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IS THE LUNA STRATIFICATION INTRINSIC?

by Jochen KUTTLER & Zinovy REICHSTEIN (*)

Abstract. — Let $G \to \mathrm{GL}(V)$ be a representation of a reductive linear algebraic group G on a finite-dimensional vector space V, defined over an algebraically closed field of characteristic zero. The categorical quotient $X = V /\!/ G$ carries a natural stratification, due to D. Luna. This paper addresses the following questions:

(i) Is the Luna stratification of X intrinsic? That is, does every automorphism of $V /\!/ G$ map each stratum to another stratum?

(ii) Are the individual Luna strata in X intrinsic? That is, does every automorphism of $V /\!/ G$ map each stratum to itself?

In general, the Luna stratification is not intrinsic. Nevertheless, we give positive answers to questions (i) and (ii) for interesting families of representations.

Résumé. — Soit $G \to \mathrm{GL}(V)$ une représentation d'un groupe algébrique réductif G, définie sur un corps algébraiquement clos de caractéristique zéro. D'après D. Luna, le quotient catégorique $X = V /\!/ G$ comporte une stratification naturelle. L'article présente les deux questions suivantes :

(i) La stratification de X est-elle intrinsèque ? Plus précisément, l'image d'une strate par un automorphisme de X quelconque est-elle avec strate ?

(ii) Les strates individuelles de X, sont-elles intrinsèques ? C'est-à-dire, est-il vrai que toute strate est invariante par tous les automorphismes de X ?

D'une manière générale, la stratification de Luna n'est pas intrinsèque. Néanmoins, pour des familles de représentations intéressantes les questions (i) et (ii) ont des réponses positives.

1. Introduction

Throughout this paper k will be an algebraically closed field of characteristic zero, $G \to \mathrm{GL}(V)$ will be a representation of a reductive linear algebraic group G on a finite-dimensional vector space V defined over k,

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and $\pi\colon V \to X = V /\!/ G$ will denote the categorical quotient map for the G-action on V. For the definition and a discussion of the properties of the categorical quotient in this setting, see, e.g., [16], [19] or [8].

There is a natural stratification on X, due to D. Luna; we shall refer to it as *the Luna stratification*. Recall that for every $p \in X$ the fiber $\pi^{-1}(p)$ has a unique closed orbit. Choose a point v_p in this orbit. Then the stabilizer subgroup $\mathrm{Stab}(v_p)$ is reductive, and its conjugacy class in G is independent of the choice of v_p. This subgroup determines the stratum of p. More precisely, the Luna stratum associated to the conjugacy class (H) of a reductive subgroup $H \subseteq G$ is defined as

$$X^{(H)} = \{p \in X \mid \mathrm{Stab}(v_p) \in (H)\}\,.$$

There are only finitely many Luna strata and each stratum is a locally closed non-singular subvariety of X. The strata are naturally partially ordered as follows: $S \preceq T$ if S is contained in $\overline{T}$. This partial ordering has a unique maximal element $X^{(H)}$, called *the principal stratum*. The subgroup H associated to the principal stratum is called *the principal stabilizer* (it is defined up to conjugacy). Moreover, if we set

$$V^{\langle H\rangle} = \{v \in V \mid \ G \cdot v \text{ is closed and } \mathrm{Stab}(v) = H\}$$

then π restricts to a principal $N_G(H)/H$-bundle $V^{\langle H\rangle} \to X^{(H)}$. For proofs of these assertions see [19, Section 6.9] or [27, Section I.5].

The Luna stratification provides a systematic approach to the problem of describing the G-orbits in V; it also plays an important role in the study of the geometry and (if $k = \mathbb{C}$) the topology of the categorical quotient $X = V /\!/ G$. In this paper we shall address the following questions.

Question 1.1. — *(i) Is the Luna stratification of X intrinsic? In other words, is it true that for every automorphism $\sigma\colon X \to X$ and every reductive subgroup $H \subset G$ there is a reductive subgroup $H' \subset G$ such that $\sigma(X^{(H)}) = X^{(H')}$?*

(ii) Are the Luna strata in X intrinsic? Here we say that $X^{(H)}$ is intrinsic if $\sigma(X^{(H)}) = X^{(H)}$ for every automorphism $\sigma\colon X \to X$.

In general, the Luna stratification is not intrinsic. Indeed, there are many examples, where $V /\!/ G$ is an affine space (cf. e.g., [19, Section 8]) and the automorphism group of an affine space is highly transitive, so that points in the same stratum can be taken by an automorphism to points in different strata. Moreover, even in those cases where the Luna stratification is intrinsic, the individual strata may not be. The purpose of this paper is to show that one can nevertheless give positive answers to Question 1.1 in many interesting situations.

The first natural case to consider is the one where G is a finite group. Recall that a non-trivial $g \in \mathrm{GL}(V)$ is called a pseudo-reflection if g has finite order and fixes (pointwise) a hyperplane in V. If $G \to \mathrm{GL}(V)$ is a representation and G is generated by elements that act as pseudo-reflections on V then by a theorem of Chevalley and Shephard-Todd, $V /\!/ G$ is an affine space. As we remarked above, in this case the Luna stratification cannot be intrinsic. To avoid this situation, assume that G contains no pseudo-reflections. In particular, this condition is automatically satisfied for representations $G \to \mathrm{SL}(V) \subset \mathrm{GL}(V)$. In this case the proof of [20, Theorem 2] can be modified to show that every automorphism of $X = V /\!/ G$ lifts to an automorphism of V. From this one easily deduces that if $G \to \mathrm{GL}(V)$ is such a finite-dimensional representation of a finite group G then the Luna stratification on $V /\!/ G$ is always intrinsic, and moreover, every stratum is intrinsic under mild additional assumptions on the representation (but not always). For details we refer the reader to [9, Section 2]. (See also Proposition 3.6 for related results on finite group actions.)

The main focus of this paper will be on representations V of (possibly infinite) reductive groups G. The following theorem gives a positive answer to Question 1.1(i) for three families of such representations.

Theorem 1.2. — *Let $G \to \mathrm{GL}(W)$ be a finite-dimensional linear representation of a reductive algebraic group G and $V = W^r$, where*

(a) $r \geqslant 2\dim(W)$, *or*

(b) *G preserves a nondegenerate quadratic form on W and* $r \geqslant \dim(W)+1$, *or*

(c) *$W = \mathfrak{g}$ is the adjoint representation of G and* $r \geqslant 3$.

Then the Luna stratification in $V /\!/ G$ is intrinsic. Moreover, every Luna stratum S in $V /\!/ G$ is precisely the smooth locus of its closure $\overline{S}$.

Note that in the setting of reductive groups there is no direct analogue of the Chevalley-Shephard-Todd theorem and we do not know under what circumstances an automorphism $\sigma\colon V /\!/ G \to V /\!/ G$ can be lifted to V; see [9, Remark 2.8]. Our proof of the fact that the Luna stratification in $V /\!/ G$ is intrinsic under the assumptions of Theorem 1.2 relies on a different (indirect) approach based on studying the singularities of the Luna strata; see Section 3.

Theorem 1.2 concerns representations V of G of a particular form, namely V is assumed to be the rth power of another representation W for sufficiently large r. This is clearly stronger than assuming that G contains no pseudo-reflections. To motivate this condition we remark that a general (and somewhat vague) principle in invariant theory says that replacing a G-variety Z by a power Z^r often "improves" the properties of the underlying action, assuming r is sufficiently large. For two unrelated recent results along these lines see [12, Corollary, p. 1606] and [18]. Theorem 1.2 may be viewed as yet another manifestation of this principle.

Our second main result gives a positive answer to Question 1.1(ii) in the following situation. Consider the natural GL_n-action on the space $V = \mathrm{M}_n^r$ of r-tuples of $n \times n$-matrices by simultaneous conjugation. The variety $X = \mathrm{M}_n^r /\!/ \mathrm{GL}_n$ has been extensively studied in the context of both invariant and PI theories; an overview of this research area can be found in [6], [21] or [5]. In [22, 23] the second author constructed a large family of automorphisms of $X = V /\!/ \mathrm{PGL}_n$ (for $r \geqslant n+1$). Every automorphism in that family preserves the Luna strata, so it is natural to conjecture that the same should be true for every automorphism of X. The following result proves this conjecture for any $r \geqslant 3$.

Theorem 1.3. — *Suppose $r \geqslant 3$. Then every Luna stratum in $X = \mathrm{M}_n^r /\!/ \mathrm{GL}_n$ is intrinsic.*

Note that Theorem 1.3 fails if $r = 1$ or $(n, r) = (2, 2)$; see Remark 8.3.

The fact that the principal Luna stratum in $X = \mathrm{M}_n^r /\!/ \mathrm{GL}_n$ is intrinsic is an immediate consequence of a theorem of Le Bruyn and Procesi [10, Theorem II.3.4], which says that it is precisely the smooth locus of X. This result motivated Theorems 1.2 and 1.3 and served as a starting point of their proofs.

2. Actions of reductive groups

In this preliminary section we collect several well-known definitions and results about actions of reductive groups on affine varieties.

2.1. The Luna Slice Theorem

Let $G \to \mathrm{GL}(V)$ be a linear representation of a reductive group G and $v \in V$ be a point with a closed G-orbit. Then by Matsushima's theorem, the stabilizer $H = \mathrm{Stab}(v)$ is a reductive subgroup of G. Consequently, the H-subrepresentation $T_v(G \cdot v)$ of the natural representation of H on the tangent space $T_v(V)$ has an H-invariant complement. We shall refer to this H-representation as *the slice representation* and denote it by $\mathrm{Slice}(v, V)$. The Luna Slice Theorem asserts that the horizontal maps in the natural diagram

$$\begin{tikzcd} \mathrm{Slice}(v,V) *_H G \arrow[r] \arrow[d] & V \arrow[d, "\pi"] \\ \mathrm{Slice}(v,V) /\!/ H \arrow[r] & V /\!/ G \end{tikzcd}$$

are étale over v and $\pi(v)$, respectively. In addition the above diagram becomes cartesian after a base change over a neighborhood of $\pi(v)$. For an arbitrary smooth affine G-variety the same is true if $\mathrm{Slice}(v, V)$ is replaced by a suitable H-invariant étale neighborhood of the origin in an H-invariant normal space to $T_v(Gv)$. For details, see [13] or [19, Section 6].

