ANNALES

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ON DIRICHLET SERIES AND PETERSSON
PRODUCTS FOR SIEGEL MODULAR FORMS

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ABSTRACT. — We prove that the Dirichlet series of Rankin–Selberg type associated with any pair of (not necessarily cuspidal) Siegel modular forms of degree n and weight $k \geqslant n/2$ has meromorphic continuation to $\mathbb{C}$. Moreover, we show that the Petersson product of any pair of square–integrable modular forms of weight $k \geqslant n/2$ may be expressed in terms of the residue at $s = k$ of the associated Dirichlet series.

RÉSUMÉ. — On démontre que la série de Dirichlet à la Rankin-Selberg associée à toute paire de formes modulaires de Siegel (non nécessairement paraboliques) de degré n et poids $k \geqslant n/2$ admet un prolongement méromorphe à $\mathbb{C}$. En outre, on montre que le produit de Petersson de toute paire de formes modulaires de carré-intégrable et de poids $k \geqslant n/2$ a une expression en termes du résidu en $s = k$ de la série de Dirichlet associée. Ces résultats sont bien connus pour les formes paraboliques. La méthode que nous adoptons généralise celle qui a été introduite par Maass (dans le cas $n = 2$) et se base sur l'utilisation de certains opérateurs différentiels invariants.

1. Introduction

The main purpose of the present paper is to consider Petersson products of not–necessarily cuspidal Siegel modular forms and to extend to them some properties which are well–known in the cuspidal case.

In 1983, Kalinin studied the analytic properties of the Rankin convolution of Siegel modular forms of degree n. He showed that given a pair F, G of Siegel modular forms of which at least one is cuspidal, their Rankin convolution $\mathcal{R}(F, G; s)$ has a meromorphic continuation to the whole complex plane. Moreover, he proved what we shall call the Petersson identity, i.e. an

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identity (up to a suitable constant) between the relevant Petersson product and the residue of the Rankin convolution at $s = k$, k being the weight of F and G. Kalinin's results generalize previous works of Rankin [25], Petersson [24] and Maaß [21] (and many others). In fact, they basically follow from an application of the Rankin–Selberg unfolding method.

The cuspidality condition seems to be essential for the Rankin–Selberg method: it is needed to ensure the convergence of the integral which represents the Rankin convolution.

We shall however extend the Rankin–Selberg method to not necessarily cuspidal Siegel modular forms by generalizing a technique introduced by Maaß [21] for degree 2 modular forms (much later a variant was also given by Mizuno [23] for degree 1 and attributed by him to Kudla). By means of suitable $\mathrm{Sp}(n, \mathbb{R})$–invariant differential operators we shall construct certain rapidly decreasing functions whose convolution with the convenient weight 0 Eisenstein series will provide us with an integral representation of the Rankin convolution. As a corollary we get the meromorphic continuation of the Rankin convolution.

Whereas Maaß was able in [21] to carry out explicitly the calculus of differential operators in degree 2, we have to adopt a more abstract strategy developed by Deitmar and Krieg. In fact, the paper [6] implicitly provides almost all the ingredients necessary to handle the differential operators for arbitrary degrees.

Unlike the cuspidal case, the Petersson identity does not follow immediately from the integral representation. Some results due to Shimura [30] on the symmetry of certain "generalized Laplacians" (which generate the algebra of $\mathrm{Sp}(n, \mathbb{R})$–invariant differential operators) will play a crucial role for the proof of this point.

It should be noted that in the case of degree (and level) 1 our results are contained in those proved by Zagier in [33] by using a suitably "renormalized" Rankin–Selberg integral for a certain class of automorphic forms (not of rapid decay). Extending Zagier's method to groups of higher rank would be extremely interesting, but it seems however a quite difficult task (see e.g. [19]).

Let us now outline the content of the paper. In Section 2 we recall some basic facts and useful results. In particular we state explicitly as Proposition 2.1 the result of Deitmar and Krieg we shall use. In Section 3 we prove the meromorphic continuation of the Rankin convolution (Theorem 3.1). We treat in Proposition 3.2 the case of forms of different weights. Section 4 is then devoted to the proof of the Petersson identity (Theorem 4.4). We

collect in Section 5 some simple consequences of our results and a few final remarks. For instance, we obtain an orthogonality property for "almost singular" theta series associated with not rationally equivalent quadratic forms. Some features of the bilinear form defined as the residue at $s = k$ of the Rankin convolution – which formally extends the Petersson product – are also discussed.

It should be possible to use the procedure we propose here to deal with similar situations. At least, this seems to be the case for the Dirichlet series introduced by Kohnen, Skoruppa and Yamazaki ([17], [32]). We plan to take up this point again in a future work.

Notation. —   If M is a square matrix we write $|M|$ and $\sigma(M)$ for the determinant and the trace of M, respectively. For $z \in \mathbb{C}$, we abbreviate $e(z) = \exp(\pi i z)$. We shall denote by

$$\mathbb{H}_n := \{Z = X + iY \in Mat(n, \mathbb{C}) \mid Z = {}^tZ, Y > 0\}$$

the Siegel upper half–space of degree n.

2. Background

Let m, n, q be positive integers. We consider the space $\mathrm{M}(\Gamma_0^n[q], k, \chi)$ of Siegel modular forms of degree n and weight $k = m/2$ for the Hecke subgroup of level q

$$\Gamma_0^n[q] = \{M = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in \mathrm{Sp}(n, \mathbb{Z}) \mid C \equiv 0 \bmod q\},$$

with respect to a Dirichlet character χ modulo q.
By definition, the *Petersson product* of a pair of modular forms $F, G \in \mathrm{M}(\Gamma_0^n[q], k, \chi)$ is

$$(2.1) \qquad \langle F, G \rangle := \frac{1}{[\mathrm{Sp}(n, \mathbb{Z}) : \Gamma_0^n[q]]} \int_{\mathcal{F}_{n,q}} F(Z)\overline{G(Z)}|Y|^k \mathrm{d}^*Z \ ,$$

whenever the integral converges. In the integral in (2.1) $\mathcal{F}_{n,q}$ is a fundamental domain for $\Gamma_0^n[q]$ and d^*Z is the $\mathrm{Sp}(n, \mathbb{R})$–invariant measure $|Y|^{-(n+1)}\mathrm{d}X\mathrm{d}Y$.

Given $F, G \in \mathrm{M}(\Gamma_0^n[q], k, \chi)$, with Fourier expansions

$$F(Z) = \sum_{N \geqslant 0} a(N)e(\sigma(NZ)), \qquad G(Z) = \sum_{N \geqslant 0} b(N)e(\sigma(NZ)),$$

the *Rankin convolution* associated with F and G is the following Dirichlet series

$$(2.2) \qquad \mathcal{R}(F, G; s) := \sum_{\{N\}} \frac{a(N)\overline{b(N)}}{\epsilon(N)|N|^s}$$

where $\{N\}$ runs over all $GL(n, \mathbb{Z})$ equivalence classes of positive even integral matrices of size n and $\epsilon(N)$ is the number of integral units of N. Standard estimates for the Fourier coefficients of Siegel modular forms imply that the series in the r.h.s. of (2.2) converges absolutely in the half–plane $\Re(s) > \frac{n+1}{2} + m$. $\mathcal{R}(F, G; s)$ is therefore holomorphic in that region. On the other hand, if at least one among F and G is a cusp form, then

