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INTEGRABLE HIERARCHIES AND THE MODULAR CLASS

by Pantelis A. DAMIANOU & Rui Loja FERNANDES (*)

ABSTRACT. — It is well-known that the Poisson-Nijenhuis manifolds, introduced by Kosmann-Schwarzbach and Magri form the appropriate setting for studying many classical integrable hierarchies. In order to define the hierarchy, one usually specifies in addition to the Poisson-Nijenhuis manifold a bi-hamiltonian vector field. In this paper we show that to every Poisson-Nijenhuis manifold one can associate a canonical vector field (no extra choices are involved!) which under an appropriate assumption defines an integrable hierarchy of flows. This vector field is the modular class of the Poisson-Nijhenhuis manifold. This class has a canonical representative which, under a cohomological assumption, is a bi-hamiltonian vector field. In many examples the associated hierarchy of flows reproduces classical integrable hierarchies. We illustrate in detail with the Harmonic Oscillator, the Calogero-Moser system, the classical Toda lattice and various Bogoyavlensky-Toda Lattices.

RÉSUMÉ. — Il est bien connu que les variétés de Poisson-Nijenhuis, introduites par Kosmannn-Schwarzbach et Magri, constituent le contexte le plus approprié pour étudier de nombreuses hiérarchies intégrables. Afin de définir une telle hiérarchie, on explicite d'habitude, en plus de la variété de Poisson-Nijenhuis, un champ de vecteurs bi-hamiltonien. Dans cet article, nous montrons qu'à toute variété de Poisson-Nijenhuis nous pouvons associer un et un seul champ de vecteurs canonique qui, sous des hypothèses convenables, définit une hiérarchie intégrable de flots. Ce champ de vecteurs est la classe modulaire de la variété de Poisson-Nijenhuis. Cette classe possède un représentant défini de manière canonique qui, sous une hypothèse cohomologique, est un champ de vecteurs bi-hamiltonien. Dans de nombreux exemples, la hiérarchie associée de flots reproduit les hiérarchies intégrables classiques. Nous illustrons en détails ce fait à l'aide de l'oscillateur harmonique, du système de Calogero-Moser, du réseau de Toda classique et des divers réseaux de Bogoyavlensky-Toda.

1. Introduction

It is well-known that the Poisson-Nijenhuis manifolds, introduced by Kosmann-Schwarzbach and Magri in [20], form the appropriate setting for

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studying many classical integrable hierarchies. In order to define the hierarchy, one usually specifies *in addition* to the Poisson-Nijenhuis manifold a bi-hamiltonian vector field. In this paper we will show that to every Poisson-Nijenhuis manifold one can associate a canonical vector field (no extra choices are involved!) which under an appropriate assumption defines an integrable hierarchy of flows. Moreover, this vector field is a very natural geometric entity, leading to a cohomological interpretation of this condition. For many classical examples we recover well-known integrable hierarchies.

In order to explain in more detail our results, let us recall that a Poisson manifold (M,π) usually does not carry a Liouville form, *i.e.*, a volume form which is invariant under the flow of all hamiltonian vector fields[(1)]. The obstruction to the existence of an invariant volume form, as was explained by J.-L. Koszul [22] and A. Weinstein [29], lies in the first Poisson cohomology group $H^1_\pi(M)$ (the Poisson vector fields modulo hamiltonian vector fields). More precisely, given a volume form μ, we can associate to it a Poisson vector field X^π_μ, called the *modular vector field*. Though this vector field depends on the choice of μ, the Poisson cohomology class $[X^\pi_\mu]\in H^1_\pi(M)$ does not, and this *modular class* is zero iff there exists some invariant measure on M. The modular vector field was used by Dufour and Haraki in [9] to classify quadratic Poisson brackets in $\mathbb{R}^3$. It was also useful in the classification of Poisson structures in low dimensions, *e.g.*, [24, 18].

Assume now that $(M,\pi_0,\mathcal{N})$ is a Poisson-Nijenhuis manifold ([20]). It is well known that we can associate to it a hierarchy of Poisson structures:

$$\pi_1 := \mathcal{N}\pi_0,\ \pi_2 := \mathcal{N}\pi_1 = \mathcal{N}^2\pi_0,\dots$$

It is easy to check that the Nijenhuis tensor $\mathcal{N}$ maps hamiltonian (respectively, Poisson) vector fields of π_0 to hamiltonian (respect., Poisson) vector fields of π_1, and more generally those of π_i to those of π_{i+1}. However, in general, for any choice of μ, it *does not* map the modular vector field X^0_μ of π_0 to the modular vector field X^1_μ of π_1. As we will show below, the difference:

$$X_{\mathcal{N}} := X^1_\mu - \mathcal{N}X^0_\mu,$$

is a Poisson vector field for π_1, which is *independent* of the choice of volume form μ. Notice that this vector field is zero if there exists a volume form μ which is invariant simultaneously under the flows of the hamiltonian vector fields for π_1 and π_0. Hence, we may think of $X_{\mathcal{N}}$ as a modular vector field

(1) We will assume that our manifolds are orientable. This is enough to cover all applications and simplifies the presentation. However, our results do extend to the non-orientable case.

of our Poisson-Nijenhuis manifold. Moreover, using the concept of *relative* modular class, introduced recently in [17, 21], we can show that the Poisson cohomology class of $X_{\mathcal{N}}$ is the relative modular class of the transpose $\mathcal{N}^*$, when viewed as a morphism of Lie algebroids. Furthermore, we will show the following result:

THEOREM 1.1. — *Let $(M, \pi_0, \mathcal{N})$ be a Poisson-Nijenhuis manifold. Then the modular vector field $X_{\mathcal{N}}$ is hamiltonian relative to π_0 with hamiltonian equal to minus one half the trace of $\mathcal{N}$:*

$$X_{\mathcal{N}} = X^0_{-\frac{1}{2}\operatorname{tr}\mathcal{N}}.$$

Therefore, the vector field $X_{\mathcal{N}}$ is hamiltonian relative to π_0 and Poisson relative to π_1. So $X_{\mathcal{N}}$ is very close to defining a bi-hamiltonian system, and hence a hierarchy of flows. Of course, the obstruction is the Poisson cohomology class $[X_{\mathcal{N}}] \in H^1_{\pi_1}(M)$, *i.e.*, the modular class of the Poisson-Nijenhuis manifold. This class is zero, for example, if there are measures μ and η invariant under both the hamiltonian flows of π_0 and π_1. Note that, in general, $X_{\mathcal{N}}$ itself will still be non-zero, in which case the two invariant measures are non-proportional. A typical situation that fits many examples is the following:

THEOREM 1.2. — *Let $(M, \pi_0, \mathcal{N})$ be a Poisson-Nijenhuis manifold and assume that $\mathcal{N}$ is non-degenerate. Then the modular vector field $X_{\mathcal{N}}$ is bi-hamiltonian and hence determines a hierarchy of flows which are given by:*

$$X_{i+j} = \pi_i^\sharp \mathrm{d}h_j = \pi_j^\sharp \mathrm{d}h_i \quad (i, j \in \mathbf{Z})$$

where

$$h_0 = -\frac{1}{2}\log(\det\mathcal{N}), \quad h_i = -\frac{1}{2i}\operatorname{tr}\mathcal{N}^i \quad (i \neq 0).$$

We will see below that most of the known hierarchies of integrable systems can be obtained in this manner, therefore providing a new approach to the integrability of those systems. Moreover, in some cases (*e.g.*, the Toda systems) it gives rise to previously unknown bi-hamiltonian formulations. Note that the fact that the traces of the powers of $\mathcal{N}$ give rise to a hierarchy of flows was noticed early in the history of integrable systems (see, *e.g.*, [4, 19]).

The paper is organized as follows. In Section 2, we recall a few basic facts concerning modular vector fields and modular classes, and we show that the modular class of the Lie algebroid associated with a Nijenhuis tensor is represented by $\mathrm{d}(\operatorname{tr}\mathcal{N})$. This basic fact, which does not seem to have been noticed before, sets up the stage for Section 3, where we consider the modular vector field of a Poisson-Nijenhuis manifold. In Section 4, we introduce integrable hierarchies related to the modular class and we prove Theorem 1.2 above. In Section 5, we show how one can recover many of the known classical integrable hierarchies using our results.

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2. Modular classes

In this section we present several results concerning modular classes that will be needed later. This will also help establishing our notation. Our main result here is Proposition 2.4, where we compute the modular class of the Lie algebroid associated with a (1,1)-tensor $\mathcal{N}$ with vanishing Nijenhuis torsion.

2.1. Modular class of a Poisson manifold

If $(M, \{\cdot,\cdot\})$ is a Poisson manifold, we will denote by $\pi \in \mathfrak{X}^2(M)$ the associated Poisson tensor which is given by

$$\pi(\mathrm{d}f, \mathrm{d}g) := \{f, g\}, \quad (f, g \in C^\infty(M))$$

and by $\pi^\sharp : T^*M \to TM$ the vector bundle map defined by

$$\pi^\sharp(\mathrm{d}h) = X_h := \{h, \cdot\},$$

where X_h is the hamiltonian vector field determined by $h \in C^\infty(M)$. Recall also that the Poisson cohomology of (M, π), introduced by Lichnerowicz [23], is the cohomology of the complex of multivector fields $(\mathfrak{X}^\bullet(M), \mathrm{d}_\pi)$, where the coboundary operator is defined by taking the Schouten bracket with the Poisson tensor:

$$\mathrm{d}_\pi A \equiv [\pi, A].$$

This cohomology is denoted by $H^\bullet_\pi(M)$. We will be mainly interested in the first Poisson cohomology space $H^1_\pi(M)$, which is just the space of Poisson vector fields modulo the hamiltonian vector fields. Note that our conventions are such that the hamiltonian vector field associated with the function h is given by:

$$X_h = -[\pi, h] = -\mathrm{d}_\pi h. \tag{2.1}$$

In this paper we follow the same sign conventions as in the book by Dufour and Zung [10], and which differ from other sign conventions such as the one in Vaisman's monograph [28] [(2)].

