. Let $\nu$ be a $\alpha$-conformal measure. Define on $W^u_\eta$ a measure $\nu_\eta$ by

$$d\nu_\eta(\xi, s) = e^{\alpha s} d\nu(\xi) ds. \tag{1.4}$$

Clearly, the family $\nu_\eta$ is a family of transverse invariant measures for the strong stable manifolds $W^{ss}$. Indeed, two points on the same stong stable manifold have the same coordinates. Moreover, the family $\nu_\eta$ is equivariant under $\Gamma$, since this action is given by $\gamma(\eta, s, \xi) = (\gamma\eta, s + B_\xi(o, \gamma^{-1}o), \gamma\xi)$ and $\nu$ is conformal.

Define a measure $\widetilde{m}_\nu$ on $T^1\widetilde{M}$ by making the product of $\nu_\eta$ with the Riemannian volume on strong stable manifolds. This makes sense since the $\nu_\eta$ are $W^{ss}$ invariant. Moreover, the measure $\widetilde{m}_\nu$ clearly is a Radon measure. By equivariance, $\widetilde{m}_\nu$ is $\Gamma$ invariant. Write $m_\nu$ for the measure on $T^1M$ which lifts to $\widetilde{m}_\nu$. Since the conditional measures of the measure $m_\nu$ are proportional to the Riemannian volume, the measure $m_\nu$ is $\Delta^{ss}$ reversible. It has been shown by Roblin ([18], Théorème 2.2), that the measure is $\Delta^{ss}$ ergodic. Our result is that, for cocompact Anosov periodic manifolds, extremal $\Delta^{ss}$ reversible measures are all of that form. More precisely:

Theorem 1.1. — *Let $M$ be a cocompact periodic Anosov manifold with period $M_0$. Assume $M$ is not simply connected. Let $\nu$ be a $\alpha$-conformal ergodic measure on $\partial\widetilde{M}$, for some $\alpha$. Then the above measure $m_\nu$ is $\Delta^{ss}$*

*reversible ergodic, and $\Delta^s + \alpha X$ reversible ergodic. Conversely, any extremal $\Delta^{ss}$ reversible measure is of that form, for some real $\alpha$, and any extremal $\Delta^s + \alpha X$ reversible measure as well. In particular, an extremal $\Delta^{ss}$ reversible measure is $\Delta^{ss}$ ergodic and $\Delta^s + \alpha X$ reversible ergodic, for some $\alpha$.*

We illustrate these results by a few examples.

Corollary 1.2 (Bowen-Marcus's unique ergodicity [2]). — *Suppose $M$ is compact negatively curved. Then, there exists a unique $\Delta^{ss}$ harmonic probability measure.*

*Proof.* — This is the case when the symmetry group is trivial. Recall that the leaves of the $\mathcal{W}^{ss}$ foliation have polynomial growth for the distance $d^{ss}$. Since the manifold $M$ is compact, this implies that $\Delta^{ss}$ harmonic measures are $\Delta^{ss}$ reversible ([12], Corollary to Theorem 4). By Theorem 1.1 any ergodic $\Delta^{ss}$ harmonic measure is therefore associated to a conformal measure on $\partial M$. But in the cocompact case, there is only one conformal measure (this is a consequence of the 'shadow property' of Sullivan; see [19], Proposition 0.2.1 for the most general statement). Therefore, all $\Delta^{ss}$ harmonic measures are proportional and finite. There is only one $\Delta^{ss}$ harmonic probability measure. □

We are also able to classify all Radon measures on $T^1M$ which are reversible and ergodic with respect to $\Delta^{ss}$, in the case when the symmetry group $\Gamma_0/\Gamma$ is abelian or nilpotent. Indeed in these cases, U. Hamenstädt ([10]) and M. Babillot ([1]) classified the Radon measures which are reversible and ergodic with respect to $\Delta^s + \alpha X$, for any $\alpha$. Therefore:

Corollary 1.3. — *Let $M$ be a cocompact periodic Anosov manifold with period $M_0$ such that the symmetry group $G$ is abelian or nilpotent. There is a one-to-one correspondence between*

1. *the (rays of) Radon measures on $T^1M$ which are reversible and ergodic with respect to $\Delta^{ss}$,*
2. *the (rays of) Radon measures which are reversible and ergodic with respect to $\Delta^s + \alpha X$, for some $\alpha$,*
3. *the (rays of) $\alpha$-conformal measures, for the same $\alpha$ as in (2) and*
4. *exponentials $e : G \mapsto \mathbb{R}^+, e(g_1g_2) = e(g_1)e(g_2)$.*

*In particular, a $\Delta^{ss}$ reversible ergodic measure on $T^1M$ $m$ is the measure $m_\nu$ associated to a $\alpha_m$-conformal measure $\nu$ and is quasiinvariant under the action of $G$; the number $e_m(\gamma\Gamma)$ is the constant $\frac{d(m\circ D\gamma)}{dm}$.*

*Proof.* — The correspondence between (1),(2) and (3) is Theorem 1.1. The correspondence between (2) and (4) is in [1], Theorem B. □

In the case when $M$ is a cocompact periodic locally symmetric space of negative curvature, $T^1M$ can be represented as the set of double cosets $\Gamma \setminus G/T$ where $G = SO(n,1), SU(n,1), Sp(n,1)$ or $\mathfrak{f}_{4(-20)}$, $T$ is a compact subgroup and $\Gamma$ is a discrete subgroup which is normal in an uniform lattice $\Gamma_0$ in such a way that the geodesic flow is the right action of a one-parameter group $A$ which commutes with $T$, the $\mathcal{W}^{ss}$ foliation is the foliation into the orbits of the right action of a nilpotent group $N$ which normalizes $T$, and the $\mathcal{W}^s$ foliation is the foliation into the orbits of the right action of the group $AN$ (see [11], Chapter X). We have the following extension of ([15], Theorem 1):

Corollary 1.4. — *Let $M$ be a cocompact periodic locally symmetric space of negative curvature, $m$ a $N$-invariant ergodic measure on $T^1M$. Then there is real $\beta$ such that $m \circ g^s = e^{\beta s} m$.*

*Proof.* — Indeed, this follows directly from equation (2.1) below, since the function $U = -\mathrm{Div}^s X$ is a constant $k(G)$ on $T^1M$. Normalize the metric on $M$ in such a way that the maximum of the sectional curvature is $-1$. Then, we have $k(SO(n,1)) = n-1$, $k(SU(n,1)) = 2n$, $k(Sp(n,1)) = 4n+2$ and $k(\mathfrak{f}_{4(-20)}) = 22$. In all cases, $\beta = \alpha - k(G)$.(3) (4) □

Consider the Laplace operator $\Delta_M$ on functions in $C^2(M)$. The collection of positive $\lambda$–eigenfunctions of $\Delta_M$ forms a cone. The extremal rays of this cone are directions generated by the *minimal* positive $\lambda$– eigenfunctions: the $\lambda$–eigenfunctions $F$ for which $\Delta_M G = \lambda G$, $0 \leqslant G \leqslant F \Rightarrow \exists c$ s.t. $G = cF$. Then:

Corollary 1.5. — *Let $M$ be a cocompact periodic locally symmetric space with negative curvature. There is a bijection between (rays of) ergodic $N$-invariant Radon measures on $T^1(M)$ and the (rays of) minimal positive eigenfunctions of $\Delta_M$. This bijection satisfies:*

(1) $m \circ g^s = e^{\beta s} m \Leftrightarrow \Delta_M F_m = \beta(\beta + k(G)) F_m$;
(2) $m \circ dD = cm \Leftrightarrow F_m \circ D = cF_m$ *for all $D$ in the symmetry group of $M$.*

(3) There is in these cases a super strong stable manifold $W^{sss}(\omega)$, the set of unit vectors $\omega'$ such that $\limsup_{t\to\infty} \frac{1}{t} \log d(g^t\omega, g^t\omega') \leqslant -2$. Then the constant $k$ is $\mathrm{Dim}W^{ss} + \mathrm{Dim}W^{sss}$.