2.2. Stability

DEFINITION 2.1. — *Let G be a reductive group and V be an affine G-variety. A point $v \in V$ is called*

stable *if its orbit $G \cdot v$ is closed in V and*

properly stable *if v is stable and $\mathrm{Stab}_G(v)$ is finite.*

We shall say that the representation V is

stable *if a point $v \in V$ in general position is stable,*

properly stable *if a point $v \in V$ in general position is properly stable,*

generically free *if a point $v \in V$ in general position has trivial stabilizer.*

As an immediate corollary to Luna's Slice Theorem we obtain the following:

LEMMA 2.2. — *Let G be a reductive group, $G \to \mathrm{GL}(V)$ be a linear representation, and $v \in V$, $H = \mathrm{Stab}(v) \subset G$ be as above. Then*

(a) *the G-representation on V is stable if and only if the H-representation on $\mathrm{Slice}(v, V)$ is stable,*

(b) *the G representation on V is generically free if and only if the H-representation on $\mathrm{Slice}(v, V)$ is generically free.* □

Note that "generically free" is not the same thing as "having trivial principal stabilizer". The reason is that when we talk about the principal stabilizer, we are only interested in $\mathrm{Stab}(v)$, where v is a stable point. For example, the natural action of the multiplicative group $\mathbb{G}_m$ on $V = \mathbb{A}^1$ is generically free, but the principal stabilizer is all of $\mathbb{G}_m$, because the only stable point in $\mathbb{A}^1$ is the origin. The precise relationship between these notions is spelled out in the following lemma.

Lemma 2.3. — *Let V be a linear representation of a reductive group G and $\pi\colon V \to V /\!/ G$ be the categorical quotient map.*

(a) *If $\pi(v) \in (V /\!/ G)^{(\{e\})}$ then v is a properly stable point in V.*

(b) *The following conditions are equivalent:*

- *V has trivial principal stabilizer,*
- *V is generically free and properly stable,*
- *V is generically free and stable.*

Proof. — (a) Let $x = \pi(v) \in (V /\!/ G)^{(\{e\})}$. We claim that $\pi^{-1}(x)$ is a single G-orbit in V. Indeed, let $C = G \cdot v_0$ be the unique closed orbit in $\pi^{-1}(x)$. Then C is contained in the closure of every orbit in $\pi^{-1}(x)$. On the other hand, by our assumption $\mathrm{Stab}(v_0) = \{e\}$; hence, $\dim(C) = \dim(G)$, and C cannot be contained in the closure of any other G-orbit. This shows that $C = \pi^{-1}(x)$. Thus every point in $\pi^{-1}(x)$ is stable (and hence, properly stable). This proves part (a). Part (b) is an immediate consequence of part (a). □

2.3. The Hilbert-Mumford criterion

Consider a linear $\mathbb{G}_m$-representation on a vector space V. Any such representation can be diagonalized. That is, there is a basis $e_1, \dots, e_n$ of V, so that $t \in \mathbb{G}_m$ acts on V by $t \cdot e_i \mapsto t^{d_i} e_i$.

In the sequel we shall use the following variant of the Hilbert-Mumford criterion; see [16, Section 2.1].

Theorem 2.4. — *Consider a linear representation of a reductive group G on a vector space V. Then*

(a) *$v \in V$ is properly stable for the action of G if and only if it is properly stable for the action of every 1-dimensional subtorus $\mathbb{G}_m \hookrightarrow G$.*

(b) *In the above notations, $v = c_1 e_1 + \dots + c_n e_n \in V$ is properly stable for the action of $\mathbb{G}_m$ if and only if there exist $i, j \in \{1, \dots, n\}$ such that $d_i < 0$, $d_j > 0$ and $c_i, c_j \neq 0$.* □

We give several simple applications of this theorem below.

Corollary 2.5. — *Suppose G is a reductive group, $G \to \mathrm{GL}(V)$ is a linear representation, $\pi_G\colon V \to V /\!/ G$ is the categorical quotient map, $v \in V$, and $\pi_G(v) \in (V /\!/ G)^{(\{e\})}$.*

(a) *If $H \subset G$ is a reductive subgroup and $\pi_H : V \to V /\!/ H$ is the categorical quotient map for the induced H-action on V then $\pi_H(v) \in (V /\!/ H)^{(\{e\})}$.*

(b) *If $f\colon V' \to V$ is a G-equivariant linear map, $\pi'\colon V' \to V' /\!/ G$ is the categorical quotient, and $f(v') = v$, then $\pi'(v') \in (V' /\!/ G)^{(\{e\})}$.*

Proof. — Recall that, by definition, $\pi_G(v) \in V /\!/ G^{(\{e\})}$ if and only if (i) $\mathrm{Stab}_G(v) = \{e\}$ and (ii) v is properly stable for the G-action on V.

(a) We need to show that (i) and (ii) remain valid if G is replaced by H. In case of (i) this is obvious, and in case of (ii), this follows from the Hilbert-Mumford criterion, since every 1-parameter subgroup of H is also a 1-parameter subgroup of G.

(b) Again, we need to check that $\mathrm{Stab}_G(v') = \{e\}$ and v' is properly stable for the G-action on V'. The former is obvious, and the latter follows from the Hilbert-Mumford criterion. □

2.4. Reductive groups whose connected component is central

Let H be a reductive group whose connected component H^0 is central. In particular, H^0 is abelian and hence, a torus. Note that this class of groups includes both tori and all finite groups (in the latter case $H^0 = \{1\}$).

Lemma 2.6. — *Let H be a reductive group whose connected component H^0 is central and $\rho\colon H \to \mathrm{GL}(W)$ be a linear representation. Then*

(a) $\mathrm{Stab}(w) = \mathrm{Ker}(\rho)$ *for $w \in W$ in general position.*

(b) *Suppose H has trivial principal stabilizer in W^s for some $s \geqslant 1$. Then H has trivial principal stabilizer in W.*

Proof. — (a) By [25, Theorem A] the H-action on W has a stabilizer in general position. That is, there is a subgroup $S \subset H$ and an open subset $U \subset W$ such that $\mathrm{Stab}(u)$ is conjugate to S for any $u \in U$. Clearly, $\mathrm{Ker}(\rho) \subset S$; we only need to prove the opposite inequality.

Since H^0 is central in H, S has only finitely many conjugates; denote them by $S = S_1, \dots, S_m$. Then U is contained in the union of finitely many linear subspaces

$$U \subset W^{S_1} \cup \dots \cup W^{S_m}\,.$$

Since U is irreducible, we see that U is contained in one of them, say, $U \subset W^S$. Then $V = W^S$, i.e., $S \subset \mathrm{Ker}(\rho)$, as claimed.

(b) Let ρ^s be the (diagonal) representation of H on W^s. Clearly $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho^s) = \{e\}$. Part (a) now tells us that since ρ^s is generically free, so is ρ.

By Lemma 2.3 it now suffices to check that the H-action ρ on W is properly stable. Let $\mathbb{G}_m \hookrightarrow H$ be a 1-dimensional subtorus. Diagonalize it in the basis $e_1, \dots, e_n$ of W, so that it acts via

$$t\colon e_i \to t^{d_i} e_i\,.$$

Note that if we diagonalize the $\mathbb{G}_m$-action on W^s then the same exponents $d_1, \dots, d_n$ will appear but each will be repeated s times. Thus by Theorem 2.4,

$$\begin{array}{rcl} W^s \text{ is properly stable} & \Longleftrightarrow & d_i > 0 \text{ and } d_j < 0 \text{ for some } i, j \in \{1, \dots, n\} \\ & & \Updownarrow \\ & & W \text{ is properly stable,} \end{array}$$

and the lemma follows. □

Remark 2.7. — Note that both parts of Lemma 2.6 fail if we only assume that H^0 is a torus (but do not assume that it is central in H). For example, both parts fail for the natural action of the orthogonal group $G = \mathrm{O}_2(k) = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ on $W = k^2$; cf. [24, Example 2.5].

2.5. Multiple representations

Lemma 2.8. — *Let G be a reductive group and $G \to \mathrm{GL}(W)$ be a linear representation. Suppose that for some $r \geqslant 1$ an r-tuple $w = (w_1, \dots, w_r)$ is chosen so that $G \cdot w$ is closed in W^r and that $w_{d+1}, \dots, w_r$ are linear combinations of $w_1, \dots, w_d$ for some $1 \leqslant d \leqslant r$. Set $H = \mathrm{Stab}(w)$ and $v = (w_1, \dots, w_d) \in W^d$. Then*

(a) $\mathrm{Stab}_G(v) = H$.

(b) *W^r has a G-subrepresentation W_0 such that $w \in W_0$ and the natural projection $p\colon W^r \to W^d$ onto the first d components restricts to an isomorphism between W_0 and W^d.*

(c) *v has a closed orbit in W^d.*

(d) $\mathrm{Slice}(w, W^r) \simeq \mathrm{Slice}(v, W^d) \oplus W^{r-d}$, *where $\simeq$ denotes equivalence of H-representations.*

Proof. — (a) is obvious. To prove (b), suppose $w_j = \sum_{i=1}^{d} \alpha_{ij} w_j$ for $j = d+1, \dots, r$. Then

$$W_0 = \{(z_1, \dots, z_r) \mid z_j = \sum_{i=1}^{d} \alpha_{ij} z_j \text{ for } j = d+1, \dots, r \}.$$

has the desired properties. (c) follows from (b), since $G \cdot v$ is the image of $G \cdot w$ under p.

(d) Since H is reductive, W_0 has an H-invariant complement W_1 in W^r, so that $W^r = W_0 \oplus W_1$. Since $W_0 \simeq W^d$, we conclude that $W_1 \simeq W^{r-d}$ (as an H-representation). The desired conclusion now follows from the fact that p is an isomorphism between W_0 and W^d. □

Corollary 2.9. — *Let G be a reductive group and $G \to \mathrm{GL}(W)$ be a linear representation of dimension n. Then*

(a) *The following are equivalent:* (i) *W^r is stable for some $r \geqslant n$,* (ii) *W^n is stable, and* (iii) *W^s is stable for every $s \geqslant n$.*

(b) *The following are equivalent:* (i) *W^r is generically free for some $r \geqslant n$,* (ii) *W^n is generically free, and* (iii) *W^s is generically free for every $s \geqslant n$.*

(c) *The following are equivalent:* (i) *W^r has trivial principal stabilizer for some $r \geqslant n$,* (ii) *W^n has trivial principal stabilizer, and* (iii) *W^s has trivial principal stabilizer for every $s \geqslant n$.*

Proof. — Suppose $r \geqslant n$. Then for $w = (w_1, \dots, w_r) \in W^r$ in general position, $w_1, \dots, w_n$ span W. Keeping this in mind, we see that

(a) the implication (i) $\Rightarrow$ (ii) follows from Lemma 2.8(c) (with $d = n$) and the implication (ii) $\Rightarrow$ (iii) follows from Lemma 2.8(b) (again, with $d = n$). (iii) $\Rightarrow$ (i) is obvious.

