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EULERIAN IDEMPOTENT AND KASHIWARA-VERGNE CONJECTURE

by Emily BURGUNDER

Abstract. — By using the interplay between the Eulerian idempotent and the Dynkin idempotent, we construct explicitly a particular symmetric solution (F, G) of the first equation of the Kashiwara-Vergne conjecture

$$x + y - \log(\mathrm{e}^y \mathrm{e}^x) = (1 - \mathrm{e}^{-\operatorname{ad} x})F(x, y) + (\mathrm{e}^{\operatorname{ad} y} - 1)G(x, y).$$

Then, we explicit all the solutions of the equation in the completion of the free Lie algebra generated by two indeterminates x and y thanks to the kernel of the Dynkin idempotent.

Résumé. — Grâce à l'interaction entre l'idempotent de Dynkin et l'idempotent eulérien, on construit une solution particulière et symétrique $(F(x, y), F(-y, -x))$ de la première équation de la conjecture de Kashiwara-Vergne :

$$x + y - \log(\mathrm{e}^y \mathrm{e}^x) = (1 - \mathrm{e}^{-\operatorname{ad} x})F(x, y) + (\mathrm{e}^{\operatorname{ad} y} - 1)G(x, y).$$

Puis on explicite toutes les solutions de cette équation dans l'algèbre de Lie libre générée par les indéterminées x et y grâce au noyau de l'idempotent de Dynkin.

Introduction

M. Kashiwara and M. Vergne [4] put forward a conjecture that implies the Duflo Theorem on the local solvability of biinvariant differential operators on arbitrary finite Lie groups as well as a more general statement on convolution of invariant distributions:

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Conjecture 1 (Kashiwara-Vergne [4]). — *For any Lie algebra* $\mathfrak{g}$ *of finite dimension, we can find Lie series* F *and* G *such that they satisfy:*

(1) $x + y - \log(\mathrm{e}^y \mathrm{e}^x) = (1 - \mathrm{e}^{-\operatorname{ad} x})F(x,y) + (\mathrm{e}^{\operatorname{ad} y} - 1)G(x,y);$

(2) F *and* G *give* $\mathfrak{g}$*-valued convergent power series on* $(x,y) \in \mathfrak{g} \times \mathfrak{g}$;

(3) $$\operatorname{tr}(\operatorname{ad} x \circ \partial_x F; \mathfrak{g}) + \operatorname{tr}(\operatorname{ad} y \circ \partial_y G; \mathfrak{g}) = \frac{1}{2}\operatorname{tr}\Big(\frac{\operatorname{ad} x}{\mathrm{e}^{\operatorname{ad} x} - 1} + \frac{\operatorname{ad} y}{\mathrm{e}^{\operatorname{ad} y} - 1} - \frac{\operatorname{ad} \Phi(x,y)}{\mathrm{e}^{\operatorname{ad} \Phi(y,x)} - 1} - 1; \mathfrak{g}\Big).$$

Here $\Phi(x,y) = \log(\mathrm{e}^x \mathrm{e}^y)$ *and* $\partial_x F$ *(resp.* $\partial_y G$*) is the* $\mathbf{End}(\mathfrak{g})$*-valued real analytic function defined by*

$$\mathfrak{g} \ni a \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}F(x + ta, y)_{|t=0} \quad \Big(\textit{resp. } \mathfrak{g} \ni a \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}G(x, y + ta)_{|t=0}\Big),$$

and tr *denotes the trace of an endomorphism of* $\mathfrak{g}$*.*

In the course of their proof they consider the first equation in the completion of the free Lie algebra generated by two indeterminates x and y, denoted $\mathrm{Lie}(V)^\wedge$, where $V = \mathbb{K}x \oplus \mathbb{K}y$.

In this paper, we exhibit explicitly all the solutions of the first equation of the Kashiwara-Vergne conjecture in $\mathrm{Lie}(V)^\wedge$.

First, we display a particular solution thanks to a splitting of the equation and the use of two idempotents: the Eulerian idempotent e, and the Dynkin idempotent γ.

The major problem to find solutions of equation (1) is that there isn't any convenient basis of the free Lie algebra that eases the calculation of the Baker-Campbell-Hausdorff series. The Eulerian idempotent (cf. [6]) is the key to explicit the Baker-Campbell-Hausdorff series in terms of permutations.

We split the Baker-Campbell-Hausdorff series into two Lie formal power series $\Phi^-(y,x)$ and $\Phi^+(y,x)$, thanks to the Dynkin idempotent, such that they are in the image of $\exp(\operatorname{ad}(-x)) - 1$ (resp. image of $\exp(\operatorname{ad} y) - 1$).

We split equation (1) into the sum of two equations:

$$\begin{aligned}\Phi^-(y,x) - x &= \big(\exp\operatorname{ad}(-x) - 1\big)F(x,y),\\ \Phi^+(y,x) - y &= \big(1 - \exp\operatorname{ad}(y)\big)G(x,y),\end{aligned}$$

and we give a unique and explicit solution $(F_0(x,y), F_0(-y,-x))$ in terms of permutations. Hence we obtain a symmetric solution of equation (1) by adding the two equations.

Any solution of equation (1) on $\mathrm{Lie}(V)^\wedge$ is the sum of a particular solution and of a solution of the homogeneous equation

$$\big(\exp \mathrm{ad}(-x) - 1\big)F(x, y) = \big(\exp \mathrm{ad}(y) - 1\big)G(x, y),$$

for Lie formal power series F and G.

We prove that any solution (F, G) of the homogenous equation can be made explicit in terms of permutations and verifies that $xF(x, y)+yG(x, y)$ is in the kernel of the Dynkin idempotent, and conversely any element of the kernel of the Dynkin idempotent determines a solution of the homogeneous equation.

Moreover these tools can be extended to prove a multilinearised version of the Kashiwara-Vergne conjecture.

There exists some solutions of the Conjecture 1 for some specific algebras by M. Kashiwara and M. Vergne [4], F. Rouvière [9] and M. Vergne [11]. A. Alekseev and E. Meinrenken [1] proved the existence of a solution in the general case using C. Torossian arguments [10] which are based on Kontsevich's work [5]. The latter has not been made explicit and it is still unknown whether it is rational.

The paper is organised as follows: in section 1 we set notations, in section 2 and 3, we recall respectively the constructions of the Dynkin idempotent and the Eulerian idempotent. Section 4 is devoted to the study of the operator $\exp \mathrm{ad}\, x - 1$. The particular solution of equation (1) is constructed in section 5. This solution admits, in a certain way, a property of unicity which is treated in section 6. Then, we solve the homogeneous equation in section 7 and give a description of any solution of equation (1) in the free Lie algebra. In section 8 we give another description of these solutions using another description of the kernel of the Dynkin idempotent due to F. Patras and C. Reutenauer. Section 9 is devoted to the multilinear Kashiwara-Vergne conjecture.

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In this paper $\mathbb{K}$ denotes a characteristic zero field, that is to say $\mathbb{K} \supset \mathbb{Q}$.

1. Definitions and properties

We recall the definitions of bialgebras, convolution, tensor bialgebra and the free Lie algebra.

1.1. Bialgebra and convolution

A *bialgebra* $\mathcal{H}$ is a vector space endowed with an associative product $\mu : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$, a unit $u : \mathbb{K} \to \mathcal{H}$, a co-associative coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, and a counit $c : \mathcal{H} \to \mathbb{K}$ such that Δ is an algebra morphism or, equivalently, such that μ is a coalgebra morphism.