(4) Observe that in the cases when $G = Sp(n,1)$ or $G = \mathfrak{f}_{4(-20)}$, the symmetry group has property $(T)$, since it is the quotient of a lattice in a property $(T)$ group. Such examples of infinite quotient of cocompact lattices in $Sp(n,1)$ or $\mathfrak{f}_{4(-20)}$ are due to Gromov (see [16], Remark IX.7.16).

*Proof.* — In the same way as in dimension 2, Theorem 1.1 establishes a bijection between (rays of) ergodic $N$-invariant Radon measures on $T^1(M)$ and (rays of) ergodic $\alpha$-conformal measures on the boundary. On the other hand, again in the same way as in dimension 2, Karpelevich's theorem ([13]) establishes a bijection between the (rays of) minimal positive $\lambda$ eigenfunctions of $\Delta_M$ and (rays of) ergodic $\alpha$-conformal measures on the boundary, where $\alpha$ is the larger solution of $\lambda = \alpha(\alpha - k(G))$. Corollary 1.5 follows. We refer to [15], Section 5 for details. □

In Section 2, we discuss $\Delta^{ss}$ reversibility of measures by calculating along the leaves of the foliation. In order to prove Theorem 1.1, it remains to show, in terms of Definition 1 below, that for the action of $\Gamma$ on $\partial\widetilde{M}$, the Busemann cocycle has the invariant measure property. This was the main result of [15], and the proof here follows the same lines. Two ingredients in [15] hold in greater generality: Sarig's cocycle reduction theorem ([20]) and the ratio set argument from Section 4.3. in [15]. The third ingredient is a 'Holonomy Lemma', see Lemma 4.2 below. The proof of the Holonomy Lemma in [15] uses the fact that $\Gamma$ is a normal subgroup of a lattice $\Gamma_0$ in $PSL(2,\mathbb{R})$ and the symbolic representation of the action of $\Gamma_0$ on the boundary $\partial\mathbb{H}^2$. In higher dimension, there is no direct symbolic representation of the action of $\Gamma_0$ on $\partial\widetilde{M}$. In Section 3, we show how one can use instead the symbolic representation of the action of $\Gamma_0$ on its Gromov boundary ([4]). The rest of the proof in Section 4 closely follows [15].

## 2. Proof of Theorem 1.1

Consider the universal cover $\widetilde{M}$ of $M$ and $\Gamma$ the covering group. By assumption, $\Gamma$ is not reduced to the identity. Moreover, $\Gamma$ is a normal subgroup of the cocompact group $\Gamma_0$ of isometries of $\widetilde{M}$ generated by $\Gamma$ and the lifts of the symmetries of $M$. Denote by the same letters the invariant manifolds on $T^1\widetilde{M}$ (these are the connected components of the lifts of the invariant manifolds on $T^1M$). Also denote by the same letter the $\Gamma$ invariant lift on $T^1\widetilde{M}$ of a measure on $T^1M$. We endow $T^1\widetilde{M}$ with the Sasake metric associated with the Levi-Civita connection: at a point $(p,v) \in T^1\widetilde{M}$, the horizontal subspace $\mathcal{H} \subset T_{(p,v)}T^1\widetilde{M}$ and the vertical subspace $\mathcal{V} \subset T_{(p,v)}T^1\widetilde{M}$ are orthogonal and the natural projections from $\mathcal{H}$ and $\mathcal{V}$ on $T_p\widetilde{M}$ are isometries.

*Step 1: From $\alpha$-conformal measures to $\Delta^s + \alpha X$ reversible measures.*

Let $\nu$ be a $\alpha$-conformal measure on $\partial\widetilde{M}$. In this step is shown that the measure $m_\nu$ is a $\Delta^s + \alpha X$ reversible measure. Recall that the measure

$\widetilde{m}_\nu$ is given by integrating the Lebesgue measure $m^{ss}$ with respect to the transverse invariant measure $e^{\alpha s}d\nu(\xi)ds$. On each stable manifold $W^s$, the measure $ds$ on the geodesic trajectories is invariant under $W^{ss}$ holonomies and the measure obtained by integrating the Riemannian volume $m^{ss}$ with respect to the transverse invariant measure $ds$ is exactly the Riemannian volume $m^s$. This means that, in local charts, the conditional measures of $m_\nu$ with respect to local $W^s$ leaves are proportional to $e^{\alpha s}m^s$. Therefore, we can apply Green formula to each leaf and obtain for any smooth $f$ with compact support,

$$\begin{aligned}\int (\Delta^s f) g dm_\nu &= \int \left( \int \Delta^s f (g e^{\alpha s}) dm^s \right) \\ &= - \int < \nabla^s f, \nabla^s g > dm_\nu - \alpha \int g X f dm_\nu .\end{aligned}$$

Therefore $\int (\Delta^s f + \alpha X f) g dm_\nu = - \int < \nabla^s f, \nabla^s g > dm_\nu$ is symmetric in $f$ and $g$, and $m_\nu$ is indeed reversible.

For later use, observe that, using in the same way a leafwise Green formula, we obtain, for any smooth $f$ with compact support, any vector field $Y$ along the stable leaves (*i.e.*, $Y_v \in T_v W^s$):

$$\begin{aligned}\int Y f dm_\nu &= \int \left( \int Y (f e^{\alpha s}) dm^s \right) - \alpha \int < Y, X > f dm_\nu \\ &= - \int \left( \int (\mathrm{Div}^s Y) f e^{\alpha s} dm^s \right) - \alpha \int < Y, X > f dm_\nu \\ &= - \int (\mathrm{Div}^s Y) f dm_\nu - \alpha \int < Y, X > f dm_\nu .\end{aligned}$$

In particular, it follows that, for all $f$ smooth with compact support

$$\int X f dm_\nu = \int f (U - \alpha) dm_\nu , \tag{2.1}$$

where $U := -\mathrm{Div}^s X = \Delta^s_y B_\xi (o, y)$.

*Step 2: $\Delta^s + \alpha X$ reversible measures are $\Delta^{ss}$ reversible measures.*

This is due to U. Hamenstädt ([9], Lemma 2.6. See also [1], Section 3). We first prove that if $m$ is $\Delta^s + \alpha X$ reversible, $m$ satisfies the relation (2.1). Let $g$ be a smooth function with compact support in $T^1 \widetilde{M}$. There exists a function $h$ with compact support such that $gX = -g\nabla^s B_\xi = -g \nabla^s h$. Writing $\int \Delta^s (gh) + \alpha X (gh) dm = 0$, we obtain:

$$\int \left( g \Delta^s h + 2 < \nabla^s g, \nabla^s h > + h \Delta^s g + \alpha h X g + \alpha g X h \right) dm = 0.$$

Using reversibility again , we obtain

$$\int (g\Delta^s h + < \nabla^s g, \nabla^s h > + \alpha g X h)\, dm = 0.$$

Since on the support of $g$, $\Delta^s h = U, \nabla^s h = -X$ and $Xh = -1$, we obtain the relation (2.1).

We claim that, with the above notations:

$$\Delta^s f = \Delta^{ss} f - UXf + XXf. \tag{2.2}$$

We note $XX$ for the second derivative in the direction of the flow, *i.e.*, the derivative in the direction of the flow of the function $Xf =: f'$. Indeed,

$$\Delta^s f = \operatorname{Div}^s \nabla f = \operatorname{Div}^s(\nabla^{ss} f + f' X) = \Delta^{ss} f + f' \operatorname{Div}^s X + X f'.$$

Using (2.2), we can see that $m$ is reversible for $\Delta^{ss}$:

$$\begin{aligned}\int ((\Delta^s + \alpha X) f)\, g dm &= \int (\Delta^{ss} f) g dm + \int (\alpha - U) f' g dm + \int (X f') g dm \\ &= \int (\Delta^{ss} f) g dm - \int X\, (f' g)\, dm + \int (X f') g dm \\ &= \int (\Delta^{ss} f) g dm - \int f' g' dm.\end{aligned}$$

We used (2.1) to write the second line. Since the Left Hand Side of the first line is symmetric in $f$ and $g$, the last line is also symmetric in $f$ and $g$, and therefore the measure $m$ is reversible w.r.t. $\Delta^{ss}$.