(1) $\mathcal{R}(F, G; s)$ has *meromorphic continuation* to $\mathbb{C}$;
(2) $\mathcal{R}(F, G; s)$ has at most a simple pole at $s = k$ and satisfies the *Petersson identity*, i.e. there is a constant $c \neq 0$ (depending on n and k) such that

$$< F, G >= c \cdot Res_{s=k} \mathcal{R}(F, G; s).$$

These facts both follow from the *Rankin–Selberg method*. One may indeed take into account the Eisenstein series for $\Gamma_0^n[q]$ of weight 0,

$$E_q(Z, s) := \sum_{\alpha \in \Gamma_\infty^n \backslash \Gamma_0^n[q]} |Y|^s |J(\alpha, Z)|^{-2s} \quad (\Re(s) > \frac{n+1}{2}),$$

where $\Gamma_\infty^n = \{ \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix} \right) \in \mathrm{Sp}(n, \mathbb{Z})\}$, and, for any $\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in \mathrm{Sp}(n, \mathbb{R})$ and $Z \in \mathbb{H}_n$, $J(\alpha, Z) := |CZ + D|$. In fact, the Eisenstein series $E_q(Z, s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = \frac{n+1}{2}$, with constant non–zero residue (see [7], Theorem 9.2, and [12]). Hence, the above mentioned properties of $\mathcal{R}(F, G; s)$ are a consequence of the integral representation

$$(2.3) \quad \int_{\mathcal{F}_{n,q}} E_q(Z, s) F(Z) \overline{G(Z)} |Y|^k \mathrm{d}^* Z = \gamma_{n,k}(s) \mathcal{R}\left(F, G; s + k - \frac{n+1}{2}\right),$$

for a certain gamma factor $\gamma_{n,k}(s)$ (see [13] for more details).

In order to extend the above results to the case of not necessarily cuspidal modular forms we shall use suitable differential operators to remove the "singular terms" which would cause the integral in (2.3) to diverge. This kind of technique has been introduced by Maaß in [21] for Siegel modular forms of degree 2.

To construct the differential operator $R = R(n, k)$ acting on smooth functions on $\mathbb{H}_n$, we have first to consider the $\mathrm{GL}(n, \mathbb{R})$–invariant operator,

$$(2.4) \qquad R_0 = |Y|^{n+1-m}|\partial_Y||Y|^{m+1-n}|\partial_Y|,$$

where, as usual, ∂_Y denotes the $n \times n$ matrix, whose entries are

$$(\partial_Y)_{(i,j)} = \frac{(1 + \delta_{(i,j)})}{2} \frac{\partial}{\partial Y_{(i,j)}},$$

where $1 \leqslant i, j \leqslant n$ and $\delta_{(i,j)}$ is Kronecker's delta. Clearly, R_0 acts on smooth functions on the set $\mathcal{P}_n$ of $n \times n$ symmetric and positive real matrices,

Then, we choose R to be an $\mathrm{Sp}(n, \mathbb{R})$–invariant differential operator such that the following two identities hold

$$(2.5)\ R[|Y|^k F(Z)\overline{G(Z)}] =$$
$$= \sum_{\substack{N_1, N_2 \geqslant 0 \\ N_1 + N_2 > 0}} a(N_1)\overline{b(N_2)}R[|Y|^k e(\sigma((N_1 - N_2)X) + i\sigma((N_1 + N_2)Y))],$$

$$(2.6) \qquad \int_{Sym_n(\mathbb{R}/\mathbb{Z})} R[|Y|^k F(Z)\overline{G(Z)}]\mathrm{d}X = |Y|^k R_0[\Xi(F, G; Y)],$$

where

$$(2.7) \qquad \Xi(F, G; Y) = \sum_{N > 0} a(N)\overline{b(N)}e(2i\sigma(NY)).$$

The existence of such an operator R is guaranteed by some results in [6] (Theorem 1.1, Theorem 1.2 and Proposition 2.1, at pages 275, 276 and 278, respectively). For the reader's convenience, let us collect such results in one statement. Consider the algebras $\mathrm{D}(\mathbb{H}_n)$ and $\mathrm{D}(\mathcal{P}_n)$ of invariant differential operators w.r.t. $\mathrm{Sp}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{R})$, respectively. Recall that if $\mathrm{S}(\mathfrak{a})$ is the symmetric algebra of the Lie algebra $\mathfrak{a} = \mathfrak{a}_n$ of the maximal $\mathbb{R}$–split torus A_n of diagonal matrices in $\mathrm{GL}(n, \mathbb{R})$ (which is a maximal $\mathbb{R}$–split torus for $\mathrm{Sp}(n, \mathbb{R})$ as well) we have the isomorphisms

$$\mathrm{D}(\mathcal{P}_n) \simeq \mathrm{S}(\mathfrak{a})^{\mathrm{W}_{\mathrm{GL}(n,\mathbb{R})}}, \quad \mathrm{D}(\mathbb{H}_n) \simeq \mathrm{S}(\mathfrak{a})^{\mathrm{W}_{\mathrm{Sp}(n,\mathbb{R})}},$$

where $\mathrm{W}_{\mathrm{GL}(n,\mathbb{R})}$ and $\mathrm{W}_{\mathrm{Sp}(n,\mathbb{R})}$ are the Weyl groups of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{Sp}(n, \mathbb{R})$ with respect to A_n. In this setting, $\mathrm{W}_{\mathrm{GL}(n,\mathbb{R})}$ is the group of all the permutations of $(a_1, \ldots, a_n) \in \mathfrak{a}$, while $\mathrm{W}_{\mathrm{Sp}(n,\mathbb{R})}$ is generated by $\mathrm{W}_{\mathrm{GL}(n,\mathbb{R})}$ and all sign changes.

Next, if we consider the injective map

$$\varphi : C^\infty(\mathcal{P}_n) \quad \rightarrow \quad C^\infty(\mathbb{H}_n),$$

defined by

$$\varphi(f)(X + iY) \quad := \quad f(Y),$$

we may associate with φ a well defined map

$$\varphi^* : \mathrm{D}(\mathbb{H}_n) \quad \rightarrow \quad \mathrm{D}(\mathcal{P}_n);$$
$$D \quad \mapsto \quad \varphi^{-1} \circ D \circ \varphi.$$

Notice indeed that the image of φ consists of the functions invariant under all one parameter subgroups of

$$U = \left\{ \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \mid B = {}^t B \in Mat(n, \mathbb{R}) \right\} \leqslant \mathrm{Sp}(n, \mathbb{R}),$$

and every $D \in \mathrm{D}(\mathbb{H}_n)$ leaves therefore that image invariant (see [6] page 274).

Let us now state the result of Deitmar and Krieg we resort to.

THEOREM 2.1. — *The map φ^* is injective and the following diagram*

$$
\begin{array}{ccc}
D(\mathbb{H}_n) & \simeq & S(\mathfrak{a})^{W_{Sp(n,\mathbb{R})}} \\
{\scriptstyle \varphi^*} \big\uparrow & & \big\downarrow {\scriptstyle i} \\
D(\mathcal{P}_n) & \simeq & S(\mathfrak{a})^{W_{GL(n,\mathbb{R})}}
\end{array}
$$

is commutative. Moreover, for any $r \in \mathbb{R}$, consider the differential operator

$$D_r = D_{r,n} := |Y|^r |\partial_Y| |Y|^{1-r} \in D(\mathcal{P}_n),$$

then

$$D_r D_{n+2-r} \in \varphi^*(D(\mathbb{H}_n)).$$

Finally, let

$$R_0^\flat \quad := \quad D_{n+1-k} D_{1+k} = |Y|^k R_0 |Y|^{-k},$$
$$R \quad := \quad (\varphi^*)^{-1}(R_0^\flat),$$

then the identities (2.5) and (2.6) hold.

