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Riemann-Roch theorem for higher bivariant K-functors

RIEMANN-ROCH THEOREM FOR HIGHER BIVARIANT K-FUNCTORS

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ABSTRACT. — One defines a Riemann-Roch natural transformation from algebraic to topological higher bivariant K-theory in the category of complex spaces.

RÉSUMÉ. — On définie une transformation naturelle de type Riemann-Roch entre les K-théories algébrique et topologique supérieures bivariantes dans la catégorie des espaces complexes.

Introduction

The general form of Riemann-Roch theorem for K_0-functors was given by Baum-Fulton-Macpherson in [3]. It asserts that there exists a uniquely determined natural transformation (called also Riemann-Roch transformation) from the Grothendieck group $K_0^{alg}(X)$ of the category of coherent sheaves on the complex variety X to the topological K-functor $K_0^{top}(X)$ (the main point being the commutativity with the operation of direct image under proper morphisms). In the above cited paper the theorem was proved in the algebraic category, i.e., in the case when X can be embedded in a regular variety $\tilde{X}$, and any coherent sheaf on X has a projective resolution on $\tilde{X}$. This theorem was generalized (again in the algebraic category) in the work of Fulton-Macpherson [5], part II, where the bivariant K-groups $K_0^{alg}(f)$ and $K_0^{top}(f)$ are defined for any morphism $f : X \to Y$ of algebraic varieties, and the corresponding Riemann-Roch transformation is constructed.

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In the same work [5], I. 10.12, it was conjectured that one can build the theory of higher bivariant K-groups $K_i^{alg}(f)$, $i \geqslant 0$, together with a natural transformation $K_i^{alg}(f) \to K_i^{top}(f)$, generalizing the transformation constructed there. The purpose of the present paper is to prove this in the analytic category, i.e. in the category of coherent holomorphic sheaves on complex spaces.

An important fact, used in the proof of Riemann-Roch type theorems in the algebraic category, is the existence of a projective resolution for any coherent sheaf on a regular algebraic variety; in this way, one can attach to such a sheaf the underlying complex of vector bundles, determining an element of the corresponding K-theory. In the analytic category this is no longer true: projective resolutions exist only locally. Because of this, the proof of R.-R. theorem, given in [9], was based on the functional-analytic approach to K-theory producing a construction which could be called an *infinite-dimensional resolution* of a coherent sheaf. Recall ([2], Appendix) that the space of continuous Fredholm complexes of Banach spaces is a classifying space for the topological K-theory. It turns out that, subject to some technical restrictions, the same is true for complexes of Frechet spaces. In [9] for any coherent sheaf on a complex space one constructs a complex of Frechet spaces such that the complex of its holomorphic sections is a resolution of this sheaf. Forgetting the holomorphic structure, we obtain a representative of the needed element of the corresponding topological K-group.

The higher algebraic K-groups of an algebraic variety were introduced by Quillen in [10]. Another (equivalent) definition, better suited to work with complexes of sheaves, was developed by Waldhausen in [12]. In both definitions the K-groups of a given category are defined as shifted homotopy groups of the classifying space of a suitable category; we will call them Waldhausen and Quillen classifying spaces of the given category. In the work [6] Gillet proved theorems of Riemann-Roch type for higher K-groups in the algebraic category.

Since the construction of [9] is functorial, it extends almost immediately to higher (monovariate) K-groups. However, the extension to the higher bivariate K-group is not so direct, and is a subject of the present paper. For this purpose, we use two equivalent definitions of the higher bivariant R.-R. functor. One of it uses the infinite dimensional resolution mentioned above and is applied to the proof of commutativity of the R.-R. functor with the operation of direct image. Our second definition, used in the proof of the commutativity of the functor with the bivariant product, is constructed

by combining the local resolutions of a coherent sheaf, and is inspired by the Forster - Knorr hyperresolution of a coherent sheaf (see [4]). Roughly speaking, if $\mathcal{L}$ is a coherent sheaf on the regular complex space X, $\{U_i\}_{i \in I}$ is an open covering of X, and $L_\bullet^i$ - a free resolution of $\mathcal{L}$ on U_i, then one can construct on $U_i \cap U_j$ a free resolution $L_\bullet^{\{i,j\}}$ of $\mathcal{L}$ containing the restrictions of $L_\bullet^i$ and $L_\bullet^j$ as subcomplexes, and so on. We come to an object, defined in section 1, which will be called *local system* of complexes of sheaves. Gluing together the underlying system of complexes of vector bundles, one obtains a globally defined complex of bundles, determining an element of topological K-group.

Note that the notion of almost complex embedding introduced in [9] permits us to drop the assumption of the existence of regular embedding, but leads to more complicated details.

The content of the paper is the following: in the first section one investigates the connections between the classifying spaces of abstractly defined categories of local and global objects. We use these results in the second section in order to prove the homotopy equivalence between the classifying spaces of various categories, defining the higher algebraic K-groups $K_i(f)$ of the morphism f. In the third section two equivalent definitions of the Riemann-Roch functor, mentioned above, are given. Finally, in section 4 we extend the definitions from [5] of the operations of direct image and product in bivariant K-theory to the higher bivariant K-functors, defined in section 2, and show that they commute with the Riemann-Roch functor, constructed in section 3. The proof of the commutativity of the Riemann-Roch functor with the direct images uses the infinite-dimensional construction and arguments from [9]. The proof of the commutativity with products uses the finite-dimensional construction and essentially repeats the arguments of Fulton-Macpherson [5].

Throughout the paper we tried to keep close to the definitions and notations of [5]. The definitions of the classifying spaces of Quillen and Waldhausen are used; we would like to mention also the enlightening role of the general part of the paper [11]. We make a systematic use of theorem 1.9.8 of that paper, asserting that any functor between two complicial biWaldhausen categories inducing an invertible functor between the corresponding derived categories determines a homotopy equivalence between Waldhausen classifying spaces. In particular, if $\mathcal{A}$ is a full biWaldhausen subcategory of the biWaldhausen category $\mathcal{B}$, and any object of $\mathcal{B}$ is weakly equivalent to some object of $\mathcal{A}$, then the K-groups of $\mathcal{A}$ and $\mathcal{B}$ coincide.

1. Categories of local systems

Now we are going to define categories of local objects, which will be used in the paper. Let $\mathcal{K}$ be a finite simplicial complex having I as a set of vertices. In other words, $\mathcal{K} \subset 2^I$ has the property that if $\alpha \in \mathcal{K}$ and $\beta \subset \alpha$, then $\beta \in \mathcal{K}$ too. Suppose that for any $\alpha \in \mathcal{K}$ an Abelian category $\mathcal{F}_\alpha$ is given, and for any couple β, α, $\beta \subset \alpha \in \mathcal{K}$, there is an exact functor $R_{\beta,\alpha} : \mathcal{F}_\beta \to \mathcal{F}_\alpha$ (restriction functor), such that for any $\gamma \subset \beta \subset \alpha \in \mathcal{K}$ the equality $R_{\gamma,\alpha} = R_{\beta,\alpha} \circ R_{\gamma,\beta}$ is satisfied. Next, suppose that for any $\alpha \in \mathcal{K}$ one has a complicial biWaldhausen category $\mathcal{A}_\alpha$, associated with the Abelian category $\mathcal{F}_\alpha$ (see [11], def. 1.2.11), and the functors $R_{\beta,\alpha}$ map $\mathcal{A}_\beta$ into $\mathcal{A}_\alpha$. To simplify the situation, we will suppose that the weak equivalences in $\mathcal{A}_\alpha$ are exactly the quasi-isomorphisms of complexes, the cofibrations are the admissible degree-wise split monomorphisms, and fibrations - the admissible epimorphisms.

Such a system $\mathcal{A}$ of categories and functors will be called a *local system of biWaldhausen categories* on $\mathcal{K}$, and a morphism from the local system of categories $\mathcal{A}$ to $\mathcal{A}'$ will be defined as a system of functors $\mathcal{T}_\alpha : \mathcal{A}_\alpha \to \mathcal{A}'_\alpha$, commuting with the restriction functors.

Remark 1.1. — In fact, the situation when we will use the definitions and results of this section is the following: $\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering of the topological space X, $\mathcal{K} = \mathcal{N}(\mathcal{U})$ is the nerve of $\mathcal{U}$ (i.e. one has $\alpha \in \mathcal{K}$ iff $U_\alpha := \cap U_i$, $i \in \alpha$ is non-empty), $\mathcal{F}_\alpha$ is a category of sheaves on U_α, and $R_{\beta,\alpha}$ is the morphism of restriction from U_β to U_α.

In the situation above one can give

DEFINITION 1.2. — *Denote by* $Loc\mathcal{A}$ *the category of local systems of* $\mathcal{A}$-*complexes, defined as follows:*

Objects $\mathcal{L} = \left\{ \{\mathcal{L}_{\bullet,\alpha}\}_{\alpha \in \mathcal{K}}, \{E_{\bullet,\beta,\alpha}\}_{\beta \subset \alpha \in \mathcal{K}} \right\}$ *of* $Loc\mathcal{A}$ *are the systems consisting of the following data:*

(1) *A family of complexes* $\mathcal{L}_{\bullet,\alpha}$, $\alpha \in \mathcal{K}$ *in* $\mathcal{F}_\alpha$, *belonging to* $\mathcal{A}_\alpha$, *for any finite non-empty subset* $\alpha \in \mathcal{K}$.

(2) *A family of monomorphic quasi-isomorphisms of complexes in* $\mathcal{A}_\alpha$ *(called in the sequel* connecting morphisms*)*

$$E_{\bullet,\beta,\alpha} : R_{\beta,\alpha}(\mathcal{L}_{\bullet,\beta}) \to \mathcal{L}_{\bullet,\alpha}$$

for any α, β *with* $\beta \subset \alpha \in \mathcal{K}$, *such that for any* $\gamma \subset \beta \subset \alpha \in \mathcal{K}$ *the equality* $E_{\bullet,\gamma,\alpha} = E_{\bullet,\beta,\alpha} \circ E_{\bullet,\gamma,\beta}$ *is satisfied, and* $E_{\bullet,\alpha,\alpha}$ *is the identity morphism for any* α.

Sometimes we will drop the requirement that $E_{\bullet,\alpha\beta}$ are quasi-isomorphisms, and will refer to such a system as non-quasi-isomorphic (nq-) local system.

A morphism F from $\mathcal{L}'$ to $\mathcal{L}''$ in the category $Loc\mathcal{A}$ consists of the family of morphisms of complexes $F_{\bullet,\alpha} : \mathcal{L}'_{\bullet,\alpha} \to \mathcal{L}''_{\bullet,\alpha}$ in $\mathcal{A}_\alpha$, commuting with the monomorphisms $E_{\bullet,\alpha,\beta}$.

Let us call the morphism F a cofibration if all $F_{\bullet,\alpha}$ are degree-wise split monomorphisms, and a weak equivalence if all $F_{\bullet,\alpha}$ are quasi-isomorphisms. Consider the Abelian category $\mathcal{F}_\mathcal{K}$ consisting on all non-quasi-isomorphic local systems of elements of $\mathcal{F}(U_\alpha)$ (considered as complexes of length one); any element of $Loc\mathcal{A}$ can be considered as a complex of elements of $\mathcal{F}$, and so $Loc\mathcal{A}$ becomes a complicial biWaldhausen category, associated with the Abelian category $\mathcal{F}_\mathcal{K}$ (in the sense of the definition in [11], cited above).

Let us call the morphism R a cofibration if all $R_{\bullet,\alpha}$ are degree-wise split monomorphisms, and is a weak equivalence if all $R_{\bullet,\alpha}$ are quasi-isomorphisms. Consider the Abelian category $\mathcal{F}$ of all non-quasi-isomorphic local systems of elements of $\mathcal{F}(U_\alpha)$; any element of $Loc\mathcal{A}$ can be considered as a complex of elements of $\mathcal{F}$, and so $Loc\mathcal{A}$ become a complicial biWaldhausen category, associated with the Abelian category $\mathcal{F}$ (see [11], def. 1.2.11).

Denote by $Glob\mathcal{F}$ the subcategory of $\mathcal{F}_\mathcal{K}$ consisting of all systems $\{F_\alpha\}_{\alpha\in\mathcal{K}}$ such that for any $\beta \subset \alpha \in \mathcal{K}$ the object $R_{\beta,\alpha}(F_\beta)$ is isomorphic to F_α in $\mathcal{F}_\alpha$. Analogously, denote by $Glob\mathcal{A}$ the subcategory of $Loc\mathcal{A}$ consisting on all local systems $\mathcal{L} = \left\{ \{\mathcal{L}_{\bullet,\alpha}\}_{\alpha\in\mathcal{K}}, \{E_{\bullet,\beta,\alpha}\}_{\beta\subset\alpha\in\mathcal{K}} \right\}$ such that all the connecting morpisms $E_{\bullet,\beta,\alpha}$ are degree-wise isomorphisms; we will call the objects of $Glob\mathcal{A}$ global systems. The elements of $Glob\mathcal{A}$ can be considered as complexes of elements of $Glob\mathcal{F}$.

Examples of categories of local systems, such as local systems of complexes of coherent sheaves, of complexes of free sheaves, etc., on a complex space, will be considered in the next section. In the rest of the present section we state some lemmas which will be used to show that the Waldhausen K-groups of all these categories coincide (and define the algebraic K-functor of this complex space).