The *primitive part* of a bialgebra is the subvector space of $\mathcal{H}$ defined as

$$\operatorname{Prim}\mathcal{H} := \big\{x \in \mathcal{H} : \Delta(x) = 1 \otimes x + x \otimes 1\big\}.$$

Let $f, g : \mathcal{H} \longrightarrow \mathcal{H}$ be two bialgebra morphisms. The *convolution* of f and g is a bialgebra morphism defined as

$$f \star g := \mu \circ (f \otimes g) \circ \Delta : \mathcal{H} \longrightarrow \mathcal{H}.$$

The convolution satisfies these easily verified propositions:

Proposition 2. — *The convolution is associative.* □

Proposition 3. — *The convolution admits $u \circ c : \mathcal{H} \to \mathcal{H}$ for unit.* □

1.2. Tensor bialgebra

Let V be a $\mathbb{K}$-vector space. The *tensor algebra* is the tensor module

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

endowed with the concatenation product

$$\mu : T(V) \otimes T(V) \longrightarrow T(V),$$
$$v_1 \otimes \cdots \otimes v_n \bigotimes v_{n+1} \otimes \cdots \otimes v_{n+p} \longmapsto v_1 \otimes \cdots \otimes v_{n+p}.$$

The tensor algebra can, moreover, be endowed with a unique coproduct (the co-shuffle) $\Delta : T(V) \to T(V) \otimes T(V)$ such that $\Delta(v) = 1 \otimes v + v \otimes 1$, making it into a co-commutative bialgebra.

Remark 4. — If the $\mathbb{K}$-vector space V is spanned by $\{x_1, \dots, x_n\}$, then the tensor algebra $T(V)$ is spanned by all the tensors $x_{i_1} \otimes \cdots \otimes x_{i_m}$ where $i_j \in \{1, \dots, n\}$, and $m \in \mathbb{N}$. By the isomorphism induced by

$$x_{i_1} \otimes \cdots \otimes x_{i_m} \longmapsto x_{i_1} x_{i_2} \cdots x_{i_m}$$

for all $x_{i_j} \in \{x_1, \dots, x_n\}$, the tensor algebra is isomorphic to the algebra of non-commutative polynomials in variables $x_1, \dots, x_n$.

Following this remark we introduce the following notation in the algebra of non-commutative polynomials in variables $x_1, \dots, x_n$.

Definition 5. — Let $p(x_1, \dots, x_n)$ be a non-commutative polynomial in the variables $x_1, \dots, x_n$. Then it determines uniquely n polynomials $b_1(x_1, \dots, x_n), \dots, b_n(x_1, \dots, x_n)$ such that

$$p(x_1, \dots, x_n) = x_1 b_1(x_1, \dots, x_n) + \cdots + x_n b_n(x_1, \dots, x_n).$$

We call the polynomial $b_i(x_1, \dots, x_n)$ the *x_i-part* of $p(x_1, \dots, x_n)$ and denote it as $(p(x_1, \dots, x_n))_{x_i}$.

Let V be any vector space. The tensor bialgebra $T(V)$ verifies moreover this well-known connectedness property:

PROPOSITION 6. — *Let $J := \mathrm{Id} - u \circ c$. For any $w \in V^{\otimes n}$, $J^{\star n}(w) = 0$.*

1.3. Action of the symmetric group

Let S_n be the symmetric group acting on $\{1, \dots, n\}$. It acts by the right on $V^{\otimes n}$ by permutation of the variables: for all $x_1, \dots, x_n \in V$ and all $\sigma \in S_n$ the action is given by

$$(x_1, \dots, x_n)^\sigma = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Let $\sigma \in S_n$ be a permutation. We denote by $\mathrm{Des}(\sigma)$ the set of *descent* of σ defined as

$$\mathrm{Des}(\sigma) := \big\{ i : \sigma(i) > \sigma(i+1) \big\}$$

and by $d(\sigma)$ the *number of descents* of σ which is the number of integers i such that $\sigma(i) > \sigma(i+1)$.

Denote by $D_{\{1,\dots,k\}}$ the set of all the permutations $\sigma \in S_n$ such that its descent set $\mathrm{Des}(\sigma)$ is exactly the set $\{1, \dots, k\}$.

1.4. Free Lie algebra

Let V be a $\mathbb{K}$-vector space. The *free Lie algebra over V*, denoted $\mathrm{Lie}(V)$, is defined by the following property:

Any map $f : V \to \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra, extends uniquely to a Lie algebra morphism $\tilde{f} : \mathrm{Lie}(V) \to \mathfrak{g}$. Diagrammatically, this would be read as the commutativity of the following diagram:

$$\begin{tikzcd} \forall\mathfrak{g}, \quad V \arrow[r, hook, "i"] \arrow[dr, "\forall f"'] & \mathrm{Lie}(V) \arrow[d, "\exists! \tilde{f}"] \\ & \mathfrak{g} \end{tikzcd}$$

It is known that $\mathrm{Lie}(V)$ can be identified with the subspace of $T(V)$ generated by V under the bracket $[x, y] = x \otimes y - y \otimes x$. So we have

$$\mathrm{Lie}(V) = V \oplus [V, V] \oplus [V, [V, V]] \oplus \cdots \oplus \underbrace{[V, [V, [\cdots [V, V] \cdots]]]}_{n \text{ times}} \oplus \cdots .$$

2. Free Lie algebra and Dynkin idempotent [8], [12]

We recall the notion of Dynkin idempotent and some useful properties of the free Lie algebra.

We define the completion of the free Lie algebra $\mathrm{Lie}(V)^\wedge$ as

$$\mathrm{Lie}(V)^\wedge := \prod_{n \geqslant 0} \mathrm{Lie}(V)_n.$$

Definition 7. — The *Dynkin idempotent* is defined as

$$\gamma : T(V) \longrightarrow \mathrm{Lie}(V) \hookrightarrow T(V),$$
$$v_1 \otimes \cdots \otimes v_n \longmapsto \frac{1}{n}[v_1, [v_2, [\cdots [v_{n-1}, v_n] \cdots]]].$$

The map γ restricted to $\mathbb{K}$ is null and its restriction to V is the identity. It is clear from the definition of γ and of $\mathrm{Lie}(V)$ that $\mathrm{Im}\,\gamma = \mathrm{Lie}(V)$. We denote $\gamma_{|V^{\otimes n}} : V^{\otimes n} \to V^{\otimes n}$ by $\gamma_n : V^{\otimes n} \to V^{\otimes n}$ the restriction of the Dynkin idempotent on $V^{\otimes n}$.

Moreover, this map satisfies the following properties.

Proposition 8. — *For any $x, x_2, \ldots, x_n \in V$, with $n \geqslant 2$, one has*

$$\gamma(x \otimes x_2 \otimes \cdots \otimes x_n) = \frac{n-1}{n}\,\mathrm{ad}(x)\gamma(x_2 \otimes \cdots \otimes x_n),$$

where $\mathrm{ad}(x) : T(V) \to T(V)$, $y \mapsto [x, y]$.

Proof. — Let $x, x_2, \dots, x_n \in V$. A direct computation gives

$$\begin{aligned}\gamma(x \otimes x_2 \otimes \cdots \otimes x_n) &= \frac{1}{n}[x, [x_2, [\cdots [x_{n-1}, x_n] \cdots]]] \quad \text{(by definition)}\\ &= \frac{n-1}{n}\big[x, \gamma(x_2 \otimes \cdots \otimes x_{n-1} \otimes x_n)\big],\end{aligned}$$

which completes the proof. □

Proposition 9 (Friedrichs-Specht-Wever, cf. [12]). — *Suppose that $\mathbb{K}$ is of characteristic zero. If $x \in V^{\otimes n}$, the following are equivalent:*

- ▹ $x \in \mathrm{Lie}(V)$,
- ▹ $\Delta(x) = 1 \otimes x + x \otimes 1$,
- ▹ $\gamma(x) = x$.