Remark 2.2. — Deitmar and Krieg actually consider in their paper a set-up which is slightly different from ours. Namely, they are concerned with theta liftings of Eisenstein series and they need therefore to apply their differential operators to certain theta–kernels. We believe it is worth showing how the arguments of Proposition 2.1 in [6] can be adapted to get (2.5). In other words, taking

$$f(Z) = |Y|^k e(\sigma((N_1 - N_2)X + i(N_1 + N_2)Y)),$$

we want to prove that $R[f] = 0$ if $|N_1 + N_2| = 0$. To this end, since R is $\mathrm{Sp}(n, \mathbb{R})$–invariant and N_1, N_2 are positive semi–definite, it is sufficient to assume

$$N_1 = \begin{pmatrix} \tilde{N_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} \tilde{N_2} & 0 \\ 0 & 0 \end{pmatrix}.$$

Consequently, if $n > 1$, we see that f lies in the image of the map

$$\tau : C^\infty(\mathbb{H}_{n-1} \times \mathbb{H}_1) \quad \to \quad C^\infty(\mathbb{H}_n),$$

defined by

$$\tau(h)\left({}^t\left(\begin{smallmatrix} I & b \\ 0 & 1 \end{smallmatrix}\right)\left(\left(\begin{smallmatrix} Z_1 & 0 \\ 0 & z_n \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & a \\ {}^t a & 0 \end{smallmatrix}\right)\right)\left(\begin{smallmatrix} I & b \\ 0 & 1 \end{smallmatrix}\right)\right) \quad := \quad h(Z_1, z_n) \quad \forall a, b \in \mathbb{R}^{n-1}.$$

The conclusion then follows by combining the fact that $\tau^{-1} \circ R \circ \tau$ is a simple tensor in

$$\mathrm{D}(\mathbb{H}_{n-1} \times \mathbb{H}_1) = \mathrm{D}(\mathbb{H}_{n-1}) \otimes \mathrm{D}(\mathbb{H}_1),$$

(which is Proposition 1.1 in [6]) with the identity

$$R(1, k)[y_n^k] = 0.$$

This can again be read off [6] (Lemma 1.1), but it can also be checked directly, since one knows that

$$R(1, k) = \Delta - k(k - 1),$$

where Δ is the usual non-Euclidean Laplacian (see [4]).

3. Meromorphic continuation

THEOREM 3.1. — *Let $n \leqslant m$ and $F, G \in M(\Gamma_0^n[q], k, \chi)$, then $\mathfrak{R}(F, G; s)$ has a meromorphic continuation to $\mathbb{C}$.*

Proof. — The function $R[|Y|^k F(Z)\overline{G(Z)}]$ is of exponential decay, because (according to (2.5)) the Fourier expansion involves only positive definite matrices; due to the invariance of R, this is true for all cusps. Hence, for $\mathfrak{R}(s) \gg 0$, we may take into account the integral

$$(3.1) \quad I(F, G; s) := \int_{\mathscr{F}_{n,q}} E_q\left(Z, s + \frac{n+1}{2} - k\right) R[|Y|^k F(Z)\overline{G(Z)}]\mathrm{d}^* Z .$$

Then, by applying the Rankin–Selberg unfolding method, one gets

$$(3.2) \quad I(F, G; s) = \int_{\mathcal{M}_n} |Y|^{s - \frac{n+1}{2}} R_0[\Xi(F, G; Y)]\mathrm{d}Y ,$$

where $\mathcal{M}_n$ denotes the space of Minkowski reduced $n \times n$ positive matrices. A direct evaluation of the r.h.s. of (3.2) then yields

$$\sum_{N>0} a(N)\overline{b(N)} \int_{\mathcal{M}_n} |Y|^{s+n-m} M_n[|Y|^{m-n+1}(-2\pi)^n |N| e(2i\sigma(NY))] \mathrm{d}^*Y,$$

where $M_n = |Y||\partial_Y|$, and d^*Y is the $\mathrm{GL}(n,\mathbb{R})$–invariant measure $|Y|^{-\frac{n+1}{2}} \mathrm{d}Y$. We may now first reduce the summation to a complete set of representatives of the equivalence classes $\{N\}$, and then consider the adjoint operator $\hat{M}_n$ obtaining

$$\sum_{\{N\}} \frac{a(N)\overline{b(N)}}{\epsilon(N)} |N| (-2\pi)^n 2 \int_{\mathcal{P}_n} |Y|^{s+n-m} M_n[|Y|^{m-n+1} e(2i\sigma(NY))] \mathrm{d}^*Y =$$

$$= \sum_{\{N\}} \frac{a(N)\overline{b(N)}}{\epsilon(N)} |N| (-2\pi)^n 2 \int_{\mathcal{P}_n} \hat{M}_n[|Y|^{s+n-m}] |Y|^{m-n+1} e(2i\sigma(NY)) \mathrm{d}^*Y.$$

We recall that, for $t \in \mathbb{R}$ and $T \in \mathcal{P}_n$, the following two well–known formulas hold (see [20], page 80, 81)

$$(3.3) \qquad\qquad\qquad \hat{M}_n[|Y|^t] = (-1)^n \phi_n(t) |Y|^t;$$

$$(3.4) \qquad \int_{\mathcal{P}_n} |Y|^t \exp\left(-\sigma(TY)\right) \mathrm{d}^*Y = \pi^{\frac{n^2-n}{4}} \Gamma_n(t) |T|^{-t},$$

where

$$\phi_n(t) := \prod_{j=0}^{n-1} \left(t - \frac{j}{2}\right), \quad \Gamma_n(t) := \prod_{j=0}^{n-1} \Gamma\left(t - \frac{j}{2}\right).$$

A straightforward computation then shows:

$$(3.5) \quad I(F,G;s) = 2^{1-ns} \pi^{\frac{n^2-n}{4} - ns} \Gamma_n(s) \phi_n(s) \phi_n(s+n-m) \mathcal{R}(F,G;s) .$$

The meromorphic continuation of $\mathcal{R}(F,G;s)$ is thus a consequence of the meromorphic continuation of $E_q(Z,s)$. $\square$

It is also possible to define the Rankin convolution (2.2) for pairs of modular forms with different weights and characters. Indeed one can define the Rankin convolution for $n = 1$ without any restriction on weights or characters. For $n > 1$, given F of weight k and character χ_F, and G of weight k' and character χ_G, the condition

$$|U|^k \chi_F(|U|) = |U|^{k'} \chi_G(|U|), \quad \forall U \in \mathrm{GL}(n,\mathbb{Z})$$

is necessary for the formal definition of the Rankin convolution $\mathcal{R}(F,G;s)$.