*Step 3: Extremal $\Delta^{ss}$ reversible measures are given by Step 1.*

More precisely, Step 3 is devoted to the proof of:

Proposition 2.1. — *Let $m$ be a Radon measure on $T^1M$ which is $\Delta^{ss}$ reversible and extremal with this property. Then there exist $\alpha$ and a $\alpha$-conformal measure $\nu$ on $\partial\widetilde{M}$ such that $m = m_\nu$.*

Recall that each transversal to $W^{ss}$ can be parametrized by the pair $(\xi, s)$, where $\xi = \sigma_v(+\infty)$ and $s$ is given by $s = B_\xi(o, \sigma_v(0))$. Lift $m$ into a $\Gamma$ invariant measure $\widetilde{m}$. Since the measure $\widetilde{m}$ is Radon and $\Delta^{ss}$ reversible, it is obtained as the integral of the Riemannian volume on strong stable manifolds with respect to some Radon measure $\mu$ on $(\xi, s)$. Now, that $\widetilde{m}$ is $\Gamma$-invariant means that the measure $\mu$ is invariant by the following action on $\partial\widetilde{M} \times \mathbb{R}$:

$$\gamma : (\xi, s) \mapsto \big(\gamma\xi, s + B_\xi(o, \gamma^{-1} o)\big). \tag{2.3}$$

Moreover, convex decomposition of a reversible measure into reversible measures corresponds to convex decompositions of the associated $\Gamma$ invariant measure into $\Gamma$ invariant measures. Therefore, the property that $\widetilde{m}$

descends to an extremal reversible measure is equivalent to the ergodicity of the associated measure $\mu$ with respect to the $\Gamma$ action (2.3). We have:

THEOREM 2.2. — *Let $\mu$ be a Radon measure on $\partial\widetilde{M} \times \mathbb{R}$ which is invariant and ergodic under the $\Gamma$ action (2.3). Then the measure $\mu$ is quasi-invariant under the translations of the $\mathbb{R}$ coordinate.*

Theorem 2.2 will be proven in Sections 3 and 4. Applying Theorem 2.2, the measure $\mu$ is $\Gamma$ ergodic and its translates by $s$ in the $\mathbb{R}$ direction are $\Gamma$ ergodic as well. For each real $s$, the density of the translated $\mu \circ g^s$ with respect to the measure $\mu$ is therefore a constant $a_s$ $\mu$-almost everywhere. Clearly, we have $a_{s+t} = a_s a_t$ and the map $s \mapsto a_s$ is measurable. It follows that there is a number $\alpha$ such that $a_s = e^{-\alpha s}$. Therefore, the measure $e^{-\alpha s}\mu$ is invariant by $\mathbb{R}$ translations. Thus, the measure $\mu$ can be written as

$$d\mu(\xi, s) = e^{\alpha s} d\nu(\xi) ds,$$

for some measure $\nu$ on $\partial\widetilde{M}$. $\Gamma$ invariance shows that the measure $\nu$ is $\alpha$-conformal and this proves Proposition 2.1.

*Step 4: Proof of Theorem 1.1, extremality.*

Let $\nu$ be an ergodic $\alpha$-conformal measure. Then, by construction, the measure $m_\nu$ is $\Delta^{ss}$ reversible and, by [18], $\Delta^{ss}$ ergodic. By Step 1, the measure $m_\nu$ is reversible with respect to $\Delta^s + \alpha X$ and by Step 2, the measure $m_\nu$ is extremal with this property, since a nontrivial decomposition of $m_\nu$ into $\Delta + \alpha X$ reversible Radon measures would yield a nontrivial decomposition of $m_\nu$ into $\Delta^{ss}$ reversible Radon measures. Conversely, suppose the Radon measure $\mu$ is extremal $\Delta^{ss}$ reversible. By step 3, there exists $\alpha$ and a $\alpha$-conformal measure $\nu$ on $\partial\widetilde{M}$ such that $\mu = m_\nu$. By construction, the measure $\nu$ has to be ergodic as an $\alpha$-conformal measure. Finally, starting from an extremal $\Delta^s + \alpha X$ reversible Radon measure $m$, we see that $m$ is $\Delta^{ss}$ reversible by Step 2. If we can decompose $m = \int_Y m_y d\pi(y)$ the measure $m$ into Radon measures $m_y$ that are extremal $\Delta^{ss}$ reversible, then by the above discussion, there are constants $\alpha_y$ and $\alpha_y$-conformal measures $\nu_y$ such that $\widetilde{m}$ is obtained as the product of the Lebesgue measure on strong stable manifolds and of the measure

$$d\mu(\xi, s) \;=\; \int_Y e^{\alpha_y s} d\nu_y(\xi) d\pi(y) ds.$$

In particular, we have equation (2.1) for $m$ and the $m_y$ for all $y$. By comparing the RHS of these equations (2.1), we get, for all $f$ smooth with compact support:

$$\alpha \int f dm = \int_Y \alpha_y \Big( \int f dm_y \Big) d\pi(y).$$

It follows that $\alpha_y = \alpha$ for $\pi$-a. e. $y \in Y$. Since the measure $m$ is extremal reversible with respect to $\Delta^s + \alpha X$, the decomposition $m = \int_Y m_y d\pi(y)$ is trivial.

To complete the proof of Theorem 1.1, we still have to establish the following statement:

*Step 5: Extremal $\Delta^s + \alpha X$ reversible measures are $\Delta^s + \alpha X$ ergodic.*

This follows from the observation that the measures which appear in the decomposition of a reversible measure into harmonic measures are themselves reversible. Therefore, the decomposition of a reversible measure into extremal reversible measures is also its ergodic decomposition.

Indeed, since the operator $\Delta^s + \alpha X$ is hypoelliptic on the leaves $W^s$, the conditional measures on local leaves have a smooth density $k_\omega$ with respect to the riemannian volume (see *e.g.*, [6]). The harmonicity relation reads as:

$$0 = \int (\Delta^s + \alpha X)(f) k_\omega dm^s = \int f(\Delta^s k_\omega - \alpha X k_\omega - \alpha k_\omega \mathrm{Div}^s X) dm^s.$$

Thus a measure is $\Delta^s + \alpha X$ harmonic if, and only if, it has densities $k_\omega$ with respect to the Riemannian volume on stable leaves satisfying:

$$\Delta^s k_\omega - \alpha X k_\omega + \alpha k_\omega U = 0. \tag{2.4}$$

The property comes from the fact that writing that the measure $m$ is $\Delta^s + \alpha X$ reversible will be compatible with only one solution of equation (2.4). Then, if $m$ is reversible, this determines the density of the conditional measures along $W^s$ leaves, and the measures of the ergodic decomposition have the same density. Indeed, using again a leafwise Green formula, we have, for all $f, g$ of class $C^2$ and with compact support

$$\int g(\Delta^s + \alpha X)(f) k_\omega dm^s = \int f\Big((\Delta^s g) k_\omega + 2(< \nabla^s g, \nabla^s k_\omega >) + g \Delta^s k_\omega\Big) dm^s + \alpha \int f\Big(U g k_\omega - g' k_\omega - g k'_\omega\Big) dm^s.$$

Writing $\int f(\Delta^s + \alpha X)(g) k_\omega dm^s = \int g(\Delta^s + \alpha X)(f) k_\omega dm^s$ yields, after simplifications using (2.4):

$$2 \int f\,(< \nabla^s g, \nabla^s k_\omega > - \alpha k_\omega X g)\, dm^s = 0. \tag{2.5}$$

Since the relation (2.5) holds for all $f$, we have for all $g$ with compact support:

$$< \nabla^s g, \nabla^s k_\omega > -\alpha k_\omega X g = 0.$$

It follows that $\nabla^s \log k_\omega = \alpha X$; the function $k_\omega$ has to be proportional to $e^{-\alpha B_\xi(o,\cdot)}$. Working backwards, if a $\Delta^s + \alpha X$ harmonic measure has densities along $W^s$ leaves proportional to $e^{-\alpha B_\xi(o,\cdot)}$, then it is $\Delta^s + \alpha X$ reversible.