Remark that, from this proposition, it becomes clear that γ is an idempotent that is to say $\gamma \circ \gamma = \gamma$.

Moreover this proposition gives another way to see $\mathrm{Lie}(V)$ as a subspace of $T(V)$.

Proposition 10. — *Let V be a $\mathbb{K}$-vector space. The primitive part of the tensor bialgebra is isomorphic to the free Lie algebra:*

$$\mathrm{Prim}\, T(V) \cong \mathrm{Lie}(V).$$

We state the following proposition with the notations of subsection 1.3.

Proposition 11 (Reutenauer, cf. [8]). — *The restricted Dynkin idempotent $\gamma_n : V^{\otimes n} \to V^{\otimes n}$ can be made explicit in terms of permutations as follows:*

$$\gamma_n(x_1 \cdots x_n) = \frac{(-1)^{n-1}}{n} \sum_{k=0}^{n-1} (-1)^k \sum_{\sigma \in D_{\{1,\dots,k\}}} (x_n \cdots x_1)^\sigma.$$

Proposition 12. — *Let V be a $\mathbb{K}$-vector space.*

- ▹ *For $n = 0$, the kernel of the Dynkin idempotent restricted to $\mathbb{K}$ is $\mathbb{K}$.*
- ▹ *For $n = 1$, the kernel of the Dynkin idempotent restricted to V is $\{0\}$.*
- ▹ *Let $n \geqslant 2$. The kernel of the Dynkin idempotent restricted to $V^{\otimes n}$ is spanned by the elements*

$$(n-1)x_1 \cdots x_n + \sum_{k=1}^{n-2} \sum_{\sigma \in D_{\{1,\dots k\}}} (-1)^{n+k} (x_n \cdots x_1)^\sigma, \quad (x_i \in V).$$

Proof. — The cases $n = 0$ and $n = 1$ follow from the definition. So, we focus on the case $n \geqslant 2$. As the Dynkin idempotent is a linear projector we can apply the usual trick which consists in writing any element $a \in T(V)$

as the following sum $(a-\gamma(a))+\gamma(a)$ belonging to $\operatorname{Ker}\gamma \oplus \operatorname{Im}\gamma$. Monomials $x_1 \cdots x_n$ of degree n are a basis for $V^{\otimes n}$, where $x_i \in V$. The kernel of the Dynkin idempotent are spanned by the $x_1 \cdots x_n - \gamma(x_1 \cdots x_n)$. To conclude it suffices to explicit the element $\gamma(x_1 \cdots x_n)$ in terms of permutations thanks to the above proposition: indeed, the only permutation σ of descent $\operatorname{Des}(\sigma) = \{1, \ldots, n-1\}$ is $\sigma = (n, n-1, \ldots, 1)$ and its sign is $(-1)^{2(n-1)} = 1$. $\square$

Adapting Patras and Reutenauer's [7] description of the kernel of the Dynkin idempotent as a span we get the following proposition.

Proposition 13 (cf. [3], [7]). — *The kernel of the Dynkin idempotent is spanned by* 1 *and the elements of the form* $\gamma(a(x,y))a(x,y)$, *where* $a(x,y)$ *is a non-commutative polynomial in the two indeterminates* x *and* y.

3. Eulerian idempotent, cf. [6]

We recall the notion of Eulerian idempotent and its link with the Baker-Campbell-Hausdorff series. Indeed, the Eulerian idempotent leads to an explicit formulation of the series in terms of permutations.

From now on, we consider $T(V)$ as a $\mathbb{K}$-bialgebra with the concatenation μ, a unit u, the co-shuffle Δ, and a counit c. The convolution product is denoted by $\star$. We will say Lie series for formal power Lie series.

Definition 14. — Define the map $J := \operatorname{Id} - u \circ c : T(V) \to T(V)$. The *Eulerian idempotent* e is the endomorphism of $T(V)$ defined by the formal power series

$$e := \log^\star(\operatorname{Id}) = \log^\star(J + u \circ c) = J - \frac{J^{\star 2}}{2} + \cdots + (-1)^{k-1}\frac{J^{\star k}}{k} + \cdots,$$

where $J^{\star n} = \underbrace{J \star J \star \cdots \star J}_{n \text{ times}}$.

Proposition 2 assures that $J^{\star n}$ is well-defined. The map $J : T(V) \to T(V)$ is the identity on $V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$ and is null on $\mathbb{K}$. Thanks to the connectedness property of Proposition 6, the restriction of e to $V^{\otimes n}$ is polynomial and is equal to

$$e_{|V^{\otimes n}} = J - \frac{J^{\star 2}}{2} + \cdots + (-1)^n \frac{J^{\star n-1}}{n-1}.$$

The restriction of the map e to $V^{\otimes n}$ is denoted $e_n : V^{\otimes n} \to V^{\otimes n}$.

Proposition 15 (cf. [6]). — *The map* $e : T(V) \to T(V)$ *verifies the following properties:*

▷ $\operatorname{Im} e = \operatorname{Lie}(V)$,

▷ *the map e is an idempotent, i.e. $e \circ e = e$.*

The Baker-Campbell-Hausdorff series is a formal power series

$$\Phi(x,y) = \sum_{n \geqslant 1} \Phi_n(x,y),$$

where $\Phi_n(x,y)$ is a homogeneous polynomial of degree n, in non-commutative variables x and y, defined by the equation

$$\exp(x)\exp(y) = \exp\big(\Phi(x,y)\big), \tag{4}$$

where exp denotes the exponential series.

This formal power Lie series can be extended to n variables, by defining $\Phi(x_1,\dots,x_n)$ as

$$\exp(x_1)\cdots\exp(x_n) = \exp\big(\Phi(x_1,\dots,x_n)\big).$$

Let $\Phi_m(x_1,\dots,x_n)$ denote the homogeneous part of $\Phi(x_1,\dots,x_n)$ of total degree m, and $\varphi_n(x_1,\dots,x_n)$ the multilinear part of $\Phi_n(x_1,\dots,x_n)$, (replace x_i^2 by 0, for all i).

Proposition 16 (Dynkin, cf. [6]). — *The following equality holds:*

$$\Phi_m(x_1,\dots,x_n) = \sum_{\substack{i_1+\cdots+i_n=m \\ i_j \geqslant 0}} \frac{1}{i_1!\dots i_n!}\varphi_m(\underbrace{x_1\dots,x_1}_{i_1},\dots,\underbrace{x_n\dots,x_n}_{i_n}).$$

The next proposition relates the Baker-Campbell-Hausdorff series to the Eulerian idempotent.

Proposition 17 (cf. [6]). — *The following equality holds:*

$$\varphi_n(x_1,\dots,x_n) = e_n(x_1,\dots,x_n).$$

3.1. Explicitation of e_n

The Eulerian idempotent $e_n : V^{\otimes n} \to V^{\otimes n}$ can be made explicit as a formal power series of permutations with the notations of subsection 1.3.