Let us assume that M is oriented and fix an arbitrary volume form $\mu \in \Omega^{\mathrm{top}}(M)$. The divergence of a vector field $X \in \mathfrak{X}(M)$ relative to μ is the unique function $\mathrm{div}_\mu(X)$ that satisfies:

$$\boldsymbol{\mathcal{L}}_X \mu = \mathrm{div}_\mu(X)\mu.$$

When (M, π) is a Poisson manifold, a volume form μ defines the modular vector field:

$$X_\mu(f) := \mathrm{div}_\mu(X_f).$$

Note that this vector field depends on the choice of μ.

More generally, a choice of volume form μ induces, by contraction, an isomorphism $\Phi_\mu : \mathfrak{X}^k(M) \to \Omega^{m-k}(M)$, where $m = \dim M$, and we define, following Koszul [22], the following operator that generalizes the divergence operator above: $D_\mu : \mathfrak{X}^k(M) \to \mathfrak{X}^{k-1}(M)$ defined by:

$$D_\mu = \Phi_\mu^{-1} \circ \mathrm{d} \circ \Phi_\mu \ ,$$

where d is the exterior derivative. It is obvious that $D_\mu^2 = 0$, so D_μ is a homological operator. Now we have:

Proposition 2.1. — *For a Poisson manifold (M, π) with a volume form μ the modular vector field is given by:*

$$X_\mu = D_\mu(\pi). \tag{2.2}$$

[(2)] In particular the Schouten bracket on multivector fields satisfies the following super-commutation, super-derivation and super-Jacobi identities:

$$[A, B] = -(-1)^{(a-1)(b-1)}[B, A]$$
$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C]$$
$$(-1)^{(a-1)(c-1)}[A, [B, C]] + (-1)^{(b-1)(a-1)}[B, [C, A]] + (-1)^{(c-1)(b-1)}[C, [A, B]] = 0$$

where $A \in \mathfrak{X}^a(M)$, $B \in \mathfrak{X}^b(M)$ and $C \in \mathfrak{X}^c(M)$.

If $(x^1, \dots, x^m)$ are local coordinates, such that $\mu = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ and $\pi = \sum_{i<j} \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ then:

$$X_\mu = \sum_{i=1}^m \left(\sum_{j=1}^m \frac{\partial \pi^{ij}}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

The proof of this proposition is standard and we refer, for example, to [10, Chapter 2.6] for details.

Some authors take expression (2.2) as the definition of the modular vector field. Recalling that the Koszul operator satisfies the basic identity:

$$D_\mu([A,B]) = [A, D_\mu(B)] + (-1)^{b-1}[D_\mu(A), B] \tag{2.3}$$

we see immediately from $[\pi,\pi] = 0$ that

$$\mathrm{d}_\pi X_\mu = [\pi, D_\mu(\pi)] = 0,$$

so the modular vector field is a Poisson vector field. Also, if we are given another volume form μ', so that $\mu' = g\mu$ for some non-vanishing function g, we find from the definition of the Koszul operator:

$$D_{g\mu}A = D_\mu A + [A, \ln|g|].$$

In particular, when $A = \pi$ this shows that under a change of volume form the modular vector field changes by an addition of a hamiltonian vector field:

$$X_{g\mu} = X_\mu - X_{\ln|g|}. \tag{2.4}$$

Therefore, the class $\mathrm{mod}(\pi) \equiv [X_\mu] \in H^1_\pi(M)$ is well-defined.

2.2. Modular class of a Lie algebroid

We will need also the modular class of a Lie algebroid, which was introduced in [11].

Let $p : A \to M$ be a Lie algebroid over M, with anchor $\rho : A \to TM$ and Lie bracket $[\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$. Lie algebroids are some kind of generalized tangent bundles, so many of the constructions from the usual tensor calculus can be extended to Lie algebroids, and we recall a few of them. First, the algebroid cohomology of A is the cohomology of the complex $(\Omega^k(A), \mathrm{d}_A)$, where $\Omega^k(A) \equiv \Gamma(\wedge^k A^*)$ and $\mathrm{d}_A : \Omega^k(A) \to \Omega^{k+1}(A)$

is the de Rham type differential:

$$\text{(2.5)}\quad \mathrm{d}_A\omega(\alpha_0,\dots,\alpha_k)=\sum_{i=0}^{k}(-1)^{i+1}\rho(\alpha_i)(\omega(\alpha_0,\dots,\check{\alpha}_i,\dots,\alpha_k))$$
$$\sum_{0\leqslant i<j\leqslant k}(-1)^{i+j}\omega([\alpha_i,\alpha_j],\alpha_0,\dots,\check{\alpha}_i,\dots,\check{\alpha}_j,\dots,\alpha_k).$$

The Lie algebroid cohomology is denoted by $H^\bullet(A)$. Given a section $\alpha\in\Gamma(A)$ (a "vector field"), there is a Lie A-derivative operator $\boldsymbol{\mathcal{L}}_\alpha$ and a contraction operator i_α defined as in the usual case of TM, but using the A-Lie bracket. It follows that we also have Cartan's magic formula (for details see, *e.g.*, [11]):

$$\boldsymbol{\mathcal{L}}_\alpha=i_\alpha\mathrm{d}_A+d_Ai_\alpha.$$

Now to define the modular class of A we proceed as follows. We assume that the line bundles $\wedge^{\mathrm{top}}A$ and $\wedge^{\mathrm{top}}T^*M$ are trivial and we choose global sections η and μ (3). Then $\eta\otimes\mu$ is a section of $\wedge^{\mathrm{top}}A\otimes\wedge^{\mathrm{top}}T^*M$, and we define $\xi_A\in C^1(A)$ to be the unique element such that:

$$(\boldsymbol{\mathcal{L}}_\alpha\eta)\otimes\mu+\eta\otimes(\boldsymbol{\mathcal{L}}_{\rho(\alpha)}\mu)=\xi_A(\alpha)\eta\otimes\mu,\quad\forall\alpha\in\Gamma(A).$$

One checks that ξ_A is indeed an A-cocycle, and that its cohomology class is independent of the choice of η and μ. Hence, there is a well-defined modular class of A denoted $\mathrm{mod}(A)\equiv[\xi_A]\in H^1(A)$.

Example 2.2. — For a Poisson manifold (M,π) we have a natural Lie algebroid structure on its cotangent bundle T^*M. For the anchor we have $\rho=\pi^\sharp$ and for the Lie bracket on sections of $A=T^*M$, *i.e.*, on one forms, we have:

$$[\alpha,\beta]=\boldsymbol{\mathcal{L}}_{\pi^\sharp\alpha}\beta-\boldsymbol{\mathcal{L}}_{\pi^\sharp\beta}\alpha-\mathrm{d}\pi(\alpha,\beta).$$

Note however that the two definitions above of the modular class differ by a multiplicative factor:

$$\mathrm{mod}(T^*M)=2\ \mathrm{mod}(\pi).$$

See also [14] for a deeper explanation of the factor 2. This factor will appear frequently in our formulas.

Example 2.3. — Let M be a manifold and $\mathcal{N}:TM\to TM$ a Nijenhuis tensor, *i.e.*, a (1,1)-tensor whose Nijenhuis torsion

$$T_{\mathcal{N}}(X,Y):=\mathcal{N}[\mathcal{N}X,Y]+\mathcal{N}[X,\mathcal{N}Y]-\mathcal{N}^2([X,Y])-[\mathcal{N}X,\mathcal{N}Y],$$

(3) Again, this orientability assumption is made only to simplify the presentation, and is not essential for what follows.

vanishes. This is equivalent to requiring that the triple $(TM, [\ ,\ ]_{\mathcal{N}}, \rho_{\mathcal{N}})$ is a Lie algebroid, where the anchor is given by

$$\rho_{\mathcal{N}}(X) := \mathcal{N}X,$$

and the Lie bracket is defined by:

$$[X, Y]_{\mathcal{N}} := [\mathcal{N}X, Y] + [X, \mathcal{N}Y] - \mathcal{N}([X, Y]).$$

Let us compute the modular class of this Lie algebroid.

Proposition 2.4. — *The modular class of* $(TM, [\ ,\ ]_{\mathcal{N}}, \rho_{\mathcal{N}})$ *is the cohomology class represented by the 1-form* $\mathrm{d}(\operatorname{tr}\mathcal{N})$.

Note that this class may not be trivial: we must consider it as a cohomology class in the Lie algebroid cohomology of $(TM, [\ ,\ ]_{\mathcal{N}}, \rho_{\mathcal{N}})$. This cohomology is computed by the complex of differential forms but with a modified differential $\mathrm{d}_{\mathcal{N}}$ that satisfies:

$$\mathrm{d}_{\mathcal{N}}\mathcal{N}^* = \mathrm{d}\mathcal{N}^*$$

(here $\mathcal{N}^* : T^*M \to T^*M$ denotes the transpose of $\mathcal{N}$).