(b) follows from Lemma 2.8(a). (c) follows from (a) and (b); see Lemma 2.3(b). □

3. Proof of Theorem 1.2: the overall strategy

Let G be a reductive algebraic group acting on a smooth affine variety Y and $\pi\colon Y \to X = Y /\!/ G$ be the categorical quotient map for this action. We will say that H is a *stabilizer subgroup* for Y (or simply a *stabilizer subgroup*, if the reference to Y is clear from the context), if $H = \mathrm{Stab}(y)$ for some *stable* point $y \in Y$. Clearly H is a stabilizer subgroup if and only if the Luna stratum $X^{(H)}$ associated to its conjugacy class is non-empty.

We will say that a Luna stratum S is *singular along its boundary* if the singular locus of its closure $\overline{S}$ is precisely $\overline{S} \setminus S$. The following lemma is the starting point for our proof of Theorem 1.2.

LEMMA 3.1. — *Suppose every Luna stratum in $X = Y /\!/ G$ is singular along its boundary. Then the Luna stratification in X is intrinsic.*

Proof. — Let σ be an automorphism of X and S be a Luna stratum in X. We will show that σ takes S to another Luna stratum S' by descending induction with respect to the natural partial order on the (finite) set of Luna strata in X.

By our assumption the principal Luna stratum X_0 is precisely the smooth locus of X; thus $\sigma(X_0) = X_0$. Now suppose $\sigma(S)$ is another Luna stratum S'. Then σ takes the smooth locus of $\overline{S} \setminus S$ to the smooth locus of $\overline{S'} \setminus S'$. The irreducible components of the smooth locus of $\overline{S} \setminus S$ are, by our assumption, precisely the Luna strata T which immediately precede S in the partial order. The automorphism σ takes these strata to the irreducible components of the smooth locus of $\overline{S'} \setminus S'$, which are the Luna strata T' immediately preceding S'. Thus σ takes every Luna stratum T immediately preceding S, into another Luna stratum T'. The induction step is now complete. $\square$

DEFINITION 3.2. — *We shall call a family Λ of finite-dimensional linear representations $G \to \mathrm{GL}(V)$ of reductive (but not necessarily connected) algebraic groups* acceptable *if it satisfies the following two conditions.*

(i) *If $G \to \mathrm{GL}(V)$ is in Λ then for every stabilizer subgroup H in G, the induced representation $N_G(H) \to \mathrm{GL}(V^H)$ is again in Λ.*

(ii) *For every representation $G \to \mathrm{GL}(V)$ in Λ, the principal stratum in $V /\!/ G$ is singular along its boundary.*

PROPOSITION 3.3. — *Suppose a linear representation $G \to \mathrm{GL}(V)$ belongs to an acceptable family. Then every Luna stratum in $X = V /\!/ G$ is singular along its boundary. In particular, the Luna stratification in X is intrinsic.*

Proof. — The second assertion follows from the first by Lemma 3.1. Hence, we only need to show that every Luna stratum in X is singular along its boundary. Let $\pi\colon V \to V /\!/ G = X$ be the categorical quotient map, H be a stabilizer subgroup for the G-action on V, and $S = X^{(H)}$ be a Luna stratum. Choose $p \in \overline{S} \setminus S$, say, $p \in X^{(K)}$, where K is a (reductive) stabilizer subgroup and $H \subsetneq K$. Our goal is to show that $\overline{S}$ is singular at p.

We will argue by contradiction. Assume, to the contrary, that $\overline{S}$ is smooth at p.

Let $N = N_G(H)$ be the normalizer of H in G and write the surjective map $\pi_{|V^H} \colon V^H \to \overline{S}$ as a composition

$$\begin{tikzcd} V^H \arrow[d, "\pi_N"] \arrow[dd, bend left=60, "\pi"] \\ V^H /\!/ N \arrow[d, "n"] \\ \overline{S}, \end{tikzcd}$$

where π_N is the categorical quotient map for the N-action on V^H. Here n is the normalization map for $\overline{S}$; cf. e.g., [19, Theorem 6.16].

Let $v \in V^H$ be an N-stable point with stabilizer K such that $\pi(v) = p$ and let $q = \pi_N(v)$. (Note that by Luna's criterion, v is N-stable if and only if it is G-stable; see [19, Theorem 6.17].) Recall that we are assuming that the G-representation on V belongs to an acceptable family Λ. Consequently, the N-representation on V^H also belongs to Λ, and thus the smooth locus of $V^H /\!/ N$ is precisely the principal stratum for the N-action on V^H. In other words, if q does not lie in the principal stratum in $V^H /\!/ N$ then q is a singular point of $V^H /\!/ N$. Since n is the normalization map and $n(q) = p$, this implies that p is a singular point of $\overline{S}$, a contradiction.

We may thus assume that q lies in the principal stratum U of $V^H /\!/ N$. Since we are assuming that p is a smooth point of $\overline{S}$, the normalization map n is an isomorphism between Zariski open neighborhoods of p and q.

We claim that the action of N/H on V^H is properly stable and generically free. By our assumption on H, V^H contains a G-stable point v with stabilizer H. By Luna's criterion (see [19, Theorem 6.17]) v is stable (and hence, properly stable) for the N/H-action on V^H. Since N/H-properly stable points form an open subset of V^H, we conclude that the N/H-action on V^H is properly stable. Moreover, v has a G-invariant Zariski open neighborhood $V_0 \subset V$ such that $\mathrm{Stab}_G(v_0)$ is conjugate to a subgroup of H for any $v_0 \in V_0$; see, e.g., [19, Theorem 6.3]. Intersecting V_0 with V^H, we see that $\mathrm{Stab}_G(v_0) = H$ for $v_0 \in V^H$ in general position. This proves the claim.

The claim implies that $\pi_N^{-1}(U) \to U$ is a principal N/H-bundle and consequently, the differential $d\pi_v$ maps $T_v(V^H)$ surjectively onto $T_p(\overline{S})$. We will now show that this is impossible. Indeed, since the quotient map

$\pi\colon V \to X$ is K-equivariant (where K acts trivially on X) and v is fixed by K, the differential $d\pi_v\colon T_v(V) \to T_p(X)$ is a K-equivariant linear map (where K acts trivially on $T_p(X)$). Consequently, $d\pi_v$ sends every non-trivial irreducible K-subrepresentation of $T_v(V)$ to 0. Since V is smooth, we have $(T_v(V))^K = T_v(V^K)$ and therefore $d\pi_v$ maps $T_v(V^K)$ onto $T_p(\overline{S})$. On the other hand, since $\pi(V^K) = \overline{X^{(K)}}$, we conclude that

$$T_p(\overline{X^{(K)}}) \supset T_p(\overline{S})\,. \tag{3.1}$$

Now recall that we are assuming that p is a smooth point of $\overline{S}$. Moreover, since $p \in X^{(K)}$, it is also a smooth point of $\overline{X^{(K)}}$. Thus (3.1) implies $\dim X^{(K)} \geqslant \dim S$, contradicting the fact that $X^{(K)}$ lies in $\overline{S} \setminus S$. □

We now record a corollary of the above argument for future reference.

Corollary 3.4. — *Suppose G is a reductive group and $G \to \mathrm{GL}(V)$ is a linear representation with principal isotropy H. Let $N = N_G(H)$. Then the natural map*

$$\begin{array}{c} V^H /\!/ N \\ \big\downarrow{\scriptstyle n} \\ V /\!/ G \end{array}$$

is an isomorphism, which identifies the principal Luna stratum in $V^H /\!/ N$ with the principal Luna stratum in $V /\!/ G$.

Proof. — The fact that n is an isomorphism is proved in [14, Corollary 4.4].

To show that n identifies the principal strata in $V^H /\!/ N$ and $V /\!/ G$, let $\pi\colon V \to V /\!/ G$ and $\pi_N\colon V^H \to V /\!/ N$ be the categorical quotient maps. Choose $p \in V^H /\!/ N$ and set $q = n(p)$. Let $v \in V^H$ be a point in the (unique) closed N-orbit in $\pi_N^{-1}(p)$. By Luna's criterion, the G-orbit of v is also closed; see [19, Theorem 6.17]. Our goal is to show that

$$\mathrm{Stab}_G(v) = H \iff \mathrm{Stab}_N(v) = H\,.$$

The $\Rightarrow$ direction is obvious, so suppose $\mathrm{Stab}_N(v) = H$ (i.e., p lies in the principal stratum in $V^H /\!/ N$), and let $\mathrm{Stab}_G(v) = K$. We want to show that $K = H$. Assume the contrary: $H \subsetneq K$. Since p lies in the principal stratum in $V^H /\!/ N$, $d\pi_N$ maps $T_v(V^H)$ surjectively to $T_p(V^H /\!/ N)$. Since n is an isomorphism and $\pi = n \circ \pi_N$ (on V^H), we see that $d\pi_v$ maps $T_v(V^H)$ surjectively onto $T_v(V /\!/ G)$. On the other hand, in the proof of Proposition 3.3 we showed that this is impossible if $H \subsetneq K$. □

Remark 3.5. — Our proof of Theorem 1.2 will be based on showing that each of the families of representations in parts (a), (b) and (c) is acceptable, i.e., satisfies conditions (i) and (ii) of Definition 3.2; the desired conclusion will then follow from Lemma 3.1 and Proposition 3.3.

To illustrate this strategy, we will apply it to the following simple example. We will say that a linear representation $G \to \mathrm{GL}(V)$ has *the codimension* 2 *property* if $\dim(V^A) - \dim(V^B) \neq 1$ for every pair of subgroups $A \triangleleft B \leqslant G$ (here A is normal in B).

PROPOSITION 3.6. — *Let Λ be the family of representations $\phi\colon G \to \mathrm{GL}(V)$, where G is finite and ϕ has the codimension 2 property. Then Λ is acceptable. In particular, if $\phi \in \Lambda$ then*

(a) *every Luna stratum in $V /\!/ G$ is singular along its boundary, and*

(b) *the Luna stratification in $V /\!/ G$ is intrinsic.*

Proof. — The family Λ clearly satisfies condition (i) of Definition 3.2. To check condition (ii), choose a representation $G \to \mathrm{GL}(V)$ in Λ. Let $X = V /\!/ G$ be the categorical quotient and $\pi\colon V \to X$ be the quotient map. Choose $v \in V$ and set $H = \mathrm{Stab}_G(v)$. Our goal is to show that if $H \neq \{e\}$ then X is singular at $\pi(v)$. Assume the contrary. Then by the Luna Slice Theorem, $T_v(V) /\!/ H$ is smooth at the origin. Since $T_v(V)$ and V are isomorphic as H-modules, $V /\!/ H$ is also smooth at the origin. By the Chevalley-Shephard-Todd theorem, this implies that H is generated by pseudo-reflections. In particular, since $H \neq \{e\}$, H contains a pseudo-reflection h. Setting $A = \{e\}$ and $B = \langle h \rangle$, we see that the codimension 2 property fails: V^B has codimension 1 in $V^A = V$. This contradiction shows that $V /\!/ G$ is singular at $\pi(v)$. Thus Λ is an acceptable family. Assertions (a) and (b) now follow from Proposition 3.3. □

Note that Proposition 3.6(b) is a special case of [9, Theorem 1.1].