Assuming one of the involved modular forms to be cuspidal and the difference of the weights to be even, Kalinin showed in [13] the meromorphic continuation of such Rankin convolutions. We can prove the same result

in the not–necessarily cuspidal case. To this purpose we need the so–called Maaß–Shimura raising operators. Let us recall that the differential operator

$$\delta_\alpha := \frac{1}{(2\pi i)^n} |Y|^{-\alpha + \frac{n-1}{2}} |\partial_Z| |Y|^{\alpha - \frac{n-1}{2}}$$

maps C^∞ Siegel modular forms of weight α to C^∞ Siegel modular forms of weight $\alpha + 2$. For any positive integer l, Courtieu and Panchishkin described explicitly in [5] the action of the iteration $\delta_k^{(l)} = \delta_{k+2l-2} \circ \cdots \circ \delta_k$ on the Fourier expansion of a modular form of weight k. Let us quote Theorem 3.12 from [5]: let $F(Z) = \sum\limits_{N \geqslant 0} a(N) e(\sigma(NZ)) \in M(\Gamma_0^n[q], k, \chi)$, then

$$(3.6) \quad \delta_k^{(l)}[F(Z)] = |4\pi Y|^{-l} \sum_N a(N) \sum_{t=0}^{l} \binom{l}{t} |2\pi N Y|^{l-t} \times$$

$$\times \sum_{|L| \leqslant nt-t} R_L(\frac{n+1}{2} - k - l) \lambda_L(2\pi NY) e(\sigma(NZ)),$$

where L runs over all the multi–indices $0 \leqslant l_1 \leqslant \cdots \leqslant l_t \leqslant n$, such that $|L| = l_1 + \cdots + l_t \leqslant nt - t$, the coefficients $R_L(\beta)$ are certain polynomials in $\mathbb{Z}[1/2][\beta]$ and, for any square matrix A

$$\lambda_L(A) = \prod_{j=1}^{t} \lambda_{l_j}(A),$$

with

$$|x I_n + A| = \sum_{i=0}^{n} \lambda_i(A) x^{n-i}.$$

PROPOSITION 3.2. — *Let $n \leqslant 2k$. For any positive integer l, given*

$$F \in M(\Gamma_0^n[q], k, \chi), \quad G \in M(\Gamma_0^n[q], k+2l, \chi),$$

the Rankin convolution $\mathcal{R}(F, G; s)$ has a meromorphic continuation to $\mathbb{C}$.

Proof. — We just sketch the main points. Consider the "test functions"

$$f_N(Z) := e(\sigma(NZ)),$$

with N even and positive semi–definite. Essentially, the formula of Courtieu and Panchishkin tells us that

$$\delta_k^{(l)}[f_N(Z)] = |4\pi Y|^{-l} \cdot p(NY) \cdot f_N(Z)$$

with a polynomial $p = p(A)$, which is a function of the principal minors of the matrix A, and is invariant under conjugation.

We let $\mu := 2k+2l$ and we consider $R = R(n, \frac{\mu}{2}) = (\phi^*)^{-1}(D_{n+1-\frac{\mu}{2}}D_{1+\frac{\mu}{2}})$. Then we get for $N, M \geqslant 0$ with $N + M \not\geqslant 0$

$$R[(\delta_k^{(l)} f_N) \cdot \overline{f_M} \cdot \mid Y \mid^{k+2l}] = 0.$$

The same reasoning of Remark 2.2 applies also here: To cover the case $|N + M| = 0$ we just have to consider $N = \left(\begin{smallmatrix} N_1 & 0 \\ 0 & 0 \end{smallmatrix} \right)$ and $M = \left(\begin{smallmatrix} M_1 & 0 \\ 0 & 0 \end{smallmatrix} \right)$ (i.e. with $|N+M| = 0$), and observe that the resulting function lies in the image of τ.

Then, for $N + M > 0$,

$$\int_{Sym_n(\mathbb{R}/\mathbb{Z})} \int_{\mathcal{P}_n} R[\delta_k^{(l)}[f_N]\overline{f_M} \mid Y \mid^{k+2l}]|Y|^{s+\frac{n+1}{2}-\frac{\mu}{2}} d^*Z$$

vanishes if $N \neq M$, and otherwise equals

$$(4\pi)^{-nl} \int_{\mathcal{P}_n} \mid Y \mid^s R_0[p(NY)e(2i\sigma(NY))]d^*Y =$$

$$(4\pi)^{-nl}|N|^{-s} \int_{\mathcal{P}_n} \mid Y \mid^s R_0[p(Y)e(2i\sigma(Y))]d^*Y.$$

Using formula (3.3) twice this becomes

$$(4\pi)^{-nl}\phi_n(s + n - \mu)\phi_n(s)|N|^{-s} \int_{\mathcal{P}_n} \mid Y \mid^s p(Y)e(2i\sigma(Y))d^*Y$$

and the last integral can be computed in a standard way, e.g. by using (3.4) and applying the differential operator $\delta_k^{(l)}$ (w.r.t. the "complexified" variable T).

In conclusion, we see that the integral
(3.7)
$$I(F,G;s) := \int_{\mathcal{F}_{n,q}} E_q(Z, s + \frac{n+1}{2} - \frac{\mu}{2})R[\delta_k^{(l)}[F(Z)]\overline{G(Z)}|Y|^{k+2l}]\mathrm{d}^*Z ,$$

converges for $\Re(s) \gg 0$, and it may be unfolded to get the integral representation

(3.8) $$I(F, G; s) = \gamma(n, k, l, s)\mathcal{R}(F, G; s) ,$$

where $\gamma(n, k, l, s)$ has the following expression

$$2^{1-n(s+2l)}\pi^{\frac{n^2-n}{4}-n(s+l)}\Gamma_n(s)\phi_n(s)\phi_n(s+n-\mu)\prod_{j=1}^{l} \phi_n(s - k - j + \frac{n+1}{2}).$$

$\square$

Remark 3.3. — One may introduce the modified Eisenstein series

$$\mathcal{E}_1(Z, s) := \xi(2s) \prod_{j=1}^{[\frac{n}{2}]} \xi(4s - 2j) E_1(Z, s),$$

where $[\frac{n}{2}]$ denotes the largest integer less than or equal to $\frac{n}{2}$, and

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s).$$

Then it is well–known that $\mathcal{E}_1(Z, s)$ satisfies the functional equation

$$(3.9) \qquad \mathcal{E}_1(Z, \frac{n+1}{2} - s) = \mathcal{E}_1(Z, s).$$

Thus, it follows from (3.9) and the integral representations (3.5), (3.8) that the modified Rankin convolution of any pair of level 1 modular forms $F \in \mathrm{M}(\mathrm{Sp}(n, \mathbb{Z}), k, 1)$, $G \in \mathrm{M}(\mathrm{Sp}(n, \mathbb{Z}), k + 2l, 1)$ (with $l \geqslant 0$)

$$\mathcal{R}'(F, G; s) := 2^{1-n(s+2l)} \pi^{\frac{n^2-n}{4} - n(s+l)} \Gamma_n(s) \prod_{j=1}^{l} \phi_n(s - k - j + \frac{n+1}{2}) \times$$

$$\times \xi(2s + n + 1 - \mu) \prod_{j=1}^{[\frac{n}{2}]} \xi(4s + 2n + 2 - 2\mu - 2j) \mathcal{R}(F, G; s)$$

satisfies the functional equation

$$(3.10) \qquad \mathcal{R}'(F, G; \mu - \frac{n+1}{2} - s) = \mathcal{R}'(F, G; s).$$

In the cuspidal case, Kalinin already showed (3.10) in [13]. It should be also noted that the polynomial $\phi_n(s)\phi_n(s + n - \mu)$ which appears in (3.5) and (3.8) satisfies the same functional equation.