Proof of Proposition 2.4. — We pick a volume form $\mu \in \Omega^{\mathrm{top}}(M)$, and we let $\eta \in \Gamma(\wedge^{\mathrm{top}} A) = \mathfrak{X}^{\mathrm{top}}(M)$ be the dual multivector field: $\langle \mu, \eta \rangle = 1$. Around any point, we can choose local coordinates $(x^1, \dots, x^m)$ such that:

$$\mu = \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m, \quad \eta = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^m}.$$

In these coordinates, we write

$$\mathcal{N} = \sum_{i,j=1}^{m} N^i_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j,$$

and for $X = \frac{\partial}{\partial x^k}$ we compute:

$$\begin{aligned}
\boldsymbol{\mathcal{L}}^{\mathcal{N}}_X \frac{\partial}{\partial x^i} &= \sum_{j=1}^{m} \left(\frac{\partial N^j_i}{\partial x^k} - \frac{\partial N^j_k}{\partial x^i} \right) \frac{\partial}{\partial x^j} \\
\boldsymbol{\mathcal{L}}_{\mathcal{N}X} \mathrm{d}x^i &= \sum_{j=1}^{m} \frac{\partial N^i_k}{\partial x^j} \mathrm{d}x^j \ ,
\end{aligned}$$

where the first Lie derivative is in the Lie algebroid sense, while the second is the usual Lie derivative. From these expressions it follows that:

$$\begin{aligned}\boldsymbol{\mathcal{L}}_X^{\mathcal{N}}\eta &= \sum_{i=1}^m \frac{\partial}{\partial x^1}\wedge\cdots\wedge\boldsymbol{\mathcal{L}}_X^{\mathcal{N}}\frac{\partial}{\partial x^i}\wedge\cdots\wedge\frac{\partial}{\partial x^m}\\ &= \sum_{i=1}^m\left(\frac{\partial N_i^i}{\partial x^k}-\frac{\partial N_k^i}{\partial x^i}\right)\eta\end{aligned}$$

and, similarly, that:

$$\begin{aligned}\boldsymbol{\mathcal{L}}_{\mathcal{N}X}\mu &= \sum_{i=1}^m \mathrm{d}x^1\wedge\cdots\wedge\boldsymbol{\mathcal{L}}_{\mathcal{N}X}\mathrm{d}x^i\wedge\cdots\wedge\mathrm{d}x^m\\ &= \sum_{i=1}^m\frac{\partial N_k^i}{\partial x^i}\mu\ .\end{aligned}$$

Therefore, we conclude that for $X=\frac{\partial}{\partial x^k}$:

$$\boldsymbol{\mathcal{L}}_X^{\mathcal{N}}\eta\otimes\mu+\eta\otimes\boldsymbol{\mathcal{L}}_{\mathcal{N}X}\mu=\sum_{i=1}^m\frac{\partial N_i^i}{\partial x^k}\eta\otimes\mu=\langle\mathrm{d}(\mathrm{tr}\,\mathcal{N}),X\rangle\eta\otimes\mu.$$

By linearity, this formula holds for every vector field X, on any coordinate neighborhood. Hence, it must hold on all of M. We conclude that $\mathrm{d}(\mathrm{tr}\,\mathcal{N})$ represents the modular class of $(TM,[\ ,\ ]_{\mathcal{N}},\rho_{\mathcal{N}})$. □

Note that we have chosen the volume forms $\eta=\mu^{-1}$. For other choices of η and μ we would obtain different representatives of the modular class. However, whenever we choose $\eta=\mu^{-1}$ we always get the same representative, independent of the choice of μ. Also, the appearance of the trace should not be a surprise in view of the interpretation of the modular class as a secondary characteristic class (see [14, 15]) associated with the trace.

2.3. Relative modular class

Let $\phi:A\to B$ be a morphism of Lie algebroids over the identity. Then we have an induced chain map $\phi^*:(\Omega^k(B),d_B)\to(\Omega^k(A),d_A)$ defined by:

$$\phi^*P(\alpha_1,\ldots,\alpha_k)=P(\phi(\alpha_1),\ldots,\phi(\alpha_k)),$$

and, hence, also a morphism at the level of cohomology:

$$\phi^*:H^k(B)\to H^k(A).$$

We can attach to this morphism a *relative modular class*. Again, we assume that $\wedge^{\mathrm{top}}A$ and $\wedge^{\mathrm{top}}B$ are trivial line bundles, so we take global sections $\eta \in \Gamma(\wedge^{\text{top}} A)$ and $\nu \in \Gamma(\wedge^{\text{top}} B^*)$. Then we can define $\xi^{\phi}_{A,B} \in C^1(A)$ to be the unique element such that:

$$(\boldsymbol{\mathcal{L}}^A_\alpha \eta) \otimes \mu + \eta \otimes (\boldsymbol{\mathcal{L}}^B_{\phi(\alpha)} \mu) = \xi^{\phi}_{A,B}(\alpha) \eta \otimes \mu, \quad \forall \alpha \in \Gamma(A).$$

One can check that $\xi^{\phi}_{A,B}$ is in fact a cocycle, and that its cohomology is independent of the choice of trivializing sections η and ν. We conclude that we have a well defined relative modular class:

$$\operatorname{mod}(A, B, \phi) \equiv \xi^{\phi}_{A,B} \in H^1(A).$$

Now we have the following basic fact (see [21, 17]):

Proposition 2.5. — *Let $\phi : A \to B$ be a morphism of Lie algebroids. Then:*

$$\operatorname{mod}(A, B, \phi) = \operatorname{mod}(A) - \phi^* \operatorname{mod}(B). \tag{2.6}$$

Moreover, if $\psi : B \to C$ is another morphism, we have

$$\operatorname{mod}(A, C, \psi \circ \phi) = \operatorname{mod}(A, B, \phi) + \phi^* \operatorname{mod}(B, C, \psi). \tag{2.7}$$

If we make any choice of sections $\eta \in \Gamma(\wedge^{\text{top}} A)$, $\nu \in \Gamma(\wedge^{\text{top}} B)$, $\mu \in \Gamma(\wedge^{\text{top}} T^* M)$, and we choose $\nu' \in \Gamma(\wedge^{\text{top}} B^*)$ to be dual to ν, (*i.e.*, $\langle \nu, \nu' \rangle = 1$), then (2.6) already holds at the level of cocycles, not just of cohomology classes: in the notation above, we have the equality

$$\xi^{\phi}_{A,B} = \xi_A - \phi^* \xi_B.$$

Similarly, (2.7) is also true at the level of cocycles.

Example 2.6. — The tangent bundle TM of any manifold is a Lie algebroid for the usual Lie bracket of vector fields and the identity map as an anchor. For this Lie algebroid, if we take a section $\nu \in \Gamma(\wedge^{\text{top}} TM)$ and its dual section $\mu \in \Gamma(\wedge^{\text{top}} TM^*)$, we see immediately that $\xi_{TM} = 0$, so its modular class vanishes. Now, given any Lie algebroid $(A, [\cdot, \cdot])$, its anchor $\rho : A \to TM$ is a Lie algebroid morphism. Hence, we conclude that

$$\operatorname{mod}(A, TM, \rho) = \operatorname{mod}(A).$$

In particular, in the case of a Poisson manifold (M, π) we find:

$$\operatorname{mod}(T^*M, TM, \pi^\sharp) = \operatorname{mod}(T^*M) = 2 \operatorname{mod}(\pi).$$

Again, this equality is true already at the level of vector fields.

3. Modular vector fields and Poison-Nijenhuis manifolds

We are now ready to look at Poisson-Nijenhuis manifolds and their modular classes.

3.1. Poisson-Nijhenhuis manifolds

Let $(M, \pi_0, \mathcal{N})$ be a Poisson-Nijhenhuis manifold. Let us recall what this means ([20]):

(i) π_0 is a Poisson structure on M;
(ii) $\mathcal{N} : TM \to TM$ is a Nijenhuis tensor;
(iii) π_0 and $\mathcal{N}$ are compatible.

The compatibility of π_0 and $\mathcal{N}$ means, first of all, that

$$\mathcal{N}\pi_0^\sharp = \pi_0^\sharp \mathcal{N}^*, \tag{3.1}$$

so that $\pi_1 = \mathcal{N}\pi_0$ is a bivector field, and secondly that the bracket on 1-forms $[\ ,\ ]_{\pi_1}$ naturally associated with π_1 (see Example 2.2):

$$[\alpha, \beta]_{\pi_1} := \boldsymbol{\mathcal{L}}_{\pi_1^\sharp \alpha}\beta - \boldsymbol{\mathcal{L}}_{\pi_1^\sharp \beta}\alpha - \mathrm{d}\pi_1(\alpha, \beta)$$

and the bracket $[\ ,\ ]_{\pi_0}^{\mathcal{N}^*}$ obtained from $[\ ,\ ]_{\pi_0}$ by twisting by $\mathcal{N}^*$ (see Example 2.3):

$$[\alpha, \beta]_{\pi_0}^{\mathcal{N}^*} := [\mathcal{N}^*\alpha, \beta]_{\pi_0} + [\alpha, \mathcal{N}^*\beta]_{\pi_0} - \mathcal{N}^*([\alpha, \beta]_{\pi_0})$$

actually coincide:

$$[\alpha, \beta]_{\pi_1} = [\alpha, \beta]_{\pi_0}^{\mathcal{N}^*}. \tag{3.2}$$

As a consequence of this definition, we have that π_1 must be a Poisson tensor and the dual of the Nijenhuis tensor:

$$\mathcal{N}^* : (T^*M, [\ ,\ ]_{\pi_1}, \pi_1^\sharp) \to (T^*M, [\ ,\ ]_{\pi_0}, \pi_0^\sharp)$$

is a morphism of Lie algebroids.