Example 3.7. — Every symplectic representation of a finite group has the codimension 2 property.

Indeed, since every symplectic representation is even-dimensional, the assertion of Example 3.7 is an immediate consequence of the following elementary lemma. This lemma will be used again in the sequel.

LEMMA 3.8. — *Let $G \to \mathrm{GL}(W)$ be a linear representation, leaving invariant a non-degenerate bilinear form b on W. Then for any reductive subgroup H of G the restriction of b to W^H is again non-degenerate.*

Proof. — Let $a \in W^H$ and suppose that the linear form $l_a\colon w \mapsto b(a, w)$ is identically zero on W^H. Since H is reductive, $W = W^H \oplus W_1 \oplus \cdots \oplus W_r$, where each W_i is a non-trivial irreducible H-subrepresentation of W. By Schur's lemma, l_a is identically zero on every W_i. Hence, l_a is identically zero on all of W. Since b is non-degenerate on W, this is only possible if $a = 0$. □

4. Non-coregular representations

The main difficulty in implementing the strategy outlined in Remark 3.5 is in checking condition (ii) of Definition 3.2. That is, given a linear representation $G \to \mathrm{GL}(V)$ of a reductive group G on a vector space V and a stable point $v \in V$, we want to show that $V /\!/ G$ is singular at $\pi(v)$. The Luna Slice Theorem reduces this problem to checking that $\mathrm{Slice}(v, V) /\!/ H$ is singular at $\pi_H(0)$, where $H = \mathrm{Stab}(v)$, and $\pi_H\colon \mathrm{Slice}(v, V) \to \mathrm{Slice}(v, V) /\!/ H$ is the categorical quotient map. In other words, we are reduced to a problem of the same type, where $v = 0$ and $G = H$.

Linear representations $G \to \mathrm{GL}(V)$ with the property that $V /\!/ G$ is smooth at $\pi(0)$ are called *coregular*. Here G is assumed to be reductive and π is the categorical quotient map $V \to V /\!/ G$. It is easy to see (cf., e.g., [19, Proposition 4.11]) that $V /\!/ G$ is smooth at $\pi(0)$ if and only if $V /\!/ G$ is smooth everywhere if and only if $V /\!/ G$ is isomorphic to an affine space $\mathbb{A}^d$ for some $d \geqslant 1$. Coregular representations have been extensively studied; for a survey of this topic and further references, see [19, Section 8]. Thus in order to implement the strategy for proving Theorem 1.2 outlined in Remark 3.5 we need a large family of representations that are known not to be coregular. The purpose of this section is to prove Proposition 4.1 below, which exhibits such a family.

Proposition 4.1. — *Let G be a reductive group and V_1, V_2 be linear representations of G, such that V_1 has trivial principal stabilizer and V_2 is not fixed pointed (see Definition 4.2 below). Then $V_1 \times V_2$ is not coregular. That is, $(V_1 \times V_2) /\!/ G$ is singular.*

We begin with preliminary results about fixed pointed representations, which will be used in the proof and in subsequent applications of Proposition 4.1.

4.1. Fixed pointed representations

Definition 4.2. — *Following Bass and Haboush* [2]*, we will say that a linear representation $G \to \mathrm{GL}(V)$ of a reductive group G is* fixed pointed *if* *the natural G-equivariant projection $\pi\colon V \to V^G$ is the categorical quotient for the G-action on V. Note that the projection π is sometimes called the Reynolds operator; cf., e.g.,* [19, Section 3.4].

Recall that the null cone of a representation V, denoted $\mathrm{NC}(V)$, is the subvariety of V cut out by all homogeneous invariants of positive degree.

LEMMA 4.3. — *Let G be a reductive group and $\rho\colon G \to \mathrm{GL}(V)$ be a linear representation. Then the following conditions are equivalent:*

(a) *ρ is fixed pointed.*

(b) *The null cone $\mathrm{NC}(V)$ is (scheme-theoretically) a vector space.*

(c) *The null cone $\mathrm{NC}(V)$ is (scheme-theoretically) smooth.*

If (a), (b) *and* (c) *hold then $V = \mathrm{NC}(V) \oplus V^G$.*

Proof. — (a) $\Rightarrow$ (b). Let $\pi\colon V \to V /\!/ G$ be the categorical quotient map. If ρ is fixed pointed then π is a linear projection, so clearly $\mathrm{NC}(V) = \pi^{-1}(0)$ is a vector space and $V = \mathrm{NC}(V) \oplus V^G$.

(b) $\Rightarrow$ (c) is obvious, since a vector space is smooth.

(c) $\Rightarrow$ (a). Assume that $\mathrm{NC}(V)$ is scheme-theoretically smooth. Since G is reductive, we may write $V = V^G \oplus W$ for some G-invariant subspace W. Then clearly $W^G = (0)$, $\mathrm{NC}(W) = \mathrm{NC}(V)$ and $V /\!/ G = V^G \times W /\!/ G$, where the quotient map π_V sends $(v, w) \in V = V^G \oplus W$ to $(v, \pi_W(w)) \in V^G \times W /\!/ G$.

Thus, after replacing V by W, we may assume without loss of generality that $V^G = (0)$. Our goal is then to show that V is fixed pointed, which, in this case, means that $V /\!/ G$ is a single point, or equivalently,

$$\mathrm{NC}(V) = V\,.$$

Indeed, assume the contrary. Then $\mathrm{NC}(V)$ is cut out by the equations $f = 0$, as f ranges over the homogeneous elements of $k[V]^G$.

Note that no nonzero homogeneous element $f \in k[V]^G$ can be of degree 1. Indeed, if there were a non-zero G-invariant linear function $f\colon V \to k$ then V would contain a copy of the trivial representation, contradicting $V^G = (0)$. We thus conclude that $\mathrm{NC}(V)$ is a subscheme of V cut out by a (possibly empty) collection of homogeneous polynomials of degree $\geqslant 2$. In particular, the tangent space to $\mathrm{NC}(V)$ at 0 coincides with all of $T_0(V)$. Since we are assuming that $\mathrm{NC}(V)$ is (scheme-theoretically) smooth, this is only possible if $\mathrm{NC}(V) = V$. $\square$

In the sequel we will primarily be interested in representations that are *not* fixed pointed. Two examples are given below.

Example 4.4. — No nontrivial stable representation $G \to \mathrm{GL}(V)$ can be fixed pointed. Indeed, in a fixed-pointed representation, the only stable points are those in V^G.

Example 4.5. — A non-trivial orthogonal representation $H \to \mathrm{O}_\phi(L)$ of a reductive group H on a vector space L (preserving a non-degenerate quadratic form ϕ) is not fixed pointed.

Proof. — Assume the contrary. Then

$$L = \mathrm{NC}(L) \oplus L^H \tag{4.1}$$

where $\mathrm{NC}(L)$ is the null-cone of L. Lemma 3.8 tells us that ϕ restricts to a non-degenerate quadratic form on L^H. Hence,

$$L = (L^H)^\perp \oplus L^H \,.$$

Now observe that L^H has a unique H-invariant complement in L (namely, the direct sum of all non-trivial H-subrepresentations in L). We thus conclude that $\mathrm{NC}(L) = (L^H)^\perp$. Since ϕ is non-degenerate on L and L^H, it is also non-degenerate on $(L^H)^\perp$. Note that since the H-action on L is non-trivial, $L^H \neq L$ and thus $\mathrm{NC}(L) = (L^H)^\perp \neq (0)$. In particular, the H-invariant regular function $L \to k$ given by $x \to \phi(x,x)$ is not constant on $(L^H)^\perp$. On the other hand, every H-invariant regular function on $\mathrm{NC}(L)$ has to be constant. This contradiction shows that L is not fixed pointed. □

4.2. Proof of Proposition 4.1

Set $V = V_1 \times V_2$. Assume the contrary: $V /\!/ G$ is smooth, i.e.. is isomorphic to an affine space $\mathbb{A}^d$. Let $\pi \colon V \to V /\!/ G$, $\pi_1 \colon V_1 \to V_1 /\!/ G$, and $\pi_2 \colon V_2 \to V_2 /\!/ G$ be the categorical quotient maps. We will denote the projection $V \to V_i$ by p_i and the induced morphism $V /\!/ G \to V_i /\!/ G$ by $\overline{p_i}$.

Let $T = \mathbb{G}_m \times \mathbb{G}_m$ be a two-dimensional torus acting on $V = V_1 \times V_2$ by

$$(s,t) \colon (v_1, v_2) \to (sv_2, tv_2) \,.$$

This action commutes with the G-action on V and hence descends to $V /\!/ G$. Clearly the T-fixed point $\pi(0,0)$ lies in the closure of every other T-orbit in $V /\!/ G \simeq \mathbb{A}^d$. Thus by [2, Corollary 10.6], the T-action on $V /\!/ G$ is isomorphic to a linear action. That is, we may assume that $V /\!/ G$ is a vector space with a linear action of T. This identifies $V_1 /\!/ G$ and $V_2 /\!/ G$

with the T-invariant linear subspaces $(V /\!/ G)^{\{1\}\times\mathbb{G}_m}$ and $(V /\!/ G)^{\mathbb{G}_m\times\{1\}}$ of $V /\!/ G$ respectively. In particular, $V_i /\!/ G$ is smooth for $i = 1, 2$. Moreover,

$$\overline{p_1}(\overline{v}) = \lim_{t\to 0}(1, t)\cdot\overline{v}$$

and

$$\overline{p_2}(\overline{v}) = \lim_{s\to 0}(s, 1)\cdot\overline{v}$$

are T-equivariant linear projections $V /\!/ G \to V_1 /\!/ G$ and $V /\!/ G \to V_2 /\!/ G$ respectively and

$$\overline{p} = (\overline{p_1}, \overline{p_2})\colon V /\!/ G \to (V_1 /\!/ G)\times(V_2 /\!/ G)$$

is a smooth map.