One can consider also the dual definition: let us call *colocal system* of $\mathcal{A}$-complexes any family $\mathcal{M}_{\bullet,\alpha}$ as in (1), endowed for any $\beta \subset \alpha$ with quasi-isomorphic epimorphisms $P_{\bullet,\beta,\alpha} : \mathcal{M}_{\bullet,\alpha} \to R_{\beta,\alpha}(\mathcal{M}_{\bullet,\beta})$ satisfying $P_{\bullet,\gamma,\alpha} = P_{\bullet,\gamma,\beta} \circ P_{\bullet,\beta,\alpha}$. Taking the same definitions as above for morphisms, cofibrations and weak equivalences, we obtain again a biWaldhausen category,

which will be denoted by $Coloc\mathcal{A}$. The global systems can be considered both as local and cololcal systems

LEMMA 1.3. — *The Waldhausen classifying spaces of the categories $Loc\mathcal{A}$ and $Coloc\mathcal{A}$ are homotopy equivalent.*

Proof. — We will construct an equivalence of exact categories $\mathcal{CO}$: $Loc\mathcal{A} \to Coloc\mathcal{A}$. Let $\mathcal{L} = \{\mathcal{L}_{\bullet,\alpha}\}_{\alpha \in \mathcal{K}}$ be a local system of complexes in $\mathcal{A}$ with connecting isomorphisms $E_{\bullet,\beta,\alpha}$. Suppose that the set I is ordered. Fix $\alpha \in \mathcal{K}$, and denote by $\overline{\alpha}$ the star of α in $\mathcal{K}$, i.e. the simplicial complex of all subset of α. Then the complexes $\{R_{\beta,\alpha}(\mathcal{L}_{\bullet,\beta})\}_{\beta \subset \alpha}$ and the maps $E_{\bullet,\beta,\alpha}$ form a simplicial system of objects of $\mathcal{A}_\alpha$ over the simplicial complex $\overline{\alpha}$. Denote by $\mathcal{COL} = \mathcal{COL}^\alpha_{\bullet,\bullet}$ the corresponding cochain complex of complexes, defined in the standard way. More precisely, let

$$\mathcal{COL}^\alpha_{\bullet,k} = \bigoplus_{|\beta|=k,\beta\subset\alpha} R_{\beta,\alpha}(\mathcal{L}_{\bullet,\beta})$$

be a bicomplex, whose second differentials $E^\alpha_{\bullet,k} : \mathcal{COL}^\alpha_{\bullet,k} \to \mathcal{COL}^\alpha_{\bullet,k+1}$ are defined in the standard way: the component of $E^\alpha_{\bullet,k}$ mapping $R_{\beta',\alpha}(\mathcal{L}_{\bullet,\beta'})$ into $R_{\beta'',\alpha}(\mathcal{L}_{\bullet,\beta''})$ is equal to $\varepsilon(\beta',\beta'')R_{\beta'',\alpha}(E_{\beta',\beta''})$. Here we have $|\beta'| = k$, $|\beta''| = k+1$, and $\varepsilon(\beta',\beta'')$ is zero unless $\beta' \subset \beta''$; in the latter case, if $\beta'' = \beta' \cup \{i\}$, then $\varepsilon(\beta',\beta'') = (-1)^l$, where l is the number of elements of β' greater than i.

Now, denote by $\mathcal{COL}^\alpha_\bullet$ the total complex of the bicomplex $\mathcal{COL}^\alpha_{\bullet,\bullet}$. This complex belongs to $\mathcal{A}_\alpha$. We shall show that the family $\{\mathcal{COL}^\alpha_\bullet\}_{\alpha \subset I}$ is a colocal system; indeed, if $\beta \subset \alpha$, then $\overline{\beta}$ is a subcomplex of $\overline{\alpha}$, and $R_{\beta,\alpha}(\mathcal{COL}^\beta_{\bullet,k})$ is a direct summand in $\mathcal{COL}^\alpha_{\bullet,k}$. The natural projection $P_{\bullet,k,\beta,\alpha} : \mathcal{COL}^\alpha_{\bullet,k} \to R_{\beta,\alpha}(\mathcal{COL}^\beta_{\bullet,k})$ is a quasi-isomorphism and commutes with the differential $E^\alpha_{\bullet,k}$; it is easy to check that $P_{\bullet,k+1,\beta,\alpha} \circ E^\alpha_{\bullet,k} = E^\alpha_{\bullet,k} \circ P_{\bullet,k,\beta,\alpha}$. This construction gives the desired exact functor $\mathcal{CO}$.

Let us point out more of its properties. There exist for $\beta \subset \alpha$ a natural embedding of complexes (not commuting with the differentials) $Q_{\beta\alpha}$: $R_{\beta\alpha}(\mathcal{COL}^\beta_\bullet) \to \mathcal{COL}^\alpha_\bullet$ which are right inverse maps to $P_{\beta\alpha}$. Next, taking the summand with $\beta = \alpha$ in the definition of $\mathcal{COL}^\alpha_\bullet$, we obtain an monomorphisms of complexes $G_\alpha : \mathcal{L}_{\bullet,\alpha} \to \mathcal{COL}^\alpha_\bullet[-|\alpha|+1]$ [1]. The natural left inverse epimorphism for G_α – the projection onto this direct summand (not commuting with the differentials) – will be denoted by H_α.

Taking the same construction with the corresponding modification, we can construct an exact functor $\widetilde{\mathcal{CO}} : Coloc\mathcal{A} \to Loc\mathcal{A}$. Let $\mathcal{M} = \{\mathcal{M}^\alpha_\bullet\}_{\alpha \subset I}$

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[1] Here $\mathcal{L}[k]_\bullet$ is the k-th right shift of $\mathcal{L}_\bullet$: $\mathcal{L}[k]_n = \mathcal{L}_{n-k}$.

be a colocal system with connecting epimorphisms $P_{\beta\alpha}$. Consider the bicomplex

$$\widetilde{\mathcal{COM}}_{\bullet,k,\alpha} = \bigoplus_{|\beta|=k,\beta\subset\alpha} \mathcal{M}_\bullet^\beta$$

with second differentials determined by $\varepsilon(\beta',\beta'') R_{\beta''\beta'} (P_{\beta'',\beta'})$, and let $\widetilde{\mathcal{COM}}_{\bullet,k,\alpha}$ be its total complex shifted left:

$$\widetilde{\mathcal{COM}}_{\bullet,\alpha} = tot\left(\widetilde{\mathcal{COM}}_{\bullet,\bullet,\alpha}\right)[-|\alpha|+1]$$

The quasi-isomorphic embeddings $\widetilde{\mathcal{COM}}_{\bullet,\alpha} \to \widetilde{\mathcal{COM}}_{\bullet,\beta}$ are defined in the natural way for any $\alpha \subset \beta$.

We will construct a quasi-isomorphic natural transformation from the functor $\widetilde{\mathcal{CO}}\circ\mathcal{CO}$ to the identity functor of the category $Loc\mathcal{A}$. Take $\mathcal{L} = \mathcal{L}_{\bullet,\alpha}$ and $\mathcal{COL} = \mathcal{COL}_\bullet^\alpha$ as above, and let $\mathcal{M} = \{\mathcal{M}_\bullet^\alpha\} = \mathcal{COL}_\bullet^\alpha$, $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{\bullet,\alpha} = \widetilde{\mathcal{CO}}\circ\mathcal{CO}\circ\mathcal{L}$.

Now, denote by $\pi_\alpha : \tilde{\mathcal{L}}_{\bullet,\alpha} \to \mathcal{L}_{\bullet,\alpha}$ the direct sum of maps $(-1)^{|\alpha|-|\beta|}E_{\beta\alpha}\circ H_\beta : \mathcal{M}_\bullet^\beta \to \mathcal{L}_{\bullet,\alpha}$ for all β, $\beta \subset \alpha$. It is easy to see that $\pi = \{\pi_\alpha\}_{\alpha\in I}$ is a morphism of local systems. We will prove that for any fixed $\alpha \subset I$ the morphism π_α is a quasi-isomorphism; indeed, denote by $\tilde{G}_\alpha : \mathcal{L}_{\bullet,\alpha} \to \tilde{\mathcal{L}}_{\bullet,\alpha}$ the superposition of G_α and the natural embedding of $\mathcal{M}_\bullet^\alpha$ in $\tilde{\mathcal{L}}_{\bullet,\alpha}$. (Note that $\tilde{G}_\alpha$ is not a morphism of local systems, i.e., does not commute with connecting morphisms.) Then $\tilde{G}_\alpha$ is a homotopy inverse for π_α. Indeed, it is easy to see that $\pi_\alpha \circ \tilde{G}_\alpha$ is equal to identity in $\mathcal{L}_{\bullet,\alpha}$. The homotopy between $\tilde{G}_\alpha \circ \pi_\alpha$ and the identity functor in $\tilde{\mathcal{L}}_{\bullet,\alpha}$ can be defined by the family of maps $Q_{\beta,\beta''}$ for $\beta' \subset \beta'' \subset \alpha$, $|\beta''| - |\beta'| = 1$.

In the same way one can construct a quasi-isomorphic natural transformation from $\mathcal{CO}\circ\widetilde{\mathcal{CO}}$ to the identity functor in $Coloc\mathcal{A}$. Now the assertion of the lemma will follow from([11], th. 1.9.8). $\square$

Note that one can define the functors $\mathcal{CO}$, $\widetilde{\mathcal{CO}}$ can be defined on the category of nq-local systems. Then the morphism $\pi : \widetilde{\mathcal{CO}} \circ \mathcal{CO} \circ \mathcal{L} \to \mathcal{L}$ is a (non-quasi-isomorphic) morphism of local systems.

Now suppose that the system of categories $\mathcal{A}_\alpha$ satisfy the following additional requirements:

    i) Any finite complex of objects of $\mathcal{F}_\alpha$, quasi-isomorphic to some complex in $\mathcal{A}_\alpha$, belongs to $\mathcal{A}_\alpha$ too.

    ii) For any $\beta \subset \alpha \in \mathcal{K}$ there exists an exact functor $S_{\beta,\alpha} : \mathcal{F}_\alpha \to \mathcal{F}_\beta$ such that the following relations are satisfied:

        1) $S_{\alpha,\beta} \circ S_{\beta,\gamma} = S_{\alpha,\gamma}$ for any $\gamma \subset \beta \subset \alpha \in \mathcal{K}$, $S_{\alpha,\alpha} = Id$.

        2) $R_{\beta,\alpha} \circ S_{\alpha,\beta} = Id$ in $\mathcal{F}_\alpha$ for any $\beta \subset \alpha \in \mathcal{K}$.

3) If $\alpha, \beta \in \mathcal{K}$, $\alpha \cap \beta \neq \emptyset$, then

$$R_{\alpha \cap \beta, \beta} \circ S_{\alpha, \alpha \cap \beta} = \begin{cases} S_{\alpha \cup \beta, \beta} \circ R_{\alpha, \alpha \cup \beta} & \text{if} \quad \alpha \cup \beta \in \mathcal{K}, \\ 0 & \text{if} \quad \alpha \cup \beta \notin \mathcal{K}. \end{cases}$$

In the case mentioned above, i.e. when $\mathcal{F}_\alpha$ are categories of sheaves on the open sets U_α and $R_{\beta\alpha}$ are the restriction functors, the functors $S_{\alpha,\beta}$ are the operators of extension by zero of sheaves from U_α to the larger set U_β.

LEMMA 1.4. — *Under the conditions i/ and ii/ above, the classifying spaces of the categories LocA and GlobA are homotopy equivalent.*

Proof. — Using ii/, one can define the "zero extension" functors S_α : $\mathcal{F}_\alpha \to Glob\mathcal{F}$ by the formula $(S_\alpha \mathcal{L})_\beta = R_{\alpha \cap \beta, \beta} \circ S_{\alpha, \alpha \cap \beta} (\mathcal{L})$ if $\alpha \cap \beta$ is non-empty and zero in the opposite case, where $\mathcal{L} \in \mathcal{F}_\alpha$.

Now we will construct a functor $\mathcal{C} : Loc\mathcal{A} \to Glob\mathcal{A}$ inducing a homotopy equivalence of classifying spaces. Let $\mathcal{L} = \{\mathcal{L}_{\bullet,\alpha}\}_{\alpha \in \mathcal{K}}$ be a local system of $\mathcal{A}$-complexes with connecting isomorphisms $E_{\bullet,\beta,\alpha}$. Consider the following bicomplex in $Glob\mathcal{F}$:

$$\mathcal{C}\mathcal{L}_{\bullet,k} = \prod_{|\alpha|=k} S_\alpha (\mathcal{L}_{\bullet,\alpha})$$

endowed with a second differential as in the definition of $\mathcal{C}\mathcal{O}\mathcal{L}^\alpha_{\bullet,k}$ in lemma 1.3. Then the Cech complex $\mathcal{C}\mathcal{L}_\bullet$ of $\mathcal{L}$ is defined as the total complex of this bicomplex. In the case when $\mathcal{F}_\alpha$ are categories of sheaves on the open sets U_α, and $\mathcal{L}$ is a sheaf globally defined on X, then the complex $\mathcal{C}\mathcal{L}_\bullet$ coincides with the canonical resolution of the sheaf $\mathcal{L}$ on X, connected with the covering $\{U_i\}$ (see [8], II.5.2).