As is well-known ([20]), we have in fact a whole hierarchy of Poisson structures:

$$\pi_1 := \mathcal{N}\pi_0,\ \pi_2 := \mathcal{N}\pi_1 = \mathcal{N}^2\pi_0, \dots$$

which are pairwise compatible:

$$[\pi_i, \pi_j] = 0, \quad \forall i, j = 0, 1, 2, \dots$$

From this it follows that if we have a bi-hamiltonian vector field:

$$X_1 = \pi_1^\sharp \mathrm{d}h_0 = \pi_0^\sharp \mathrm{d}h_1,$$

then we have a whole hierarchy of commuting flows $X_1, X_2, X_3, \dots$ where the higher order flows are given by:

$$X_i = \pi_i^\sharp \mathrm{d}h_0 = \pi_{i-1}^\sharp \mathrm{d}h_1.$$

Hence, one usually thinks of an integrable hierarchy as being specified by a Poisson-Nijenhuis manifold *and* a bi-hamiltonian vector field. Here we would like to show that, under a *natural* assumption, there is a *canonical hierarchy* associated with a Poisson-Nijenhuis manifold, which does not involve other choices such as a specification of a bi-hamiltonian vector field. The source of this hierarchy is the modular class of a Poisson-Nijenhuis manifold.

3.2. Modular vector field of a Poisson-Nijhenhuis manifold

Let $(M, \pi_0, \mathcal{N})$ be a Poisson-Nijhenhuis manifold. It is clear from the definition that $\mathcal{N}$ maps the hamiltonian vector field X_f^0 (relative to π_0) to the hamiltonian vector field X_f^1 (relative to π_1). Similarly, it is easy to see that $\mathcal{N}$ maps Poisson vector fields of π_0 to Poisson vector fields of π_1. More generally, $\mathcal{N}$ induces a map at the level of multivector fields, denoted by the same letter $\mathcal{N} : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet(M)$, which is defined by:

$$\mathcal{N}A(\alpha_1, \ldots, \alpha_a) = A(\mathcal{N}^*\alpha_1, \ldots, \mathcal{N}^*\alpha_a).$$

We have:

Proposition 3.1. — *The map* $\mathcal{N} : (\mathfrak{X}^\bullet(M), \mathrm{d}_{\pi_0}) \to (\mathfrak{X}^\bullet(M), \mathrm{d}_{\pi_1})$ *is a morphism of complexes:*

$$\mathcal{N}\mathrm{d}_{\pi_0} = \mathrm{d}_{\pi_1}\mathcal{N}.$$

Proof. — We need simply to observe that we have a Lie algebroid morphism:

$$\mathcal{N}^* : (T^*M, [\ ,\ ]_{\pi_1}, \pi_1^\sharp) \to (T^*M, [\ ,\ ]_{\pi_0}, \pi_0^\sharp)$$

so it induces a morphism between the complexes of forms of these Lie algebroids, in the opposite direction. Of course, this map is just the map $\mathcal{N} : (\mathfrak{X}^\bullet(M), \mathrm{d}_{\pi_0}) \to (\mathfrak{X}^\bullet(M), \mathrm{d}_{\pi_1})$ introduced above. □

It follows that we have an induced map in cohomology

$$\mathcal{N} : H^\bullet_{\pi_0}(M) \to H^\bullet_{\pi_1}(M).$$

Note, however, that in general $\mathcal{N}$ *does not* map the modular class of π_0 to the modular class of π_1. For a choice of volume form $\mu \in \Omega^{\mathrm{top}}(M)$, let us denote denote by X_μ^1 and by X_μ^0 the modular vector fields associated with π_1 and π_0 respectively.

Lemma 3.2. — *If μ and μ' are any two volume forms then:*

$$X_\mu^1 - \mathcal{N}X_\mu^0 = X_{\mu'}^1 - \mathcal{N}X_{\mu'}^0.$$

Moreover, this vector field is Poisson relative to π_1.

Proof. — Let $g \in C^\infty(M)$ be a non-vanishing function such that $\mu' = g\mu$. By relation (2.4), we have

$$\begin{aligned} X^1_{\mu'} - \mathcal{N}X^0_{\mu'} &= X^1_{g\mu} - \mathcal{N}X^0_{g\mu} \\ &= X^1_{\mu} - X^1_{\ln|g|} - \mathcal{N}(X^0_{\mu} - X^0_{\ln|g|}) \\ &= X^1_{\mu} - \mathcal{N}X^0_{\mu}, \end{aligned}$$

where we used that $X^1_f = \mathcal{N}X^0_f$, for any function f.

The modular vector field X^1_μ is a Poisson vector field relative to π_1. On the other hand, $\mathcal{N}$ maps the vector field X^0_μ, which is Poisson relative to π_0, to a Poisson vector field relative to π_1. Hence, the sum $X^1_\mu - \mathcal{N}X^0_\mu$ is a Poisson vector field relative to π_1. □

Let us set $X_\mathcal{N} := X^1_\mu - \mathcal{N}X^0_\mu$ (which, by the lemma, is independent of the choice of μ). Note that $X_\mathcal{N}$ is a vector field intrinsically associated with the Poisson-Nijenhuis manifold $(M, \pi_0, \mathcal{N})$.

DEFINITION 3.3. — *The vector field $X_\mathcal{N}$ is called the **modular vector field of the Poisson-Nijenhuis** manifold* $(M, \pi_0, \mathcal{N})$.

The modular vector field $X_\mathcal{N}$ of $(M, \pi_0, \mathcal{N})$ will play a fundamental role in the sequel. Our next proposition gives further justification for this name and explains the possible failure in $X_\mathcal{N}$ being a hamiltonian vector field:

PROPOSITION 3.4. — *Let $(M, \pi_0, \mathcal{N})$ be a Poisson-Nijenhuis manifold. The Poisson cohomology class $[X_\mathcal{N}] \in H^1_{\pi_1}(M)$ equals half the relative modular class of the Lie algebroid morphism:*

$$\mathcal{N}^* : (T^*M, [\ ,\ ]_{\pi_1}, \pi_1^\sharp) \to (T^*M, [\ ,\ ]_{\pi_0}, \pi_0^\sharp).$$

Proof. — By Proposition 2.5 and Example 2.2 we find:

$$\begin{aligned} \operatorname{mod}(T^*M_{\pi_1}, T^*M_{\pi_0}, \mathcal{N}^*) &= \operatorname{mod}(T^*M_{\pi_1}) - (\mathcal{N}^*)^* \operatorname{mod}(T^*M_{\pi_0}) \\ &= 2 \operatorname{mod}(\pi_1) - 2\mathcal{N}\operatorname{mod}(\pi_0) \\ &= 2\,[X^1_\mu] - 2\mathcal{N}[X^0_\mu] \\ &= 2\,[X^1_\mu - \mathcal{N}X^0_\mu] = 2\,[X_\mathcal{N}], \end{aligned}$$

for any volume form μ. □

We emphasize that $X_\mathcal{N}$ is a canonical representative of the relative modular class of $\mathcal{N}^*$, which does not depend on any choice of measure.

3.3. Hamiltonian character of the modular vector field

As we saw above, the modular vector field $X_{\mathcal{N}}$ of a Poisson-Nijenhuis manifold $(M,\pi_0,\mathcal{N})$ is a Poisson vector field relative to π_1, which may fail to be hamiltonian. Let us now look at its behavior relative to π_0. We have:

THEOREM 3.5. — *Let $(M,\pi_0,\mathcal{N})$ be a Poisson-Nijenhuis manifold. Then the modular vector field $X_{\mathcal{N}}$ is hamiltonian relative to π_0 with hamiltonian equal to minus one half the trace of $\mathcal{N}$:*

$$X_{\mathcal{N}} = X^0_{-\frac{1}{2}\operatorname{tr}\mathcal{N}}. \tag{3.3}$$

Before we prove this theorem, let us observe that this result is intimately related to Proposition 2.4, where we showed that the modular class of the Lie algebroid of a Nijenhuis tensor $\mathcal{N}$ is represented by the 1-form $\mathrm{d}(\operatorname{tr}\mathcal{N})$. In fact, observe that the compatibility condition of a Poisson-Nijenhuis structure states that the two Lie algebroids

$$\begin{aligned} T^*M_{\pi_1} &:= (T^*M,[\ ,\ ]_{\mathcal{N}\pi_0},\rho=\mathcal{N}\pi_0^\sharp) \\ T^*M_{\pi_0}^{\mathcal{N}^*} &:= (T^*M,[\ ,\ ]_{\pi_0}^{\mathcal{N}^*},\rho=\pi_0^\sharp\mathcal{N}^*) \end{aligned}$$

actually coincide. Therefore they have the same modular classes, and from the general Lie algebroid version of Proposition 2.4 we obtain:

$$\operatorname{mod}(T^*M_{\pi_1}) = \operatorname{mod}(T^*M_{\pi_0}^{\mathcal{N}^*}) = [\mathrm{d}_{\pi_0}(\operatorname{tr}\mathcal{N})] + \mathcal{N}^*\operatorname{mod}(T^*M_{\pi_0}).$$

Using Proposition 2.5, this leads immediately to the statement:

$$\begin{aligned} 2[X_{\mathcal{N}}] &= \operatorname{mod}(T^*M_{\pi_1},T^*M_{\pi_0},\mathcal{N}^*) \\ &= \operatorname{mod}(T^*M_{\pi_1}) - \mathcal{N}^*\operatorname{mod}(T^*M_{\pi_0}) \\ &= [\mathrm{d}_{\pi_0}(\operatorname{tr}\mathcal{N})] = -[(\pi_0^\sharp)\mathrm{d}(\operatorname{tr}\mathcal{N})]. \end{aligned}$$

By working at the level of representatives of these cohomology classes, one can give a proof of Theorem 3.5. However, we prefer to give a local coordinate proof which is a direct translation of this argument.