This achieves the reduction of the proof of Theorem 1.1 to the proof of Theorem 2.2.

*Remark.* — Extremal $\Delta^{ss}$ reversible measures are $\Delta^{ss}$ ergodic.

As recalled in the introduction, $\Delta^{ss}$ reversible measures have constant densities along strong stable manifolds. The same reasoning as in Step 5 above shows that extremal $\Delta^{ss}$ reversible measures are $\Delta^{ss}$ ergodic, without using Theorem 1.1 or [18].

## 3. Markov coding

Definition 3.1. — *Let $\Gamma$ be a countable group, $X$ a compact metric $\Gamma$-space and $\Phi : (\Gamma \times X) \mapsto \mathbb{R}$ a Hölder continuous cocycle (i.e., $\Phi(\gamma, x)$ is a Hölder continuous function of $x$ for all $\gamma$, and satisfies $\Phi(\gamma'\gamma, x) = \Phi(\gamma', \gamma x) + \Phi(\gamma, x)$). We say that $(X, \Gamma, \Phi)$ satisfies the invariant measure property if any Radon measure $m$ on $X \times \mathbb{R}$ which is invariant and ergodic under the $\Gamma$ action*

$$\gamma : (x, s) \mapsto \big(\gamma x, s + \Phi(\gamma, x)\big) \tag{3.1}$$

*is quasiinvariant under the translations of the $\mathbb{R}$ coordinate.*

Clearly, in this language, Theorem 2.2 is saying that $(\partial\widetilde{M}, \Gamma, B_\xi(o, \gamma^{-1}o))$ satisfies the invariant measure property. In this section, we reduce the discussion of the invariant measure property to a symbolic model.

Definition 3.2. — *Let $\Gamma_0$ be a countable group of homeomorphisms of a compact metric space $X$. We say that $(X, \Gamma_0)$ admits a Markov coding if there is $(\Sigma, \pi)$ with the following properties:*

1. *$\Sigma$ is a transitive subshift of finite type: there is a finite set $S$ of states and a transition matrix $A = (t_{ij})_{S\times S}, t_{ij} = \{0, 1\}$ such that $\Sigma := \{x = (x_i)_{i\geqslant 1} \in S^{\mathbb{N}} : \forall i,\ t_{x_i x_{i+1}} = 1\}$ and for all $i, j \in S \times S$, there is a $n$ such that the entry $t^{(n)}_{ij}$ of $A^n$ is positive. We denote by $f$ the shift transformation on $\Sigma$. $\Sigma$ is endowed with the metric $d(x, y) = \sum_{k\geqslant 1} \frac{1}{2^k}(1 - \delta_{x_k, y_k})$.*

(2) $\Gamma_0$ *acts on* $\Sigma$ *in such a way that for all* $s \in S$, *the map* $f$ *restricted to* $[s] := \{x \in \Sigma : x_1 = s\}$ *is given by the restriction to* $[s]$ *of the action of an element* $\gamma_s$ *of* $\Gamma_0$.

(3) $\pi : \Sigma \mapsto X$ *is a bounded-to-one* $\Gamma_0$*-equivariant Hölder continuous map.*

DEFINITION 3.3. — *A homeomorphism* $\gamma$ *on a compact space* $X$ *is called a North-South homeomorphism if there is an open cover* $U_-, U_+$ *of* $\Sigma$ *such that:*

$$\gamma U_+ \cap U_- = \varnothing, \ \bigcap_{n\in\mathbb{Z}} \gamma^n U_+ = \{\gamma_+\}, \ \bigcap_{n\in\mathbb{Z}} \gamma^n U_- = \{\gamma_-\}.$$

The action of a group $\Gamma$ on a compact space $X$ is said to be North-South if each element of $\Gamma$ distinct from the identity acts by a North-South homeomorphism. We want to prove:

THEOREM 3.4. — *Assume* $(X, \Gamma_0)$ *admits a Markov coding, that* $\Gamma$ *is a normal subgroup of* $\Gamma_0$ *and that* $\Phi$ *is a Hölder cocycle for the action of* $\Gamma$ *on* $X$. *Denote by* $\tau(\gamma) := \Phi(\gamma, \gamma_+)$ *for* $\gamma \neq \mathbb{I}d$, $\tau(\mathbb{I}d) = 0$ *the periods of the cocycle* $\Phi$. *Assume that the action of* $\Gamma$ *is minimal and North-South on* $X$, *that the cocycle* $\Phi$ *admits a Hölder continuous extension as a cocycle for the action of* $\Gamma_0$ *and that the periods satisfy:*
(Even) *for all* $\gamma \in \Gamma$, $\tau(\gamma) = \tau(\gamma^{-1})$ *and*
(NonA) *the set* $\tau(\Gamma)$ *generates* $\mathbb{R}$ *as a closed additive group.*
*Then* $(X, \Gamma, \Phi)$ *satisfies the invariant measure property.*

Theorem 2.2 follows from Theorem 3.4. Indeed, let $\widetilde{M}$ be a simply connected Anosov manifold such that $\widetilde{M}$ admits a discrete torsionless cocompact group of isometries $\Gamma_0$. Let $\Gamma$ be a normal subgroup of $\Gamma_0$, different from the identity group. We want to prove that $(\partial\widetilde{M}, \Gamma, B_\xi(o, \gamma^{-1}o))$ has the invariance measure property, where $B$ is the Busemann function defined in (1.3). Observe that the elements of $\Gamma$ other than the identity are hyperbolic isometries of $\widetilde{M}$.[(5)] Therefore the action of $\Gamma$ is North-South on the boundary $\partial\widetilde{M}$, and $\gamma \in \Gamma, \gamma \neq \mathbb{I}d$, has two fixed points $\gamma_\pm$ on $\partial\widetilde{M}$. Let us recall the definition of the metric on $\partial\widetilde{M}$: fix a point $o \in \widetilde{M}$ and define the Gromov product on $\partial\widetilde{M}$ by

$$|\xi, \eta|_o \ = \lim_{x\to\xi, y\to\eta} \frac{1}{2}\left(d(o,x) + d(o,y) - d(x,y)\right).$$

[(5)] Let $l(\gamma) := \inf_{x\in\widetilde{M}} d(x, \gamma x)$ and $\{x_n\}$ be a sequence of points of $\widetilde{M}$ such that $l(\gamma) = \lim d(x_n, \gamma x_n)$. There are $\gamma_n \in \Gamma_0$ such that $\gamma_n x_n \in \widetilde{M}_0$, for a fixed relatively compact fundamental domain $\widetilde{M}_0$, and a subsequence $n_k$ such that $\gamma_{n_k} x_{n_k} \to y \in \overline{\widetilde{M}_0}$. Then $l(\gamma) = \lim_{k\to\infty} d(y, \gamma_{n_k}^{-1}\gamma\gamma_{n_k} y)$ is positive because the orbit of $y$ is discrete.