Proposition 18 (cf. [6]). — *The Eulerian idempotent has the following explicit formula in terms of permutations:*

$$e_n = \sum_{\sigma \in \mathcal{S}_n} c_\sigma (.)^\sigma, \quad \textit{where} \quad c_\sigma = \frac{(-1)^{d(\sigma)}}{n}\binom{n-1}{d(\sigma)}^{-1}.$$

Here $\binom{n}{p}$ denotes the binomial number.

If we restrict ourselves to $T(V)$, with $V = \mathbb{K}x \oplus \mathbb{K}y$, then the above formulas lead to:

Proposition 19. — *Let $V = \mathbb{K}x \oplus \mathbb{K}y$. The Baker-Campbell-Hausdorff series can be made explicit in terms of permutations:*

$$\Phi(x,y) = \sum_{n\geqslant 1} \sum_{\substack{i+j=n\\ i,j\geqslant 1}} \frac{1}{i!}\frac{1}{j!} \sum_{\sigma\in\mathcal{S}_n} c_\sigma(\underbrace{x,\dots,x}_{i},\underbrace{y,\dots,y}_{j})^\sigma,$$

where $c_\sigma = \frac{(-1)^{d(\sigma)}}{n}\binom{n-1}{d(\sigma)}^{-1}$.

Proof. — By definition $\Phi(x,y) = \sum_{n\geqslant 1}\Phi_n(x,y)$. Restricting Propositions 17 and 16 to the two variables x and y, we get that

$$\Phi(x,y) = \sum_{n\geqslant 1} \sum_{\substack{i+j=n\\ i,j\geqslant 1}} \frac{1}{i!}\frac{1}{j!} e_n(\underbrace{x,\dots,x}_{i},\underbrace{y,\dots,y}_{j}).$$

Proposition 18 completes the proof. □

3.2. Symmetric properties of the Eulerian idempotent

Let ω denote the permutation

$$\omega(1,\dots,n) := (n, n-1,\dots,2,1).$$

With the above explicitation of the Eulerian idempotent, we remark :

Proposition 20. — *The Eulerian idempotent verifies the symmetry*

$$e_n = (-1)^{n+1} e_n \circ (.)^\omega.$$

Proof. — We first note that $(-1)^{n+1}c_\sigma = c_{\omega\circ\sigma}$. Indeed, we have:

$$\begin{aligned} c_{\omega\circ\sigma} &= \frac{(-1)^{d(\omega\circ\sigma)}}{n}\binom{n-1}{d(\omega\circ\sigma)}^{-1} = \frac{(-1)^{n-1}}{n}(-1)^{d(\sigma)}\binom{n-1}{n-1-d(\sigma)}^{-1}\\ &= \frac{(-1)^{n-1}}{n}(-1)^{d(\sigma)}\binom{n-1}{d(\sigma)}^{-1} = (-1)^{n-1}c_\sigma, \end{aligned}$$

as $d(\omega\circ\sigma) = n-1-d(\sigma)$. This gives the expected property. As a consequence we have

$$\begin{aligned} e_n &= \sum_{\sigma\in\mathcal{S}_n} c_\sigma(.)^\sigma = \sum_{\omega\circ\sigma\in\mathcal{S}_n} c_{\omega\circ\sigma}(.)^{\omega\circ\sigma}\\ &= (-1)^{n-1}\sum_{\sigma\in\mathcal{S}_n} c_\sigma(.)^{\omega\circ\sigma} = (-1)^{n+1}e_n\circ(.)^\omega. \end{aligned}$$ □

As a consequence we can prove that:

Lemma 21. — *The x-part of the Eulerian idempotent verifies the symmetry property*

$$\big(e_n(\underbrace{x,\dots,x}_{i},\underbrace{y,\dots,y}_{j})\big)_x = -(e_n(\underbrace{-y,\dots,-y}_{j},\underbrace{-x,\dots,-x}_{i}))_x.$$

Proof. — Let ω denote the permutation

$$\omega(1,\dots,n) := (n, n-1, \dots, 2, 1).$$

Proposition 20 gives a symmetry property of the Eulerian idempotent $e_n = (-1)^{n+1} e_n \circ (.)^\omega$. From this symmetry property we deduce

$$\begin{aligned}
e_n\big(\underbrace{x,\dots,x}_{i},\underbrace{y,\dots,y}_{j}\big) &= (-1)^{n+1} e_n \circ \big(\underbrace{x,\dots,x}_{i},\underbrace{y,\dots,y}_{j}\big)^\omega\\
&= (-1)^{n+1} e_n\big(\underbrace{y,\dots,y}_{j},\underbrace{x,\dots,x}_{i}\big)\\
&= -e_n\big(\underbrace{-y,\dots,-y}_{j},\underbrace{-x,\dots,-x}_{i}\big)\\
&= -\Big(x\big(e_n(\underbrace{-y,\dots,-y}_{j},\underbrace{-x,\dots,-x}_{i})\big)_x\\
&\qquad\qquad + y\big(e_n(\underbrace{-y,\dots,-y}_{j},\underbrace{-x,\dots,-x}_{i})\big)_y\Big).
\end{aligned}$$

By identifying the monomials starting with x, we get

$$\big(e_n(\underbrace{x,\dots,x}_{i},\underbrace{y,\dots,y}_{j})\big)_x = -\big(e_n(\underbrace{-y,\dots,-y}_{j},\underbrace{-x,\dots,-x}_{i})\big)_x.$$

This ends the proof. □

4. The operator $\mathrm{E}(x)$

This section is devoted to a study of the image and the kernel of the operator

$$\exp \operatorname{ad} x - 1 : T(V) \longrightarrow T(V),$$

restricted to $\exp \operatorname{ad} x - 1 : \mathrm{Lie}(V) \to \mathrm{Lie}(V)$, which is needed in severals proofs.

Let $\mathbb{K}$ be a characteristic zero field, i.e. $\mathbb{K} \supset \mathbb{Q}$. From now on, V is the following $\mathbb{K}$-vector space $V = \mathbb{K}x \oplus \mathbb{K}y$, and $T(V)$ the tensor bialgebra defined in sections 1.2 and 3. Recall that this tensor bialgebra is isomorphic to the non-commutative polynomial bialgebra in two variables x and y. For conveniency, we will say Lie series for Lie formal power series.

We will use the notation:

Notation 22. — We denote $\mathrm{E}(x) : \mathrm{Lie}(V) \to \mathrm{Lie}(V)$ the map

$$\mathrm{E}(x) := \exp(\operatorname{ad} x) - 1 = \sum_{n\geqslant 1} \frac{(\operatorname{ad} x)^n}{n!}\cdot$$

Notation 23. — Let B_n be the n-th Bernoulli number defined as:

$$B_n := (-1)^n \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \binom{2n+1}{k} \sum_{r=0}^{k} r^{2n}.$$

Notation 24. — We denote $\mathrm{Ber}(x) := \mathrm{ad}(x)/\mathrm{E}(x) : \mathrm{Lie}(V) \to \mathrm{Lie}(V)$ the map defined as

$$\mathrm{Ber}(x) := \sum_{k\geqslant 0} B_k \frac{\mathrm{ad}(x)^k}{k!}\cdot$$

Proposition 25. — *The maps* $\mathrm{Ber}(x), \mathrm{E}(x) : T(V) \to T(V)$ *verify*

$$\mathrm{Ber}(x) \circ \mathrm{E}(x) = \mathrm{E}(x) \circ \mathrm{Ber}(x) = \mathrm{ad}(x)\cdot$$