Proof of Theorem 3.5. — Note that it is enough to prove that the two sides of (3.3) agree in any local coordinate system. Hence, let us choose local coordinates $(x^1,\dots,x^m)$, so that:

$$\begin{aligned} \pi_0 &= \sum_{i<j}\pi_0^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}, \\ \mathcal{N} &= \sum_{i,j}N^i_j\frac{\partial}{\partial x^i}\wedge\mathrm{d}x^j, \\ \mu &= f\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^m. \end{aligned}$$

In these local coordinates, the compatibility condition (3.2) for a Poisson-Nijenhuis structure reads:

$$\begin{aligned}0 &= [\mathrm{d}x^i, \mathrm{d}x^j]_{\pi_1} - [\mathrm{d}x^i, \mathrm{d}x^j]^{\mathcal{N}^*}_{\pi_0}\\ &= \sum_{k,l}\left(\pi_0^{lj}\frac{\partial N^i_k}{\partial x^l} + \pi_0^{il}\frac{\partial N^j_k}{\partial x^l} - \pi_0^{lj}\frac{\partial N^i_l}{\partial x^k} - N^l_k\frac{\partial \pi_0^{ij}}{\partial x^l} + N^j_l\frac{\partial \pi_0^{il}}{\partial x^k}\right)\mathrm{d}x^k.\end{aligned}$$

If in each coefficient of $\mathrm{d}x^k$ we contract j and k, we see that the two last terms cancel out, and we obtain:

$$\sum_{k,l}\left(2\pi_0^{lk}\frac{\partial N^i_k}{\partial x^l} + \pi_0^{il}\frac{\partial N^k_k}{\partial x^l}\right) = 0, \qquad (i = 1, \ldots, m).$$

Using this identity and Proposition 2.1, we conclude that:

$$\begin{aligned}X_{\mathcal{N}} &= X^1_\mu - \mathcal{N}X^0_\mu\\ &= \sum_{i,j}\frac{\partial \pi_1^{ij}}{\partial x_j}\frac{\partial}{\partial x_i} - \mathcal{N}\sum_{i,j}\frac{\partial \pi_0^{ij}}{\partial x_j}\frac{\partial}{\partial x_i}\\ &= \sum_{i,j,k}\left(\frac{\partial \pi_0^{ik}N^j_k}{\partial x_j} - N^i_k\frac{\partial \pi_0^{kj}}{\partial x_j}\right)\frac{\partial}{\partial x_i}\\ &= \sum_{i,l,k}\pi_0^{lk}\frac{\partial N^i_k}{\partial x^l}\frac{\partial}{\partial x_i}\\ &= -\frac{1}{2}\sum_{i,l,k}\pi_0^{il}\frac{\partial N^k_k}{\partial x^l}\frac{\partial}{\partial x_i} = X^0_{-\frac{1}{2}\mathrm{tr}\,\mathcal{N}}.\end{aligned}$$

□

Remark 3.6. — We recall (see [20, page 58]) that one has a commutative diagram of morphisms of Lie algebroids:

$$\begin{tikzcd}(T^*M, [\cdot,\cdot]_{\pi_1}) \arrow[r, "\mathcal{N}^*"] \arrow[d, "\pi_0^\sharp"'] \arrow[dr, "\pi_1^\sharp"] & (T^*M, [\cdot,\cdot]_{\pi_0}) \arrow[d, "\pi_0^\sharp"]\\ (TM, [\cdot,\cdot]_{\mathcal{N}}) \arrow[r, "\mathcal{N}"] & (TM, [\cdot,\cdot])\end{tikzcd}$$

The relative modular class of the morphism $\mathcal{N}$ on the bottom horizontal arrow is represented by $\mathrm{d}(\mathrm{tr}\,\mathcal{N})$. On the other hand, Theorem 3.5 states that the relative modular class of the morphism $\mathcal{N}^*$ on the top arrow is represented by $-(\pi_0^\sharp)\mathrm{d}(\mathrm{tr}\,\mathcal{N}) = (\pi_0^\sharp)^*\mathrm{d}(\mathrm{tr}\,\mathcal{N})$. Hence this diagram codifies nicely the relationship between all (relative and absolute) modular classes involved in a Poisson-Nijenhuis manifold.

4. Integrable hierarchies and the modular class

We now consider integrable hierarchies of hamiltonian systems and observe that they are closely related to the modular vector field introduced above.

4.1. The hierarchy of a non-degenerate PN manifold

Theorem 3.5 above shows that for any non-degenerate Poisson-Nijenhuis manifold the modular vector field $X_{\mathcal{N}}$ is hamiltonian relative to π_0 and Poisson relative to π_1. So, the question arises whether this vector field is also Hamiltonian relative to π_1 with respect to another function of $\mathcal{N}$, *i.e.*, whether it is a bi-hamiltonian vector field. Of course, the obstruction is the Poisson cohomology class $[X_{\mathcal{N}}]$, the modular class of the Poisson-Nijenhuis manifold. An important case where this class vanishes and which fits many examples is the following:

THEOREM 4.1. — *Let $(M, \pi_0, \mathcal{N})$ be a Poisson-Nijenhuis manifold and assume that $\mathcal{N}$ is non-degenerate. Then the modular vector field $X_{\mathcal{N}}$ is bi-hamiltonian and hence determines a hierarchy of flows which is given by:*

$$X_{i+j} = \pi_i^\sharp \mathrm{d}h_j = \pi_j^\sharp \mathrm{d}h_i, \quad (i, j \in \mathbf{Z}) \tag{4.1}$$

where

$$h_0 = -\frac{1}{2}\log(\det \mathcal{N}), \quad h_i = -\frac{1}{2i}\operatorname{tr}\mathcal{N}^i, \quad (i \neq 0). \tag{4.2}$$

Proof. — Let us start by verifying that $X_{\mathcal{N}}$ is bi-hamiltonian:

$$X_{\mathcal{N}} = X^1_{-\frac{1}{2}\log(|\det \mathcal{N}|)} = X^0_{-\frac{1}{2}\operatorname{tr}\mathcal{N}}. \tag{4.3}$$

By Theorem 3.5, we just need to prove the first equality.

We claim that the first equality holds on the open dense set of common regular points of π_0 and π_1. In fact, for any such regular point we can choose an open neighborhood U where both π_0 and π_1 admit invariant volume forms μ_0 and μ_1. It is easy to see that we can take these two volume forms to be related by $\mathcal{N}$:

$$\mu_1 = \mathcal{N}^{-1}\mu_0 = \frac{1}{|\det(\mathcal{N})|^{\frac{1}{2}}}\mu_0.$$

where $n = \frac{1}{2}\dim M$. It follows from relation (2.4) that the modular vector fields for π_1 relative to these two n-forms are related by:

$$0 = X^1_{\mu_1} = X^1_{\mu_0} + X^1_{\frac{1}{2}\log(|\det\mathcal{N}|)}.$$

Since $X^0_{\mu_0} = 0$, we can compute the modular vector field $X_{\mathcal{N}}$ as follows:

$$X_{\mathcal{N}} = X^1_{\mu_0} - \mathcal{N}X^0_{\mu_0} = X^1_{-\frac{1}{2}\log(|\det\mathcal{N}|)}.$$

This proves our claim, so $X_{\mathcal{N}}$ is bi-hamiltonian.

Now, it remains to prove the multi-hamiltonian structure for the higher flows. This follows by an iterative procedure. For example, let us check the multi-hamiltonian structure of the 2nd flow in the hierarchy:

$$X_2 = X^2_{-\frac{1}{2}\log(\det\mathcal{N})} = X^1_{-\frac{1}{2}\operatorname{tr}\mathcal{N}} = X^0_{-\frac{1}{4}\operatorname{tr}\mathcal{N}^2}. \tag{4.4}$$

First, we note that, from what we just proved, we have:

$$X_{\mathcal{N}^2} = X^2_{\mu} - \mathcal{N}^2 X^0_{\mu} = X^0_{-\frac{1}{2}\operatorname{tr}\mathcal{N}^2} = X^2_{-\frac{1}{2}\det(\log\mathcal{N}^2)}.$$

which shows equality of two terms in (4.4). On the other hand, we have:

$$\begin{aligned}
X^1_{-\frac{1}{2}\operatorname{tr}\mathcal{N}} &= X^2_{\mu} - \mathcal{N}X^1_{\mu} \\
&= X^2_{\mu} - \mathcal{N}^2 X^0_{\mu} + \mathcal{N}^2 X^0_{\mu} - \mathcal{N}X^1_{\mu} \\
&= X_{\mathcal{N}^2} - NX_{\mathcal{N}} \\
&= X^0_{-\frac{1}{2}\operatorname{tr}\mathcal{N}^2} - NX^0_{-\frac{1}{2}\operatorname{tr}\mathcal{N}} = X^0_{-\frac{1}{2}\operatorname{tr}\mathcal{N}^2} - X^1_{-\frac{1}{2}\operatorname{tr}\mathcal{N}}
\end{aligned}$$

which gives:

$$X^1_{-\frac{1}{2}\operatorname{tr}\mathcal{N}} = X^0_{-\frac{1}{4}\operatorname{tr}\mathcal{N}^2},$$

so giving equality with the remaining term in (4.4).

By iteration, looking at the vector fields $X_{\mathcal{N}^i}$, one obtains the multi-hamiltonian formulation of the remaining higher order flows. For negative values of the index we apply the proposition to $\mathcal{N}^{-1}$ to obtain

$$X^1_{-\frac{1}{2}\log(|\det\mathcal{N}|)} = X^0_{\frac{1}{2}\operatorname{tr}\mathcal{N}^{-1}},$$

and then we proceed as in the case of positive indices. □

Remark 4.2. — F. Magri pointed out to us that both relations:

$$k\,\mathcal{N}^*\mathrm{d}(\mathcal{N}^{k-1}) = (k-1)\mathrm{d}(\operatorname{tr}\mathcal{N}^k),$$
$$\mathcal{N}^*\mathrm{d}(\log|\det\mathcal{N}|) = \mathrm{d}(\operatorname{tr}\mathcal{N}),$$

which are well known to people working in integrable systems (see, *e.g.*, [4, 19]), also lead to the bi-hamiltonian hierarchy of the theorem. The two relations above appear also in [2] and [3] respectively as was pointed out by the anonymous referee.