An estimate for Fourier coefficients of level 1 modular forms can be then obtained as a consequence of the integral representation and the functional equation for the Rankin convolution. Basically, given a modular form

$$F(Z) = \sum_{N \geqslant 0} a(N) e(\sigma(NZ)) \in \mathrm{M}(\mathrm{Sp}(n, \mathbb{Z}), k, 1),$$

we may follow the same argument of Section 5 in [3], and apply Landau's Theorem (as given by Sato and Shintani in Section 3 of [27]) to the function

$$\psi(s) = (2\pi)^{-ns} \pi^{-(2[\frac{n}{2}]+1)s} \zeta(2s + n + 1 - m)$$

$$\prod_{j=1}^{[\frac{n}{2}]} \zeta(2(2s + n + 1 - m - j)) \mathcal{R}(F, F; s).$$

In our case, the region of absolute convergence of the series $\psi(s)$ is in general different from that in [3]. This leads us to make a slightly different choice for some parameters: namely, if we use the same notation of [3] (and [27]), we set

$$\mu_1 = \mu_2 = \begin{cases} k + \varepsilon & \text{for } n = m \\ m - \frac{n+1}{2} + \varepsilon & \text{for } n < m \end{cases} \quad \text{(for any } \varepsilon > 0\text{)}.$$

In this way we get finally the estimates

$$a(N) = O(|N|^{\frac{k}{2} - \frac{1}{2n+2+4[\frac{n}{2}]+\frac{2}{n+1}} + \varepsilon}),$$

for $n = 2k$ or $n = 2k - 1$, and

$$a(N) = O(|N|^{k - \frac{n+1}{4} - \frac{1}{2n+2+4[\frac{n}{2}]+\frac{2}{4k-(n+1)}} + \varepsilon}),$$

for $n \leqslant 2k - 1$. It seems, however, that such estimates are of some interest only for small weights (see [14], [20]).

4. Petersson identity

We denote by

$$L^2[\mathrm{M}(\Gamma_0^n[q], k, \chi)] := \{F \in \mathrm{M}(\Gamma_0^n[q], k, \chi) \mid \langle f, f \rangle < \infty\}$$

the space of square–integrable modular forms with respect to the Petersson product. It is well–known that cusp forms are square–integrable and that in general they form a proper subspace of $L^2[\mathrm{M}(\Gamma_0^n[q], k, \chi)]$. Indeed, Satake, and more generally Weissauer, gave in [26] and [31] respectively, a complete characterization of $L^2[\mathrm{M}(\Gamma_0^n[q], k, \chi)]$. For example, it follows from their characterization that non–cuspidal square integrable Siegel modular forms may occur only if $n > k$. Besides, they prove that $L^2[\mathrm{M}(\Gamma_0^n[q], k, \chi)] = \mathrm{M}(\Gamma_0^n[q], k, \chi)$ if $n \geqslant m$. We point out that even though Satake and Weissauer only treated modular forms of integral weight it is in fact possible to extend their arguments to include the half integral weight case as well.

LEMMA 4.1. — Let $n \leqslant m$ and $F, G \in L^2[\mathrm{M}(\Gamma_0^n[q], k, \chi)]$, then

$$(4.1) \qquad \int_{\mathscr{F}_{n,q}} R[|Y|^k F(Z)\overline{G(Z)}]\mathrm{d}^*Z = h \cdot \int_{\mathscr{F}_{n,q}} |Y|^k F(Z)\overline{G(Z)}\mathrm{d}^*Z ,$$

where $h = h(n, k) = \phi_n(k)\phi_n(n - k)$.

Proof. — It is well known that the ring $D(\mathbb{H}_n)$ of $\mathrm{Sp}(n, \mathbb{R})$–invariant differential operators is isomorphic to a polynomial ring in n variables. Moreover, Shimura shows in [29] that $D(\mathbb{H}_n)$ is actually generated by certain canonically defined "generalized Laplacians" $\mathcal{L}_i$ ($i \in \{1, \ldots, n\}$). We may thus consider the following decomposition:

$$(4.2) \qquad R = h \cdot I + p(\mathcal{L}_1, \ldots, \mathcal{L}_n) \;,$$

where h is a suitable constant (depending on n and k), I is the identity, and $p(X_1, \ldots, X_n)$ is a polynomial of positive degree.

Also, it is one of the main result of [30] (namely, Theorem 2.1) that, under suitable convergence conditions, the operators $\mathcal{L}_i$ are *symmetric*.

In our case the above mentioned convergence conditions are satisfied. Indeed, F, G are square–integrable and the square–integrability is preserved by the action of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $\mathrm{Sp}(n, \mathbb{R})$ on the automorphic forms F and G (see Corollary 4.7.2 of [11], Theorem 1 of [9] and their proofs). Thus, we get

$$\int_{\mathscr{F}_{n,q}} p(\mathcal{L}_1, \ldots, \mathcal{L}_n)[|Y|^k F(Z)\overline{G(Z)}]\mathrm{d}^*Z = 0.$$

In order to compute the constant h, we take into account the function identically equal to 1 and we observe that

$$h = R[1] = R_0^\natural[1].$$

Now the claim follows from the formula

$$|Y|^\nu M_n |Y|^{-\nu}[1] = \phi_n(\nu), \quad (\nu \in \mathbb{R}).$$

$\square$

Remark 4.2. — We point out that in the statement of Lemma 4.1 we have $h = \phi_n(k)^2 \neq 0$ for $n = m$, while $h = 0$ for $k < n < m$. This vanishing will imply some complications in the proof of Theorem 4.4. The following lemma will be needed to overcome such complications.

LEMMA 4.3. — *Following the same notation as above,*

$$R[E_q(Z, s)] = \phi_n(s + \frac{n-1}{2} - k)\phi_n(s - \frac{n+1}{2} + k)E_q(Z, s) \;.$$

In particular, $s = k$ is not a pole for the function $R[E_q(Z, s + \frac{n+1}{2} - k)]$ if $k < n < m$.

Proof. — Since R is $\mathrm{Sp}(n, \mathbb{R})$–invariant, the claim follows from

$$R[|Y|^s] = R_0^\natural[|Y|^s] = \phi_n(s + \frac{n-1}{2} - k)\phi_n(s - \frac{n+1}{2} + k)|Y|^s \;.$$

$\square$

THEOREM 4.4. — *Let $k < n \leqslant m$ and consider $F, G \in M(\Gamma_0^n[q], k, \chi)$ such that $\langle F, G \rangle$ is defined. There exists a constant $c \neq 0$ (depending on n and k) such that*

(4.3) $< F, G >= c \cdot Res_{s=k} \mathfrak{R}(F, G; s).$

Proof. — Let us first consider the case $n = m$. Computing the residue of $I(F, G; s)$ at $s = k$, yields

$$Res_{s=\frac{n+1}{2}} \int_{\mathcal{F}_{n,q}} E_q(Z, s) R[|Y|^k F(Z) \overline{G(Z)}] \mathrm{d}^* Z =$$

(4.4) $= K \cdot Res_{s=k} \mathfrak{R}(F, G; s) ,$

where

$$K = 2^{1-nk} \pi^{\frac{n^2}{4} - n(\frac{1}{4}+k)} \Gamma_n(k) \phi_n^2(k).$$

By combining (4.4) and (4.1) we can write,

(4.5) $(Res_{s=\frac{n+1}{2}} E_q(Z, s)[\mathrm{Sp}(n, \mathbb{Z}) : \Gamma_0^n[q]]) \cdot \langle F, G \rangle = \dfrac{K}{h} \cdot Res_{s=k} \mathfrak{R}(F, G; s).$

We now turn to the more delicate range $n < m$. To handle the extra difficulties appearing in this case (see Remark 4.2), we adopt an approximation technique. For this reason, we take into account the sequence of functions

$$\psi_\nu(Z) := \exp(-\frac{\|E_1(Z, n+1)\|}{\nu}) \in C^\infty(\mathcal{F}_{n,q}).$$

We notice that for any "differential operator with polynomial coefficients" D the following properties hold (see [22]):

   (1) $\psi_\nu(Z)$ and $D[\psi_\nu](Z)$ are of rapid decay as Z approaches the boundary of $\mathcal{F}_{n,q}$;
   (2) $\psi_\nu \to 1$ as $\nu \to \infty$;
   (3) $D[\psi_\nu] \to 0$ uniformly as $\nu \to \infty$.