Proof. — The composition of series of operators is the multiplication of the series. As series $t/(\exp(t) - 1)$ defined as $\sum_{n\geqslant 0} B_k t^k/k!$ admits the property

$$\big(\exp(t) - 1\big)\frac{t}{\exp(t) - 1} = \frac{t}{\exp(t) - 1}\big(\exp(t) - 1\big) = t,$$

this ends the proof. □

Proposition 26. — *The image of* $\mathrm{E}(x) : \mathrm{Lie}(V) \to \mathrm{Lie}(V)$ *is*

$$\operatorname{Im} \mathrm{E}(x) = \operatorname{Im} \operatorname{ad} x.$$

Proof. — Clearly $\operatorname{Im} \mathrm{E}(x) \subset \operatorname{Im} \operatorname{ad} x$. Moreover, any element in $\operatorname{Im} \operatorname{ad} x$ is of the form $[x, \alpha] \in \operatorname{Im} \operatorname{ad} x$, where $\alpha \in \mathrm{Lie}(V)$. Then, $\beta := \mathrm{Ber}(x)(\alpha) = \sum_{n\geqslant 0} B_n/n!\,(\operatorname{ad} x)^n(\alpha) \in \mathrm{Lie}(V)$ is such that

$$\mathrm{E}(x)(\beta) = \mathrm{E}(x) \circ (\mathrm{Ber}\, x)(\beta) = \operatorname{ad} x(\alpha) = [x, \alpha].$$

□

Lemma 27. — *The kernel of the operator* $\operatorname{ad} x : T(V) \to T(V)$ *is spanned by the elements* x^n*:*

$$\operatorname{Ker} \operatorname{ad} x = \mathbb{K}[x].$$

Proof. — Let $p(x,y)$ be a non-commutative polynomial in the kernel of $\mathrm{ad}\,x$, i.e. such that $\mathrm{ad}\,x\,p(x,y) = 0$. This non-commutative polynomial can be decomposed as the sum of its homogeneous part $p(x,y) = \sum_{n\geqslant 0} p_n(x,y)$, and each $p_n(x,y)$ verifies $\mathrm{ad}\,x\,p_n(x,y) = 0$.

The proof is based on induction on the total degree n of the homogeneous polynomial.

▹ If $n = 0$, then $p_0(x,y) = \lambda$, where $\lambda \in \mathbb{K}$, verifies the equation.

▹ If $n = 1$, then a generic non-commutative homogeneous polynomial of degree one can be made explicit on the basis of V as $p_1(x,y) = \lambda x + \mu y$, where $\lambda, \mu \in \mathbb{K}$. Then, in order to verify the equation $\mathrm{ad}\,x\,p_n(x,y) = 0$, we must take $\mu = 0$ and $\lambda \in \mathbb{K}$.

▹ For the degree n, suppose by induction that $p_k(x,y) = \lambda x^k$, for any $k \leqslant n-1$. The homogeneous non-commutative polynomial $p_n(x,y)$ can be split into

$$p_n(x,y) = xq_{n-1}(x,y) + yr_{n-1}(x,y),$$

where $q_{n-1}(x,y)$, $r_{n-1}(x,y)$ are respectively the x-part and the y-part of $p_n(x,y)$, i.e. non-commutative polynomials in variables x and y of total degree $n-1$ verifying the above equation. And so the equation $\mathrm{ad}\,x\,p_n(x,y) = 0$ can be rewritten as

$$xxq_{n-1}(x,y) - xq_{n-1}(x,y)x + xyr_{n-1}(x,y) - yr_{n-1}(x,y)x = 0, \quad \text{or}$$
$$x(xq_{n-1}(x,y) - q_{n-1}(x,y)x + yr_{n-1}(x,y)) - yr_{n-1}(x,y)x = 0.$$

As the above identity is the nullity of a non-commutative polynomial, by identification we have that $r_{n-1}(x,y) = 0$ and we are left with the equation

$$xq_{n-1}(x,y) - q_{n-1}(x,y)x = 0 \quad \text{or} \quad \mathrm{ad}(x)q_{n-1}(x,y) = 0$$

to solve. Applying the induction hypothesis gives that $p_n(x,y) = \lambda x^n$, where $\lambda \in \mathbb{K}$, and it ends the proof. □

LEMMA 28. — *The kernel of the operator* $\mathrm{ad}\,x : \mathrm{Lie}(V) \to \mathrm{Lie}(V)$ *is*

$$\mathrm{Ker}\,\mathrm{ad}\,x = \mathbb{K}x.$$

Proof. — As the kernel of $\mathrm{ad}\,x : T(V) \to T(V)$ is the polynomial algebra in indeterminate x, the kernel of $\mathrm{ad}\,x : \mathrm{Lie}(V) \to \mathrm{Lie}(V)$ is $\mathrm{Lie}(V) \cap \mathbb{K}[x] = \mathbb{K}x$. This concludes the proof. □

LEMMA 29. — *Let $S(x,y)$ and $P(x,y)$ be two Lie polynomials. If these polynomials verify*

$$\mathrm{E}(-x)S(x,y) = \mathrm{ad}(-x)P(x,y), \tag{5}$$

then for a certain $\lambda \in \mathbb{K}$

$$S(x,y) = \mathrm{Ber}(-x)P(x,y) + \lambda x.$$

Proof. — The fact that $(S(x,y) = \mathrm{Ber}(-x)P(x,y) + \lambda x, P(x,y))$ verifies equation (5) is straigthforward. Conversely, suppose that the pair $(S(x,y), P(x,y))$ verifies equation (5):

$$\mathrm{E}(-x)S(x,y) = \mathrm{ad}(-x)P(x,y).$$

Then, multiplying on the left side by the operator $\mathrm{Ber}(-x)$ leads to

$$\mathrm{ad}(-x)S(x,y) = \mathrm{Ber}(-x)\,\mathrm{ad}(-x)P(x,y).$$

The following identity follows from the commutativity of $\mathrm{ad}(-x)$ and $\mathrm{Ber}(-x)$:

$$\mathrm{ad}(-x)S(x,y) = \mathrm{ad}(-x)\,\mathrm{Ber}(-x)P(x,y),$$

therefore the Lie series $S(x,y) - \mathrm{Ber}(-x)P(x,y)$ is in the kernel of the morphism $\mathrm{ad}(-x)$ exhibited in Proposition 28. This ends the proof as there exists $\lambda \in \mathbb{K}$ such that $S(x,y) = \mathrm{Ber}(-x)P(x,y) + \lambda x$. □

Proposition 30. — *The kernel of the operator* $\mathrm{E}(x) : \mathrm{Lie}(V) \to \mathrm{Lie}(V)$ *is*

$$\mathrm{Ker}\,\mathrm{E}(x) = \mathbb{K}x.$$

Proof. — By Proposition 26 for any Lie polynomial $P(x,y)$ there exists a Lie polynomial $p(x,y)$ such that

$$\mathrm{E}(x)P(x,y) = \mathrm{ad}(x)p(x,y). \tag{6}$$

By Proposition 29 there exists $\mu \in \mathbb{K}$ such that

$$P(x,y) = \mathrm{Ber}(x)p(x,y) + \mu x.$$

Remark that to determine the kernel of the operator $\mathrm{E}(x)$ we will use the kernel of the adjunction, i.e. if $P(x,y) \in \mathrm{Ker}\,\mathrm{E}(x)$ then we have $p(x,y) \in \mathrm{Ker}\,\mathrm{ad}(x)$ and by Proposition 27 we conclude that $p(x,y) = \lambda x$ and therefore we have

$$P(x,y) = \mathrm{Ber}(x)(\lambda x) + \mu x.$$

By expanding the inverse operator we have,

$$P(x,y) = \sum_{k \geqslant 0} \frac{B_k}{k!}\,\mathrm{ad}(k)^n(\lambda x) + \mu x = \frac{B_0}{0!}(\lambda x) + \mu x = (\lambda + \mu)x,$$

which concludes the proof. □

5. A particular solution of equation (1)

In this section, we construct an explicit symmetric solution to equation (1) of the Kashiwara-Vergne conjecture by taking the Dynkin idempotent of a split of the Eulerian idempotent.