Note that $\mathcal{N}$ always has double eigenvalues. For the hierarchy to be completely integrable, we need $n = \frac{1}{2}\dim M$ independent spectral invariants $\det \mathcal{N}, \operatorname{tr} \mathcal{N}, \operatorname{tr} \mathcal{N}^2, \ldots$ This will follow if $\mathcal{N}$ has $n = \frac{1}{2}\dim M$ independent eigenvalues.

4.2. Master symmetries and modular vector fields

When $\mathcal{N}$ is degenerate the results in the previous paragraph do not apply. In this situation, there is a procedure due to Oevel [26] to produce integrable hierarchies from master symmetries, and it is natural to look how the modular vector fields fit into this scheme. We start with the following result which is of independent interest:

PROPOSITION 4.3. — *Let π_0 and π_1 be Poisson tensors such that $\pi_1 = \boldsymbol{\mathcal{L}}_Z \pi_0$, for some vector field Z. Also fix a volume form $\mu \in \Omega^{top}(M)$. Then their modular vector fields are related by*

$$X^1_\mu = \boldsymbol{\mathcal{L}}_Z X^0_\mu + X^0_{\operatorname{div}_\mu(Z)}. \tag{4.5}$$

Proof. — The proof is straightforward if we use the definition of the modular class in terms of the homological operator D_μ:

$$\begin{aligned} X^1_\mu &= D_\mu(\pi_1) \\ &= D_\mu([Z, \pi_0]) \\ &= [Z, D_\mu(\pi_0)] - [D_\mu(Z), \pi_0] \\ &= \boldsymbol{\mathcal{L}}_Z X^0_\mu + X^0_{\operatorname{div}_\mu(Z)}. \end{aligned}$$

□

Now, in Oevel's approach, one assumes that we have a bi-hamiltonian system defined by the Poisson tensors π_0 and π_1 and the hamiltonians h_1 and h_0:

$$X_1 = X^0_{h_1} \equiv \pi_0^\sharp \mathrm{d}h_1 = X^1_{h_0} \equiv \pi_1^\sharp \mathrm{d}h_0. \tag{4.6}$$

If, additionally, π_0 is symplectic, one can define the recursion operator in the usual way:

$$\mathcal{N} = \pi_1^\sharp \circ (\pi_0^\sharp)^{-1},$$

the higher flows $X_i := \mathcal{N}^{i-1} X_1$, and the higher order Poisson tensors $\pi_i := \mathcal{N}^i \pi_0$. Note that $\mathcal{N}$ can now be degenerate. Now one can generate master-symmetries by the following method:

THEOREM 4.4 ([26]). — *Suppose that Z_0 is a conformal symmetry for both π_0, π_1 and h_0, i.e., for some scalars λ, μ, and ν we have*

$$\mathcal{L}_{Z_0}\pi_0 = \lambda\pi_0, \quad \mathcal{L}_{Z_0}\pi_1 = \mu\pi_1, \quad \mathcal{L}_{Z_0}h_0 = \nu h_0.$$

Then the vector fields

$$Z_i = \mathcal{N}^i Z_0$$

are master symmetries and we have

$$\mathcal{L}_{Z_i}h_j = (\nu + (j-1+i)(\mu-\lambda))h_{i+j}, \tag{4.7}$$

$$\mathcal{L}_{Z_i}\pi_j = (\mu + (j-i-2)(\mu-\lambda))\pi_{i+j}, \tag{4.8}$$

$$[Z_i, Z_j] = (\mu-\lambda)(j-i)Z_{i+j}. \tag{4.9}$$

To simplify the notation, we will set:

$$c_{i,j} = (\mu + (j-i-2)(\mu-\lambda))$$

so, for example, $[Z_i, \pi_j] = c_{i,j}\pi_{i+j}$. Also, we fix a volume form $\mu \in \Omega^{\text{top}}(M)$, so the jth Poisson bracket in the hierarchy has the modular vector field

$$X^j_\mu = D_\mu(\pi_j).$$

The following proposition establishes relations among these modular vector fields:

THEOREM 4.5. — *For the hierarchy above:*

$$\begin{aligned}[X^j_\mu, Z_i] &= c_{i,j}X^{i+j}_\mu + X^j_{f_i},\\ \mathcal{L}_{X^i_\mu}\pi_j &= -\mathcal{L}_{X^j_\mu}\pi_i,\end{aligned}$$

where $f_i = D_\mu(Z_i) = \mathrm{div}_\mu(Z_i)$.

Proof. — To prove the first relation, one simply applies Proposition 4.3 repeatedly. For the second relation, we observe that:

$$\begin{aligned}\mathcal{L}_{X^i_\mu}\pi_j &= [X^i_\mu, \pi_j]\\ &= \frac{1}{c_{j-i,i}}\left[X^i_\mu, [Z_{j-i}, \pi_i]\right].\end{aligned}$$

Using the super–Jacobi identity for the Schouten bracket and the fact that X_μ^i is Poisson relative to π_i, the last term reduces to $\left[Z_{j-i}, X_\mu^i], \pi_i\right]$. Therefore

$$\begin{aligned}\boldsymbol{\pounds}_{X_\mu^i}\pi_j &= \frac{1}{c_{j-i,i}}\left[[Z_{j-i}, X_\mu^i], \pi_i\right]\\ &= -\frac{1}{c_{j-i,i}}\left[[Z_i, X_\mu^{i-j}], \pi_i\right]\\ &= -\frac{1}{c_{j-i,i}}\left[c_{j-i,i}X_\mu^j + X^i_{f_{j-i}}, \pi_i\right]\\ &= -[X_\mu^j, \pi_i] = -\boldsymbol{\pounds}_{X_\mu^j}\pi_i.\end{aligned}$$

□

Note that, even when $\mathcal{N}$ is non-degenerate, there is no reason for the hierarchy of flows produced by this method to coincide with the hierarchy of flows canonically associated with the Poisson-Nijenhuis manifold. In general, one would obtain two distinct hierarchies. However, as we shall see in the next section, in most of the examples it is often the case that this two hierarchies coincide. This is due to the fact that, in many examples, the initial bi-hamiltonian system (4.6) has a multiple of $\operatorname{tr}\mathcal{N}$ as one of the hamiltonians.

5. Examples

In this section we will illustrate the results of this paper on some well-known integrable systems such as the Harmonic oscillator, the Calogero-Moser system and various versions of the Toda lattice.

5.1. Harmonic oscillator

This classical integrable system has a well-known bi-hamiltonian structure which we now recall (see [5]).

On $\mathbb{R}^{2n}$ with the standard symplectic structure and canonical coordinates (q_i, p_i), consider the following hamiltonian function:

$$h_1 = \sum_{i=1}^{n} \frac{1}{2}(p_i^2 + q_i^2)\ .$$

The resulting hamiltonian system is completely integrable with the following integrals of motion in involution:

$$I_i = \frac{1}{2}(p_i^2 + q_i^2), \quad (i = 1, \dots, n).$$

For its bi-hamiltonian structure one takes the Poisson structure associated with the canonical symplectic form:

$$\pi_0 = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i},$$

and the new Poisson structure:

$$\pi_1 = \sum_{i=1}^{n} I_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} \ .$$

These form a compatible pair of Poisson structures, and we also have:

$$X_1 = \sum_{i=1}^{n} p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i} = \pi_0^\sharp \mathrm{d}h_1 = \pi_1^\sharp \mathrm{d}h_0, \tag{5.1}$$

where

$$h_0 = \log I_1 + \cdots + \log I_n.$$

It is easy to see that this is the bi-hamiltonian formulation of the first flow in the integrable hierarchy of the Poisson-Nijenhuis manifold $(\pi_0, \mathcal{N})$, where the Nijenhuis tensor is the diagonal (1,1) tensor:

$$\mathcal{N} = \mathrm{diag}(I_1, \dots, I_n, I_1, \dots, I_n).$$

In fact, with this definition, we find $\pi_1 = \mathcal{N}\pi_0$ and:

$$\det \mathcal{N} = \prod_{i=1}^{n} I_i^2,$$

so that:

$$\frac{1}{2}\log(\det \mathcal{N}) = \log I_1 + \cdots + \log I_n = h_0,$$
$$\frac{1}{2}\mathrm{tr}\, \mathcal{N} = I_1 + \cdots + I_n = h_1.$$

Hence, the bi-hamiltonian formulation (5.1) coincides with the one of the first flow of the hierarchy (4.1) canonically associated with the Poisson-Nijenhuis manifold $(\pi_0, \mathcal{N})$.

In this example, we have a master symmetry Z such that $\boldsymbol{\mathcal{L}}_Z \pi_0 = \pi_1$ which is given by:

$$Z = -\sum_{i=1}^{n} \frac{1}{4} I_i \left(q_i \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_i} \right).$$

If we let $\mu_0 = \mathrm{d}p_1 \wedge \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}p_n \wedge \mathrm{d}q_n$, which is a Liouville form for π_0, we compute:

$$-\mathrm{div}_{\mu_0}(Z) = \frac{1}{2}\,\mathrm{tr}\,\mathcal{N} = h_1,$$

as expected.