Next, we put

$$h' := h'(n, m, q) = \lim_{s \to k} R[E_q(Z, s + \frac{n+1}{2} - k)]$$

$$= \lim_{s \to k} \frac{\phi_n(s) \phi_n(s + n - m)}{s - k} \cdot Res_{s=\frac{n+1}{2}} E_q(Z, s) .$$

By standard arguments, we have

$$[\mathrm{Sp}(n,\mathbb{Z}):\Gamma_0^n[q]]\cdot\langle F,G\rangle = \int_{\mathscr{F}_{n,q}} F(Z)\overline{G(Z)}|Y|^k \mathrm{d}^* Z =$$

$$= \lim_{\nu\to\infty}\int_{\mathscr{F}_{n,q}} \psi_\nu(Z)F(Z)\overline{G(Z)}|Y|^k \mathrm{d}^* Z$$

$$= \frac{1}{h'}\lim_{\nu\to\infty}\int_{\mathscr{F}_{n,q}} \psi_\nu(Z)F(Z)\overline{G(Z)}|Y|^k \lim_{s\to k} R[E_q(Z,s+\frac{n+1}{2}-k)]\mathrm{d}^* Z$$

(4.6)

$$= \frac{1}{h'}\lim_{\nu\to\infty}\lim_{s\to k}\int_{\mathscr{F}_{n,q}} \psi_\nu(Z)F(Z)\overline{G(Z)}|Y|^k R[E_q(Z,s+\frac{n+1}{2}-k)]\mathrm{d}^* Z.$$

We observe that we can now move the differential operator R from the Eisenstein series to the function $\psi_\nu(Z)F(Z)\overline{G(Z)}|Y|^k$. Applying any holomorphic or antiholomorphic differential operator D on such a function yields the function of rapid decay

$$(4.7)\quad D[\psi_\nu(Z)F(Z)\overline{G(Z)}|Y|^k] = \sum_j D^{(j)}[\psi_\nu(Z)]\tilde{D}^{(j)}[F(Z)\overline{G(Z)}|Y|^k]$$

for suitable differential operators $D^{(j)}$ and $\tilde{D}^{(j)}$. As it is stated by Shimura ([30], page 150) this condition is sufficient for the symmetry of the generalized Laplacians which appear in the decomposition (4.2) of R. Thus we may rewrite (4.6) as

$$(4.8)\quad \frac{1}{h'}\lim_{\nu\to\infty}\lim_{s\to k}\int_{\mathscr{F}_{n,q}} R[\psi_\nu(Z)F(Z)\overline{G(Z)}|Y|^k]E_q(Z,s+\frac{n+1}{2}-k)\mathrm{d}^* Z.$$

We may now proceed further and interchange the limits in (4.8) and then move the limit with respect to ν inside the integral to obtain

$$\frac{1}{h'}\lim_{s\to k}\lim_{\nu\to\infty}\int_{\mathscr{F}_{n,q}} R[\psi_\nu(Z)F(Z)\overline{G(Z)}|Y|^k]E_q(Z,s+\frac{n+1}{2}-k)\mathrm{d}^* Z$$

$$= \frac{1}{h'}\lim_{s\to k} I(F,G;s).$$

Indeed, it is possible to perform the previous operations since

$$\lim_{s \to k} \int_{\mathcal{F}_{n,q}} R[\psi_\nu(Z)F(Z)\overline{G(Z)}|Y|^k]E_q(Z,s+\frac{n+1}{2}-k)\mathrm{d}^*Z$$

$$= \lim_{s \to k} \int_{\mathcal{F}_{n,q}} \psi_\nu(Z)F(Z)\overline{G(Z)}|Y|^k R[E_q(Z,s+\frac{n+1}{2}-k)]\mathrm{d}^*Z$$

exists and is uniform with respect to ν and

$$\lim_{\nu \to \infty} \int_{\mathcal{F}_{n,q}} R[\psi_\nu(Z)F(Z)\overline{G(Z)}|Y|^k]E_q(Z,s+\frac{n+1}{2}-k)\mathrm{d}^*Z$$

$$= \int_{\mathcal{F}_{n,q}} \lim_{\nu \to \infty} \psi_\nu(Z)R[F(Z)\overline{G(Z)}|Y|^k]E_q(Z,s+\frac{n+1}{2}-k)]\mathrm{d}^*Z+$$

$$+ \sum_j \int_{\mathcal{F}_{n,q}} \lim_{\nu \to \infty} R^{(j)}[\psi_\nu(Z)]\tilde{R}^{(j)}[F(Z)\overline{G(Z)}|Y|^k]E_q(Z,s+\frac{n+1}{2}-k)]\mathrm{d}^*Z$$

$$= \int_{\mathcal{F}_{n,q}} R[F(Z)\overline{G(Z)}|Y|^k]E_q(Z,s+\frac{n+1}{2}-k)\mathrm{d}^*Z + 0$$

where we have used a notation similar to (4.7) and the uniform convergence of the derivatives of ψ_ν. Summing up, we obtain

$$[\mathrm{Sp}(n,\mathbb{Z}):\Gamma_0^n[q]] \cdot \langle F,G \rangle = \frac{1}{h'} \lim_{s \to k} I(F,G;s)$$

$$= \frac{1}{h'} \lim_{s \to k} 2^{1-ns}\pi^{\frac{n^2}{4}-n(\frac{1}{4}+s)}\Gamma_n(s)\phi_n(s)\phi_n(s+n-m)\mathcal{R}(F,G;s)$$

$$= \frac{K'}{h'} Res_{s=k}\mathcal{R}(F,G;s) \ ,$$

where

$$K' = \lim_{s \to k} 2^{1-ns}\pi^{\frac{n^2}{4}-n(\frac{1}{4}+s)}\Gamma_n(s)\frac{\phi_n(s)\phi_n(s+n-m)}{s-k}.$$

In conclusion, for any $n \leqslant m$, we get the identity

$$\langle F,G \rangle = \frac{2^{1-nk}\pi^{\frac{n^2}{4}-n(\frac{1}{4}+k)}\Gamma_n(k)}{[\mathrm{Sp}(n,\mathbb{Z}):\Gamma_0^n[q]]Res_{s=\frac{n+1}{2}}E_q(Z,s)} \cdot Res_{s=k}\mathcal{R}(F,G;s).$$

$$\square$$

5. Applications and remarks

As a first simple consequence of the above results, we wish to mention the following statement about the orthogonality of certain theta series.

COROLLARY 5.1. — *Let S_1 and S_2 be two even positive definite quadratic forms of rank n and level dividing q. Assume that S_1 and S_2 are not rationally equivalent. Then, if the corresponding theta series*

$$\vartheta(S_j, Z) = \sum_{G \in \mathbb{Z}^{(n,n)}} e(\sigma(S_j[G]Z)) \quad (j = 1, 2)$$

belong to the same space of modular forms $M(\Gamma_0^n[q], n/2, \chi)$, they are orthogonal with respect to the Petersson inner product. Of course, the same statement holds by definition if $\vartheta(S_j, Z) \in M(\Gamma_0^n[q], n/2, \chi_j)$ and $\chi_1 \neq \chi_2$.