Now the complex $\mathcal{C}\mathcal{O}\mathcal{L}^\alpha_{\bullet,k}$ is a direct summand in $(\mathcal{C}\mathcal{L}_{\bullet,k})_\alpha$, the natural projection $P_\alpha : \mathcal{C}\mathcal{L}_{\bullet,\alpha} \to \mathcal{C}\mathcal{O}\mathcal{L}^\alpha_\bullet$ is a quasi-isomorphism, and by i/ we have $(\mathcal{C}\mathcal{L}_\bullet)_\alpha \in \mathcal{A}_\alpha$ and therefore $\mathcal{C}\mathcal{L}_\bullet \in Glob\mathcal{A}$. Considering $Glob\mathcal{A}$ as a subcategory of $Coloc\mathcal{A}$, we obtain two functors, $\mathcal{C}$ and $\mathcal{C}\mathcal{O}$, from $Loc\mathcal{A}$ to $Coloc\mathcal{A}$, and a quasi-isomorphic natural transformation $P = \{P_\alpha\}$: $\mathcal{C} \to \mathcal{C}\mathcal{O}$. Since, by 1.3, the functor $\mathcal{C}\mathcal{O}$ determines a homotopy equivalence of derived categories, then the same is true for the functor $\mathcal{C}$. ☐

In the same way one can construct the Cech complex of a colocal system, determining an equivalence of categories $Co\mathcal{C} : Coloc\mathcal{A} \to Glob\mathcal{A}$.

Remark 1.5. — We will need a relative version of the constructions from 1.3 and 1.4. Take simplicial complexes $\mathcal{K}$ and $\mathcal{P}$ with sets of vertices I and J corr., and denote by $\mathcal{K} \times \mathcal{P}$ the set of all subsets $\gamma \subset I \times J$ such that there exist $\alpha \in \mathcal{K}$, $\beta \in \mathcal{P}$ with $\gamma \subset \alpha \times \beta$.

Suppose one has a system of Abelian categories $\mathcal{F}_\gamma$, biWaldhausen categories $\mathcal{A}_\gamma$, and exact functors $R_{\gamma',\gamma}$, $\gamma' \subset \gamma \in \mathcal{K} \times \mathcal{P}$ as in the beginning of the section. Then for any fixed $\beta \in \mathcal{P}$ the set of categories $\mathcal{A}^\beta = \{A_{\alpha \times \beta}\}_{\alpha \in \mathcal{K}}$ and functors $\mathcal{R}_{\alpha' \times \beta, \alpha \times \beta}$ for $\alpha' \subset \alpha \in \mathcal{K}$ form a local system of categories on $\mathcal{K}$. For any $\beta' \subset \beta \in \mathcal{P}$ the functors $\mathcal{R}_{\alpha \times \beta', \alpha \times \beta}$, $\alpha \in \mathcal{K}$, define a morphism of local system of categories from $\mathcal{A}^{\beta'}$ to $\mathcal{A}^\beta$. The categories of global objects $\{Glob\mathcal{A}^\beta\}_{\beta \in \mathcal{P}}$ form a local system of categories on $\mathcal{P}$.

Denote by $\mathcal{CO}_\beta \mathcal{L}^\alpha_\bullet$ and $\mathcal{C}_\beta \mathcal{L}_\bullet$ respectively the colocal system and the Cech complex for the system $\mathcal{A}^\beta$, constructed as in 1.3 and 1.4. Since all the constructions are compatible with the connecting monomorphisms of the system with respect to β, then $\mathcal{CO}_\beta \mathcal{L}^\alpha_\bullet$ is a local system with respect to β and a colocal system with respect to α, and $\mathcal{C}_\beta \mathcal{L}_\bullet$ is a local system of objects in $\{Glob\mathcal{A}^\beta\}_{\beta \in \mathcal{P}}$. As above, these functors induce homotopy equivalences between the corresponding categories. In particular, the functor $\widetilde{\mathcal{CO}}_\beta$ is adjoint to the functor $\mathcal{CO}_\beta$.

Let $\mathcal{A} = \{\mathcal{A}_\alpha\}_{\alpha \in \mathcal{K}}$ be a local system of categories as above. Denote by $\mathcal{A}^0_\alpha$ the category of all finite complexes of projective objects from $\mathcal{F}_\alpha$, and let $\mathcal{A}^0 = \{\mathcal{A}^0_\alpha\}_{\alpha \in \mathcal{K}}$. Suppose that any complex $\mathcal{L}_\bullet$ from $\mathcal{A}_\alpha$ has a finite projective resolution, i.e., a complex $L_\bullet$ from $\mathcal{A}^0_\alpha$ and a quasi-isomorphism $R_\bullet : L_\bullet \to \mathcal{L}_\bullet$. Then we have:

LEMMA 1.6. — *Under the conditions above any (nq-) local system from $Loc\mathcal{A}$ is quasi-isomorphic to some (nq-) local system from $Loc\mathcal{A}^0$.*

Proof. — Denote the category of all non-quasi-isomorphic local systems from $\mathcal{A}$ by $\mathcal{B}$, and let $\mathcal{B}^0$ be its subcategory, consisting of all systems, composed by complexes from $\mathcal{A}^0$. We will prove that any system $\mathcal{L}$ from $\mathcal{B}$ is quasi-isomorphic to some system L from $\mathcal{B}^0$. If the system $\mathcal{L}$ is in $Loc\mathcal{A}$, i.e. all its connecting morphisms are quasi-isomorphisms, then the same will be true for the system L, which is the statement of the lemma.

Take the local system $\mathcal{L} = \left\{ \{\mathcal{L}_{\bullet,\alpha}\}_{\alpha \in \mathcal{K}} , \{E_{\bullet,\beta,\alpha}\}_{\beta \subset \alpha \in \mathcal{K}} \right\} \in \mathcal{B}$. We will construct a non-quasi-isomorphic local system $L = \{L_{\bullet,\alpha}\}_{\alpha \in \mathcal{K}} \in \mathcal{B}^0$ and a morphism of local systems $G = \{G_{\bullet,\alpha}\}_{\alpha \in \mathcal{K}} : L_{\bullet,\alpha} \to \mathcal{L}_{\bullet,\alpha}$, such that any of the morphisms of complexes $G_{\bullet,\alpha}$, $\alpha \in \mathcal{K}$, induces an epimorphisms in all homology groups. Then, applying the inductive procedure from [11], 1.9.5, and SGA 6, I.1.4, after a finite number of steps one obtains a system from $\mathcal{B}^0$, quasi-isomorphic to the original system $\mathcal{L}$.

In order to construct the system L, take for any $\alpha \in \mathcal{K}$ a complex $\widetilde{L}_{\bullet,\alpha}$ from $\mathcal{A}^0_\alpha$, and a quasi-isomorphism of complexes $\widetilde{G}_{\bullet,\alpha} : \widetilde{L}_{\bullet,\alpha} \to \mathcal{L}_{\bullet,\alpha}$.

Put $L_{\bullet,\alpha} = \prod_{\beta \subset \alpha} R_{\beta,\alpha}\left(\widetilde{L}_{\bullet,\beta}\right)$, and define the morphism of complexes $G_{\bullet,\alpha} : L_{\bullet,\alpha} \to \mathcal{L}_{\bullet,\alpha}$ as a direct sum of morphisms $E_{\bullet,\beta,\alpha} \circ R_{\beta,\alpha}\left(\widetilde{G}_{\bullet,\beta}\right) :$ $R_{\beta,\alpha}\left(\widetilde{L}_{\bullet,\alpha}\right) \to \mathcal{L}_{\bullet,\alpha}$, $\beta \subset \alpha$. For $\alpha' \subset \alpha''$ the complex $R_{\alpha',\alpha''}\left(L_{\bullet,\alpha'}\right)$ is a direct summand in $L_{\bullet,\alpha''}$. Taking the corresponding natural embeddings as connecting maps, one can consider $\{L_{\bullet,\alpha}\}_{\alpha \in \mathcal{K}}$ as a non-quasi-isomorphic local system, and the set of morphisms $G = \{G_{\bullet,\alpha}\}_{\alpha \in \mathcal{K}}$ is a morphism of local system. It is easy to see that any $G_{\bullet,\alpha}$ induces an epimorphisms in all homology groups. $\qquad \square$

Remark 1.7. — If $\mathcal{L} = \{\mathcal{L}_{\bullet,\alpha}\}$ is a (nq-) local system of complexes from $Loc\mathcal{A}$, and for any $\alpha \in \mathcal{K}$ one has a complex $\overline{L}_{\bullet,\alpha} \in \mathcal{A}_{\alpha}^0$ and a quasi-isomorphism $\overline{G}_{\bullet,\alpha} : \overline{L}_{\bullet,\alpha} \to \mathcal{L}_{\bullet,\alpha}$, then there exists a (nq-) local system $L = \{L_{\bullet,\alpha}\}_{\alpha \in \mathcal{K}}$ in $\mathcal{A}^0$, quasi-isomorphism of local systems $G = \{G_{\bullet,\alpha}\}_{\alpha \in \mathcal{K}} :$ $L \to \mathcal{L}$, and a family of quasi-isomorphic embeddings of complexes $i_{\bullet,\alpha} :$ $\overline{L}_{\bullet,\alpha} \to L_{\bullet,\alpha}$ such that for any $\alpha \in \mathcal{K}$ one has $\overline{G}_{\bullet,\alpha} = G_{\bullet,\alpha} \circ i_{\bullet,\alpha}$.

We will need also a substitute for the covering morphisms property of the resolutions for the local systems constructed above. Suppose that the assumptions of the lemma above are satisfied. Take the nq-local systems $\mathcal{L} = \{\mathcal{L}_{\bullet,\alpha}\}$, $\mathcal{M} = \{\mathcal{M}_{\bullet,\alpha}\}$ of complexes from $\mathcal{A}$, and a morphism of local systems $\varphi = \{\varphi_{\bullet,\alpha}\} : \mathcal{L} \to \mathcal{M}$. Let $L = \{L_{\bullet,\alpha}\}$ and $M = \{M_{\bullet,\alpha}\}$ be objects of $Loc\mathcal{A}^0$, and $\epsilon_L : L \to \mathcal{L}$, $\epsilon_M : M \to \mathcal{M}$ be quasi-isomorphisms of nq-local systems. Denote, as in 1.2, $\tilde{L} = \left\{\widetilde{\mathcal{CO}} \circ \mathcal{COL}\right\}_{\bullet,\alpha}$, and $\tilde{M} = \left\{\widetilde{\mathcal{CO}} \circ \mathcal{COM}\right\}_{\bullet,\alpha}$. Let $\pi_L : \tilde{L} \to L$, $\pi_M : \tilde{M} \to M$ be the epimorphic morphisms of local systems constructed in the proof of lemma 1.3. We have

LEMMA 1.8. — *Under the assumptions above there exists a morphism of local systems* $\tilde{\varphi} = \{\tilde{\varphi}_{\bullet,\alpha}\} : \tilde{L} \to \tilde{M}$, *such that the diagram*

$$
\begin{array}{ccccc}
\tilde{L} & \overset{\pi_L}{\to} & L & \overset{\epsilon_L}{\to} & \mathcal{L} \\
\tilde{\varphi} \downarrow & & & & \varphi \downarrow \\
\tilde{M} & \overset{\pi_M}{\to} & M & \overset{\epsilon_M}{\to} & \mathcal{M}
\end{array}
$$

is commutative.

In particular, any two systems from $Loc\mathcal{A}^0$, quasi-isomorphic to the same system from $Loc\mathcal{A}$, are homotopy equivalent.

Proof. — In the notations of 1.3, it will be sufficient to construct a series of morphisms of complexes $\tilde{\varphi}_{\bullet}^{\alpha} : \mathcal{COL}_{\bullet}^{\alpha} \to \mathcal{COM}_{\bullet}^{\alpha}$ commuting with

connecting projections $P_{\alpha,\beta}$ such that its restrictions to $L_{\bullet,\alpha}$ are covering morphisms for the morphisms $\varphi_{\bullet,\alpha}$.

One can suppose that $\tilde{\varphi}_{\bullet}^{\alpha}$ are constructed for all α with $|\alpha| < n$, and we have to construct $\tilde{\varphi}_{\bullet}^{\alpha}$ for a given α with $|\alpha| = n$. For this, let us denote by $\mathcal{CO}'L_{\bullet}^{\alpha}$ the factor-complex of $\mathcal{COL}_{\bullet}^{\alpha}$ consisting of all summands $L_{\bullet,\beta}$ with $\beta \neq \alpha$. This complex coincides with the total complex of the bicomplex, obtained from $\mathcal{COL}_{\bullet,\bullet}^{\alpha}$ truncating its rightmost column, coinciding with $L_{\bullet,\alpha}$. Then, we have a canonical morphism of complexes $\tau_{\alpha} : \mathcal{CO}'L_{\bullet}^{\alpha} \to L_{\bullet,\alpha}$, and $\mathcal{COL}_{\bullet}^{\alpha}$ coincides with the cone of this morphism. The morphisms $\tilde{\varphi}_{\bullet}^{\beta}, \beta \subset \alpha, \beta \neq \alpha$, constructed by the inductive assumption, determine a morphism of complexes $\tilde{\varphi'}_{\bullet}^{\alpha} : \mathcal{CO}'L_{\bullet}^{\alpha} \to \mathcal{CO}'M_{\bullet}^{\alpha}$.