Consider the commutative diagram

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, "\pi"] & V_1\times V_2 \arrow[ddl, "\pi_1\times\pi_2"] \\ V /\!/ G \arrow[d, "\overline{p}"] & \\ V_1 /\!/ G\times V_2 /\!/ G & \end{tikzcd}$$

Choose a stable point $v_1\in V_1$ such that $\pi_1(v_1)$ lies in the principal stratum of $V_1 /\!/ G$ (V_1 has a dense open subset consisting of such points; cf. Lemma 2.3) and let $x = (\pi_1(v_1), \pi_2(0)) \in V_1 /\!/ G\times V_2 /\!/ G$. The (scheme-theoretic) preimage of x under the map $\pi_1\times\pi_2$ is clearly $G\cdot v_1\times \mathrm{NC}(V_2)$, where $\mathrm{NC}(V_2)$ is the null cone in V_2. Since we are assuming that the G-action on V_2 is not fixed pointed, $\mathrm{NC}(V_2)$ is singular; cf. Lemma 4.3. Thus

$$(\pi_1\times\pi_2)^{-1}(x) \text{ is singular.}$$

On the other hand, as we saw above, the map $\overline{p}$ is smooth. The map π is smooth over the principal Luna stratum in $V /\!/ G$. By Corollary 2.5(b), every point in $\overline{p}^{-1}(x)$ lies in the principal stratum of $V /\!/ G$. Hence, the composition map $\pi_1\times\pi_2 = \overline{p}\pi\colon V\to V_1 /\!/ G\times V_2 /\!/ G$ is smooth over some Zariski open neighborhood of x. Consequently,

$$(\pi_1\times\pi_2)^{-1}(x) \text{ is smooth.}$$

This contradiction shows that $V /\!/ G$ cannot be smooth, thus completing the proof of Proposition 4.1. $\square$

Corollary 4.6. — *Suppose V_1, V_2 and V_3 are three linear representations of a reductive group G, where V_1 has trivial principal stabilizer, V_2 is not fixed pointed, and V_3 is arbitrary. Then $V_1\oplus V_2\oplus V_3$ is not coregular.*

Proof. — The G-representation $V_1' = V_1 \oplus V_3$ has trivial principal stabilizer; see Corollary 2.5(a). Now apply Proposition 4.1 to $V_1' \oplus V_2$. □

5. Proof of Theorem 1.2(a) and (b)

We will follow the strategy outlined in Remark 3.5 by exhibiting acceptable families Λ_a and Λ_b which include the representations in parts (a) and (b) of Theorem 1.2, respectively.

Elements of Λ_a are representations of the form $V = W^r$, where $r \geqslant 2\dim(W)$ and $G \to \mathrm{GL}(W)$ is a representation of a reductive group G.

Elements of Λ_b are representations of the form $V = W^r$, where $r \geqslant \dim(W) + 1$ and $G \to \mathrm{O}(W)$ is an orthogonal representation of a reductive group G. That is, G preserves some non-degenerate quadratic form on W.

In view of Proposition 3.3 it suffices to show that Λ_a and Λ_b are acceptable families. We begin by checking condition (i) of Definition 3.2.

(a) Suppose $V = W^r$ is in Λ_a. Then $V^H = (W^H)^r$ is again in Λ_a, because $r \geqslant 2\dim(W) \geqslant 2\dim(W^H)$.

(b) Suppose $V = W^r$, where W is an orthogonal representation of G and $r \geqslant \dim(W) + 1$. Once again, $V^H = (W^H)^r$, where $r \geqslant \dim(W) + 1 \geqslant \dim(W^H) + 1$. Moreover, in view of Lemma 3.8 the $N_G(H)$-representation on W^H is orthogonal. Thus V^H belongs to Λ_b, as claimed.

It remains to show that Λ_a and Λ_b satisfy condition (ii) of Definition 3.2. That is, given a representation $V = W^r$ in Λ_a or Λ_b, we want to show that $X = V /\!/ G$ is singular at every point x away from the principal stratum. We begin with two reductions.

First we claim that V may be assumed, without loss of generality, to have trivial principal stabilizer. Indeed, suppose the principal stabilizer in V is $H \subset G$. Set $N = N_G(H)$ and $\overline{N} = N/H$. Then by Corollary 3.4

$$\begin{tikzcd} V^H /\!/ \overline{N} \arrow[r, equal] & V^H /\!/ N \arrow[d, "n"] \\ & X \end{tikzcd}$$

is an isomorphism which takes the principal stratum in $V^H /\!/ N$ to the principal stratum in X. Thus it suffices to prove that $V^H /\!/ N$ is singular away from its principal stratum. As we just showed, the representation $N \to \mathrm{GL}(V^H)$ lies in Λ_a in part (a) and in Λ_b in part (b). Hence, so does $\overline{N} \to \mathrm{GL}(V^H)$. Since the latter representation has trivial principal stabilizer, this proves the claim.

From now on, we will assume that the principal stabilizer subgroup of G in $V = W^r$ (but not necessarily in W) is $\{e\}$. Suppose x is represented by an element $w = (w_1, \dots, w_r) \in W^r$ whose G-orbit is closed. Let $H = \mathrm{Stab}_G(w)$. Note that since x does not lie in the principal stratum in $V /\!/ G$, $H \neq \{e\}$. After permuting the components of W^r if necessary, we may assume that $w_{n+1}, \dots, w_r$ are linear combinations of $w_1, \dots, w_n$.

Recall that by the Luna Slice Theorem $(W^r /\!/ G, x)$ is étale isomorphic to

$$(\mathrm{Slice}(w, W^r) /\!/ H, \pi(0))\,;$$

cf., e.g., [19, Section 6]. Thus it suffices to prove that $\mathrm{Slice}(w, W^r) /\!/ H$ is singular, i.e., that the H-representation on $\mathrm{Slice}(w, W^r)$ is not coregular.

Let $v = (w_1, \dots, w_n) \in W^n$. By Lemma 2.8 $\mathrm{Stab}(v) = H$, $G \cdot v$ is closed in W^n, and

$$\mathrm{Slice}(w, W^r) \simeq \mathrm{Slice}(v, W^n) \oplus W^{r-n}\,.$$

Recall that we are assuming that the G-action (and hence, the H-action) on W^r has trivial principal stabilizer; cf. Corollary 2.5(a). Since $r \geqslant n$ (both in part (a) and in part (b)), Corollary 2.9(c) tells us that the H-action on W^n also has trivial principal stabilizer. By Lemma 2.2 this implies that the H-action on $\mathrm{Slice}(v, W^n)$ has trivial principal stabilizer as well.

We will now consider the families Λ_a and Λ_b separately.

(a) Suppose V is in Λ_a. Recall that we are assuming $r \geqslant 2n$, i.e. $r - n \geqslant n$. Thus by Corollary 2.9(c) W^{r-n} also has trivial principal stabilizer. Consequently, it is not fixed pointed; cf. Example 4.4. By Proposition 4.1 the H-representation

$$\mathrm{Slice}(w, W^r) \simeq \underbrace{\mathrm{Slice}(v, W^n)}_{\text{trivial principal stabilizer}} \oplus \underbrace{W^{r-n}}_{\text{not fixed pointed}}$$

is not coregular.

(b) Since G preserves the non-degenerate quadratic form $q \oplus \cdots \oplus q$ ($r - n$ times) on W^{r-n} and $r \geqslant n+1$, Example 4.5 tells us that the H-representation on W^{r-n} is not fixed pointed. Proposition 4.1 now tells us that the H-representation

$$\mathrm{Slice}(w, W^r) \simeq \underbrace{\mathrm{Slice}(v, W^n)}_{\text{trivial principal stabilizer}} \oplus \underbrace{W^{r-n}}_{\text{not fixed pointed}}$$

is not coregular. $\square$

6. Proof of Theorem 1.2(c)

Once again, we will follow the strategy outlined in Remark 3.5, by defining a suitable family Λ_c of representations of reductive groups and then checking that Λ_c is acceptable. In the context of Theorem 1.2(c) the natural candidate for Λ_c is the family of representations of the form W^r, where $W = \mathrm{Lie}(G)$ is the adjoint representation for some reductive group G. Unfortunately, this family is not acceptable, because it does not satisfy condition (i) of Definition 3.2. To make the argument go through, we need to consider a slightly larger family which we now proceed to define.

Definition 6.1. — *Let G be a reductive group. We will say that a linear representation $\rho\colon G \to \mathrm{GL}(W)$ is* almost adjoint *if $\mathrm{Ker}(\rho)$ contains a normal closed subgroup K of G such that W is isomorphic to the Lie algebra of G/K and ρ can be written as a composition*

$$\rho\colon G \to G/K \overset{\mathrm{Ad}_{G/K}}{\to} \mathrm{GL}(W)\,, \tag{6.1}$$

where $G \to G/K$ is the natural quotient map and $\mathrm{Ad}_{G/K}$ is the adjoint representation. Note that the groups G, K or G/K are assumed to be reductive but not necessarily connected.

We are now ready to define Λ_c. Fix an integer $r \geqslant 3$ and let Λ_c be the family of representations of the form W^r, where $G \to \mathrm{GL}(W)$ is almost adjoint. Following the strategy of Remark 3.5, in order to prove Theorem 1.2(c), it suffices to check that Λ_c is an acceptable family.

We begin by checking condition (i) of Definition 3.2. Suppose $\rho\colon G \to \mathrm{GL}(W)$ is an almost adjoint representation, with $K \lhd G$ as in (6.1) and $H \subset G$ is a stabilizer subgroup. Since $(W^r)^H = (W^H)^r$, it suffices to show that the natural representation of the normalizer $N = N_G(H)$ on W^H is again almost adjoint.

Note that H contains K; since $N/K = N_{G/K}(H/K)$, we may, after replacing G by G/K, assume without loss of generality that $K = \{e\}$, i.e., $W = \mathrm{Lie}(G)$ is the Lie algebra of G and ρ is the adjoint representation.

In this situation $W^H = \mathrm{Lie}(G)^H$ is a reductive Lie algebra. In fact, it is the Lie algebra of $Z = Z_G(H)$, the centralizer of H in G. Note that both N and Z are reductive; cf. [14, Lemma 1.1]. We claim that the natural representation

$$\rho\colon N \to \mathrm{GL}(\mathrm{Lie}(Z))$$

is almost adjoint. Indeed, let (H, H) be the commutator subgroup of H. Since H acts trivially on $\mathrm{Lie}(Z)$, $N^0 = Z^0H^0$ (cf. e.g., [14, p. 488]), and $(H, H) \cap Z$ is finite, we see that $\mathrm{Lie}(Z)$ is also the Lie algebra of $\overline{N} =$

$N/(H,H)$ and ρ descends to the adjoint action of $\overline{N}$ on its Lie algebra. This proves the claim. Condition (i) is now established.

To check condition (ii) of Definition 3.2, we will need the following lemma.

Lemma 6.2. — *Let G be a reductive (but not necessarily connected) group, with Lie algebra W and let $G \to \mathrm{GL}(W)$ be the adjoint representation. Then*

(a) *the G-action on W^s is stable for any $s \geqslant 1$,*

(b) *the action of $\rho(G) = G/Z_G(G^0)$ on W^s has trivial principal stabilizer for any $s \geqslant 2$.*

Parts (a) and (b) may be viewed as a variants of [26, Theorem 6.1] and [26, Theorem 4.1], respectively.