We adopt notation 22 and use the Baker-Campbell-Hausdorff series (Proposition 19) to rewrite the Kashiwara-Vergne first equation.

Proposition 31. — *Equation* (1) *of the Kashiwara-Vergne conjecture is equivalent to*

$$\sum_{n\geqslant 2}\Phi_n(y,x)=\mathrm{E}(-x)F(x,y)-\mathrm{E}(y)G(x,y). \tag{7}$$

Proof. — By equation (4),

$$\log(\exp y\exp x)=\log\exp\Phi(y,x)=\Phi(y,x)=\sum_{n\geqslant 1}\Phi_n(y,x).$$

Moreover, by Proposition 18, $\Phi_1(x,y)=e_1(x)+e_1(y)=x+y$. So the left part of the Kashiwara-Vergne first equation,

$$x+y-\log(\mathrm{e}^y\,\mathrm{e}^x)=(1-\mathrm{e}^{-\operatorname{ad}x})F(x,y)+(\mathrm{e}^{\operatorname{ad}y}-1)G(x,y),$$

can be rewritten as

$$\sum_{n\geqslant 2}\Phi_n(y,x)=\big(\exp(-\operatorname{ad}x)-1\big)F(x,y)-\big(\exp(\operatorname{ad}y)-1\big)G(x,y).$$

The use of notation 22 completes the proof. □

With the notations of section 3.1, we define the two following polynomials.

Definition 32. — Define $\Phi_x(x,y)$ as the sum of all the monomials of $\sum_{n\geqslant 2}\Phi_n(x,y)$ beginning with the indeterminate x:

$$\Phi_x(x,y):=x\Big(\sum_{n\geqslant 2}\big(\Phi_n(x,y)\big)_x\Big),$$

with the notation of Definition 5. We denote

$$\Phi^+(x,y):=\gamma\big(\Phi_x(x,y)\big)\quad\text{and}\quad\Phi^-(x,y):=\gamma\big(\Phi_y(x,y)\big).$$

Remark that $\Phi_x(x,y)$ and $\Phi_y(x,y)$ are non-commutative series and not Lie series. Taking their Dynkin idempotent forces them to be Lie series and to verify the following proposition.

Proposition 33. — *The formal power series* $\Phi^+(x,y)$ *and* $\Phi^-(x,y)$ *satisfy the following properties:*

$$\Phi^+(y,x) \in \operatorname{Im} \mathrm{E}(y), \quad \Phi^-(y,x) \in \operatorname{Im} \mathrm{E}(-x).$$

Proof. — By Definition 32 the Lie series $\Phi^+(x,y)$ is defined as $\Phi^+(y,x) = \gamma(y(\Phi(y,x))_y)$. Proposition 8 assures that this Lie series is in the image of $\mathrm{ad}(y)$. Then applying Proposition 26 ends the proof. The other property is proved analogously. □

These formal power series can be made explicit in terms of permutations as follows:

Proposition 34. — *The formal power series defined above have the following expression in terms of permutations:*

$$\Phi^+(x,y) := \sum_{n\geqslant 2} \sum_{\substack{i+j=n \\ i,j\geqslant 1}} \frac{1}{i!}\frac{1}{j!} \sum_{\substack{\sigma\in S_n \\ \sigma^{-1}(1)\in\{1,\dots,i\}}} \gamma\circ c_\sigma(\underbrace{x,\dots,x}_{i},\underbrace{y,\dots,y}_{j})^\sigma,$$

$$\Phi^-(x,y) := \sum_{n\geqslant 2} \sum_{\substack{i+j=n \\ i,j\geqslant 1}} \frac{1}{i!}\frac{1}{j!} \sum_{\substack{\sigma\in S_n \\ \sigma^{-1}(1)\in\{i+1,\dots,n\}}} \gamma\circ c_\sigma(\underbrace{x,\dots,x}_{i},\underbrace{y,\dots,y}_{j})^\sigma,$$

where $c_\sigma = \frac{(-1)^{d(\sigma)}}{n}\binom{n-1}{d(\sigma)}^{-1}$.

Proof. — Remark that $\sum_{n\geqslant 2}\Phi_n(x,y)$ can be made explicit thanks to its link with the Eulerian idempotent (cf. Proposition 19). Taking its x-part is to restrict the explicit version of $\sum_{n\geqslant 2}\Phi_n(x,y)$ to the permutations σ such that $\sigma^{-1}(1) \in \{1,\dots,i\}$ which guarantees that the monomial will start with a x. □

We could make this formula even more explicit by using Proposition 11.

These polynomials split the left part of the Kashiwara-Vergne conjecture (7):

Proposition 35. — *The formal power series defined in Definition 32 verify the property:*

$$\Phi_n(x,y) = \Phi_n^+(x,y) + \Phi_n^-(x,y).$$

Proof. — By definition of $\Phi_n^+(x,y)$ and $\Phi_n^-(x,y)$, we have:

$$\Phi_n^+(x,y) + \Phi_n^-(x,y) = \gamma\big(x\big(\Phi_n(x,y)\big)_x\big) + \gamma\big(y\big(\Phi_n(x,y)\big)_y\big),$$

which is equal to

$$\Phi_n^+(x,y) + \Phi_n^-(x,y) = \gamma\big(\Phi_n(x,y)\big) = \Phi_n(x,y),$$

as $\Phi_n(x,y)$ is a Lie polynomial. □

This splitting admits moreover a certain symmetry:

LEMMA 36. — *The split Baker-Campbell-Hausdorff series verify the anti-symmetric property*

$$\Phi^+(x,y) = -\Phi^-(-y,-x).$$

Proof. — We use the definition $\Phi_x(x,y) = x(\sum_{n\geqslant 2}(\Phi_n(x,y))_x)$, to prove the symmetry property. By Proposition 19 the formal power series $\sum_{n\geqslant 2}\Phi(x,y)$ can be made explicit thanks to its link with the Eulerian idempotent:

$$\Phi^+(x,y) = \sum_{n\geqslant 2}\sum_{\substack{i+j=n\\ i,j\geqslant 1}} \frac{1}{i!}\frac{1}{j!}\gamma\big(x\big(e_n(\underbrace{x,\dots,x}_{i},\underbrace{y,\dots,y}_{j})\big)_x\big).$$

By Proposition 21, we have

$$\Phi^+(x,y) = \sum_{n\geqslant 2}\sum_{\substack{i+j=n\\ i,j\geqslant 1}} \frac{1}{i!}\frac{1}{j!}\gamma\big(-x\big(e_n(\underbrace{-y,\dots,-y}_{j},\underbrace{-x,\dots,-x}_{i})\big)_x\big).$$

And the symmetry is proven as

$$-\Phi^-(-y,-x) = \sum_{n\geqslant 2}\sum_{\substack{i+j=n\\ i,j\geqslant 1}} \frac{1}{i!}\frac{1}{j!}\gamma\big(-x\big(e_n(\underbrace{-y,\dots,-y}_{i},\underbrace{-x,\dots,-x}_{j})\big)_x\big).$$

□

In order to simplify the particular solution of equation (1) we construct the following polynomial.