Now, choose arbitrary morphism of complexes $\Phi_{\bullet,\alpha} : L_{\bullet,\alpha} \to M_{\bullet,\alpha}$ covering the given morphism $\varphi_{\bullet,\alpha}$. It is easy to see that the diagram of complexes

$$
\begin{array}{ccc}
\mathcal{CO}'L_{\bullet,\alpha} & \overset{\tau_{\alpha}}{\to} & L_{\bullet,\alpha} \\
\tilde{\varphi'}_{\bullet}^{\alpha} \downarrow & & \Phi_{\bullet,\alpha} \downarrow \\
\mathcal{CO}'M_{\bullet,\alpha} & \overset{\tau_{\alpha}}{\to} & M_{\bullet,\alpha}
\end{array}
$$

is commutative up to homotopy. Choosing a homotopy $S_{\bullet,\alpha} : \mathcal{CO}'L_{\bullet,\alpha} \to M_{\bullet,\alpha}[-1]$ for it, we can see that the triple $\tilde{\varphi'}_{\bullet}^{\alpha}, \Phi_{\bullet,\alpha}, S$ determines a morphism between the cones of the horizontal morphisms of the diagram, i.e., morphism $\tilde{\varphi}_{\bullet}^{\alpha} : \mathcal{COL}_{\bullet}^{\alpha} \to \mathcal{COM}_{\bullet}^{\alpha}$, satisfying our requirements. $\square$

We will need also the following generalization of the lemma above. Suppose that the assumptions of lemma 1.6 are satisfied. Take the nq-local systems $\mathcal{L}_k = \{\mathcal{L}_{k,\bullet,\alpha}\}_{\alpha \in \mathcal{K}}$, $k = 1, \ldots, K$ of complexes from $\mathcal{A}$, and morphisms of local systems $\delta_k = \{\delta_{\bullet,\alpha}\} : \mathcal{L}_k \to \mathcal{L}_{k+1}$, satisfying $\delta_{k+1} \circ \delta_k = 0$. In other words, we have a complex $\mathcal{L}_{\bullet} = 0 \to \mathcal{L}_1 \to \ldots \to \mathcal{L}_K \to 0$ of nq-local systems. One can form its total complex $tot(\mathcal{L}_{\bullet}) = \{tot(\mathcal{L}_{\bullet,\bullet,\alpha})\}$ which is again a nq-local system of complexes from $\mathcal{A}$. We will denote by $\sigma_k \mathcal{L}_{\bullet}$ the "brutal" truncation of the complex $\mathcal{L}_{\bullet}$, i.e. the complex $\sigma_k \mathcal{L}_{\bullet} : 0 \to \mathcal{L}_k \to \ldots \to \mathcal{L}_K \to 0$. Then $tot(\sigma_k \mathcal{L}_{\bullet})$ form a decreasing filtration of $tot(\mathcal{L}_{\bullet})$.

Suppose also that for any k there are fixed local systems of complexes $L_k = \{L_{k,\bullet,\alpha}\}_{\alpha \in \mathcal{K}}$ from $Loc\mathcal{A}^0$, and quasi-isomorphisms of local systems $\epsilon_k : L_k \to \mathcal{L}_k$. For any k, denote $\tilde{L}_k = \left\{\widetilde{\mathcal{CO}} \circ \mathcal{CO} \, L\right\}_k$, and let $\pi_k : \tilde{L}_k \to L_k$ be the natural quasi-isomorphism of local systems, constructed in the proof of lemma 1.3. We have

LEMMA 1.9. — *In the conditions above there exist:*

(1) *A nq-local system $E = \{E_{\bullet,\alpha}\}$ from $Loc\mathcal{A}^0$ endowed with a decreasing filtration by local subsystems $E = E_1 \supset \ldots \supset E_K \supset E_{K+1} = \{0\}$ such that $E_k/E_{k+1} \approx \tilde{L}_k$, and*

(2) *A quasi-isomorphism $e : E \to tot(\mathcal{L}_\bullet)$ of local systems such that e maps E_k into $tot(\sigma_k \mathcal{L}_\bullet)$, and the morphism from $\tilde{L}_k \approx E_k/E_{k+1}$ to*
$$\mathcal{L}_k \approx tot(\sigma_k\mathcal{L})/tot(\sigma_{k+1}\mathcal{L}), \text{ induced by } e, \text{ coincides with } \epsilon_k \circ \pi_k.$$

Proof. — Suppose that the complex of local systems E_{k+1} and the restriction e_{k+1} of the morphism e to E_{k+1} are already constructed. We will find a morphism $\partial_k : \tilde{L}_k[-1] \to E_{k+1}$ such that the diagram:

$$
\begin{array}{ccc}
\tilde{L}_k[-1] & \xrightarrow{\partial_k} & E_{k+1} \\
\epsilon_k \circ \pi_k \downarrow & & e_{k+1} \downarrow \\
\mathcal{L}_k[-1] & \xrightarrow{\delta_k} & tot(\sigma_{k+1}\mathcal{L}_\bullet)
\end{array}
$$

is commutative. Then one can define E_k as the cone[2] of ∂_k, and the morphism $e_k : E_k \to tot(\sigma_k)$ – as the morphism between the cones of the horizontal arrows of the diagram above, determined by the vertical arrows. It is easy to check that the properties (1) and (2) are fulfilled.

As a first approximation to the construction of ∂_k, take a morphism of complexes of local systems, constructed in 1.7: $\widehat{\partial}_k : \tilde{L}_k \to \tilde{L}_{k+1}$ such that the diagram:

$$
\begin{array}{ccc}
\tilde{L}_k & \xrightarrow{\widehat{\partial}_k} & \tilde{L}_{k+1} \\
\epsilon_k \circ \pi_k \downarrow & & \epsilon_{k+1} \circ \pi_{k+1} \downarrow \\
\mathcal{L}_k & \xrightarrow{\delta_k} & \mathcal{L}_{k+1}
\end{array}
$$

is commutative.

Now we will construct the morphism $\partial_k : \tilde{L}_k[-1] \to E_{k+1}$. One may suppose that the morphisms ∂_l with the properties as above are already constructed for $l > k$. Since E_{k+1} is defined as the cone of the morphism $\partial_{k+1} : \tilde{L}_{k+1}[-1] \to E_{k+2}$, then there exists a projection of complexes $P_{k+1} : E_{k+1} \to \tilde{L}_{k+1}$, and one can find ∂_k such that $\widehat{\partial}_k[-1] = P_k \circ \partial_k$. Indeed, the superposition $\partial_{k+1} \circ \widehat{\partial}_k[-1] : \tilde{L}_k[-1] \to E_{k+2}$ covers the zero morphism $\delta_{k+1} \circ \delta_k : \mathcal{L}_k \to tot(\sigma_{k+2}\mathcal{L})$ and therefore is homotopicaly equivalent to zero. Let $\tilde{S}_k : \tilde{L}_k[-1] \to E_{k+2}[-1]$ be the corresponding homotopy. Then the map $\partial_k = \left(\widehat{\partial}_k, \tilde{S}_k\right)$ is a morphism of complexes from

----

[2] Here and below we choose the enumeration of the stages of the cone of a morphism $L_\bullet \to M_\bullet$ in such a manner that the stages of $M_\bullet$ keep its initial enumeration and the numbers of the stages of $L_\bullet$ decrease by one.

$\tilde{L}_k [-1]$ to the cone of ∂_{k+1}, i.e. to E_{k+1}, and satisfies the requirement above. □

Remark 1.10. —

a) The construction of the covering morphism $\tilde{\varphi}$ in lemma 1.7 depends on the choice of the covering morphisms $\Phi_{\bullet,\alpha}$ and the homotopies $S_{\bullet,\alpha}$. Since this choice is unique up to a homotopy, then the construction of $\tilde{\varphi}$ is unique up to a homotopy equivalence. The same arguments show that the construction of the local system E in lemma 1.8 is unique up to a homotopy equivalence.

b) Suppose there are given two complexes $\mathcal{L}'_\bullet$, $\mathcal{L}''_\bullet$ of local systems in $\mathcal{A}$, morphism of complexes of local systems $\varphi = \{\varphi_\bullet\} : \mathcal{L}'_\bullet \to \mathcal{L}''_\bullet$, and fixed resolutions L'_k, L''_k for $\mathcal{L}'_k$ and $\mathcal{L}''_k$ corresp. Denote by E', E'' the corresponding complexes of local systems of $\mathcal{A}^0$ constructed in the lemma above. Then, using the same arguments as in the proof of the lemma, one can construct a morphism $E\varphi : E' \to E''$ of complexes of local systems, covering the morphism φ.

2. Higher bivariant algebraic K-theory for complex spaces

Let X, Y be complex spaces, and f - a closed morphism from X to Y. The higher bivariant algebraic K-groups $K_i \left(X \xrightarrow{f} Y \right)$, or simply $K_i(f)$, can be defined in the same way as the absolute one, using the equivalent definitions of Quillen [10] and Waldhausen [12] (for a detailed review of all results in higher K-theory, used here, see [11]). In this section we give a list of biWaldhausen categories (i.e., categories with cofibrations and weak equivalences), consisting of complexes of sheaves, and producing the bivariant K-groups as homotopy groups of the corresponding Waldhausen classifying space. As usual, the cofibrations in these categories will be taken to be the set of all monomorphisms (or the monomorphisms, having a complementable image - see [11], 1.9.2), and the weak equivalences to be the quasi-isomorphisms.

Recall (SGA 6 and [5],II.1.) that a complex of sheaves $\mathcal{L}_\bullet$ on X is called *f-perfect*, if for any sufficiently small open sets W in X, V in Y with $f(W) \subset V$, and a suitable closed embedding $W \xrightarrow{e} U$ of W in the regular domain U, the direct image $(e, f)_* \mathcal{L}_\bullet$ is quasi-isomorphic on the space $U \times V$ to a finite complex of finite-dimensional free $\mathcal{O}_{U \times V}$-modules. A sheaf on X will be called f-perfect, if it is perfect as a complex, concentrated in degree 0. We shall adopt the following notations:

C1: by $Per^0(f)$ we will denote the category of all f-perfect sheaves on X;

C2: by $CPer^0(f)$ we will denote the category of all finite complexes of f-perfect sheaves;

C3: by $Per(f)$ we will denote the category of all f-perfect complexes of nuclear Frechet sheaves on X.

As shown in [6] (see for detailed proof [11] 1.11.7), the natural inclusion of $Per^0(f)$ in $CPer^0(f)$ induces a homotopy equivalence of corresponding Waldhausen classifying spaces; the homotopy groups of these spaces will be denoted by $K_i(f)$. As we shall see below, $Per(f)$ produces the same K-groups; in order to prove the equivalence and to describe the operations in bivariant K-theory, we shall consider some equivalent localized categories. We will define it using an embedding in a regular space, although the corresponding definitions could be made independently.

Let us fix a regular closed embedding $\varrho : X \to \tilde{X}$ of the complex space X in the complex manifold $\tilde{X}$, a contractible Stein covering $\{W_i\}_{i\in I}$ of X, and contractible Stein domains $U_i \subset \tilde{X}$ such that $U_i \bigcap X = W_i$. Then the set $\mathcal{U} = \{W_i, \varrho_{W_i}, U_i\}_{i\in I}$ forms an atlas for the complex space X. As usual for any non-empty finite subset $\alpha = (i_1 \ldots i_k) \in I$, we will denote $W_\alpha = W_{i_1} \bigcap \ldots \bigcap W_{i_k}$, resp. $U_\alpha = U_{i_1} \bigcap \ldots \bigcap U_{i_k}$. Let us fix also a contractible Stein covering $\{V_j\}_{j\in J}$ of the complex space Y, and suppose that the covering $\{U_i\}$ is subordinated to $\left\{f^{-1}(V_j) \times \tilde{X}\right\}$. Then $V_j \times U_i$ cover $Y \times \tilde{X}$.

Having this, one can define new categories of complexes on $Y \times \widetilde{X}$ or on $V_\beta \times U_\alpha$, $\beta \subset J$, $\alpha \subset I$. Consider the embedding $(f \times \varrho) : X \to Y \times \tilde{X}$, and denote by J_X the sheaf of ideals in $\mathcal{O}_{Y \times \tilde{X}}$ such that $\varrho_* \mathcal{O}_X = \mathcal{O}_{Y \times \tilde{X}}/J_X$. We will say that the complex $L_\bullet$ of free finite-dimensional $\mathcal{O}_{V_\beta \times U_\alpha}$-modules is *supported on X* iff all its sheaves of homologies are annihilated by the ideal J_X, i.e., are direct images of sheaves on X under the embedding above.

Now one can extend the notion of the perfect complex. Let $\mathcal{L}_\bullet$ be a complex of $\mathcal{O}_{Y \times \tilde{X}}$-modules on $Y \times \tilde{X}$. The complex $\mathcal{L}_\bullet$ will be called f-perfect, if for any $\alpha \subset I$, $\beta \subset J$ its restriction on $V_\beta \times U_\alpha$ is quasi-isomorphic to a suitable finite complex $L_\bullet$ of free finite-dimensional $\mathcal{O}_{V_\alpha \times U_\alpha}$ modules, supported on X.

We shall introduce the notation:

C4: $\widetilde{Per}(f)$ - the category of all f-perfect complexes of nuclear Frechet sheaves of $\mathcal{O}_{Y \times \tilde{X}}$-modules on $Y \times \tilde{X}$.

It is easy to see that the complex $\mathcal{L}_\bullet$ is f-perfect on X if and only if its direct image under ϱ is f-perfect on $Y \times \tilde{X}$, so the category $Per(f)$ can be considered as a full subcategory of $\widetilde{Per}(f)$.

Let us fix, as above, the covering $\{V_j \times U_i\}_{j \in J, i \in I}$ of $Y \times \tilde{X}$. We shall use the following categories of local (and colocal) systems on the nerve of this covering:

C5: $LocFfd(f)$ - the category, consisting of local systems of finite complexes of free finite-dimensional $\mathcal{O}_{Y \times \tilde{X}}$-modules on $Y \times \tilde{X}$.