Proof. — (a) Since $R = \mathrm{Rad}(G)$ acts trivially on W, the G-action on W^s descends to an action of its semisimple quotient G/R. Thus, by a theorem of Popov [17] (cf. also [19, p. 236]) it suffices to show that $\mathrm{Stab}_{G/R}(w_1, \dots, w_s)$ is reductive for $(w_1, \dots, w_s) \in W^s$ in general position. Clearly $\mathrm{Stab}_{G^0/R}(w_1, \dots, w_s)$ is reductive if and only if $\mathrm{Stab}_{G^0}(w_1, \dots, w_s)$ is reductive. On the other hand, $\mathrm{Stab}_{G^0}(w_1, \dots, w_s)$ is contained in $\mathrm{Stab}_{G^0}(w_1)$, which is a maximal torus of G^0, assuming $w_1 \in W$ is in general position. Since any algebraic subgroup of a torus is reductive, this completes the proof of part (a).

(b) By part (a) the G-action on W^s is stable; hence, we only need to check that the $\rho(G)$-action on W^s is generically free. We may assume without loss of generality that $s = 2$.

Choose $w_1, w_2 \in W$ in general position and suppose $g \in \mathrm{Stab}_G(w_1, w_2)$. We want to show that $g \in Z_G(G^0)$ or equivalently that $\mathrm{Ad}_G(g) = \mathrm{id}$ on W.

Recall that a Lie subalgebra W' of W is called *algebraic* if W' is the Lie algebra of a closed subgroup G' of G. In particular, the subalgebra $W^{\langle g \rangle}$ of fixed points of $\mathrm{Ad}_G(g)$ in W is algebraic, because it is the Lie algebra of $Z_{G^0}(\langle g \rangle)$. On the other hand, by [26, Lemma 3.3(b)], the only algebraic Lie subalgebra containing $w_1, w_2 \in W$ is W itself (provided that $w_1, w_2 \in W$ are in general position). Thus $W^{\langle g \rangle} = W$. Equivalently, $\mathrm{Ad}_G(g) = \mathrm{id}$ on W, and part (b) follows. □

We are now ready to prove that the family of representations Λ_c defined at the beginning of this subsection, satisfies condition (ii) of Definition 3.2. More precisely, we want to show that if $\rho \colon G \to \mathrm{GL}(W)$ is an almost adjoint representation and $r \geqslant 3$ then $W^r /\!/ G$ is singular away from its principal stratum.

Suppose $K \triangleleft G$ is as in Definition 6.1. After replacing G by G/K (this doesn't change the quotient $W^r /\!/ G$ or the Luna strata in it), we may assume that $W = \mathrm{Lie}(G)$ and ρ is the adjoint representation. It now suffices to prove the following lemma.

Lemma 6.3. — *Let G be a reductive (but not necessarily connected) group, W be the Lie algebra of G and $G \to \mathrm{GL}(W)$ be the adjoint representation. Then $W^r /\!/ G$ is singular away from its principal stratum for any $r \geqslant 3$.*

Proof. — Let π be the quotient map $W^r \to W^r /\!/ G$, $x \in W^r /\!/ G$ be a point away from the principal stratum and $v \in W^r$ be a point with closed G-orbit, such that $x = \pi(v)$. Our goal is to show that $W^r /\!/ G$ is not smooth at x. As in the proof of Theorem 1.2(a) and (b) in the previous section, we shall do this by showing that the slice representation of $H = \mathrm{Stab}_G(v)$ on $\mathrm{Slice}(v, W^r)$ is not coregular. Our strategy will be to express this representation as a direct sum of three H-representations,

$$\mathrm{Slice}(v, W^r) = V_1 \oplus V_2 \oplus V_3\,,$$

where V_1 has trivial principal stabilizer and V_2 is not fixed pointed, then appeal to Corollary 4.6.

Since we are assuming that x does not lie in the principal stratum, $H = \mathrm{Stab}_G(v) \neq \{e\}$. As an H-representation, the tangent space $T_v(G \cdot v)$ is isomorphic to $W/\mathrm{Lie}(H)$ (recall that here $W = \mathrm{Lie}(G)$). Thus the complement $\mathrm{Slice}(v, W^r)$ to $T_v(G \cdot v)$ in W^r can be written as

$$W^{r-1} \oplus \mathrm{Lie}(H) \oplus S$$

for some linear representation S of H. By Lemma 6.2(b), the principal stabilizer for the $\rho(G)$-action on $W \oplus W$ is trivial. Hence, the same is true of the H-action on $W \oplus W$, since $\rho(H)$ is a reductive subgroup of $\rho(G)$; see Corollary 2.5(a). We will now consider two cases.

Case 1. H acts non-trivially on $\mathrm{Lie}(H)$. By Lemma 6.2(a), the H-action on $\mathrm{Lie}(H)$ is stable. Since we are assuming that this action is non-trivial, it is not fixed pointed; see Example 4.4. By Corollary 4.6, the H-representation

$$\mathrm{Slice}(v, W^r) = \underbrace{W^{r-1}}_{\text{trivial principal stabilizer}} \oplus \underbrace{\mathrm{Lie}(H)}_{\text{not fixed pointed}} \oplus\ S$$

is not coregular, as desired. (Recall that we are assuming throughout that $r \geqslant 3$.)

Case 2. H acts trivially on $\mathrm{Lie}(H)$. Since H is reductive this is only possible if H^0 is a central torus in H. By Lemma 6.2(b) the G-action (and hence, the H-action) on W^2 has trivial principal stabilizer. Lemma 2.6 now tells us that the H-action on W also has trivial principal stabilizer. Since $H \neq \{e\}$, no such action can be fixed pointed. Thus by Corollary 4.6

$$\mathrm{Slice}(v, W^r) = \underbrace{W}_{\text{trivial principal stabilizer}} \oplus \underbrace{W}_{\text{not fixed pointed}} \oplus (W^{r-3} \oplus \mathrm{Lie}(H) \oplus S)$$

is not coregular. □

The above lemma shows that the family Λ_c satisfies condition (ii) of Definition 3.2. Thus Λ_c is an acceptable family. The proof of Theorem 1.2(c) is now complete. □

Remark 6.4. — Lemma 6.3 may be viewed as a variant of a result of Richardson [26, Theorem 8.1], which asserts that $\mathrm{Lie}(G)^r /\!/ G$ is singular away from its principal stratum for any $r \geqslant 2$, if G is connected, semisimple and has no factors of rank 1. These assumptions cannot be made in our setting, because even if we make them for G, they may not remain valid for the quotient of $N_G(H)$ which comes up at the next step in the induction process. For this reason we were not able to appeal to [26, Theorem 8.1] directly. However, our proof of Lemma 6.3 is very much in the same spirit.

7. Representation types

Consider the action of the general linear group GL_n on the variety

$$V_{l,n,r} = \mathbb{A}^{lr} \times \mathrm{M}_n^r$$

given by

$$g \cdot (a_1, \ldots, a_l, A_1, \ldots, A_r) \mapsto (a_1, \ldots, a_l, gA_1g^{-1}, \ldots, gA_rg^{-1})$$

for any $a_1, \ldots, a_l \in k$ and any $A_1, \ldots, A_r \in \mathrm{M}_n$. Let $X_{l,n,r}$ be the quotient variety $V_{l,n,r} /\!/ \mathrm{GL}_n$. Our goal in the next two sections will be to prove Theorem 1.3 in the following slightly more general from.

Theorem 7.1. — *Suppose $r \geqslant 3$. Then every Luna stratum in $X_{l,n,r} = V_{l,n,r} /\!/ \mathrm{GL}_n$ is intrinsic.*

Of course, the case where $l = 0$ is of greatest interest to us; in this case Theorem 7.1 reduces to Theorem 1.3. For $l \geqslant 1$ the variety $X_{l,n,r}$ is only marginally more complicated than $X_{0,n,r}$. Indeed, since GL_n acts trivially on $\mathbb{A}^{lr}$, $X_{l,n,r}$ is isomorphic to $\mathbb{A}^{lr} \times X_{0,n,r}$, and every Luna stratum in $X_{l,n,r}$ is of the form

$$S = \mathbb{A}^{lr} \times S_0 \,, \tag{7.1}$$

where S_0 is a Luna stratum in $X_{0,n,r}$. We allow $l \geqslant 1$ in the statement of Theorem 7.1 to facilitate the induction step (on n) in the proof.

The Luna stratification in $X_{l,n,r}$, has a natural combinatorial interpretation, which we will now recall, in preparation for the proof of Theorem 7.1 in the next section. Let $v = (a_1, \dots, a_l, A_1, \dots, A_r) \in V_{l,n,r}$ be a point in the unique closed GL_n-orbit in the fiber over $x \in X_{l,n,r}$. Here each $a_i \in k$ and each $A_j \in \mathrm{M}_r$. We will view an r-tuple $(A_1, \dots, A_r) \in \mathrm{M}_n^r$ of $n \times n$-matrices as an n-dimensional representation

$$\rho\colon k\{x_1, \dots, x_r\} \to \mathrm{M}_n$$

of the free associative algebra $k\{x_1, \dots, x_r\}$ on r generators, sending x_i to A_i. By a theorem of Artin [1, (12.6)], the orbit of v is closed in M_n^r if and only if ρ is semisimple. (Strictly speaking, Artin's theorem only covers the case where $l = 0$; but since $V_{l,n,r} = \mathbb{A}^{lr} \times V_{0,n,r}$ and GL_n acts trivially on $\mathbb{A}^{lr}$, the general case is an immediate consequence.) If ρ can be written as $\rho_1^{e_1} \oplus \dots \oplus \rho_s^{e_s}$, where

$$\rho_i\colon k\{x_1, \dots, x_r\} \to \mathrm{M}_{d_i}$$

is an irreducible d_i-dimensional representation, we will say that the *representation type* of x is

$$\tau = [(d_1, e_1), \dots, (d_r, e_r)] \,. \tag{7.2}$$

The square brackets [ ] are meant to indicate that τ is an *unordered* collection of pairs (d_i, e_i); permuting these pairs does not change the representation type. Note also that $d_i, e_i \geqslant 1$ for every $i = 1, \dots, s$. Following Le Bruyn and Procesi [10], we shall denote the set of representation types (7.2) with $d_1e_1 + \dots + d_se_s = n$ by RT_n. If $\tau = [(d_1, e_1), \dots, (d_s, e_s)] \in \mathrm{RT}_n$ and $\mu = [(d_1', e_1'), \dots, (d_{s'}', e_{s'}')] \in \mathrm{RT}_{n'}$ then we will sometimes denote the representation type

$$[(d_1, e_1), \dots, (d_s, e_s), (d_1', e_1'), \dots, (d_{s'}', e_{s'}')] \in \mathrm{RT}_{n+n'}$$

by $[\tau, \mu]$.