Definition 37. — Define the Lie series

$$a(x,y) := \sum_{n\geqslant 1}\frac{n}{n+1}\sum_{\substack{i+j=n\\ i,j\geqslant 1}} \frac{1}{(i+1)!}\frac{1}{j!}\gamma\big(\big(e_n(\underbrace{x,\dots,x}_{i},\underbrace{y,\dots,y}_{j})\big)_x\big).$$

As before, though e_n is a Lie polynomial, its x-part is not a Lie polynomial in general. So, in order to force $a(x,y)$ to become a Lie series, we have applied the Dynkin idempotent γ to $e_n(x,\dots,x,y,\dots,y)_x$.

Proposition 18 gives an explicit version of the Eulerian idempotent $e : T(V) \to T(V)$ which permits us to define explicitly these two Lie polynomials:

PROPOSITION 38. — *Let $\sigma \in S_n$, we denote $\widetilde{\sigma}$ the image of the $n-1$ last variables:*

$$\sigma(1,\dots,n) = \big(\sigma(1),\widetilde{\sigma}(2,\dots,n)\big).$$

Then, $(e_n(\underbrace{x,\ldots,x}_{i},\underbrace{y,\ldots,y}_{j}))_x$ *can be made explicit in the following way:*

$$\big(e_n(\underbrace{x,\ldots,x}_{i},\underbrace{y,\ldots,y}_{j})\big)_x = \sum_{\substack{\sigma\in S_n\\ \sigma^{-1}(1)\in\{1,\ldots,i\}}} c_\sigma \gamma\circ(\underbrace{x,\ldots,x}_{i-1},\underbrace{y,\ldots,y}_{j})^{\widetilde{\sigma}}. \qquad \square$$

This formula could be made more explicit by Proposition 11.

Now we can give the definition of the particular solution of equation (1).

Definition 39. — Let $\mathbb{K}$ be a characteristic zero field, and V the vector space defined by $V = \mathbb{K}x \oplus \mathbb{K}y$. Let $a(x,y)$ be the Lie series defined in Definition 37. We define the Lie series

$$F_0(x,y) := -\sum_{n\geqslant 0}\frac{B_n}{n!}(-1)^n(\operatorname{ad}x)^n\circ a(-x,-y).$$

Note that the Lie series is well defined as restricted to elements of degree n, $a_n(-x,-y)$ is polynomial. Therefore it is polynomial when restricted to a degree n.

Thanks to the two polynomials $\Phi^+(x,y)$ and $\Phi^-(x,y)$ defined from the Baker-Campbell-Hausdorff series (cf. Definition 32) we split the Kashiwara-Vergne first equation into the following equation.

Proposition 40. — *The equation*

$$\Phi^-(y,x) = \mathrm{E}(-x)F(x,y). \tag{8}$$

admits the Lie series $F_0(x,y)$ *defined in Definition* 39 *as solution on* $\mathrm{Lie}(V)$.

Proof. — Equation (8) is well-defined by Proposition 33.

The solution can be rewritten as $F_0(x,y) = -\operatorname{Ber}(-x)\circ a(-x,-y)$ by notation 23. It verifies the following equalities by Proposition 25, Definition 32, Proposition 8 and Proposition 21 respectively:

$$\begin{aligned}
\mathrm{E}(-x)F_0(x,y) &= -\,\mathrm{E}(-x)\operatorname{Ber}(-x)a(-x,-y) = \operatorname{ad}x\circ a(-x,-y)\\
&= \operatorname{ad}x\sum_{n\geqslant 2}\frac{n-1}{n}\sum_{\substack{i+j=n\\ i\geqslant 2, j\geqslant 1}}\frac{1}{i!}\frac{1}{j!}\gamma\big((-1)^{n-1}(e_{n-1}(\underbrace{x,\ldots,x}_{i-1},\underbrace{y,\ldots,y}_{j}))_x\big)\\
&= \sum_{n\geqslant 2}\sum_{\substack{i+j=n\\ i\geqslant 2, j\geqslant 1}}\frac{1}{i!}\frac{1}{j!}(-1)^{n-1}\gamma\big(x(e_{n-1}(\underbrace{x,\ldots,x}_{i-1},\underbrace{y,\ldots,y}_{j}))_x\big)\\
&= \sum_{n\geqslant 2}\sum_{\substack{i+j=n\\ i\geqslant 2, j\geqslant 1}}\frac{1}{i!}\frac{1}{j!}\,\gamma\big(x(e_{n-1}(\underbrace{y,\ldots,y}_{j},\underbrace{x,\ldots,x}_{i-1}))_x\big)
\end{aligned}$$

$$= \sum_{n\geqslant 2} \sum_{\substack{i+j=n\\ i,j\geqslant 1}} \frac{1}{i!}\frac{1}{j!} \sum_{\substack{\sigma\in S_n\\ \sigma^{-1}\in\{i+1,\dots,n\}}} \gamma\big(c_\sigma(\underbrace{y,\dots,y}_{i},\underbrace{x,\dots,x}_{j})^\sigma\big) = \Phi^-(y,x).$$

This ends the proof. □

Thanks to Lemma 36, we can prove:

Proposition 41. — *Let $F(x,y)$ be a Lie series which is a solution of the split equation*

$$\Phi^-(y,x) = \mathrm{E}(-x)F(x,y).$$

Then, $F(-y,-x)$ is a solution of

$$\Phi^+(y,x) = -\,\mathrm{E}(y)G(x,y).$$

Proof. — Let $F(x,y)$ be a Lie polynomial solution of the split equation $\Phi^-(y,x) = \mathrm{E}(-x)F(x,y)$. By exchanging x and $-y$ we have

$$\Phi^-(-x,-y) = \mathrm{E}(y)F(-y,-x).$$

Then by Lemma 36, we get $\Phi^-(-x,-y) = \Phi^+(y,x)$, and so

$$\Phi^+(y,x) = \mathrm{E}(y)F(-y,-x).$$

Then $G(x,y) = F(-y,-x)$ is a solution of $\Phi^+(y,x) = -\,\mathrm{E}(y)G(x,y)$, which ends the proof. □

Now, we can state the main result:

Theorem 42. — *Let $\mathbb{K}$ be a characteristic zero field, and V the vector space defined by $V = \mathbb{K}x \oplus \mathbb{K}y$. Let $F_0(x,y)$ be the Lie series of Definition* 39. *On* $\mathrm{Lie}(V)$, *the Lie series $F_0(x,y)$ and $G_0(x,y) = F_0(-y,-x)$ verify equation* (1) *of the Kashiwara-Vergne conjecture*

$$\begin{aligned} &x + y - \log\big(\exp(y)\exp(x)\big)\\ &\qquad\qquad = \big(1-\exp(-\operatorname{ad}x)\big)F(x,y) + \big(\exp(\operatorname{ad}y)-1\big)G(x,y). \end{aligned}$$

Remark 43. — Note that the theorem is true over any field $\mathbb{K}$ of characteristic zero, i.e. such that $\mathbb{K}$ contains $\mathbb{Q}$, and not only for the two fields $\mathbb{R}$ and $\mathbb{C}$, as asked in the Kashiwara-Vergne conjecture 1.