There is however one point that we overlooked: strictly speaking these results are true only on the manifold $M = \mathbb{R}^{2n} - \cup_{i=1}^{n}\{I_i = 0\}$ where $\mathcal{N}$ is invertible. In fact, on $\mathbb{R}^{2n}$ the relative modular vector field $X_{\mathcal{N}}$ is not hamiltonian relative to π_1! In fact, if $X_{\mathcal{N}} = \pi_1^{\sharp}\mathrm{d}H$ for some smooth function $H \in C^{\infty}(\mathbb{R}^{2n})$ then, on points away from $I_i = 0$, H must differ from $h_0 = \log I_1 + \cdots + \log I_n$ by a constant, and this is clearly impossible. Therefore, on $\mathbb{R}^{2n}$ the relative modular class $[X_{\mathcal{N}}]$ is non-trivial, and there is no canonical bi-hamiltonian hierarchy.

Note that this examples is *universal*: any integrable hierarchy associated with a non-degenerate Poisson-Nijenhuis manifold $(M, \pi_0, \mathcal{N})$ locally (in action-variables coordinates) looks like this one.

5.2. The rational Calogero-Moser system

The Calogero-Moser system is a well-known finite-dimensional integrable system (in fact, super-integrable). One can define this system on $\mathbb{R}^{2n}$, with the standard symplectic structure and canonical coordinates (q_i, p_i), by the hamiltonian function:

$$h_2 = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{g^2}{2}\sum_{j\neq i}\frac{1}{(q_i - q_j)^2}.$$

The Calogero-Moser system admits a Lax pair formulation where the Lax matrix L is given by

$$L_{ij} = p_i\delta_{ij} + g\frac{i(1-\delta_{ij})}{q_i - q_j}.$$

The system is then completely integrable with involutive first integrals given by:

$$F_i = \mathrm{tr}\,(L^i), \quad (i = 1, \ldots, n).$$

Moreover, following Ranada [27], consider also the functions $G_i = \mathrm{tr}(QL^{i-1})$, where Q is the diagonal matrix $\mathrm{diag}(q_1, \ldots, q_n)$. It was shown in [6] that these functions are independent and lead to the algebraic linearization of the system. Using these functions as coordinates, we can write the hamiltonian vector field in the form:

$$X_1 = \sum_{i=1}^{n} F_i \frac{\partial}{\partial G_i}.$$

The original Poisson structure becomes:

$$\pi_0 = \sum_{i=1}^{n} \frac{\partial}{\partial F_i} \wedge \frac{\partial}{\partial G_i},$$

and there exists a second compatible Poisson structure given by:

$$\pi_1 = \sum_{i=1}^{n} F_i \frac{\partial}{\partial F_i} \wedge \frac{\partial}{\partial G_i},$$

providing a bi-hamiltonian formulation given by

$$X_1 = \pi_0^\sharp \mathrm{d}h_i = \pi_1^\sharp \mathrm{d}h_{i-1} \quad (i = 2, \dots, n)$$

where

$$h_j = \frac{1}{2j} \mathrm{tr}\left(\pi_1^\sharp \circ (\pi_0^\sharp)^{-1}\right)^j = \frac{1}{2j} \sum_k (F_k)^j, \quad (j = 1, \dots, n).$$

Now we observe that if we let $h_0 := \log(F_1 \cdots F_n)$, then we can write the system in the form:

$$X_1 = \pi_0^\sharp \mathrm{d}h_1 = \pi_1^\sharp \mathrm{d}h_0.$$

If we set $\mathcal{N} := \pi_1^\sharp \circ (\pi_0^\sharp)^{-1}$, then one checks easily that:

$$h_0 = \frac{1}{2} \log(\det \mathcal{N}), \quad h_1 = \frac{1}{2} \mathrm{tr}\, \mathcal{N},$$

so X_1 is in fact the first flow of the hierarchy (4.1) canonically associated with the Poisson-Nijenhuis manifold $(\mathbb{R}^{2n}, \pi_0, \mathcal{N})$.

5.3. Toda lattice in Moser coordinates

Our next example is related to the Toda hierarchy in the so-called Moser coordinates. The hierarchy of Poisson tensors is due to Faybusovich and Gekhtman [12], and can be defined as follows. Consider $\mathbb{R}^{2n}$ with coordinates $(\lambda_1, \dots, \lambda_n, r_1, \dots, r_n)$ and define the Poisson structures:

$$\pi_0 = \sum_{i=1}^{n} r_i \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial r_i},$$

$$\pi_1 = \sum_{i=1}^{n} \lambda_i r_i \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial r_i}.$$

We also set $h_2 = \frac{1}{2}\sum_{j=0}^{n}\lambda_j^2$ as the hamiltonian. Using the π_0 bracket, we obtain the following set of Hamilton's equations

$$\dot{\lambda}_i = 0, \qquad \dot{r}_i = \lambda_i r_i, \qquad (i = 1, \ldots, n).$$

This system is bi-hamiltonian, since

$$\pi_0^\sharp \mathrm{d}h_2 = \pi_1^\sharp \mathrm{d}h_1,$$

where $h_1 = \sum_{j=1}^{n}\lambda_j$. Again, if we define

$$\mathcal{N} := \pi_1^\sharp \circ (\pi_0^\sharp)^{-1} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n, \lambda_1, \ldots, \lambda_n),$$

we have:

$$\operatorname{tr}\mathcal{N} = 2\sum_{j=1}^{n}\lambda_j = 2\,h_1, \qquad \operatorname{tr}\mathcal{N}^2 = 4\sum_{j=1}^{n}\lambda_j^2 = 4\,h_2.$$

It follows that our system is in fact the *second* flow in the hierarchy (4.1) canonically associated with the Poisson-Nijenhuis manifold $(\mathbb{R}^{2n}, \pi_0, \mathcal{N})$. The first flow of this hierarchy is [(4)]:

$$X_1 = \pi_0^\sharp \mathrm{d}h_1 = \pi_1^\sharp \mathrm{d}h_0$$

where $h_0 = \frac{1}{2}\log(\det\mathcal{N}) = \log\lambda_1 + \cdots + \log\lambda_n$, and in coordinates is simply:

$$\dot{\lambda}_i = 0, \qquad \dot{r}_i = r_i, \qquad (i = 1, \ldots, n).$$

In this example, there is a master symmetry connecting π_0 and π_1 given by:

$$Z = -\frac{1}{2}\sum_{j=1}^{n}\lambda_j^2\frac{\partial}{\partial\lambda_j}$$

so that $\boldsymbol{\mathcal{L}}_Z\pi_0 = \pi_1$. Then, as expected, we find:

$$\operatorname{div}(Z) = -\sum_{j=1}^{n}\lambda_j = -h_1.$$

This example also falls in Oevel's scheme of Section 4.2. The vector field:

$$Z_0 = \sum_{j=1}^{n}\lambda_j\frac{\partial}{\partial\lambda_j},$$

is a conformal symmetry of π_0, π_1 and h_1:

$$\boldsymbol{\mathcal{L}}_{Z_0}\pi_0 = -\pi_0, \quad \boldsymbol{\mathcal{L}}_{Z_0}\pi_1 = 0, \quad \boldsymbol{\mathcal{L}}_{Z_0}h_1 = h_1.$$

---

(4) Note that, just like in the case of the harmonic oscillator we should exclude the points with some $\lambda_i = 0$, where $\det\mathcal{N}$ vanishes.

Note here h_1, instead of h_0. This means that it is the second flow in the hierarchy (*i.e.*, the original flow) that falls into Oevel scheme! Recalling now that $Z_i = \mathcal{N}^i Z_0$, we find:

$$Z_{-1} = \sum_{j=1}^{n} \frac{\partial}{\partial \lambda_j}, \qquad Z_1 = -2\,Z.$$

If we let μ be the standard volume on $\mathbb{R}^{2n}$, we see that Z_{-1} coincides with the modular vector field for π_0 relative to μ:

$$X_\mu^0 = \sum_{j=1}^{n} \frac{\partial}{\partial \lambda_j} = Z_{-1}.$$

On the other hand

$$X_\mu^1 = \sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial \lambda_j} - \sum_{j=1}^{n} r_j \frac{\partial}{\partial r_j},$$

so that:

$$\mathfrak{X}_{\mathcal{N}} = X_\mu^1 - \mathcal{N} X_\mu^0 = X_\mu^1 - \mathcal{N} Z_{-1} = X_\mu^1 - Z_0 = X_{h_0}^1,$$

is indeed the first flow in the hierarchy.

5.4. Bogoyavlensky-Toda systems

We consider now the example of the C_n Toda system. The B_n and D_n Toda systems are similar and details on the computations can be found in [8, 7].

To define the C_n system one considers $\mathbb{R}^{2n}$ with the canonical symplectic structure and the hamiltonian function:

$$H_2 = \frac{1}{2} \sum_{1}^{n} p_j^2 + e^{q_1 - q_2} + \cdots + e^{q_{n-1} - q_n} + e^{2q_n}.$$

Let us consider the Flaschka-type change of coordinates:

$$\begin{aligned} a_i &= \frac{1}{2}\, e^{\frac{1}{2}(q_i - q_{i+1})}, && (i = 1, \dots, n-1) \\ a_n &= \frac{1}{\sqrt{2}}\, e^{q_n}, && \\ b_i &= -\frac{1}{2}\, p_i, && (i = 1, \dots, n). \end{aligned}$$

The equations for the flow in (a_i, b_i) coordinates become

$$\begin{aligned} \dot{a}_i &= a_i\,(b_{i+1} - b_i), && (i = 1, \ldots, n-1),\\ \dot{a}_n &= -2\,a_n b_n, &&\\ \dot{b}_i &= 2\,(a_i^2 - a_{i-1}^2), && (i = 1, \ldots, n), \end{aligned}$$

with the convention that $a_0 = 0$. These equations can also be written as a Lax pair $\dot{L} = [B, L]$, where the Lax matrix L is given by:

$$L = \begin{pmatrix} b_1 & a_1 & & & & & & \\ a_1 & \ddots & \ddots & & & & & \\ & \ddots & \ddots & a_{n-1} & & & & \\ & & a_{n-1} & b_n & a_n & & & \\ & & & a_n & -b_n & -a_{n-1} & & \\ & & & & -a_{n-1} & \ddots & \ddots & \\ & & & & & \ddots & \ddots & -a_1 \\ & & & & & & -a_1 & -b_1 \end{pmatrix},$$

and B is the skew-symmetric part of L.