C6: $LocCPer^0(f)$ - the category of local systems of finite complexes of perfect sheaves on X.

C7: $LocPer(f)$ - the category of local systems of f-perfect complexes of nuclear Frechet sheaves on X.

C8: $\widetilde{LocPer}(f)$ - the category of all local systems of f-perfect complexes of nuclear Frechet sheaves on $Y \times \tilde{X}$.

Evidently, any globally defined complex can be considered as a local system of complexes; for this, one can take the set of restrictions of this complex on the subsets $V_\beta \times U_\alpha$, with isomorphisms as connecting maps. In this way, **C3** and **C4** can be considered as full subcategories of **C7** and **C8** respectively.

In general, we have the following diagram, where all the arrows are embeddings of full subcategories:

$$
\begin{array}{ccccccc}
\mathbf{C1} & \longrightarrow & \mathbf{C2} & \longrightarrow & \mathbf{C3} & \longrightarrow & \mathbf{C4} \\
& & & & \downarrow & & \downarrow \\
& & \mathbf{C6} & \longrightarrow & \mathbf{C7} & \longrightarrow & \mathbf{C8} & \longleftarrow & \mathbf{C5}
\end{array}
$$

The rest of this section is devoted to the proof of

PROPOSITION 2.1. — *The Waldhausen classifying spaces of all the categories* **C1** – **C8** *listed above have the same homotopy type.*

Proof. — Lemma 1.6 shows that the embedding **C5** → **C8** induces a homotopy equivalence of classifying spaces.

Next. applying lemma 1.4, one can see that the global-local embeddings **C3** → **C7** and **C4** → **C8** are homotopy equivalences also.

It remains to prove the homotopy equivalence of the embeddings **C2** → **C3** → **C4** and **C6** → **C7** → **C8**. For this, we shall use the Waldhausen Fibration theorem, ([12], 1.6.4), and the equivalence of Waldhausen and Quillen definitions in the case when weak equivalences coincide with isomorphisms ([12], 1.9, and [6], Th. 6.2). The following assertion is an immediate consequence of these theorems:

LEMMA 2.2. — *Suppose $\mathcal{A}$ and $\mathcal{B}$ are complicial Waldhausen categories, and $\mathcal{A}^e$ and $\mathcal{B}^e$ are its subcategories, consisting of all exact complexes. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be an exact functor, such that the induced maps of Quillen classifying spaces*

$$BQ\mathcal{F} : BQ\mathcal{A} \to BQ\mathcal{B} \quad , \quad BQ\mathcal{F}^e : BQ\mathcal{A}^e \to BQ\mathcal{B}^e$$

are homotopy equivalences. Then $\mathcal{F}$ induces a homotopy equivalence between Waldhausen classifying spaces of $\mathcal{A}$ and $\mathcal{B}$.

Let us consider the embedding **C2** → **C4** (the same arguments hold for **C2** → **C3**). Denote by $\mathcal{A}$ the category of all perfect complexes on $Y \times \widetilde{X}$ with the property that the kernels of all the differentials are complemented subsheaves. Let $\widetilde{\mathcal{A}} = \mathcal{A} \cap \mathbf{C2}$ be the category of all complexes of perfect sheaves with this property. Denote by $\mathcal{B}$ the category of complexes from $\mathcal{A}$ with fixed subsheaves, complementing the kernels of the differentials, and again $\widetilde{\mathcal{B}} = \mathcal{B} \cap \mathbf{C2}$. Let $\mathcal{C}$ be the category with the same set of objects as $\mathcal{B}$ with morphisms all the morphisms from $\mathcal{B}$, preserving the complements of kernels of differentials. Put $\widetilde{\mathcal{C}} = \mathcal{C} \cap \mathbf{C2}$. Finally, let $\mathcal{D}$ be the subcategory of **C2** consisting of all complexes with zero differentials. We have the following commuting diagram of functors and categories:

$$
\begin{array}{ccccccccc}
\mathcal{D} & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathbf{C4} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{D} & \longrightarrow & \widetilde{\mathcal{C}} & \longrightarrow & \widetilde{\mathcal{B}} & \longrightarrow & \widetilde{\mathcal{A}} & \longrightarrow & \mathbf{C2}
\end{array}
$$

It is sufficient to prove that all horizontal arrows are homotopy equivalences. The arguments used here will be identical for both rows of the diagram. Let first consider the embedding $\mathcal{A} \longrightarrow \mathbf{C4}$. For any complex, consisting of sheaves L_n and differentials $d_n : L_n \to L_{n+1}$, there exists a filtration by subcomplexes $\tau'^k L$, $\tau''^k L$, defined in the following way:

for $n < k$, $\left(\tau'^k L \right)_n = \left(\tau''^k L \right)_n = L_n$

$\left(\tau'^k L \right)_k = im \, d_{k-1}$, $\left(\tau''^k L \right)_k = ker \, d_k$

for $n > k$, $\left(\tau'^k L \right)_n = \left(\tau''^k L \right)_n = 0$

If the complex L_n is perfect, or a finite complex of perfect sheaves, then the same is true for the complexes $\tau'^k L$, $\tau''^k L$. We have $\cdots \subset \tau'^k L \subset \tau''^k L \subset \tau'^{k+1} L \subset \dots$. The factor-complex $\tau''^k L \,/\, \tau'^k L$ is non-zero only at the stage k, and the next factor $\tau'^{k+1} L \,/\, \tau''^k L$ is exact and concentrated in the stages $k, k+1$. Obviously these factor-complexes belong to $\mathcal{A}$. Therefore, any object in **C4**, resp. $\mathbf{C4}^e$, possesses a finite filtration, whose factor-objects belong to $\mathcal{A}$, resp. to $\mathcal{A}^e$. Then we can use the Quillen

devissage theorem ([10], th. 5.4) asserting that the embeddings $\mathcal{A} \to \mathbf{C4}$, $\mathcal{A}^e \to \mathbf{C4}^e$ induce homotopy equivalences of Quillen classifying spaces. Now, using lemma 2.3, one sees that $\mathcal{A} \to \mathbf{C4}$ is a homotopy equivalence of Waldhausen categories. The same argument shows that $\widetilde{\mathcal{A}} \to \mathbf{C2}$ is a homotopy equivalence also.

Next, it is clear that the forgetful functors $\mathcal{B} \to \mathcal{A}$, $\widetilde{\mathcal{B}} \to \widetilde{\mathcal{A}}$ are homotopy equivalences. To prove the homotopy equivalence of the embeddings $\mathcal{C} \to \mathcal{B}$, $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{B}}$, it is sufficient to note that any morphism between two complexes with fixed complements of the kernels of differentials is homotopic to a morphism, preserving these complements, and to apply theorem 1.9.8 of [11].

Finally, $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ or $\widetilde{\mathcal{C}}$. For any complex $\mathcal{L}_\bullet$ from $\mathcal{C}$ there exists an epimorphic quasi-isomorphism $\mathcal{L}_\bullet \to \mathcal{H}_\bullet$ where $\mathcal{H}_\bullet$ denotes the complex of all homology sheaves of $\mathcal{L}_\bullet$ with zero differentials. This determines an exact functor from $\mathcal{C}$ to $\mathcal{D}$ which is left adjoint and homotopy right adjoint to the embedding. Therefore, the embeddings $\mathcal{D} \to \mathcal{C}$, $\mathcal{D} \to \widetilde{\mathcal{C}}$ are homotopy equivalences. Thus, the homotopy equivalence of the embeddings $\mathbf{C2} \to \mathbf{C3} \to \mathbf{C4}$ is proved.

The proof of the homotopy equivalence of the embeddings $\mathbf{C6} \to \mathbf{C7} \to \mathbf{C8}$ can be proved following the same lines. Indeed, for the local systems of complexes on can introduce filtrations similar to these considered above. We omit the details. $\qquad\square$

3. The Riemann-Roch functor

Given compact topological spaces A, B, $B \subset A$, denote by $CBun(A, B)$ ($CBun(A)$ if $B = \emptyset$) the complicial biWaldhausen category of all finite continuous complexes of finite-dimensional vector bundles on A, exact on B, and by $CBun(f)$ - the category $CBun(Y \times \tilde{X}, Y \times \tilde{X}/X)$. We shall describe briefly a natural map from the Waldhausen K-groups $K_i(CBun(A, B))$ of this category to the topological K-groups $K^i(A, B)$. Let $\mathcal{E}_A$ be the (discrete) category of all finite-dimensional vector bundles on A. Denote by $K_*^Q(A, B)$ the Quillen relative K-groups of the pair $(\mathcal{E}_A, \mathcal{E}_B)$; that is, $K_i^Q(A, B)$ is the $i + 1$-th homotopy group of the homotopy fibre of the restriction map $BQ\mathcal{E}_A \to BQ\mathcal{E}_B$. Now one can apply the results of [6] p.6, establishing the equality of Quillen and Waldhausen K-groups, to the pair of categories $(\mathcal{E}_A, \mathcal{E}_B)$; one obtains a natural map $K_i(CBun(A, B)) \to K_i^Q(A, B)$ (see [6], th. 6.2).

The way to pass from the groups $K_*^Q(A, B)$ to the topological K-groups is described in another paper of Gillet - [7], p.6.1, where the equivalence between Q and +-constructions is used: indeed, there is a natural transformation of functors

$$BGL_d(C(A))^+ \to BGL(C(A))$$

where $GL_d(C(A))$ denotes the group of invertible matrices over $C(A)$ endowed with the discrete topology, and $GL(C(A))$ - the same with the compact-open topology. It induces a natural map of groups $K_i^Q(A) \to K^i(A)$, and, in the relative version, $K_i^Q(A, B) \to K^i(A, B)$. We have:

LEMMA 3.1. —

(1) *To any exact functor α from the given complicial biWaldhausen category $\mathcal{A}$ to the category $CBun(A, B)$ there corresponds a map $\alpha_* : K_*(\mathcal{A}) \to K^*(A, B)$ from the Waldhausen K-groups of $\mathcal{A}$ to the topological K-groups of (A, B).*

(2) *Let $I = [0, 1]$, and denote by*

$$R^j : CBun(A \times I, B \times I) \to CBun(A, B), \ j = 0, 1,$$

the functors determined by the restrictions to the corresponding endpoint of I. Suppose that $\alpha^j : \mathcal{A} \to CBun(A, B)$, $j = 0, 1$, are two exact functors, and there exists an exact functor $\alpha : \mathcal{A} \to CBun(A \times I, B \times I)$ such that $\alpha^j = R^j \circ \alpha$, $j = 0, 1$. Then α^0 and α^1 induce identical maps to topological K-groups: $\alpha_*^1 = \alpha_*^0$.

Proof. — (1): one defines α_i as a composition of the map $K_i(\mathcal{A}) \to K_i(CBun(A, B))$, induced by α, with the maps $K_i(CBun(A, B)) \to K_i^Q(A, B) \to K^i(A, B)$ defined above. To prove (2), it is sufficient to note that $\alpha_*^j = R_*^j \circ \alpha_*$, $j = 0, 1$. □

We shall now define a Riemann-Roch functor, acting from some of categories of complexes of sheaves, considered in the previous section, to a suitable category, equivalent to $CBun(f)$. We will need two equivalent definitions for this functor, the first using the finite-dimensional resolutions, while the second one - the infinite-dimensional resolutions constructed by the use of Koszul complexes.

3.1. Definition 1

Let $LocCBun(A, B)$, resp. $LocCBun(f)$ be the category of all local systems of complexes of finite-dimensional vector bundles corresponding to the

covering $\mathcal{U}$. (We will suppose that all the elements of $\mathcal{U}$ and all their finite intersections are contractible.) Then $CBun(f)$ and $LocCBun(f)$ are complicial Waldhausen categories, and there is a natural inclusion of $CBun(f)$ in $LocCBun(f)$. We will show that it induces a homotopy equivalence between the classifying spaces. For this, in view of theorem 1.9.8 of [11], it is sufficient to prove that any local system is quasi-isomorphic to some globally defined complex of vector bundles.

We will say that the local system of complexes of bundles $\{L_{\bullet,\alpha}\}_{\alpha \subset I}$, connected with the covering $\mathcal{U} = \{U_i\}_{i \in I}$ of the topological space K, is *transversal*, if for any two subsets $\alpha, \beta \subset I$, any integer n, and any $x \in U_\alpha \bigcap U_\beta$, the intersection of $L_{n,\alpha}(x)$ and $L_{n,\beta}(x)$ in $L_{n,\alpha \bigcup \beta}(x)$ coincides with $L_{n,\alpha \bigcap \beta}(x)$ [3]. It is easy to see that any local system of bundles is quasi-isomorphic to a transversal system. Indeed, for any local system of bundles $L_{\bullet,\alpha}$, the system $\widetilde{\mathcal{CO}} \circ \mathcal{CO} \circ L_{\bullet,\alpha}$ (see lemma 1.3) is transversal.

In order to establish the homotopy equivalence from above, it is sufficient to prove:

LEMMA 3.2. — *Let $\{L_{\bullet,\alpha}\}_{\alpha \subset I}$ be a transversal local system of complexes of vector bundles connected with the covering $\mathcal{U} = \{U_i\}_{i \in I}$ of the topological space K. Then there exists a continuous complex $H_\bullet$ of vector bundles on K, and an embedding of local systems $L_{\bullet,\alpha} \to H_\bullet$, such that on any domain U_α we have $H_\bullet = L_{\bullet,\alpha} \bigoplus M_{\bullet,\alpha}$, where $M_{\bullet,\alpha}$ is a suitable continuous and exact complex of vector bundles on U_α.*

Proof. — First, let us note that if such a complex $H_\bullet$ is constructed, one can assume that the corresponding vector bundles are trivial in all the stages except in one, say in degree zero. Indeed, this can be done by addition of exact summands.