For $\alpha_0$ sufficiently small, we can (see [7], page 124) define a distance on $\partial\widetilde{M}$ equivalent to

$$d(\xi,\eta) = e^{-\alpha_0|\xi,\eta|_o}. \tag{3.2}$$

The function $\xi \mapsto B_\xi(o,\gamma^{-1}o)$ is Hölder continuous with respect to the distance (3.2).(6)

The group $\Gamma_0$, being cocompact, is finitely generated. Fix $A$ a finite symmetric set of generators for $\Gamma_0$ and consider the Cayley graph on $\Gamma_0$ defined by $A$. Endowed with the graph metric, it is quasi-isometric to $\widetilde{M}$ and therefore, it is hyperbolic and there is a $\Gamma_0$-equivariant Hölder homeomorphism $\pi_0$ between the boundary $\partial\Gamma_0$, endowed with its own Gromov metric (which is defined by a formula analogous to (3.2)), and $\partial\widetilde{M}$ (see [7], Chapter 7). Theorem 2.2 is the statement that $(\partial\Gamma_0,\Gamma,\Phi'(\gamma,\xi))$ has the invariant measure property, where

$$\Phi'(\gamma,\xi) = B_{\pi_0(\xi)}(o,\gamma^{-1}o).$$

In the rest of this section, we verify that the $(\partial\Gamma_0,\Gamma,\Phi'(\gamma,\xi))$ satisfy the hypotheses of Theorem 3.4. We first have:

Proposition 3.5. — *([4]) Let $\Gamma_0$ be a Gromov hyperbolic group. Then the action of $\Gamma_0$ on its boundary $\partial\Gamma_0$ admits a Markov coding.*

*Proof.* — This is the precise formulation, due to M. Coornaert and A. Papadopoulos ([4]) of a result of Gromov about Markov coding for hyperbolic groups ([8], Section 8.5.Q). Namely, following [4], call a horofunction on $\Gamma_0$ a function $h : \Gamma_0 \mapsto \mathbb{Z}$ which is quasi-convex and such that for all $n \leqslant h(\gamma)$, the graph distance from the point $\gamma$ to the level set $h^{-1}(\{n\})$ is exactly $h(\gamma)-n$. By [4], Proposition 4.3., if two points $(\gamma_1,\gamma_2)$ are adjacent in $\Gamma_0$ (*i.e.*, $\gamma_1^{-1}\gamma_2$ is a generator), then $h(\gamma_1)-h(\gamma_2)$ has value $0,-1$ or $1$.

Two horofunctions are called equivalent if they differ by a constant. Let $X_0$ be the set of equivalence classes. $\Gamma_0$ acts continuously on $X_0$ by $(\gamma h)(\gamma') = h(\gamma^{-1}\gamma')$. For $[h]\in X_0$, define a *gradient lines* as sequences $(\gamma_n)_n, \gamma_n\in\Gamma_0$, such that, for all $n$, $h(\gamma_{n+1}) = h(\gamma_n)-1$. Gradient lines are geodesics converging to the same point at infinity, and this defines a map

(6) For $\xi,\eta\in\partial\widetilde{M}$, denote by $\sigma_{\xi,\eta}$ the unit speed geodesic going from $\xi$ to $\eta$. Then we have, from (1.3),

$$B_\xi(o,\gamma^{-1}o) - B_\eta(o,\gamma^{-1}o) = \int_{-\infty}^{+\infty}\frac{d}{dt}\Big(d(\sigma_{\xi,\eta}(t),o) - d(\sigma_{\xi,\eta}(t),\gamma^{-1}o)\Big)dt.$$

We have $\left|\frac{d}{dt}\Big(d(\sigma_{\xi,\eta}(t),o)-d(\sigma_{\xi,\eta}(t),\gamma^{-1}o)\Big)\right| \leqslant |\sin(ang_{\sigma_{\xi,\eta}(t)}(o,\gamma^{-1}o))|$. By comparaison $|\sin(ang_{\sigma_{\xi,\eta}(t)}(o,\gamma^{-1}o))| \leqslant C\exp(-d(\sigma_{\xi,\eta}(t),o))$. The Hölder property follows.

$\pi : X_0 \mapsto \partial\Gamma_0$. Clearly, the map $\pi$ is $\Gamma_0$-equivariant and surjective. Moreover, the map $\pi$ is bounded-to-one ([4], Proposition 4.5). Fix an order relation on the set of generators $A$, and consider the map $\alpha : X_0 \mapsto X_0$ defined by: $\alpha(h) = a^{-1}h$, where $a$ is the smallest element of $A$ with $h(\mathbb{I}d) - h(a) = 1$. Then,

Theorem 3.6. — ([4]) *The action of a on $X_0$ is topologically conjugate to a subshift of finite type $(\Sigma, f)$.*

We still have to verify Property (2) of definition 3.2. Let $P : X_0 \mapsto \Sigma$ be the conjugating map, $[s]$ the set of sequences with first coordinate $s$. It turns out that the action of $P^{-1}fP$ on $P^{-1}[s]$ is indeed given by the action of an element $w(s)$ of $\Gamma_0$ ( [4], page 455). □

Then, we know that the action of $\Gamma_0$ on $\partial\Gamma_0$ is minimal (*cf.*, [7], page 153). As soon as $\Gamma$ is not the subgroup reduced to the identity, it acts minimally as well. Indeed, since $\Gamma$ is a nontrivial normal subgroup of $\Gamma_0$, the limit set of $\Gamma$ acting on $\Gamma_0$ is a $\Gamma_0$ invariant closed subset of $\partial\Gamma_0$. So the limit set of $\Gamma$ is the whole $\partial\Gamma_0$. Since $\Gamma \subset \Gamma_0$, $\partial\Gamma_0$ is also the boundary of $\Gamma$ for the induced metric and $\Gamma$ acts minimally on its boundary. We have moreover seen that the function $\xi \mapsto B_\xi(o, \gamma^{-1}o)$ is Hölder continuous. Therefore, Theorem 2.2 follows from Theorem 3.4 if the set of $\tau(\gamma), \gamma \in \Gamma$ satisfies the properties (Even) and (NonA), where

$$\tau(\gamma) \ = \ B_{\gamma_+}(o, \gamma^{-1}(o)).$$

Property (Even) is clear, since

$$\tau(\gamma) = \lim_{n\to+\infty} \frac{1}{n} d(o, \gamma^{-n}o) = \lim_{n\to+\infty} \frac{1}{n} d(o, \gamma^{n}o) = \tau(\gamma^{-1}).$$

If Property (NonA) were not true, we would have $\tau(\gamma) \in c\mathbb{Z}$ for all $\gamma \in \Gamma$ and some real $c$. Since the group $\Gamma$ is not reduced to the identity, the limit set of $\Gamma$ is a closed nonempty $\Gamma_0$-invariant subset of the boundary $\partial\widetilde{M}$. By the minimality of $\Gamma_0$ on $\partial\widetilde{M}$, the limit set of $\Gamma$ is the whole boundary $\partial\widetilde{M}$, a nonempty and connected set. Then, following [5], Corollaire 1.3, [17], we see that then the crossratio of four distinct points $(\xi, \eta, \zeta, \rho)$ on $\partial\widetilde{M}$:

$$(\xi, \eta, \zeta, \rho) = |\xi, \zeta|_o - |\xi, \rho|_o + |\eta, \rho|_o - |\eta, \zeta|_o. \tag{3.3}$$

would take values in $c\mathbb{Z}$ as well. The crossratio is continuous in its arguments and therefore has to be constant. On the other hand, the crossratio can take arbitrarily large values, a contradiction.