In the new variables (a_i, b_i), the canonical Poisson bracket on $\mathbb{R}^{2n}$ is transformed into a bracket π_1 which is given by

$$\begin{aligned} \{a_i, b_i\} &= -a_i, && (i = 1, \ldots, n-1)\\ \{a_i, b_{i+1}\} &= a_i, && (i = 1, \ldots, n-1)\\ \{a_n, b_n\} &= -2a_n. && \end{aligned}$$

We follow the tradition of denoting this bracket by π_1 (instead of π_0) being a linear bracket of degree one. This will lead to a shift in degrees, when compared to the formulas in the rest of the paper (to obtain the same formulas we should denote this bracket by π_0). The same comments applies to the first integrals of the system which, following the tradition, will be denoted by $H_2, H_4, \ldots, H_{2n}$, where:

$$H_{2i} = \frac{1}{2i} \operatorname{tr} L^{2i}.$$

In order to obtain a bi-hamiltonian formulation (see [25]), one introduces a cubic Poisson bracket π_3, defined by:

$$\begin{aligned}
\{a_i, a_{i+1}\} &= a_i a_{i+1} b_{i+1}, && (i = 1, \dots, n-2)\\
\{a_{n-1}, a_n\} &= 2\, a_{n-1} a_n b_n, &&\\
\{a_i, b_i\} &= -a_i b_i^2 - a_i^3, && (i = 1, \dots, n-1)\\
\{a_n, b_n\} &= -2\, a_n b_n^2 - 2\, a_n^3, &&\\
\{a_i, b_{i+2}\} &= a_i a_{i+1}^2, && (i = 1, \dots, n-2)\\
\{a_i, b_{i+1}\} &= a_i b_{i+1}^2 + a_i^3, && (i = 1, \dots, n-1)\\
\{a_{n-1}, b_n\} &= a_{n-1}^3 + a_{n-1}(b_n^2 - a_n^2), &&\\
\{a_i, b_{i-1}\} &= -a_{i-1}^2 a_i, && (i = 1, \dots, n)\\
\{a_n, b_{n-1}\} &= -2\, a_{n-1}^2 a_n, &&\\
\{b_i, b_{i+1}\} &= 2\, a_i^2 (b_i + b_{i+1}), && (i = 1, \dots, n-1).
\end{aligned}$$

This leads immediately ([25]) to a bi-hamiltonian system:

$$\pi_3^\sharp \mathrm{d}h_2 = \pi_1^\sharp \mathrm{d}h_4.$$

However, this is not the original system. For the original system, we follow [8] and define $\pi_{-1} = \pi_1 \pi_3^{-1} \pi_1$. Then the C_n Toda system has the bi-hamiltonian formulation:

$$\pi_1^\sharp \mathrm{d}h_2 = \pi_{-1}^\sharp \mathrm{d}h_4.$$

We now give a new bi-hamiltonian formulation using our Theorem 4.1. We have the Nijenhuis tensor:

$$\mathcal{N} := \pi_3^\sharp \circ (\pi_1^\sharp)^{-1},$$

and we set

$$H_0 := \frac{1}{2} \log(\det \mathcal{N}).$$

We need to check that the hamiltonian vector field of H_0 with respect to the second bracket π_3 satisfies:

$$\pi_3^\sharp \mathrm{d}H_0 = \pi_1^\sharp \mathrm{d}H_2, \tag{5.2}$$

so that this yields a bi-hamiltonian formulation for the C_n-Toda. In fact, this follows easily from the Lenard relations for the eigenvalues of the Lax matrix L:

$$\pi_3^\sharp \mathrm{d}\lambda_i = \lambda_i^2 \pi_1^\sharp \mathrm{d}\lambda_i,$$

which lead to:

$$\begin{aligned}\pi_3^\sharp \mathrm{d}H_0 &= \pi_3^\sharp \mathrm{d}\frac{1}{2}\log(\det L)\\ &= \frac{1}{2}\sum_{1=1}^{n}\frac{1}{\lambda_I}\pi_3^\sharp \mathrm{d}\lambda_i\\ &= \frac{1}{2}\sum_{i=1}^{n}\lambda_i\pi_1^\sharp \mathrm{d}\lambda_i\\ &= \sum_{i=1}^{n}\lambda_i^2\pi_1^\sharp \mathrm{d}\frac{\lambda_i^2}{2} = \pi_1^\sharp \mathrm{d}H_2.\end{aligned}$$

Of course, this computation can be avoided by invoking Theorem 4.1.

The situation in the other simple Lie algebras of type B_N and D_N is entirely similar. Therefore we have the following result:

THEOREM 5.1. — *Consider the B_n, C_n and D_n Toda systems. In each case we define*

$$\mathcal{N} := \pi_3^\sharp \circ (\pi_1^\sharp)^{-1},$$

where π_1 is the Lie-Poisson bracket and π_3 is the cubic Poisson bracket. Also, let $H_0 = \log(\det(L))$ and $H_{2i} = \frac{1}{2i}\operatorname{tr} L^{2i}$, if $i \neq 0$. Then we have the following new bi-hamiltonian formulation for these systems:

$$\pi_3^\sharp \mathrm{d}H_0 = \pi_1^\sharp \mathrm{d}H_2 \quad \Longleftrightarrow \quad X^1_{\operatorname{tr}\mathcal{N}} = X^3_{\log\det\mathcal{N}}.$$

The function $\sqrt{\det\mathcal{N}}$ equals the determinant of L for C_N and D_N and the product of the non-zero eigenvalues of L for B_N, while the function $\frac{1}{2}\operatorname{tr}\mathcal{N} = H_2$ is the original hamiltonian. Finally,

$$\pi^\sharp_{2k+1}\mathrm{d}H_{2-2k} = \pi^\sharp_{2k-1}\mathrm{d}H_{4-2k}, \qquad (k\in\mathbb{Z}). \tag{5.3}$$

5.5. Finite, non-periodic Toda lattice

The case of the A_n Toda lattice was already considered in [1], using specific properties of this system. We use our general approach to show how one can quickly recover those results.

The hamiltonian defining the Toda lattice is given in canonical coordinates (p_i, q_i) of $\mathbb{R}^{2n}$ by

$$h_2(q_1,\ldots,q_n,p_1,\ldots,p_n) = \sum_{i=1}^{n}\frac{1}{2}p_i^2 + \sum_{i=1}^{n-1}e^{q_i-q_{i+1}}. \tag{5.4}$$

For the integrability of the system we refer to the classical paper of Flaschka [16].

Let us recall the bi-hamiltonian structure given in [13]. The first Poisson tensor in the hierarchy is the standard canonical symplectic tensor, which we denote by π_0, and the second Poisson tensor is:

$$\pi_1 = \begin{pmatrix} A_n & -B_n \\ B_n & C_n \end{pmatrix},$$

where A_n, B_n and C_n are $n \times n$ skew-symmetric matrices defined by

$$\begin{aligned} a_{ij} &= 1 = -a_{ji}, \qquad (i < j) \\ b_{ij} &= p_i \delta_{ij}, \\ c_{i,j} &= e^{q_i - q_{i+1}} \delta_{i,j+1} = -c_{j,i}, \qquad (i < j). \end{aligned}$$

Then setting $h_1 = 2\,(p_1 + p_2 + \cdots + p_n)$, we obtain the bi-hamiltonian formulation:

$$\pi_0^\sharp \mathrm{d}h_2 = \pi_1^\sharp \mathrm{d}h_1.$$

If we set, as usual,

$$\mathcal{N} := \pi_1^\sharp \circ (\pi_0^\sharp)^{-1},$$

then a small computation shows that Theorem 4.1 gives the following multi-hamiltonian formulation:

Proposition 5.2. — *The A_n Toda hierarchy admits the multi-hamiltonian formulation:*

$$\pi_j^\sharp \mathrm{d}h_2 = \pi_{j+2}^\sharp \mathrm{d}h_0,$$

where $h_0 = \frac{1}{2}\log(\det \mathcal{N})$ and h_2 is the original hamiltonian (5.4).

If we change to Flaschka coordinates, $(a_1, \ldots, a_{n-1}, b_1, \ldots, b_n)$, then there is no recursion operator anymore (recall that this is a singular change of coordinates, where we loose one degree of freedom). Nevertheless, the multi-hamiltonian structure does reduce ([13]). One can then compute the modular vector fields of the reduced Poisson tensors π_j relative to the standard volume form:

$$\mu = \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_{n-1} \wedge \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_n.$$

It turns out that the modular vector fields X_μ^j are hamiltonian vector fields with hamiltonian function

$$h = \log(a_1 \cdots a_{n-1}) + (j-1)\log(\det(L)),$$

where L is the Lax matrix. For a discussion of this result we refer to [1]. Note that the analogue of (5.3) in this case of the Toda chain is

$$\pi_j^\sharp \mathrm{d}h_{2-j} = \pi_{j-1}^\sharp \mathrm{d}h_{3-j}, \quad j \in \mathbf{Z}$$

where $h_j = \frac{1}{j}\mathrm{tr}\, L^j$ for $j \neq 0$ and $h_0 = \ln(\det(L))$.

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