In the proof we shall use an induction on the number n of elements of the covering $\mathcal{U}$. Denote $K' = U_1 \bigcup \ldots \bigcup U_{n-1}$, and let $K = K' \bigcup U_n$. Let $H'_\bullet$ and $M'_{\bullet,\alpha}$ be the complexes constructed by the inductive assumption on K' for α not containing n. One can introduce an accorded system of scalar products in all the bundles involved such that the bundles $M'_{n,\alpha}$ coincide with the orthogonal complements of $L_{n,\alpha}$ in H'_n.

Let $K'' = K' \bigcap U_n$. Take the system $L''_{\bullet,\alpha} = L_{\bullet,\alpha} \bigoplus M'_{\bullet,\alpha'}$ for $\alpha = \alpha' \bigcup \{n\}$ on K'', corresponding to the covering $U_i \bigcap U_n$, $i \neq n$, of K''. It follows from the transversality of the original local system that $L''_{\bullet,\alpha}$ is a transversal local system also. Indeed, take $\{n\} \subset \beta \subset \alpha$, $\alpha' = \alpha \backslash \{n\}$, $\beta' =$

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[3] If $L_\bullet$ is a complex of vector bundles on the space X, and $x \in X$, then we will denote by $L_\bullet(x)$ the corresponding complex of vector spaces.

$\beta \setminus \{n\}$. Then we have $L_{\bullet,\beta} \bigcap L_{\bullet,\alpha'} = L_{\bullet,\beta'}$ and therefore we have a natu-
ral embedding of factor-bundles $L_{\bullet,\beta}/L_{\bullet,\beta'} \subset L_{\bullet,\alpha}/L_{\bullet,\alpha'}$. Now, identifying
$L_{\bullet,\alpha} \bigoplus M'_{\bullet,\alpha'}$ with $L_{\bullet,\alpha}/L_{\bullet,\alpha'} \bigoplus H'_{\bullet}$, resp. $L_{\bullet,\beta} \bigoplus M'_{\bullet,\beta'}$ with $L_{\bullet,\beta}/L_{\bullet,\beta'} \bigoplus$
$H'_{\bullet}$, one obtains the connecting monomorphisms.

Denote by $H''_{\bullet}$ and $M''_{\bullet,\alpha}$ the complexes constructed by the inductive
assumption on K'' starting from $L''_{\bullet,\alpha}$. Then $H'_{\bullet}$ is a subcomplex of $H''_{\bullet}$
on K''. The factor-complex $H''_{\bullet}/H'_{\bullet}$ is exact. By the assumption above, all
its components H''_n/H'_n for $n \neq 0$ are stably trivial; therefore, the same is
true for $n = 0$. Adding a suitable finite complex of trivial bundles to $H''_{\bullet}$,
one can suppose that the components of $H''_{\bullet}/H'_{\bullet}$ are trivial bundles, and
therefore this complex can be extended as an exact complex on the whole
K'.

In the same way, the exact complex $H''_{\bullet}/L_{\bullet,\{n\}}$ can be extended up to
an exact complex on U_n. Then, taking the direct sum of $H'_{\bullet}$ with the first
extension on K', and the direct sum of $L_{\bullet,\{n\}}$ with the second extension
on U_n, we obtain a globally defined on K complex, which extends $H''_{\bullet}$ on
K and satisfies the conditions of the lemma. $\square$

Now, one can construct the Riemann-Roch homomorphism from the
category $LocFfd(f)$ of local systems of finite complexes of free finite-
dimensional modules to the category $CBun(f)$. Indeed, forgetting the an-
alytic structure, one can consider any finite complex of finite-dimensional
free $\mathcal{O}_{Y \times \tilde{X}}$-modules as a complex of trivial bundles, which gives us an exact
functor $\alpha(f) : LocFfd(f) \to LocCBun(f)$. This functor, as it was pointed
out in lemma 3.1(1), determines a mapping from the K-groups of the cate-
gory $LocFfd(f)$ (or any of the equivalent categories considered in the pre-
vious paragraph) to the topological K-groups of the pair $Y \times \tilde{X}, Y \times \tilde{X}/X$,
providing the necessary homomorphism of the algebraic into the topological
K-groups.

3.2. Definition 2

We shall describe briefly the main construction from [9] with suitable
modifications. Let U be a bounded Stein domain in the regular n-dimen-
sional complex space $\tilde{X}$, $\mathcal{L}$ is a Frechet sheaf of $\mathcal{O}_{\tilde{X}}$-modules on $\tilde{X}$ and the
functions $z(x) = z_1(x), \dots, z_n(x)$, defined in a neighborhood of $\overline{U}$, form a
coordinate system on U. The coordinate functions on U can be considered
as sections of the sheaf $\mathcal{O}_U$. The operators $(M_{z_1}, \dots, M_{z_n})$ of multiplication
by the coordinate functions form a commuting n-tuple of operators acting

on the Frechet space $\Gamma_U(\mathcal{L})$ of the sections of the sheaf $\mathcal{L}$ on U. We will denote the Koszul complex of the operators $(M_{z_1} - \lambda_1 I), \dots, (M_{z_n} - \lambda_n I)$ by $K_\bullet(U, \mathcal{L})(\lambda)$; it is a finite complex of Frechet spaces with differentials holomorphically depending on the coordinates $\lambda \in \mathbb{C}^n$, and exact out of the domain $\widetilde{U} = z(U) \subset \mathbb{C}^n$. Recall that the m-th stage $K_m(U, \mathcal{L})(\lambda)$ is equal to the direct sum of $\binom{n}{m}$ copies of the Frechet space $\Gamma_U(\mathcal{L})$. The corresponding complex $K_\bullet(U, \mathcal{L})(x) := K_\bullet(U, \mathcal{L})(z(x))$ is holomorphic on some neighborhood of of $\overline{U}$ in X. One can extend the vector-function $z(x)$ to the whole $\widetilde{X}$ as a smooth function with values in $\mathbb{C}^n \setminus \widetilde{U}$; then the complex $K_\bullet(U, \mathcal{L})(x)$ defined above is smooth, holomorphic near $\overline{U}$, and exact out of U. As in [9], 2.3, one can see that if $\mathcal{O}_{\widetilde{X}} K_\bullet(U, \mathcal{L})(x)$ denotes the complex of sheaves of holomorphic sections of the complex $K_\bullet(U, \mathcal{L})(x)$, then the natural epimorphism of evaluation at the point x: $K_\bullet(U, \mathcal{O}_U)(x) \to \mathbb{C}$ determines a quasi-isomorphism of complexes of sheaves $\mathcal{O}_{\widetilde{X}} K_\bullet(U, \mathcal{L})(x) \to \mathcal{L}|_U$, where the sheaf $\mathcal{L}|_U$ coincides with the sheaf $\mathcal{L}$ on the domain U and is zero outside it. If V is another complex space, then on $V \times \widetilde{X}$ one has a quasi-isomorphism of complexes of sheaves $\mathcal{O}_{V \times \widetilde{X}} K_\bullet(U, \mathcal{L})(x) \to \mathcal{O}_V \widehat{\otimes} \mathcal{L}\big|_{V \times U}$. In particular, if $\mathcal{L}$ is free and finite dimensional, i.e. $\mathcal{L} = (\mathcal{O}_X)^p$, then the latter sheaf coincides with $(\mathcal{O}_{V \times U})^p$.

Suppose that $\widetilde{X}$ is a product of the spaces X_1 and X_2 of dimension n and m correspondingly, U_1, U_2 are open in X_1, X_2, $U = U_1 \times U_2$, and the tuples of functions $z(x) = z_1(x), \dots, z_n(x)$ resp. $w(x) = w_1(x), \dots, w_m(x)$ define coordinate systems on U_1 resp. U_2. Denote by p_1 the projection of U on U_1. Then the Koszul complex of the operators $(M_{z_1} - \lambda_1 I), \dots, (M_{z_n} - \lambda_n I), (t M_{w_1} - \mu_1), \dots, (t M_{w_m} - \mu_m)$ defined for $(\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m) \in \mathbb{C}^{n+m}$ and $t \in [0, 1]$, determines a homotopy on $\mathbb{C}^{n+m}$ between the complex $K_\bullet(U, \mathcal{L})(\lambda, \mu)$ and the Koszul-Thom transform from $\mathbb{C}^n$ to $\mathbb{C}^{n+m}$ of the complex $K_\bullet(U_1, p_* \mathcal{L})(\lambda)$.

Let X be a complex space, $\varrho : X \to \widetilde{X}$ – a regular embedding of X into the regular space $\widetilde{X}$. Fix the locally finite covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $\varrho(X)$ by bounded Stein domains such that for any finite set $\alpha = (i_1, \dots, i_k) \subset I$ the domain $U_\alpha = \bigcap_{j=1}^k U_{i_j}$ is also Stein (if it is non-empty). For any such α we will fix also a coordinate system $z^\alpha(x)$ defined in a neighborhood of $\overline{U_\alpha}$. To glue together the local resolutions $K_\bullet(U_\alpha, \mathcal{L})(x)$ of the sheaf $\mathcal{L}$, corresponding to a different coordinate systems, we will use the *correcting maps* $r_{m, \alpha', \alpha}$ defined in [9], lemma 3.3. Suppose we have for any $\alpha' \subset \alpha$ a

system of maps of the type

$$r_{m,\alpha',\alpha}(x) \,=\, F_{m,\alpha',\alpha}(x) \circ R_{\alpha',\alpha} : K_m\left(U_{\alpha'},\mathcal{L}\right)(x)$$
$$\rightarrow K_{m-|\alpha|+|\alpha'|+1}\left(U_\alpha,\mathcal{L}\right)(x)$$

where $R_{\alpha',\alpha}$ is the restriction operator from the space $\Gamma_{U_{\alpha'}}\left(\mathcal{L}\right)$ to $\Gamma_{U_\alpha}\left(\mathcal{L}\right)$, and $F_{m,\alpha',\alpha}(x)$ are suitable matrix-functions of the parameter $x \in \tilde{X}$, whose entries for any fixed x are operators of multiplication in $\Gamma_{U_{\alpha'}}\left(\mathcal{L}\right)$ by some element of $\Gamma_{U_{\alpha'}}\left(\mathcal{O}_X\right)$.

Denote

$$K\mathcal{C}_m\left(\mathcal{U},\mathcal{L}\right) := \bigoplus_{\alpha \subset I, p=m-|\alpha|} K_p\left(U_\alpha,\mathcal{L}\right)$$

and consider the maps from $K\mathcal{C}_m\left(\mathcal{U},\mathcal{L}\right)$ to $K\mathcal{C}_{m+1}\left(\mathcal{U},\mathcal{L}\right)$, determined by the operators $r_{m,\alpha',\alpha}(x)$ and the differentials of the complexes $K_\bullet\left(U_\alpha,\mathcal{L}\right)(x)$. We will call the operators $r_{m,\alpha',\alpha}(x)$ *correcting maps* if the Freshet spaces and the differentials described above form a complex, i.e. if the product of any two consecutive differentials is zero. A simple diagram chase (see [9], lemma 3.3) show that one can choose the matrix-functions $F_{m,\alpha',\alpha}(x)$, smooth on $\tilde{X}$ and holomorphic in a neighborhood of U_α, such that the corresponding $r_{m,\alpha',\alpha}(x)$ are correcting maps. Moreover, it is sufficient to choose such the matrix-functions when $\mathcal{L} = \mathcal{O}_X$; the maps $r_{m,\alpha',\alpha}(x)$ obtained in this way are correcting for any $\mathcal{L}$. So the construction above provides a complex $K\mathcal{C}_\bullet\left(\mathcal{U},\mathcal{L}\right)(x)$ of Frechet spaces on $\tilde{X}$. Any morphism $\Phi : \mathcal{L}' \rightarrow \mathcal{L}''$ of sheaves of $\mathcal{O}_X$-modules induces a constant morphism of complexes $K\Phi_\bullet : K\mathcal{C}_\bullet\left(\mathcal{U},\mathcal{L}'\right)(x) \rightarrow K\mathcal{C}_\bullet\left(\mathcal{U},\mathcal{L}''\right)(x)$ and therefore $K\mathcal{C}_\bullet\left(\mathcal{U},\mathcal{L}\right)(x)$ is an exact functor of the sheaf $\mathcal{L}$. Locally this complex splits into a direct sum of an smooth exact complex and an holomorphic complex (let us call such a complex *essentially holomorphic complex*). The evaluation morphisms defined above induce a quasi-isomrphism between the complex of sheaves of holomorphic sections of $K\mathcal{C}_\bullet\left(\mathcal{U},\mathcal{L}\right)(x)$, and the Cech complex for the covering $\mathcal{U}$ and the sheaf $\mathcal{L}$. If $\mathcal{L}_\bullet$ is a complex of sheaves on X, we obtain a bicomplex $K\mathcal{C}_\bullet^\alpha\left(\mathcal{U},\mathcal{L}_\bullet\right)(x)$, and its total complex will be denoted by $K\mathcal{C}_\bullet^\alpha\left(\mathcal{U},\mathcal{L}\right)(x)$; the properties stated above remain true.