## 4. Proof of Theorem 3.4

We first have the following proposition, which reduces Theorem 3.4 to the Markov case:

PROPOSITION 4.1. — *Let $(X, \Gamma, \Phi)$ and $(X', \Gamma, \Phi')$ be as in Definition 1, and assume that there is a $\Gamma$ equivariant, Hölder continuous, finite-to-one map $\pi$ from $X$ to $X'$ such that $\Phi = \Phi' \circ \pi$. Then if $(X, \Gamma, \Phi)$ has the invariant measure property, then $(X', \Gamma, \Phi')$ has the invariant measure property as well.*

*Proof.* — Let indeed $m'$ be a Radon measure on $X' \times \mathbb{R}$ invariant ergodic under the action of $\Gamma$ by:

$$\gamma : (x', s) \mapsto \big(\gamma x', s + \Phi'(\gamma, x')\big).$$

By ergodicity, observe that the function $\mathrm{Card}(\pi^{-1}(x'))$ is $m'$-almost everywhere constant. There is a unique measure $m$ on $X \times \mathbb{R}$ obtained by integrating in $m'$ the uniform distribution on the finite sets $\pi^{-1}(x')$. The measure $m$ is clearly Radon and $\Gamma$ invariant. We may write $m = a_1 m_1 + a_2 m_2 + \cdots + a_N m_N$, where $m_i, i = 1, \ldots, N$ are the ergodic components of the measure $m$, and the $a_i, i = 1, \ldots, N$, are multiple of $(\mathrm{Card}(\pi^{-1}))^{-1}$. Each of the $m_i$ projects on $m'$ and is quasi-invariant under $\mathbb{R}$ translations by the invariant measure property of $(X, \Gamma, \Phi)$. The measure $m$ is $\mathbb{R}$-quasi-invariant as well and $(X', \Gamma, \Phi')$ has the invariant measure property. □

By Proposition 4.1, we only have to prove the invariant measure property for $(\Sigma, \Gamma, \Phi' \circ \pi)$. Recall that $\Sigma$ is a subshift of finite type, based on the alphabet $S$. For $s = s_1 s_2 \ldots s_q$ a finite admissible word, write $[s]$ for the cylinder based on $s$, *i.e.*, the set of points $x \in \Sigma$ such that $x_1 = s_1, x_2 = s_2, \ldots, x_q = s_q$. In particular, for $s \in S$, denote $I_s := [s] = \{x, x \in \Sigma; x_1 = s_1\}$. Let $N$ be the maximum cardinality of $\pi^{-1}(x), x \in X$. The action of $\Gamma$ on $\Sigma$ is not necessarily North-South anymore, but we still have an open cover $U_-, U_+$ of $\Sigma$ such that

$$\gamma U_+ \cap U_- = \varnothing,\ \gamma^{\pm} U_{\pm} \subset U_{\pm},\ \bigcap_{n \in \mathbb{Z}} \gamma^n U_+ = A_+,\ \bigcap_{n \in \mathbb{Z}} \gamma^n U_- = A_-,$$

where $A_+ := \pi^{-1}(\{\gamma_+\}), A_- := \pi^{-1}(\{\gamma_-\})$ are finite subsets with at most $N$ elements. Denote $\gamma_z, z \in \mathcal{Z}, \mathcal{Z} = \mathcal{Z}_+ \cup \mathcal{Z}_-$ the different points of $A_{\pm}$. For $Q = N!$, all points $\gamma_z, z \in \mathcal{Z}$ are fixed by $\gamma^Q$. We can decompose $U_{\pm}$ into disjoint $U_z, z \in \mathcal{Z}$, where the points in $U_z$ are attracted to $\gamma_z$ under one of $\gamma^Q$ or $\gamma^{-Q}$.

Let $m$ be a Radon measure on $\Sigma \times \mathbb{R}$ invariant ergodic under the action of $\Gamma$

$$\gamma : (x, s) \mapsto \big(\gamma x, s + \Phi(\gamma, x)\big), \tag{4.1}$$

where $\Phi(\gamma, x) = \Phi'(\gamma, \pi(x))$. Denote $g^s, s \in \mathbb{R}$ the translation on the $\mathbb{R}$ coordinate: $g^s(x, t) = (x, t + s)$. Set $H_m := \{s \in \mathbb{R} : m \circ g^s \sim m\}$. This is a closed subgroup of $\mathbb{R}$, and our goal is to show that $H_m = \mathbb{R}$.

We have reduced the proof of Theorem 3.4 to a form where we can reproduce the proof of Theorem 1 in [15]. We now give the analog of section 4.2 in [15], since there are small differences. The $\Gamma$-action defines a *full pseudogroup* $[\Gamma]$: $\kappa \in [\Gamma]$ if $\kappa : B \subset \Sigma \mapsto \kappa(B) \subset \Sigma$ is Borel one-to-one and for each $x \in B$, there is $\gamma_x \in \Gamma$ such that $\kappa x = \gamma_x x$, and this $\gamma_x$ is unique unless $x$ is a fixed point of some $\gamma \in \Gamma$. Extend the cocycle $\Phi$ to $[\Gamma]$ by setting $\Phi(\kappa, x) = \Phi(\gamma_x, x)$ (one can make any measurable choice on the countable set where this definition is ambiguous). Then the transformation $\overline{\kappa} : B \times \mathbb{R} \mapsto \kappa(B) \times \mathbb{R}$ defined by:

$$\overline{\kappa}(x, s) = (\kappa x, s + \Phi(\kappa, x))$$

preserves the measure $m$.[7] Such transformations are called holonomies. We have the following Lemma:

Lemma 4.2 (Holonomy Lemma). — *Let $N_\varepsilon(\cdot)$ denote the $\varepsilon$–neighborhood of a set, and $\Gamma$, $\Gamma_0$, and $m$ be as above. Let $[\underline{a}] \subset X$ be a cylinder and $I$ be a compact interval such that $m([\underline{a}] \times I) \neq 0$. For every $\tau_0 \in \tau(\Gamma)$ and $\varepsilon > 0$, there exists a 1–1 measure preserving Borel holonomy $\overline{\kappa}$ such that $\overline{\kappa}([\underline{a}] \times I) \subset [\underline{a}] \times N_\varepsilon(I + Q\tau_0) \bmod m$.*

*Proof.* — By ergodicity, there is certainly a $b \in S$ such that the $f$–orbit of a.e. $x \in \Sigma$ enters $I_b$ infinitely many times, more precisely: if $\Omega_b \subset \Sigma$ is the set of points with this property, then $m[(\Omega_b \times \mathbb{R})^c] = 0$. Now fix some $[\underline{a}], I, \tau_0, \varepsilon$ as in the statement. By the definition of $\tau(\Gamma)$, there is $\gamma \in \Gamma$ with the attracting fixed points of $\gamma^Q$ noted $\gamma_z, z \in \mathcal{Z}_+$ and the repelling fixed points $\gamma_z, z \in \mathcal{Z}_-$ satisfying $\Phi(\gamma^Q, \gamma_z) = Q\tau_0$ for $z \in \mathcal{Z}_+$, $\Phi(\gamma^{-Q}, \gamma_z) = Q\tau_0$ for $z \in \mathcal{Z}_-$. We may assume w.l.o.g. that some $\gamma_z \in I_b$. Otherwise, choose some $h \in \Gamma$ such that $h(A_+) \cap I_b \neq \varnothing$ and work with $h \circ g \circ h^{-1}$ (such $h$ exists because $\Gamma$ acts minimally on $X = \pi\Sigma$). Divide $I_b$ into open subsets

[7] This is clear if $m(\{x : \exists \gamma, \gamma x = x\}) = 0$. Otherwise, the set of points which are fixed by some $\gamma \in \Gamma$ is countable, and for any such point $x_0$ with $m(x_0 \times \mathbb{R}) > 0$, the corresponding measure $m_{x_0}$ on $\mathbb{R}$ is a Radon measure invariant under the translations by the group $\Gamma_{x_0}$ generated by $\{\Phi(\gamma, x_0); \gamma x_0 = x_0\}$. For any choice of $\gamma'$ such that $\gamma' x_0 = \kappa x_0$, the translation of $m_{x_0}$ by $\Phi(\gamma', x_0)$ is the same $m_{\kappa x_0}$ and the invariance follows.