The Luna strata in $X_{l,n,r} = (\mathbb{A}^{lr} \times \mathrm{M}_n^r) /\!/ \mathrm{GL}_n$ are in a 1-1 correspondence with RT_n; cf., e.g., [10, Section 2]. If $x \in X_{l,n,r}$ has representation type τ, as in (7.2), then the associated stabilizer subgroup

$$H_\tau = \mathrm{Stab}_G(v) \simeq \mathrm{GL}_{e_1} \times \dots \times \mathrm{GL}_{e_s} \,, \tag{7.3}$$

embedded into GL_n as follows. Write

$$k^n = V_1 \otimes W_1 \oplus \cdots \oplus V_s \otimes W_s \,,$$

where $\dim(V_i) = d_i$ and $\dim(W_i) = e_i$ and let GL_{e_i} act on W_i. Then $H_\tau = \mathrm{GL}_{e_1} \times \cdots \times \mathrm{GL}_{e_s}$ is embedded in GL_n via

$$(g_1, \ldots, g_s) \mapsto (I_{d_1} \otimes g_1 \oplus \cdots \oplus I_{d_s} \otimes g_s) \,.$$

where I_d denotes the $d \times d$ identity matrix; cf. [10, Section 2]. For notational convenience we shall denote the Luna strata in $X_{l,n,r} = \mathrm{M}_n^r /\!/ \mathrm{GL}_n$ by $X_{n,r}^\tau$, rather than $X_{n,r}^{(H_\tau)}$. Note that if $\tau = [(d_1, e_1), \ldots, (d_s, e_s)] \in \mathrm{RT}_n$ then

$$\dim X_{l,n,r}^\tau = (r-1)(d_1^2 + \cdots + d_s^2) + s + lr \tag{7.4}$$

for any $r \geqslant 2$; cf. [10, p. 158].

Definition 7.2. — *An* elementary refinement *of*

$$\tau = [(d_1, e_1), \ldots, (d_s, e_s)] \in \mathrm{RT}_n$$

consists in either

(1) *replacing one of the pairs (d_i, e_i) by two pairs, (a_i, e_i) and (b_i, e_i), where $a_i, b_i \geqslant 1$ and $a_i + b_i = d_i$*

or

(2) *replacing two pairs (d_i, e_i) and (d_j, e_j), with $d_i = d_j$, by the single pair $(d_i, e_i + e_j)$.*

Given two representation types τ and τ', we will say that $\tau' \prec \tau$ if τ' can be obtained from τ by a sequence of elementary refinements. This defines a partial order $\preceq$ on RT_n.

Note that while operations (1) and (2) are defined in purely combinatorial terms, they are, informally speaking, designed to reflect the two ways a representation

$$\rho = \rho_1^{e_1} \oplus \cdots \oplus \rho_s^{e_s} : k\{x_1, \ldots, x_r\} \to \mathrm{M}_n$$

can "degenerate". Here $\rho_1, \ldots, \rho_s$ are distinct irreducible representations of dimensions $d_1, \ldots, d_s$ respectively. In case (1), one of the representations ρ_i "degenerates" into a direct sum of irreducible subrepresentations of degree a_i and b_i (each with multiplicity e_i). In case (2) we "degenerate" ρ by making ρ_i and ρ_j isomorphic (of course, this is only possible if their dimensions d_i and d_j are the same). The following lemma gives this a precise meaning.

Lemma 7.3. — *$X_{l,n,r}^\mu$ lies in the closure of $X_{l,n,r}^\tau$ if and only if $\mu \preceq \tau$.*

Proof. — In view of (7.1), we may assume $l = 0$. In this case Lemma 7.3 is proved in [10, Theorem II.1.1]. $\square$

Remark 7.4. — Note that $V_{l,n,r} = W^r$, where $W = \mathbb{A}^l \times \mathrm{M}_n$ is the Lie algebra of $G = (\mathrm{GL}_1)^l \times \mathrm{GL}_n$. The $(\mathrm{GL}_1)^l$-factor acts trivially on W (via the adjoint action), so we may drop it without changing the quotient $W^r /\!/ G$ or the Luna strata in it. Theorem 1.2(c) now tells us that the Luna stratification in $X_{l,n,r} = V_{l,n,r} /\!/ \mathrm{GL}_n = W^r /\!/ G$ is intrinsic (provided $r \geqslant 3$). Thus an automorphism σ of $X_{l,n,r}$ induces an automorphism σ^* of the set RT_n of Luna strata in $X_{l,n,r}$ given by $\sigma^*(\tau) = \nu$ if $\sigma(X^\tau) = X^\nu$. By Lemma 7.3 σ^* respects the partial order $\preceq$ on RT_n. Theorem 7.1 asserts that σ^* is always trivial. In some cases this can be deduced from the fact that the partially ordered set $(\mathrm{RT}_n, \preceq)$ has no non-trivial automorphisms. For example, the partially ordered sets RT_n, for $n = 1$, 2 and 3, pictured below have no non-trivial automorphisms.

$$\mathrm{RT}_1 \qquad\qquad \mathrm{RT}_2 \qquad\qquad \mathrm{RT}_3$$

$$\begin{tikzcd}[row sep=small]
{[(1,1)]} & {[(2,1)]} \arrow[d, dash] & {[(3,1)]} \arrow[d, dash] \\
 & {[(1,1),(1,1)]} \arrow[d, dash] & {[(2,1),(1,1)]} \arrow[d, dash] \\
 & {[(1,2)]} & {[(1,1),(1,1),(1,1)]} \arrow[d, dash] \\
 & & {[(1,2),(2,1)]} \arrow[d, dash] \\
 & & {[(1,3)]}
\end{tikzcd}$$

This proves Theorem 7.1 for $n \leqslant 3$.

Note however, that for larger n the partially ordered set $(\mathrm{RT}_n, \preceq)$ does have non-trivial automorphisms. For example, a quick look at RT_4 (pictured on p. 156 in [10]), shows that the permutation α of RT_4 interchanging

$$\tau = [(1,1),(1,1),(1,1),(1,1)] \text{ and } \nu = [(2,1),(1,2)]$$

and fixing every other element, does, indeed, respect the partial order on RT_4. For this reason we cannot hope to prove Theorem 7.1 by purely combinatorial arguments, without taking into account the geometry of the strata $X^{\tau}_{l,n,r}$. Nevertheless, the following combinatorial proposition will play a key role in the proof of Theorem 7.1 in the next section.

Proposition 7.5. — (a) *Let α be an automorphism of RT_n (as a partially ordered set). If $\alpha([(1,1),\mu]) = [(1,1),\mu]$ for every $\mu \in \mathrm{RT}_{n-1}$ then $\alpha = \mathrm{id}$.*

(b) *Let $l \geqslant 0$, $n \geqslant 1$ and $r \geqslant 3$. Suppose we know that Luna strata of the form $X^{[(1,1),\mu]}_{l,n,r}$ are intrinsic in $X_{l,n,r}$ for any $\mu \in \mathrm{RT}_{n-1}$. Then every Luna stratum in $X_{l,n,r}$ is intrinsic.*

Proof. — Part (b) is an immediate consequence of part (a) and Remark 7.4; we shall thus concentrate on proving part (a). Given a representation type $\tau = [(d_1, e_1), \ldots, (d_s, e_s)]$, let $m(\tau)$ denote the minimal value of $d_i + e_i$, as i ranges from 1 to s. Note that since $d_i, e_i \geqslant 1$ for each i, we have $m(\tau) \geqslant 2$. We will now show that $\alpha(\tau) = \tau$ by induction on $m(\tau)$. By our assumption this is the case if $m(\tau) = 2$, since in this case $d_i = e_i = 1$ for some i.

For the induction step, assume that $m(\tau) = m \geqslant 3$ and $\alpha(\nu) = \nu$ for every $\nu \in \mathrm{RT}_n$ with $m(\nu) < m$. After renumbering the pairs (d_i, e_i), we may assume that $d_1 + e_1 = m$. Suppose

$$\alpha(\tau) = \tau' = [(d'_1, e'_1), \ldots, (d'_t, e'_t)]\,. \tag{7.5}$$

Our goal is to show that $\tau = \tau'$. We will consider two cases, where $d_1 \geqslant 2$ and $e_1 \geqslant 2$ separately. Since we are assuming that $d_1 + e_1 = m \geqslant 3$, these two cases cover every possibility.

Case 1: $d_1 \geqslant 2$. Let

$$\tau_0 = [(1, e_1), (d_1 - 1, e_1), (d_2, e_2), \ldots, (d_s, e_s)]\,.$$

Since $m(\tau_0) = e_1 + 1 < d_1 + e_1 = m$, the induction assumption tells us that $\alpha(\tau_0) = \tau_0$. Now observe that τ_0 immediately precedes τ in the partial order on RT_n, i.e., τ_0 is obtained from τ by a single elementary refinement; see Definition 7.2. Consequently, τ_0 can also be obtained from τ' by a single elementary refinement. Schematically,

$$\begin{tikzcd}
\tau \arrow[rr, bend left, "\alpha"] \arrow[dr, dashed] & & \tau' \arrow[dl, dashed] \\
& \tau_0 \,, &
\end{tikzcd}$$

where the broken arrows denote elementary refinements. Now observe that

$$(d'_i, e'_i) \neq (1, e_1) \quad \text{or} \quad (d_1 - 1, e_1) \quad \forall \quad i = 1, \ldots, s\,.$$

Indeed, otherwise we would have $m(\tau') < m$ and thus $\alpha(\tau') = \tau'$ by the induction assumption. Combining this with (7.5), we obtain $\alpha(\tau) = \tau' = \alpha(\tau')$. Since α is a permutation of RT_n, we conclude that $\tau = \tau'$, which is impossible, since $m(\tau') < m = m(\tau)$.

To sum up, τ_0 "contains" two pairs, $(1, e_1)$ and $(d_1 - 1, e_1)$, that are not "present" in τ'. It now follows from Definition 7.2 that the only possible elementary refinement taking τ' to τ_0 is of type (1), consisting of "splitting up" (d_1, e_1) into $(d_1 - 1, e_1)$ and $(1, e_1)$. That is, $\tau' = [(d_1, e_1), \mu] = \tau$, as claimed.