This construction can be transferred to the local systems. Let $\mathcal{L} = \{\mathcal{L}_{\bullet,\alpha}\}_{\alpha \subset I}$ be a local system on $\tilde{X}$ of complexes of $\mathcal{O}_x$-modules with connecting maps $E_{\bullet,\alpha',\alpha} : \mathcal{L}_{\bullet,\alpha'} \rightarrow \mathcal{L}_{\bullet,\alpha}$ for $\alpha' \subset \alpha$. Denote by $K_\bullet\left(U_\alpha,\mathcal{L}_\alpha\right)(x)$ the total complex of the bicomplex $K_\bullet\left(U_\alpha,\mathcal{L}_{\bullet,\alpha}\right)(x)$. So, $K_\bullet\left(U_\alpha,\mathcal{L}_\alpha\right)(x)$ is a holomorphic complex on U_α, and all its stages are direct sums of finitely many copies of the Frechet space $\Gamma_{U_\alpha}\left(\mathcal{L}_\alpha\right)$. Then for $\alpha' \subset \alpha$ the mapping $\Gamma_{U_\alpha}E_{\bullet,\alpha',\alpha}$ between the complexes of Frechet spaces $\Gamma_{U_\alpha}\left(\mathcal{L}_{\bullet,\alpha'}\right)$ and

$\Gamma_{U_\alpha}(\mathcal{L}_{\bullet,\alpha})$ induces constant morphism of complexes $KE_{\bullet,\alpha',\alpha}$: $K_\bullet(U_\alpha, R_{\alpha,\alpha'}\mathcal{L}_{\alpha'})(x) \to K_\bullet(U_\alpha, \mathcal{L}_\alpha)(x)$.

Consider for any $\alpha' \subset \alpha$ the maps $rc_{m,\alpha',\alpha}(x) = F_{m,\alpha',\alpha}(x) \otimes KE_{\bullet,\alpha',\alpha} \circ R_{\alpha',\alpha}$ from $K_m(U_{\alpha'}, \mathcal{L}_{\alpha'})(x)$ to $K_{m-|\alpha|+|\alpha'|+1}(U_\alpha, \mathcal{L}_\alpha)(x)$. Denote by $K\mathcal{C}_\bullet(\mathcal{U}, \mathcal{L})(x)$ the complex constructed by the use of the maps $rc_{m,\alpha',\alpha}$ in the same way as the maps $r_{m,\alpha',\alpha}$ in the case when $\mathcal{L}$ is a sheaf; it is easy to see that the product of two consecutive differentials of this complex again is equal to zero. The complex of sheaves of germs of holomorphic sections of this complex is quasi-isomorphic via the evaluation map to the Cech complex of the local system $\mathcal{L}$.

The complex $K\mathcal{C}_\bullet(X, \mathcal{U}, \mathcal{L})(x)$ defines a Riemann-Roch functor in the absolute case $(Y = pt)$, and we are going to extend it to the bivariant case. Take a covering $\mathcal{U} = \{V_j \times U_i\}_{i \in I, j \in J}$ of $Y \times \tilde{X}$ as above. Let $\{\mathcal{L}_{\bullet,\alpha,\beta}\}_{\alpha \subset I, \beta \subset J}$ be a local system of finite complexes of sheaves on $Y \times \tilde{X}$. We will restrict to the case when all sheaves $\mathcal{L}_{m,\alpha,\beta}$ are sheaves of the type $\mathcal{O}_{V_\beta \times U_\alpha}(F_{m,\alpha,\beta})$, i.e. the sheaves of germs of holomorphic functions with values in the (finite or infinite dimensional) Frechet space $F_{m,\alpha,\beta}$. Then the connecting maps of the system $\mathcal{L}$ can be considered as holomorphic operator-valued functions $E_{\bullet,(\alpha',\beta'),(\alpha,\beta)}(x, y) : \mathcal{L}_{\bullet,\alpha,\beta} \to \mathcal{L}_{\bullet,\alpha',\beta'}$, $x \in U_\alpha, y \in V_\beta$, acting between the corresponding Frechet spaces.

Fix $\beta \subset J$, and take the restricted local system on $V_\beta \times \tilde{X}$. The construction above gives us a complex of Frechet spaces $K\mathcal{C}_{\bullet,\beta}(X, \mathcal{U}, \mathcal{L})(x, y)$, holomorphically depending on $y \in V_\beta$ and essentialy holomorfic for $\tilde{X}$. Via the evaluation morphism it is quasi-isomorphic to the complex of sheaves $\mathcal{C}_\beta \mathcal{L}_\bullet$ (see remark 1.5).

Take a pair $\beta' \subset \beta \subset J$. Then the set of connecting maps $\{E_{\bullet,(\beta',\alpha),(\beta,\alpha)}(x, y)\}_{\alpha \subset I}$, defined for $y \in V_\beta$, determines a quasi-isomorphic monomorphism of complexes

$$KE_{\bullet,\beta',\beta}(y) : K\mathcal{C}_{\bullet,\beta'}(X, \mathcal{U}, \mathcal{L})(x, y) \to K\mathcal{C}_{\bullet,\beta}(X, \mathcal{U}, \mathcal{L})(x, y)$$

and it is easy to see that these morphisms form a local system of complexes of Frechet spaces with respect to β. This leads us to the following definition:

DEFINITION 3.3. — **(C9)**: *Define the category $LocF(f)$ in the following way: the objects of this category are local systems of essentially holomorphic complexes of nuclear Frechet spaces on $Y \times \tilde{X}$, connected with the covering $\left\{V_j \times \tilde{X}\right\}_{j \in J}$. The connecting morphisms of these local system are required to be constants with respect to $\tilde{X}$, i.e., are quasi-isomorphism of complexes of Frechet spaces depending holomorphically only on Y-coordinate of the parameter. The morphisms in this category are again bounded*

morphisms of local systems of complexes depending holomorphically only on Y-coordinate.

The construction above defines an exact functor $K\mathcal{C}$ from the category of local systems of complexes of locally free finite dimensional modules to the category $LocF(f)$. The evaluation morphism defines a quasi-isomorphic natural transformation between this functor and the Cech functor $\mathcal{CL}_\bullet$ defined in 1.3. (It is easy to see from this that $K\mathcal{C}$ is a homotopy equivalence.)

We will use the notion of uniformly Fredholm continuous complex of Frechet spaces, defined in [9] as follows. A continuous morphism $\varphi_\bullet(\lambda)$: $X_\bullet(\lambda) \to Y_\bullet(\lambda)$ between finite complexes of Frechet spaces, continuously depending on the parameter, will be called an *uniform quasi-isomorphism* if it induces a quasi-isomorphism of sheaves of continuous sections of these complexes. The complex $X_\bullet(\lambda)$ is called *uniformly Fredholm*, if it is uniformly quasi-isomorphic to a continuous complex of finite-dimensional vector bundles. It is shown in [9] 1.4 that any holomorphic perfect complex of Frechet spaces is uniformly Fredholm, and any morphism inducing quasi-isomorphism between complexes of sheaves of holomorphic sections is an uniform quasi-isomorphism. Consider the next category:

DEFINITION 3.4. — **(C10)**. *Denote by $LocFred(A, B)$ the category of all local systems, corresponding to the covering $\mathcal{U}$ of A, of uniformly Fredholm complexes of Frechet spaces on the space A, exact on its subspace B. (We shall suppose that all the elements of the covering $\mathcal{U}$ and their intersections with the set B are contractible.) The connecting morphisms in this category are all the uniformly quasi-isomorphic monomorphisms with closed image. Denote by $LocFred(f)$ the category $LocFred(Y \times \tilde{X}, Y \times \tilde{X}/X)$.*

The category $LocFred(f)$ can be used for a second definition of the Riemann-Roch functor. Indeed, denote by $LocCBun^0(A, B)$ the category $LocCBun(A, B)$ with an additional structure on any object: a fixed trivialization of all the bundles involved in the given local system on the corresponding element of the covering $\mathcal{U}$. It is easy to see that the forgetting functor from $LocCBun^0(A, B)$ to $LocCBun(A, B)$ is a homotopy equivalence. We have:

LEMMA 3.5. — *The natural embedding*

$$LocCBun^0(A, B) \to LocFred(A, B)$$

is a homotopy equivalence.

Proof. — The proof uses again the Waldhausen Approximation Theorem and is similar to the proof of statements 1.6. First, one can show that for any local system $X_{\bullet,\alpha}(x)$ of uniformly Fredholm continuous complex of Frechet space there exists a system $L_{\bullet,\alpha}(x)$ of complexes of vector bundles and a morphism from $L_{\bullet,\alpha}(x)$ to $X_{\bullet,\alpha}(x)$ inducing the epimorphism in the spaces of homologies for any x. Then the proof can be completed by the inductive construction in the same way as in 1.6. $\qquad\square$

Forgetting the holomorphic structure, we obtain an exact functor from $LocF(f)$ to $LocFred(f)$. Now, taking the superposition of the functor $K\mathcal{C}$ with this forgetting functor, we obtain the second definition of the Riemann-Roch functor $LocFfd(f) \to LocFred(f)$. The arguments, used in [9], show that the definition is independent on the choice of the atlas $\mathcal{U}$ and the ambient space $\widetilde{X}$.

PROPOSITION 3.6. — *The definitions* **1** *and* **2** *of the Riemann-Roch functor are equivalent.*

Proof. — We will define an intermediate functor from the category $LocFfd(f)$ of local system of complexes of free finite-dimensional sheaves to the category of colocal systems of perfect complexes.
Take $\{\mathcal{L}_{\bullet,\alpha,\beta}\}_{\alpha\subset I,\beta\subset J} \in LocFfd(f)$. For $(x,y) \in V_\beta \times U_\alpha$ denote by $K\mathcal{C}^\alpha_{\bullet,\beta}(\mathcal{L})(x,y)$ the complex of Frechet spaces on $V_\beta \times U_\alpha$ defined by the formula

$$K\mathcal{C}^\alpha_{m,\beta}(\mathcal{U},\mathcal{L}) := \bigoplus_{\alpha'\subset\alpha,p=m-|\alpha'|} K_p(U_{\alpha',\beta},\mathcal{L})$$

with differentials determined by the correcting maps $r_{m,(\alpha',\beta),(\alpha'',\beta)}(x,y)$, and the differentials of the complexes $K_\bullet(V_\beta \times U_{\alpha'},\mathcal{L})(x)$.

The complexes $K\mathcal{C}^\beta_{\bullet,\alpha}(\mathcal{L})(x,y)$ form a local system with respect to β and a colocal system with respect to α; the evaluation morphism gives an epimorphic quasi-isomorphism between this system and the system $\mathcal{CO}_\beta\mathcal{L}^\alpha_\bullet$ (see remark 1.5), and therefore $K\mathcal{C}^\beta_{\bullet,\alpha}(\mathcal{L})(x,y)$ is a colocal system of perfect complexes.

On the other side, for $(x,y) \in V_\beta \times U_\alpha$ the natural projection $K\mathcal{C}_{\bullet,\beta}(\mathcal{L})(x,y) \to K\mathcal{C}^\alpha_{\bullet,\beta}(\mathcal{L})(x,y)$ is a quasi-isomorphism; indeed, applying to both sides the evaluation morphism, we obtain the quasi-isomorphic projection $P_\alpha : \mathcal{C}_\beta\mathcal{L}_{\bullet,\alpha} \to \mathcal{CO}_\beta\mathcal{L}^\alpha_\bullet$ (see the proof of 1.4 and remark 1.5). Hence we have a quasi-isomorphic natural transformation between the functors $K\mathcal{C}$ and $\mathcal{CO}_\beta$. Forgetting the holomorphic structure, one can take the same transformations in the category $LocFred(f)$, which proves the assertion. $\qquad\square$

4. Commutativity of the Riemann-Roch functor
with the operations

To complete the proof of the theorem, it is sufficient to prove that the Riemann-Roch functor defined above commutes with the operations of taking the direct image and products, which are defined both in the algebraic and topological bivariant K-theory. The operations in bivariant K_0-theory are described in [5]. Now we shall show that they can be extended on the higher K-functors, and still commute with the Riemann-Roch.

4.1. Definition of direct image

Suppose now that $f : X \to Z$, $g : Y \to Z$ are morphisms of complex spaces, and let $h : X \to Y$ be a proper morphism such that $f = g \circ h$. If $\varrho_X : X \to \tilde{X}$ and $\varrho_Y : Y \to \tilde{Y}$ are regular embeddings, then $\varrho = (\varrho_X, \varrho_Y \circ h)$ is a regular embedding of X in $\tilde{X} \times \tilde{Y}$ and the projection of $\tilde{X} \times \tilde{Y}$ onto $\tilde{Y}$ extends the map h. Therefore, one can define the category $\widetilde{LocPer}(f)$ by the use of the embedding ϱ.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an atlas on X, connected with ϱ_X, $\mathcal{V} = \{V_j\}_{j \in J}$ - an atlas on Y, connected with ϱ_Y, and $\mathcal{F} = \{F_l\}_{l \in L}$ - a Stein covering of Z. Then $\{F_l \times V_j \times U_i\}$ is a covering of $Z \times \tilde{Y} \times \tilde{X}$.