Remark 5.2. — We would like to observe that in the singular range (i.e. for $n > m$) there is a result analogous to Theorem 4.4, at least if the modular forms in question are (linear combinations of) theta series attached to quadratic forms which are rationally equivalent. Of course one has first to define an appropriate substitute for the Rankin convolution in the case of singular modular forms. In fact, keeping in mind that singular modular forms are generated by theta series, one may obtain a proof of the relevant theorem by directly evaluating the scalar product of pairs of theta series (see [2] for further details). There seems to be not much hope to apply the Rankin–Selberg method directly in this case.

Remark 5.3. — Even though in the present paper we have considered only modular forms with respect to the Hecke subgroups, we wish to stress the fact that the given proofs work for general congruence subgroups $\Gamma \leqslant \Gamma^n := \mathrm{Sp}(n, \mathbb{Z})$ as well. Namely, given $F \in M(\Gamma, k, \chi)$ and $G \in M(\Gamma, k + 2l, \chi)$ $(l \geqslant 0)$ with Fourier expansion

$$F(Z) = \sum_{N \geqslant 0} a(N)e(\sigma(NZ)), \qquad G(Z) = \sum_{N \geqslant 0} b(N)e(\sigma(NZ)),$$

(where the summations extend over the set of *rational*, symmetric, positive semi–definite matrices of size n) one may define the Rankin–convolution as

$$\mathcal{R}(F, G; s) := \frac{1}{[\mathrm{GL}(n, \mathbb{Z}) : U_\Gamma]} \sum_{\{N\}} \frac{a(N)\overline{b(N)}}{\epsilon(N)|N|^s}$$

where

$$U_\Gamma = \{ \begin{pmatrix} U & 0 \\ 0 & {}^t U^{-1} \end{pmatrix} \in \Gamma \mid U \in \mathrm{GL}(n, \mathbb{Z}) \},$$

$\{N\}$ runs over the U_Γ–equivalence classes of rational, symmetric, positive definite matrices of size n and $\epsilon(N)$ is the order of the stabilizer of N in U_Γ. Moreover, the following Eisenstein series has to be taken into account

$$E_\Gamma(Z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} |Y|^s |J(\alpha, Z)|^{-2s} \quad (Z \in \mathbb{H}_n) \ ,$$

where $\Gamma_\infty = \{ (\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}) \in \Gamma \}$. It is shown in [7] that all the Eisenstein series $E_\Gamma(Z,s)$ have a simple pole with constant residue in $s = \frac{n+1}{2}$. Theorems 3.1 and 4.4 can then be stated and proved along the same lines we followed for the Hecke subgroups. We may also observe that the results which are obtained in this way do not depend on the level. Indeed,

$$(5.1) \qquad [\Gamma_\infty : \Gamma'_\infty] E_\Gamma(Z,s) = \sum_{\gamma \in \Gamma' \backslash \Gamma} E_{\Gamma'}(\gamma\langle Z\rangle, s) \ ,$$

for any $\Gamma' \leqslant \Gamma$, of finite index, and the factor $\frac{[\Gamma_\infty^n : \Gamma_\infty]}{[\mathrm{GL}(n,\mathbb{Z}) : U_\Gamma]}$ appears in the general unfolding procedure.

Orthogonality relations as those in Corollary 5.1 could also be easily derived for pairs of theta series belonging to the same space of modular forms $\mathrm{M}(\Gamma, n/2, \chi)$, which do not have Fourier coefficients relevant to rationally equivalent matrices.

Let Γ be an arbitrary congruence subgroup of $\mathrm{Sp}(n, \mathbb{Z})$. We can define an Hermitian form $\{,\}$ on $\bigcup\limits_{\Gamma \leqslant \mathrm{Sp}(n,\mathbb{Z})} \mathrm{M}(\Gamma, k, \chi)$ by

$$(5.2) \qquad \{F, G\} = Res_{s=k} \mathcal{R}(F, G; s), \quad (F, G \in \mathrm{M}(\Gamma, k, \chi))$$

This is a natural extension of the Petersson product. The signature of $\{,\}$ is however not clear from the definition (it could even be degenerate!).
We notice moreover that, at least when the weight is large enough $(k > n)$, the value $\{F, G\}$ of the Hermitian form equals up to a non–zero constant the integral

$$(5.3) \qquad \frac{1}{[\mathrm{Sp}(n, \mathbb{Z}) : \Gamma]} \int_{\Gamma \backslash \mathbb{H}_n} R[\mid Y \mid^k F(Z)\overline{G(Z)}]d^*Z.$$

Depending on the properties to be investigated, we may freely switch between these two definitions (if k is large enough).
Definition (5.3) of $\{,\}$ is easily seen to share with the Petersson product on cusp forms all the invariance properties described e.g. in the books of Lang [18] or Freitag [8]; in particular, the Hecke operators "away from the level" are self-adjoint (or normal in the case of nebentypus) for $\{,\}$. Therefore eigenforms with different eigenvalues are orthogonal to each other. In

particular, (say for $\Gamma^n = \mathrm{Sp}(n, \mathbb{Z})$, $k > 2n$ and $\chi = 1$ for simplicity) the Klingen decomposition, [15],

$$\mathrm{M}(\Gamma^n, k, 1) = \oplus \mathrm{M}(\Gamma^n, k)^{n,r}$$

is an *orthogonal* decomposition, because (using an argument originally due to Harris [10]) the subspaces $\mathrm{M}(\Gamma^n, k)^{n,r}$ can be separated by eigenvalues. We recall that by definition the space $\mathrm{M}(\Gamma^n, k)^{n,r}$ is generated by Klingen–Eisenstein series attached to cusp forms of degree r.

Remark 5.4. — It seems more complicated (though desirable) to obtain these properties by using the definition of $\{,\}$ in terms of the Rankin convolution. In case of level 1 one can use the functional equation to consider the Rankin convolution at $s = k - \frac{n+1}{2}$ rather than $s = k$; then the self–adjointness of the Hecke operators can also be obtained by considering the explicit action of Hecke operators on the Fourier expansions.

For the special case $n = 1$, $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ Zagier showed in [33] that $\{,\}$ is always non-degenerate and it is positive definite if and only if $k \equiv 2 \bmod 4$; for k divisible by 4 the Eisenstein series has negative norm. We show here that this is a special phenomenon for the full modular group.

For simplicity we consider here the congruence subgroups of type

$$\Gamma_1[q] = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid c \equiv 0 \bmod q, \quad a \equiv d \equiv 1 \bmod q \}.$$

It is well known that $\mathrm{M}(\Gamma_1[q], k, 1) = \oplus_\chi \mathrm{M}(\Gamma_0^1[q], k, \chi)$, where χ runs over all characters mod q.