Case 2: $e_1 \geqslant 2$. The argument here is very similar (or more precisely, "dual"; in the sense of Remark 7.6) to the argument in Case 1. Let

$$\tau_1 = [(d_1, 1), (d_1, e_1 - 1), (d_2, e_2), \ldots, (d_s, e_s)]\,.$$

Since $m(\tau_1) = d_1 + 1 < d_1 + e_1 = m$, the induction assumption tells us that $\alpha(\tau_1) = \tau_1$. The relationship between τ, τ' and τ_1 is shown in the following diagram

$$\begin{tikzcd}
& \tau_1 \arrow[dl, dashed] \arrow[dr, dashed] & \\
\tau \arrow[rr, bend right, "\alpha"'] & & \tau' \,,
\end{tikzcd}$$

where the broken arrows denote elementary refinements. Once again, we see that τ_1 "contains" two pairs, $(d_1, 1)$ and $(d_1, e_1 - 1)$ both of which "disappear" after we perform an elementary refinement (and obtain τ'). This is only possible if the elementary refinement taking τ_1 to τ' is of type (2) (cf. Definition 7.2) and consists of replacing $(d_1, 1)$ and $(d_1, e_1 - 1)$ by (d_1, e_1). This shows that $\tau = \tau'$, thus completing the proof of Proposition 7.5. ☐

Remark 7.6. — The two elementary refinement operations of Definition 7.2 are dual to each other in the following sense. Given a representation type

$$\tau = [(d_1, e_1), \ldots, (d_s, e_s)]$$

let $\overline{\tau} = [(e_1, d_1), \ldots, (e_s, d_s)]$. Then α is obtained from β by an elementary refinement of type (1) (respectively, of type (2)) if and only if $\overline{\beta}$ is obtained from $\overline{\alpha}$ by an elementary refinement of type (2) (respectively, of type (1)). Consequently, the map $\tau \mapsto \overline{\tau}$ is an isomorphism between the partially ordered sets $(\mathrm{RT}_n, \succeq)$ and $(\mathrm{RT}_n, \preceq)$. The statement and the proof of Proposition 7.5 are invariant with respect to this map (in particular, Case 2 is dual to Case 1).

8. Proof of Theorem 1.3

In this section we will prove Theorem 7.1. This will immediately yield Theorem 1.3 (for $l = 0$). We will continue to use the notations introduced in the previous section; in particular, $V_{l,n,r}$ stands for $\mathbb{A}^{lr} \times \mathrm{M}_n$ and $X_{l,n,r}$ denotes the categorical quotient $V_{l,n,r} /\!/ \mathrm{GL}_n$.

We will argue by induction on n. The base cases, $n = 1$ and 2, are proved in Remark 7.4. (Theorem 7.1 is also proved there for $n = 3$ but we shall not need that here.) For the induction step assume $n \geqslant 3$ and σ is an automorphism of $X_{l,n,r}$. Recall that σ maps each stratum in $X_{l,n,r}$ to another stratum; cf. Remark 7.4. In particular, σ preserves the maximal (principal) stratum $X_{l,n,r}^{[(n,1)]}$ (which is the unique stratum of maximal dimension) and permutes the "submaximal" strata $X_{l,n,r}^{[(d,1),(n-d,1)]}$, $1 \leqslant d \leqslant \frac{n}{2}$, among themselves. By the dimension formula (7.4),

$$\begin{aligned} \dim X^{[(d,1),(n-d,1)]} &= rl + 2 + (r-1)(d^2 + (n-d)^2) \\ &= rl + 2 + 2(r-1)((d - \frac{n}{2})^2 + \frac{n^2}{4}) \,. \end{aligned}$$

Thus the submaximal strata $X_{l,n,r}^{[(d,1),(n-d,1)]}$ have different dimensions for different values of d between 1 and $\frac{n}{2}$. Hence, σ preserves each one of them.

Of particular interest to us is the submaximal stratum $X_{l,n,r}^{[(1,1),(n-1,1)]}$. Since σ preserves this stratum, it preserves its closure $\overline{X_{l,n,r}^{[(1,1),(n-1,1)]}}$ and thus lifts to an automorphism $\tilde{\sigma}$ of the normalization of $\overline{X_{l,n,r}^{[(1,1),(n-1,1)]}}$. The rest of the argument will proceed as follows. We will identify the normalization of $\overline{X_{l,n,r}^{[(1,1),(n-1,1)]}}$ with $X_{l+1,n-1,r}$ and relate Luna strata in $X_{l+1,n-1,r}$

and $X_{l,n,r}$ via the normalization map. By the induction assumption $\tilde{\sigma}$ preserves every Luna stratum in $X_{l+1,n-1,r}$; using the normalization map we will be able to conclude that σ preserves certain Luna strata in $X_{l,n,r}$. Proposition 7.5 will then tell us that, in fact, σ preserves every Luna stratum in $X_{l,n,r}$, thus completing the proof.

We now proceed to fill in the details of this outline. First we will explicitly describe the normalization map for $\overline{X_{l,n,r}^{[(1,1),(n-1,1)]}}$. The stabilizer H_τ corresponding to $\tau = [(1,1),(n-1,1)]$ consists of diagonal $n \times n$ matrices of the form

$$\mathrm{diag}(a, \underbrace{b, \dots, b}_{n-1 \text{ times}}),$$

where $a, b \in k^*$; cf. (7.3). The natural projection

$$V_{l,n,r}^{H_\tau} /\!/ N_G(H_\tau) \to \overline{X_{l,n,r}^{\tau}}$$

is the normalization map for $\overline{X_{l,n,r}^{\tau}}$; cf. [19, Theorem 6.16]. Here

$$V_{l,n,r}^{H_\tau} = \mathbb{A}^{lr} \times \mathrm{M}_1^r \times \mathrm{M}_{n-1}^r = \mathbb{A}^{(l+1)r} \times \mathrm{M}_{n-1}^r = V_{l+1,n-1,r}$$

and

$$N_G(H_\tau) = \mathrm{GL}_1 \times \mathrm{GL}_{n-1}\,, \tag{8.1}$$

where GL_1 acts trivially on $\mathbb{A}^{(l+1)r} \times \mathrm{M}_{n-1}^r$ and GL_{n-1} acts on the second factor by simultaneous conjugation. Since GL_1 acts trivially, we may replace $\mathrm{GL}_1 \times \mathrm{GL}_{n-1}$ by GL_{n-1} without changing the categorical quotient or the Luna strata in it. That is, the normalization $V_{l,n,r}^{H_\tau} /\!/ N_G(H_\tau)$ of $\overline{X_{l,n,r}^{[(1,1),(n-1,1)]}}$ is (canonically) isomorphic to $V_{l+1,n-1,r} /\!/ \mathrm{GL}_{n-1} = X_{l+1,n-1,r}$.

The following lemma gives a summary of this construction. Here, as before, we identify an r-tuple $A = (A_1, \dots, A_r)$ of $n \times n$-matrices with the n-dimensional representation $\rho_A \colon k\{x_1, \dots, x_r\} \to \mathrm{M}_n$ of the free associative k-algebra $k\{x_1, \dots, x_n\}$, taking x_i to A_i.

Lemma 8.1. — *Suppose $n \geqslant 3$ and $r \geqslant 2$ and let $f \colon V_{l+1,n-1,r} \to V_{l,n,r}$ be the morphism given by*

$$f \colon (t_1, \dots, t_{(l+1)r}, \rho) \mapsto (t_1, \dots, t_{lr}, \rho_t \oplus \rho))\,,$$

where $t = (t_{lr+1}, t_{lr+2}, \dots, t_{(l+1)r}) \in \mathrm{M}_1^r$. Then

(a) *f descends to the normalization map*

$$\overline{f} \colon X_{l+1,n-1,r} \to \overline{X_{l,n,r}^{[(1,1),(n-1,1)]}}\,,$$

where $X_{l,n,r} = V_{l,n,r} /\!/ \mathrm{GL}_n$.

(b) $\overline{f}$ *maps* $\overline{X^{\mu}_{l+1,n-1,r}}$ *onto* $\overline{X^{[(1,1),\mu]}_{l,n,r}}$ *for every* $\mu \in \mathrm{RT}_{n-1}$.

Proof. — Part (a) follows from the discussion before the statement of the lemma. To prove part (b), observe that every semisimple representation $k\{x_1, \dots, x_r\} \to \mathrm{M}_n$ of type $[(1,1),\mu]$ can be written in the form $\rho_0 \oplus \rho$, where ρ is an $n-1$-dimensional representation of type μ and ρ_0 is a 1-dimensional representation (of type $(1,1)$). This shows that the image of $X^{\mu}_{l+1,n-1,r}$ contains $X^{[(1,1),\mu]}_{l,n,r}$. Since

$$\dim X^{\mu}_{l+1,n-1,r} = \dim X^{[(1,1),\mu]}_{l,n,r} ,$$

see (7.4), and $\overline{f}$ is a finite map (in particular, $\overline{f}$ takes closed sets to closed sets), part (b) follows. $\square$

Remark 8.2. — Lemma 8.1 uses, in a crucial way, the assumption that $n \geqslant 3$. If $n = 2$ then (8.1) fails; instead we have $N_G(H_\tau) = (\mathrm{GL}_1 \times \mathrm{GL}_1) \rtimes \mathrm{S}_2$, and the entire argument falls apart.

We are now ready to finish the proof of Theorem 7.1. Restricting σ to the closure of the stratum $X^{[(1,1),(n-1,1)]}_{l,n,r}$ and lifting it to an automorphism $\tilde{\sigma}$ of the normalization, we obtain the following commutative diagram:

$$\begin{tikzcd}
X_{l+1,n-1,r} \arrow[r, "\tilde{\sigma}"] \arrow[d, "\overline{f}"'] & X_{l+1,n-1,r} \arrow[d, "\overline{f}"] \\
\overline{X^{[(1,1),(n-1,1)]}_{l,n,r}} \arrow[r, "\sigma"] & \overline{X^{[(1,1),(n-1,1)]}_{l,n,r}} .
\end{tikzcd}$$

By our induction assumption $\tilde{\sigma}$ preserves every Luna stratum $X^{\mu}_{l+1,n-1,r}$ in $X_{l+1,n-1,r}$. Hence, by Lemma 8.1(b), σ preserves the closure of every Luna stratum in $X_{l,n,r}$ of the form $X^{[(1,1),\mu]}_{l,n,r}$. Since $X^{[(1,1),\mu]}_{l,n,r}$ is the unique stratum of maximal dimension in its closure, we conclude that σ preserves $X^{[(1,1),\mu]}_{l,n,r}$ for every $\mu \in \mathrm{RT}_{n-1}$. By Propostion 7.5(b), we conclude that σ preserves every Luna stratum in $X_{l,n,r}$. This concludes the proof of Theorem 7.1 and thus of Theorem 1.3. $\square$

Remark 8.3. — Theorem 1.3 fails if (a) $r = 1$ or (b) $(n,r) = (2,2)$, because in this case the ring of invariants $R = k[\mathrm{M}^r_n]^{\mathrm{GL}_n}$ is a polynomial ring or equivalently, $X = \mathrm{M}^r_n /\!/ \mathrm{GL}_n$ is an affine space. In case (a) R is freely generated by the coefficients of the characteristic polynomial of $A \in \mathrm{M}_n$, viewed as GL_n-invariant polynomials $\mathrm{M}^1_n \to k$ and in case (b) by the five GL_2-invariants $\mathrm{M}^2_n \to k$ given by $(A_1, A_2) \mapsto \mathrm{tr}(A_1)$, $\mathrm{tr}(A_2)$, $\det(A_1)$, $\det(A_2)$, and $\det(A_1 + A_2)$, respectively; cf. [7, VIII, Section 136].

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