Now let $\{\mathcal{L}_{\bullet,\tilde{\alpha}}\}$, $\tilde{\alpha} = (\alpha, \beta, \gamma)$ with $\alpha \subset L, \beta \subset J, \gamma \subset I$ be a local system of f-perfect complexes on $Z \times \tilde{Y} \times \tilde{X}$. Denote by $\mathcal{C}_{\alpha\beta}\mathcal{L}_\bullet$ the Cech complex of this system with respect to γ, constructed as in remark 1.5. For any $\alpha' \subset \alpha$, $\beta' \subset \beta$ there exists a natural quasi-isomorphic embedding of complexes $\mathcal{C}_{\alpha'\beta'}\mathcal{L}_\bullet \to \mathcal{C}_{\alpha\beta}\mathcal{L}_\bullet$. Then, the complexes $\mathcal{C}_{\alpha\beta}\mathcal{L}_\bullet$ form a local system of perfect complexes on $Z \times \tilde{Y} \times \tilde{X}$, connected with the covering $\{F_i \times V_j \times \tilde{X}\}$.

Now, one can define $h_! \mathcal{L}_{\bullet,\tilde{\alpha}} := h_* \mathcal{C}_{\alpha\beta}\mathcal{L}_\bullet$, which is a local system on $Z \times \tilde{Y}$. Combining the Forster-Knorr proof of Grauert theorem and the argument from SGA III.4.8, one can see that the complexes involved in this system are g-perfect, and we obtain an exact functor

$$h_! : \widetilde{LocPer}(f) \to \widetilde{LocPer}(g).$$

4.2. Commutativity with direct image

Suppose that in the definition above $\{\mathcal{L}_{\bullet,\tilde{\alpha}}\} \in LocFfd(f)$ (**C5**), i.e. all $\{\mathcal{L}_{k,\tilde{\alpha}}\}$ are free and finite-dimensional. Then the sheaves in all the stages

of complexes in $h_! \mathcal{L}_{\bullet, \tilde{\alpha}}$ are free infinite-dimensional, i.e. sheaves of germs of holomorphic functions on $F_\alpha \times V_\beta$ with values in a suitable Frechet space. Therefore, the second definition of the Riemann-Roch functor is applicable to it.

The proof of the commutativity of the Riemann-Roch functor with the functors of direct image, given in [9], holds without changes, and we will recall the main steps used there. The space $\tilde{X}$ can be embedded in $\tilde{Y} \times \mathbb{C}^N$ such that its normal bundle posses a complex structure (what was called an *almost complex embedding* in [9]). Then, multiplying the operators of multiplication by coordinates of $\tilde{X}$ by a parameter $t \in [0,1]$ as described above, and extending the correcting maps $r_{m,\alpha',\alpha}$ up to maps $r_{m,\alpha',\alpha,t}$ for $t \in [0,1]$, one obtains a continuous homotopy between the Koszul - Thom transform of $K\mathcal{C}_\bullet (X \times Y, \mathcal{U}, \mathcal{L})$ in $\tilde{Y} \times \mathbb{C}^N$ and the Koszul - Thom transform of $K\mathcal{C}_\bullet (Y, \mathcal{U}, h!\mathcal{L})$ for the embedding $\tilde{Y} \times \{0\}$ in $\tilde{Y} \times \mathbb{C}^N$. It is important to note that all the elements used in the construction of the homotopy are *functorial* with respect of the sheaf $\mathcal{L}$. As it was shown in the second definition of the Riemann - Roch functor above, one can perform this construction in the bivariant case, and replacing the sheaf $\mathcal{L}$ with a local system for complexes of free finite-dimensional sheaves. Then, by 3.1 (2), one obtains the commutativity of the Riemann-Roch functor with the operation of the direct image:

PROPOSITION 4.1. — *The functors $h_* \circ \alpha(f)$ and $\alpha(g) \circ h_!$ from $LocFfd(f)$ to $LocFred(g)$ induce identical mappings to topological K-groups.*

4.3. Definition of products

Suppose that $f : X \to Y$, $g : Y \to Z$ are morphisms of complex spaces. Take the regular embeddings $\varrho : X \to \tilde{X}$, $\tau : Y \to \tilde{Y}$. Let $\mathcal{F} = \{F_l\}_{l \in L}$ be an open Stein covering of Z. Let $\mathcal{U} = \{(W_i, \varphi_i, U_i)\}_{i \in I}$ be an atlas on X, corresponding to the embedding ϱ, and $\mathcal{V} = \{(V_j, \theta_j, \tilde{V}_j)\}_{j \in J}$ be an atlas on Y, corresponding to τ. One may suppose that the covering $\{\tilde{V}_j\}$ is subordinated to $\{F_l \times \tilde{Y}\}$, and $\{\tilde{U}_i\}$ – to $\{V_j \times \tilde{X}\}$.

One has the following commutative diagram (see [5], 3.5):

$$X \xrightarrow{(f,\rho)} Y \times \tilde{X} \xrightarrow{(g,\tau)\times Id} Z \times \tilde{Y} \times \tilde{X}$$

$$f \searrow \qquad \downarrow \qquad \qquad \downarrow p$$

$$Y \xrightarrow{(g,\tau)} Z \times \tilde{Y}$$

$$g \searrow \qquad \downarrow q$$

$$Z$$

Let $L = \{L_{\bullet,\alpha,\beta}\}_{\beta \subset J, \alpha \subset I}$ be a local system from $LocFfd(f)$. L is a local system of complexes of free sheaves on the elements of the covering $\mathcal{V} \times \mathcal{U}$ of $Y \times \tilde{X}$. We will denote by the same symbol its direct image on $Z \times \tilde{Y} \times \tilde{X}$ via the map $(g,\tau) \times Id_{\tilde{X}}$. Let $\mathcal{M}$ be a g-perfect sheaf on $Y \times \tilde{X}$, supported on Y. Then the local system $L \bigotimes p^*\mathcal{M} = \{L_{\bullet,\alpha,\beta} \bigotimes_{Y \times \tilde{X}} p^*\mathcal{M}\}_{\beta \subset J, \alpha \subset I}$ is a local system of $f \circ g$-perfect complexes of sheaves on the elements of the covering $\{Z \times \tilde{V}_j \times \tilde{U}_i\}$ of $Z \times \tilde{Y} \times \tilde{X}$.

Since all the sheaves $L_{k,\alpha,\beta}$ are free on $Y \times \tilde{X}$, the operation of tensor product of local systems is exact with respect of both factors and determines a biexact functor:

$$LocFfd(f) \times Per^0(g) \to Loc\widetilde{Per}(f \circ g),$$

where the category on the right hand is considered as a category of all local system of perfect sheaves on $Z \times \tilde{Y} \times \tilde{X}$ with respect to the covering $\{F_l \times \tilde{V}_j \times \tilde{U}_i\}_{i \in I, j \in J}$. Then, by the multiplicative construction of Waldhausen ([12], p.342), this functor gives rise to the multiplication map in K-groups:

$$K_i(f) \times K_j(g) \to K_{i+j}(f \circ g)$$

4.4. Commutativity with products

We will adapt the construction from [5]. Let, as above, $L = \{L_{\bullet,\alpha,\beta}\}_{\beta \subset J, \alpha \subset I}$ be a local system from $LocFfd(f)$, and let $\mathcal{M}$ be a globally defined g-perfect sheaf on Y. Fix a local system $M = \{M_{\bullet,\beta,\gamma}\} \in LocFfd(g)$ of complexes free finite-dimensional modules on $Z \times \tilde{Y}$, quasi-isomorphic to $\mathcal{M}$. Then L and p^*M are local systems of complexes of sheaves, supported on $Y \times \tilde{X} \subset Z \times \tilde{Y} \times \tilde{X}$, and corresponding to the covering $\{F_l \times \tilde{V}_j \times \tilde{U}_i\}_{i \in I, j \in J}$. To abbreviate the notations, we will denote it by $L = \{L_{\bullet,\tilde{\alpha}}\}$ and $M = \{M_{\bullet,\tilde{\alpha}}\}$, $\tilde{\alpha} = (\alpha, \beta, \gamma) \subset I \times J \times L$.

We will denote by $\overline{L}_{n,\tilde{\alpha}}$ the standard extensions of the free sheaves $L_{n,\tilde{\alpha}}$ on the corresponding open subsets of $Z \times \tilde{Y} \times \tilde{X}$. For any fixed n the sheaves

$L_{n,\tilde{\alpha}} \bigotimes p^*\mathcal{M}$ form a nq-local system, and for any n and $\tilde{\alpha} \subset I \times J \times L$ the complex of free sheaves $\overline{L}_{n,\tilde{\alpha}} \bigotimes p^*M_{\bullet,\tilde{\alpha}}$ is quasi-isomorphic to $L_{n,\tilde{\alpha}} \bigotimes p^*\mathcal{M}$. The system of complexes $\overline{L}_{n,\tilde{\alpha}} \bigotimes p^*M_{\bullet,\tilde{\alpha}}$ is not a local system in the sense of definition 1.2. However, using 1.6 and 1.7, one can find for any n a nq-local system $K_n = \{K_{n,\bullet,\tilde{\alpha}}\}$ of complexes of free finite-dimensional sheaves on the elements of the covering $\mathcal{F} \times \mathcal{V} \times \mathcal{U}$, quasi-isomorphic to the nq-system $L_{n,\tilde{\alpha}} \bigotimes p^*\mathcal{M}$ and containing for any $\tilde{\alpha}$ the resolution $\overline{L}_{n,\tilde{\alpha}} \bigotimes M_{\bullet,\tilde{\alpha}}$ as a subcomplex.

Next, applying lemma 1.9, one obtains a filtered local system of finite complexes of free sheaves $E = E_1 \supset \ldots$, $E = \{E_{\bullet,\tilde{\alpha}}\}$, quasi-isomorphic to $L \bigotimes p^*\mathcal{M}$, such that $E_n/E_{n+1} \approx \widetilde{K_n}$. Note that since $L \bigotimes p^*\mathcal{M}$ is a (quasi-isomorphic) local system, then so is E.

Now consider the bicategory $\mathcal{E}$, consisting of the data $\{L, \mathcal{M}, M, K_n, E\}$ satisfying all the conditions above. One can define in an obvious manner the horizontal, vertical, and bi-morphisms in this category such that the forgetful map $\Phi : \mathcal{E} \rightarrow LocFfd(f) \bigotimes Per^0(g)$, mapping the data above into $L, \mathcal{M}$, is a functor between bicategories.

LEMMA 4.2. — *The functor Φ is a homotopy equivalence.*

Proof. — Fix $\mathcal{M} \in Per^0(g)$, denote by $\mathcal{E}_{\mathcal{M}}$ the corresponding underlying category of the bicategory $\mathcal{E}$, and by $\Phi_{\mathcal{M}} : \mathcal{E}_{\mathcal{M}} \rightarrow LocFfd(f)$ the corresponding restriction of the functor Φ. Then it is sufficient to prove that for any $\mathcal{M}$ the corresponding functor $\Phi_{\mathcal{M}}$ is a homotopy equivalence. Indeed, the statements 1.6 - 1.10 show that all the objects and morphisms in $LocFfd(f)$ belong to the image of the functor $\Phi_{\mathcal{M}}$, and the entities $\{M, K_n, E\}$ are unique up to a homotopy equivalence. Therefore, the requirements of theorem 1.9.8 of [11] (or of the Waldhausen approximation theorem) are satisfied for the functor $\Phi_{\mathcal{M}}$. □

Further, the proof of Fulton-Macpherson in lemma 4.1, *loc. cit.*, still works in our case. Indeed, represent $Y \times \tilde{X}$ as a homotopy retract of some its neighborhood in $Z \times \tilde{Y} \times \tilde{X}$. It determines a canonical extension of the local system L up to a local system $\overline{L} = \{\overline{L}_{\bullet,\tilde{\alpha}}\}$ of continuous complexes of finite-dimensional spaces, defined in this neighborhood.

Now we have two biexact functors from the category $\mathcal{E}$ to the category $LocCBun(g \circ f)$ (see 3.2) of local systems of continuous complexes of finite - dimensional vector spaces, defined in a neighborhood of X in $Z \times \tilde{Y} \times \tilde{X}$ and exact off X. The first one, say α_1, is defined by the complex of local systems E, while the second one, α_2 - as the total complex of the double complex $\overline{L} \bigotimes p^*M$. Taking the filtration, corresponding to the the truncation in the direction of the differentials of L, the latter complex can be

considered as a filtered complex. As it follow from the construction of E and K_n, there exists a continuous embedding $\varphi = \{\varphi_{\bullet, \tilde{\alpha}}\} : \overline{L} \bigotimes p^* M \to E$, inducing quasi-isomorphisms of local systems on the corresponding graduated complexes. The only obstacle to completing the proof as in [5], prop. 3.1.3, is the fact that the linear homotopy between the complexes is no more a complex. However, the procedure of collapsing can reduce the proof to the case of complexes of length two. Indeed, one may suppose that from the beginning all the free sheaves, involved in the construction, are endowed with Hermitian metrics; since all Hermitian metrics are homotopic, this gives equivalent categories. Then, collapsing both complexes up to complexes of length two, and taking the linear homotopy, we obtain a canonical homotopy between the functors α_1 and α_2, and one can apply 3.1 (2).

Taking the product with the inverse map of the homotopy equivalence $\Phi : \mathcal{E} \to LocFfd(f) \bigotimes Per^0(g)$, one sees that α_1 is homotopy equivalent to the functor $(L, \mathcal{M}) \to \alpha(g \circ f)(L \bigotimes \mathcal{M})$. On the other hand, the functor α_2 by definition coincides with $\alpha(f)(L) \bigotimes \alpha(g)(\mathcal{M})$ (in both cases we use the first definition of the Riemann-Roch functor α). The homotopy equivalence stated above proves the commutativity of the Riemann-Roch functor with the products.

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