PROPOSITION 5.5. — *Assume that $k \geqslant 3$ and let p be any odd prime. The space of Eisenstein series for $\Gamma_1[p]$ has dimension $2\frac{p-1}{2}$; as an Hermitian space for $\{,\}$ it is an orthogonal sum of $\frac{p-1}{2}$ hyperbolic planes.*

Proof. — For primitive Dirichlet characters χ mod p with $\chi(-1) = (-1)^k$ we consider the two Eisenstein series of nebentypus χ defined by

$$E_\chi(z) := A(\chi) + \sum_{n \geqslant 1} \sigma_{k-1}(n, \chi) e(2nz)$$

and

$$F_\chi(z) := \sum \sigma_{k-1}^*(n, \chi) e(2nz)$$

with

$$\sigma_r(n, \chi) = \sum_{d|n} \chi(d) d^r, \qquad \sigma_r^*(n, \chi) = \sum_{d|n} \chi(\frac{n}{d}) d^r;$$

the exact shape of the constant $A(\chi)$ is not interesting for us.

The Dirichlet series attached to these modular forms (in the sense of Hecke) are

$$\mathcal{L}(s,\chi) = \sum \sigma_{k-1}(n,\chi)n^{-s} = \zeta(s) \cdot L(s-k+1,\chi)$$

and

$$\mathcal{L}^*(s,\chi) = \sum \sigma_{k-1}^*(n,\chi)n^{-s} = \zeta(s-k+1) \cdot L(s,\chi).$$

Then we have to consider three Rankin convolutions; their factorizations can be obtained from the factorizations above by a general principle (see e.g. [28], Lemma 1); for two primitive characters χ_1, χ_2 mod p with appropriate parity

$$\mathcal{R}(E_{\chi_1}, E_{\chi_2}; s) = \sum \sigma_{k-1}(n,\chi_1) \cdot \sigma_{k-1}(n,\overline{\chi_2})n^{-s} =$$
$$\frac{\zeta(s) \cdot L(s-k+1,\chi_1) \cdot L(s-k+1,\overline{\chi_2}) \cdot L(s-2k+2,\chi_1\overline{\chi_2})}{L(2s-2k+2,\chi_1\overline{\chi_2})}$$

and

$$\mathcal{R}(F_{\chi_1}, F_{\chi_2}; s) = \sum \sigma_{k-1}^*(n,\chi_1) \cdot \sigma_{k-1}^*(n,\overline{\chi_2})n^{-s} =$$
$$\frac{\zeta(s-2k+2) \cdot L(s-k+1,\chi_1) \cdot L(s-k+1,\overline{\chi_2}) \cdot L(s,\chi_1\overline{\chi_2})}{L(2s-2k+2,\chi_1\overline{\chi_2})}$$

and

$$\mathcal{R}(E_{\chi_1}, F_{\chi_2}; s) = \sum \sigma_{k-1}(n,\chi_1) \cdot \sigma_{k-1}^*(n,\overline{\chi_2})n^{-s} =$$
$$\frac{\zeta(s-k+1) \cdot L(s-2k+2,\chi_1) \cdot L(s,\overline{\chi_2}) \cdot L(s-k+1,\chi_1\overline{\chi_2})}{L(2s-2k+2,\chi_1\overline{\chi_2})}$$

Then the first two convolutions do not have a pole at $s = k$; the last one has a pole at $s = k$ if and only if $\chi_1 = \chi_2$.

For the principal character we prefer a slightly modified treatment: we put (again with a suitable constant A)

$$E_{\chi_0}(z) := A + \sum \sigma_{k-1}(n)e(2nz)$$

and

$$F_{\chi_0}(z) = \sum \sigma_{k-1}^*(n)e(2nz).$$

Of course E_{χ_0} is of level one and $\sigma_r^*(n) = \sum_{d|n} d^r$ where $\frac{n}{d}$ has to be coprime to p.

By similar reasons as before E_{χ_0} and F_{χ_0} are orthogonal to the Eisenstein series with nontrivial character.

We get

$$\mathcal{R}(E_{\chi_0}, E_{\chi_0}; s) = \frac{\zeta(s) \cdot \zeta(s-k+1)^2 \cdot \zeta(s-2k+2)}{\zeta(2s-2k+2)},$$

$$\mathcal{R}(F_{\chi_0}, F_{\chi_0}; s) = \frac{\zeta_p(s) \cdot \zeta_p(s-k+1)^2 \cdot \zeta(s-2k+2)}{\zeta_p(2s-2k+2)},$$

$$\mathcal{R}(E_{\chi_0}, F_{\chi_0}; s) = \frac{\zeta_p(s) \cdot \zeta_p(s-k+1) \cdot \zeta(s-k+1) \cdot \zeta(s-2k+2)}{\zeta_p(2s-2k+2)}.$$

The Gram matrix for this two-dimensional space then equals

$$\{E_{\chi_0}, E_{\chi_0}\} \times \begin{pmatrix} 1 & \frac{(1-p^{-1})(1-p^{-k})}{1-p^{-2}} \\ \frac{(1-p^{-1})(1-p^{-k})}{1-p^{-2}} & \frac{(1-p^{-1})^2(1-p^{-k})}{1-p^{-2}} \end{pmatrix}.$$

The determinant of this matrix is negative (for $k > 2$), therefore we get another hyperbolic plane! □

Remark 5.6. — The proof above also shows that for $p = 2$ the space of Eisenstein series is a hyperbolic plane for $\{,\}$.

Remark 5.7. — There is another argument to show that $\{,\}$ is in general not positive definite (without obtaining any result about the signature): by arguing as in Kohnen's paper [16] the definiteness of $\{,\}$ would give estimates for Hecke eigenvalues for all eigenforms; these estimates would in general be too sharp for noncuspidal eigenforms. This argument only works if we include the case of congruence subgroups.

It may be of some interest to study (in the same way as was done above for degree 1) the explicit arithmetic form of Rankin-Selberg convolutions of Eisenstein series of higher degree. We describe some formulas for degree 2; here the Rankin-Selberg convolutions are of the form

$$\sum_D \gamma(D) D^{-s}$$

where D runs over all positive numbers satisfying $D \equiv 3 \mod 4$. These D can be written as $D = D_0 f^2$ where $-D_0$ is a discriminant of an imaginary quadratic field. In most cases $\gamma(D_0)$ has a simple form, and $\gamma(D)$ can be expressed (in a complicated way) in terms of $\gamma(D_0)$.

Example 5.8. — F is arbitrary, but G is in the Maaß space. Then

$$\gamma(D_0) = \overline{b(D_0)} \sum_N \frac{a(N)}{\epsilon(N)}$$

where N runs over representatives of binary quadratic forms of determinant D_0 and $b(D_0) = b(N)$ (any such N) is a Fourier coefficient of a modular

form (of half-integral weight) corresponding to G. Let us look at special cases:

Case 1 F is also in the Maaß space. Then

$$\gamma(D_0) = h(D_0)a(D_0)\overline{b(D_0)}$$

where $h(D_0)$ is the class number of the imaginary quadratic number field of discriminant $-D_0$ (and again $a(D_0) := a(T)$ for any such T).

In other words, the Rankin convolution for F and G is then a kind of *Rankin triple convolution for three modular forms of half-integral weight*.

Case 2 $F = E_{2,1}(f)$, where f is a (normalized) cuspidal Hecke eigenform of degree 1 and $E_{2,1}(f)$ denotes the Klingen Eisenstein series attached to it. From the explicit formulas for the Fourier coefficients given in [1] we can get easily that

$$\sum_N \frac{a(N)}{\epsilon(N)} \sim D_0^{k-1} L(f, \chi_{D_0}, k-1)$$

where χ_{D_0} is the quadratic character associated to D_0; then

$$\gamma(D_0) \sim D_0^{k-1} L(f, \chi_{D_0}, k-1)\overline{b(D_0)}$$

which is a somewhat unfamiliar object.

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