$I_b^z$ for $\gamma_z \in I_b^z$ such that $\gamma^{\pm Q} I_b^z \subset I_b^z, z \in \mathcal{Z}_\pm$. Set $I_b^z = \varnothing$ for $z$ such that $\gamma_z \notin I_b$. Therefore, if we set:

$$\underline{\gamma}(x) := \begin{cases} \gamma^{Q}(x) & x \in I_b^z, z \in \mathcal{Z}_+ \\ \gamma^{-Q}(x) & x \in I_b^z, z \in \mathcal{Z}_- \end{cases}$$

then $\underline{\gamma}(I_b) \subset I_b$ and for all $z \in \mathcal{Z}_- \cup \mathcal{Z}_+$ such that $\gamma_z \in I_b$, $\Phi(\underline{\gamma}, \gamma_z) = Q\tau_0$.

Sublemma 1. — *Fix $\ell$ (to be determined later) and set $[b_z^\ell] := \gamma^{\pm \ell Q}(I_b^z)$. Then, for almost every $x \in [\underline{a}]$, the $f$–orbit of $x$ enters $\cup_z [b_z^\ell] = \underline{\gamma}^\ell(I_b)$ infinitely many times.*

*Proof.* — Assume by way of contradiction that this is not the case. In this case the function $N(x) := 1_{[\underline{a}]}(x) \max\{0, n : f^n(x) \in \underline{\gamma}^\ell(I_b)\}$ is finite for $m$–a.e. $(x,s)$.

By choice of $b$, the $f$–orbit of a.e. $x$ enters $I_b$ infinitely many times. Denote these times by $n_1(x) < n_2(x) < \cdots$. For $i$ larger than the length of $[\underline{a}]$ and for every $x$, let $[s_1, \ldots, s_{n_i(x)}]$ be the cylinder which contains $x$. Then define $\gamma_{i,x}$ on $[s_1, \ldots, s_{n_i(x)}]$ by

$$\gamma_{i,x} := (f^{n_i(x)}|_{[s_1,\ldots,s_{n_i(x)}]})^{-1} \circ \underline{\gamma}^\ell \circ f^{n_i(x)}|_{[s_1,\ldots,s_{n_i(x)}]}.$$

Then, $\gamma_{i,x}$ is a locally constant element of $\Gamma$, because $f$ acts by elements of $\Gamma_0$ and $\Gamma$ is normal in $\Gamma_0$. Set $\kappa_i(x) = \gamma_{i,x}(x)$, $\Phi(\kappa_i, x) = \Phi(\gamma_{i,x}, x)$. The map $\kappa_i$ belongs to the full pseudogroup $[\Gamma]$, and consider the associated holonomy

$$\overline{\kappa}_i(x,s) = (\kappa_i x, s + \Phi(\kappa_i, x)).$$

Then,

(i) $\overline{\kappa}_i$ *is injective,* because $\kappa_i$ is injective (it is piecewise injective and the images of the pieces are disjoint).

(ii) $\overline{\kappa}_i$ *is measure preserving,* because it is a holonomy of the orbit relation of the action of $\Gamma$ on $\Sigma \times \mathbb{R}$.

(iii) *$\exists M_0$ such that $\overline{\kappa}_i([\underline{a}] \times I) \subset [\underline{a}] \times N_{M_0}(I)$,* as we shall see below.

(iv) *For a.e. $(x,s) \in [\underline{a}] \times I$, $\overline{\kappa}_i(x,s) \in \{y : N(y) \geqslant i\} \times N_{M_0}(I)$,* because by construction, $N(\kappa_i(x)) \geqslant n_i(x) \geqslant i$.

In order to verify property (iii), we may write:

$$\begin{aligned}\Phi(\kappa_i, x) &= \Phi((f^{n_i(x)}|_{[s_1,\ldots,s_{n_i(x)}]})^{-1} \circ \underline{\gamma}^\ell \circ f^{n_i(x)}|_{[s_1,\ldots,s_{n_i(x)}]}, x) \\ &= \sum_{j=0}^{n_i(x)-1} \Phi(\gamma_{s_j(x)}^{-1}, f^{j+1}y) + \Phi(\underline{\gamma}^\ell, f^{n_i(x)}x) + \sum_{j=0}^{n_i(x)-1} \Phi(\gamma_{s_j(x)}, f^j x),\end{aligned}$$

where $y = \kappa_i x$. Note that $x$ and $y$ have the same $n_i$ first addresses in $\Sigma$. The cocycle relation yields $\Phi(\gamma_{s_j(x)}^{-1}, f^{j+1}y) = -\Phi(\gamma_{s_j(x)}, f^j y)$ and, by the

Hölder property, since $f^j x$ and $f^j y$ have the same $n_i - j$ first addresses in $\Sigma$,

$$\left|\Phi(\gamma_{s_j(x)}, f^j x) - \Phi(\gamma_{s_j(x)}, f^j y)\right| \leqslant C_{s_j}\theta^{n_i - j}$$

for some $\theta < 1$. Set $C := \sup_{s\in S} C_s$. We therefore have:

$$|\Phi(\kappa, x)| \leqslant \max_{z\in I_b} \Phi(\underline{\gamma}^\ell, z) + \frac{C}{1-\theta} =: M_0.$$

Now, $\{y : N(y) \geqslant i\} \times N_{M_0}(I)$ is a decreasing sequence of sets whose intersection is negligible (because $N < \infty$ a.e.). These are subsets of the finite measure set $[\underline{a}] \times N_{M_0}(I)$, so their measure must tend to zero. By (iv), $(m \circ \overline{\kappa}_i)([\underline{a}] \times I) \xrightarrow[i\to\infty]{} 0$. But this contradicts (ii). □

By Sublemma 1, for any $\ell$, the orbit of a.e. $x \in [\underline{a}]$ enters $\underline{\gamma}^\ell(I_b)$ infinitely often. It follows that $[\underline{a}]$ is (up to measure zero) of the form

$$[\underline{a}] = \biguplus_{i=1}^{\infty} [\underline{p}_i] \cap f^{-\ell_i}(\underline{\gamma}^\ell I_b)$$

where $[\underline{p}_i]$ are cylinders of length $\ell_i + 1$ and $f^{\ell_i}[\underline{p}_i] = I_b$. Define a map $\kappa$ on $[\underline{a}]$ by

$$\kappa|_{[\underline{p}_i]\cap f^{-\ell_i}(\underline{\gamma}^\ell I_a)} = (f^{\ell_i}|_{[\underline{p}_i]})^{-1} \circ \underline{\gamma} \circ f^{\ell_i}|_{[\underline{p}_i]}.$$

(i) *$\kappa$ is injective and $\kappa[\underline{a}] \subset [\underline{a}]$*: Indeed, $\kappa$ maps $[\underline{p}_i] \cap f^{-\ell_i}(\gamma^\ell I_b)$ bijectively onto $[\underline{p}_i] \cap f^{-\ell_i}(\underline{\gamma}^{\ell+1} I_b) \subset [\underline{p}_i] \cap f^{-\ell_i}(\underline{\gamma}^\ell I_b)$.

(ii) *$\kappa$ belongs to the full pseudo-group of the $\Gamma$ action on $X$*: because $f$ acts by elements of $\Gamma_0$ and $\Gamma$ is normal in $\Gamma_0$.

(iii) $\sup |\Phi(\kappa, x) - Q\tau_0| \xrightarrow[\ell\to\infty]{} 0$ on $[\underline{a}]$. See below.

Before checking (iii), we explain how it can be used to complete the construction. Fix, using (iii), $\ell$ large enough that $|\Phi(\kappa, x) - Q\tau_0| < \varepsilon$. As before,

$$\overline{\kappa} : (x, s) \mapsto (\kappa x, s + \Phi(\kappa, x)).$$

makes sense, is measure preserving, and maps $[\underline{a}] \times I$ into $[\underline{a}] \times N_\varepsilon(I + Q\tau_0)$.

We check (iii). By the cocycle relation, we have:

$$\begin{aligned}
|\Phi(\kappa, x) - Q\tau_0| &= \left|\Phi(f^{\ell_i}, x) + \Phi(\underline{\gamma}, f^{\ell_i} x) + \Phi(f^{-\ell_i}, \underline{\gamma} f^{\ell_i} x) - Q\tau_0\right| \\
&\leqslant \left|-\Phi(f^{-\ell_i}, f^{\ell_i} x) + \Phi(f^{-\ell_i}, \underline{\gamma} f^{\ell_i} x)\right| \\
&\quad + \left|\Phi(\underline{\gamma}, f^{\ell_i} x) - \Phi(\underline{\gamma}, \gamma_z)\right|,
\end{aligned}$$

where $z$ is chosen so that the points $f^{\ell_i}x, \underline{\gamma} f^{\ell_i}x$ and $\gamma_z$ all lie in the same $\underline{\gamma}^\ell I_b^z$. Then,

$$\begin{aligned}\left|\Phi(f^{-\ell_i},\underline{\gamma} f^{\ell_i}x)-\Phi(f^{-\ell_i},f^{\ell_i}x)\right| &\leqslant \sum_{j=0}^{\ell_i-1}\left|\Phi(f^{-1},f^{-j}\underline{\gamma} f^{\ell_i}x)-\Phi(f^{-1},f^{-j}f^{\ell_i}x)\right| \\ &\leqslant C\sum_{j=0}^{\ell_i-1}(d(f^{-j}\underline{\gamma} f^{\ell_i}x, f^{-j}f^{\ell_i}x))^\beta \\ &\leqslant C\sum_{j=0}^{\ell_i-1}\theta^j(d(\underline{\gamma} f^{\ell_i}x, f^{\ell_i}x))^\beta \\ &\leqslant C'(\max_z \operatorname{Diam}\underline{\gamma}^\ell I_b^z)^\beta.\end{aligned}$$

Since we also have $\left|\Phi(\underline{\gamma}, f^{\ell_i}x)-\Phi(\underline{\gamma},\gamma_z)\right| \leqslant C(\max_z \operatorname{Diam}\underline{\gamma}^\ell I_b^z)^\beta$, we indeed have a constant $K$ such that

$$|\Phi(\kappa,x)-Q\tau_0| \leqslant K(\max_z \operatorname{Diam}\underline{\gamma}^\ell I_b^z)^\beta.$$

Since every $\underline{\gamma}^\ell I_b^z$ decreases to a point as $\ell\to\infty$, for every $\varepsilon$, one can choose $\ell$ such that for all $x\in[\underline{a}]$, $|\Phi(\kappa,x)-Q\tau_0|<\varepsilon$, which is Property (iii). $\square$

Recall that we want to show that $H_m=\mathbb{R}$. We say that the invariant measure $m$ is continuous on $\Sigma$ if $m(\{x\}\times\mathbb{R})=0$ for all $x\in\Sigma$. By ergodicity, if $m$ is not continuous on $\Sigma$, there is a countable set $\Sigma_0$ such that $m\left((\Sigma_0\times\mathbb{R})^c\right)=0$. Then, we claim that $H_m=\mathbb{R}$. Indeed, fix $x\in\Sigma_0, \tau_0\in\tau(\Gamma)$ and $I$ an interval such that $m(\{x\}\times I)>0$. Applying Lemma 4.2 to a decreasing family of cylinders $[\underline{a}]_n$ such that $\bigcap[\underline{a}]_n=\{x\}$ and a sequence $\varepsilon_n\to 0$, we obtain, for all $n$,

$$m(\{x\}\times I)\leqslant m([\underline{a}]_n\times N_{\varepsilon_n}(I+Q\tau_0)).$$

As $n\to\infty$, it follows that $m(\{x\}\times I)\leqslant m(\{x\}\times(I+Q\tau_0))$. Since this holds for all intervals $I$, we have that $-Q\tau_0\in H_m$. By our hypothesis (NonA), $H_m$ is the whole group $\mathbb{R}$.

So we may assume that $m$ is continuous on $\Sigma$ and, by way of contradiction, that $H_m\neq\mathbb{R}$. Since $H_m$ is a closed subgroup of $\mathbb{R}$, $H_m=c\mathbb{Z}$ for some $c$. The rest of the proof reproduces Section 4.3 in [15].

## BIBLIOGRAPHY

[1] M. Babillot, "On the classification of invariant measures for horospherical foliations on nilpotent covers of negatively curved manifolds", in *Random walks and geometry* (V.A. Kaimanovich, Ed.) de Gruyter, Berlin, 2004, p. 319-335.

[2] R. Bowen & B. Marcus, "Unique ergodicity for horocycle foliations", *Israel J. Math.* **26** (1977), p. 43-67.

[3] É. Cartan, "Leçons sur la géométrie des espaces de Riemann", 1928, Paris.

[4] M. Coornaert & A. Papadopoulos, "Horofunctions and symbolic dynamics on Gromov hyperbolic groups", *Glasgow Math.J.* **43** (2001), p. 425-456.

[5] F. Dal'bo, "Remarques sur le spectre des longueurs d'une surface et comptages", *Bol. Soc. Brasil. Mat. (N.S.)* **30** (1999), no. 2, p. 199-221.

[6] L. Garnett, "Foliations, the ergodic theorem and Brownian motion", *J. Funct. Anal.* **51** (1983), p. 285-311.

[7] E. Ghys & P. de la Harpe, "Sur les groupes hyperboliques, d'après Mikhael Gromov", in *Progress in math.*, vol. 83, Birkhäuser, 1990.

[8] M. Gromov, "Hyperbolic Groups", in *Essays in Group Theory*, vol. 8, Math. Sci. Res. Inst. Publ., 1987, p. 75-263.

[9] U. Hamenstädt, "Harmonic measures for compact negatively curved manifolds", *Acta Mathematica* **178** (1997), p. 39-107.

[10] ———, "Ergodic properties of Gibbs measures on nilpotent covers", *Ergod. Th. & Dynam. Sys.* **22** (2002), p. 1169-1179.

[11] S. Helgason, "Differential Geometry, Lie Groups and Symmetric spaces", in *Graduate Studies in Mathematics*, Academic Press, New York, 2001.

[12] V. A. Kaimanovich, "Brownian motion on foliations: Entropy, invariant measures, mixing", *Funct. Anal. Appl.* **22** (1989), p. 326-328.

[13] F. I. Karpelevich, "The geometry of geodesics and the eigenfunctions of the Laplacian on symmetric spaces", *Trans. Moskov. Math. Soc.* **14** (1965), p. 48-185.

[14] A. Katok & B. Hasselblatt, "Introduction to the modern theory of dynamical systems", CUP, 1995, Cambridge.

[15] F. Ledrappier & O. Sarig, "Invariant measures for the horocycle flow on periodic hyperbolic surfaces", to appear, Isr. J. Math.

[16] G. A. Margulis, "Discrete subgroups of semi-simple groups", Ergebnisse, Band 17, Springer-Verlag, 1991.

[17] J.-P. Otal, "Sur la géométrie symplectique de l'espace des géodésiques d'une variété à courbure négative", *Rev. Mat. Iberoamericana* **8** (1992), p. 441-456.

[18] T. Roblin, *Ergodicité et équidistribution en courbure négative*, Mémoires, vol. 95, S.M.F., 2003.

[19] ———, "Un théorème de Fatou pour les densités conformes avec applications aux revêtements galoisiens en courbure négative", *Isr. J. Math.* **147** (2005), p. 333-357.

[20] O. Sarig, "Invariant measures for the horocycle flow on Abelian covers", *Inv. Math.* **157** (2004), p. 519-551.



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