Proof. — The proof of the theorem is done in three steps. First, we rewrite the Kashiwara-Vergne conjecture, thanks to the Baker-Campbell-Hausdorff series and its relation with the Eulerian idempotent (cf. Proposition 31). Secondly, we split the equation into two parts:

$$\Phi^-(x,y) = \mathrm{E}(-x)F(x,y), \quad \Phi^+(y,x) = -\,\mathrm{E}(y)G(x,y).$$

By Theorem 40 the first equation of this split equation admits $F_0(x,y)$ as a solution, and Proposition 41 gives $F_0(-y,-x)$ as a solution of the second equation. Then adding the two equations and applying Proposition 35, proves the theorem. □

Remark 44. — The first terms of the symmetric solution $F_0(x,y)$ (cf. Definition 39) are:

$$\begin{aligned} F_0(x,y) = {} & \tfrac{1}{4}y + \tfrac{1}{24}xy - \tfrac{1}{24}yx \\ & - \tfrac{1}{48}xxy + \tfrac{1}{24}xyx + \tfrac{1}{48}xyy - \tfrac{1}{48}yxx - \tfrac{1}{24}yxy + \tfrac{1}{48}yyx \\ & - \tfrac{1}{180}xxxy + \tfrac{1}{60}xxyx + \tfrac{1}{480}xxyy - \tfrac{1}{60}xyxx - \tfrac{1}{240}xyxy \\ & + \tfrac{1}{360}xyyy + \tfrac{1}{180}yxxx + \tfrac{1}{240}yxyx - \tfrac{1}{120}yxyy - \tfrac{1}{480}yyxx \\ & + \tfrac{1}{120}yyxy - \tfrac{1}{360}yyyx \\ & + \tfrac{1}{2880}xxxxy - \tfrac{1}{720}xxxyx - \tfrac{7}{2880}xxxyy + \tfrac{1}{480}xxyxx + \tfrac{7}{1440}xxyxy \\ & + \tfrac{7}{2880}xxyyx + \tfrac{1}{720}xxyyy - \tfrac{1}{720}xyxxx - \tfrac{7}{720}xyxyx - \tfrac{1}{240}xyxyy \\ & + \tfrac{7}{2880}xyyxx + \tfrac{1}{240}xyyxy - \tfrac{1}{360}xyyyx + \tfrac{1}{2880}yxxxx + \tfrac{7}{1440}yxyxx \\ & + \tfrac{1}{240}yxyyx - \tfrac{1}{240}yyxyx - \tfrac{7}{2880}yyxxx + \tfrac{1}{720}yyyxx \\ & + \text{higher order terms.} \end{aligned}$$

This can be written (non-uniquely) in terms of bracket:

$$\begin{aligned} F_0(x,y) = {} & \tfrac{1}{4}y + \tfrac{1}{24}[x,y] \\ & - \tfrac{1}{48}(-[x,[x,y]] + [y,[y,x]]) \\ & - \tfrac{1}{180}[x,[x,[x,y]]] - \tfrac{1}{480}[x,[y,[x,y]]] - \tfrac{1}{360}[y,[y,[y,x]]] \\ & - \tfrac{1}{240}[x,[x,[y,[x,y]]]] - \tfrac{7}{720}[x,[y,[x,[x,y]]]] + \tfrac{1}{144}[x,[y,[x,[x,y]]]] \\ & - \tfrac{1}{240}[y,[x,[y,[x,y]]]] + \tfrac{1}{240}[y,[y,[x,[x,y]]]] + \text{higher order terms.} \end{aligned}$$

6. Unicity of the solution

Moreover it can be proven that up to λx, where $\lambda \in \mathbb{K}$, the solution of the split equation (8) is unique.

Proposition 45. — *Let $F_0(x,y)$ be the Lie series defined in Definition 39. Any Lie series $H(x,y)$ which is a solution of* (8)

$$\Phi^-(y,x) = \mathrm{E}(-x)H(x,y)$$

is of the form $H(x,y) = F_0(x,y) + \lambda x$.

Proof. — Let $H(x,y)$ be a Lie series solution of the split equation (8). By Proposition 40, the Lie series defined in Definition 39 is a solution of (8).

Then substracting the two equations, it comes out that $H(x,y) - F_0(x,y)$ is in the kernel of $\mathrm{E}(-x)$. Applying Proposition 30 ends the proof. □

So we can conclude a unicity property for solutions of the split equation defined in the proof of Theorem 42:

PROPOSITION 46. — *Let $F_0(x,y)$ be the Lie series defined in Definition 39. Let $(F(x,y), G(x,y))$ be a solution of equation* (8)*:*

$$\Phi^+(y,x) = -\,\mathrm{E}(y)G(x,y), \quad \Phi^-(y,x) = \mathrm{E}(-x)F(x,y).$$

Then, there exists $\lambda_1, \lambda_2 \in \mathbb{K}$ such that: $F(x,y) = F_0(x,y) + \lambda_1 x$ and $G(x,y) = F_0(-y,-x) + \lambda_2 y$. Conversely the pair $(F(x,y), G(x,y))$ is solution of equations (8). □

Though split equation (8) is unique up to the first degree term, equation (1) is not unique even if restricted to symmetric solutions, that is to say a pair solution $(F(x,y), F(-y,-x))$. And we show that any non-symmetric solution of (1) can be symmetrised thanks to the symmetry property verified by the left part of the Kashiwara-Vergne conjecture which is highlighted by Baker-Campbell-Hausdorff series.

PROPOSITION 47. — *The Baker-Campbell-Hausdorff series satisfy the symmetry property*

$$-\sum_{n\geqslant 2} \Phi_n(-y,-x) = \sum_{n\geqslant 2} \Phi_n(x,y).$$

Proof. — By Proposition 35, the following identity holds:

$$\sum_{n\geqslant 2} \Phi_n(x,y) = \Phi^+(x,y) + \Phi^-(x,y).$$

Moreover Lemma 36 ensures that $\Phi^+(x,y) = -\Phi^-(-y,-x)$. Therefore,

$$\begin{aligned}\sum_{n\geqslant 2}\Phi_n(x,y) &= \Phi^+(x,y) + \Phi^-(x,y)\\ &= -\Phi^-(-y,-x) - \Phi^+(-y,-x) = -\sum_{n\geqslant 2}\Phi_n(-y,-x).\end{aligned}$$

The proof is completed. □

From this proposition we deduce that any non-symmetric solution produces a symmetrised solution:

PROPOSITION 48. — *Let $(F(x,y), G(x,y)) \in \mathrm{Lie}(V)^2$ be a non-symmetric solution of equation* (1) *of the Kashiwara-Vergne conjecture. The* *solution can be symmetrised in another solution:*

$$F_1(x,y) := \tfrac{1}{2}\big(F(x,y) + G(-y,-x)\big) + \lambda x,$$
$$G_1(x,y) := \tfrac{1}{2}\big(G(x,y) + F(-y,-x)\big) - \lambda y,$$

where $\lambda \in \mathbb{K}$.

Proof. — As $(F(x,y), G(x,y))$ is a solution of equation (1) of the Kashiwara-Vergne conjecture it satisfies, by Proposition 47,

$$\sum_{n\geqslant 2} \Phi_n(y,x) = \mathrm{E}(-x)F(x,y) - \mathrm{E}(y)G(x,y).$$

By exchanging x and $-y$ we get

$$\sum_{n\geqslant 2} -\Phi_n(-x,-y) = -\mathrm{E}(y)F(-y,-x) + \mathrm{E}(-x)G(-y,-x).$$

We obtain the next equation by adding the two preceding ones.

$$2\sum_{n\geqslant 2} \Phi_n(y,x) = \mathrm{E}(-x)(F(x,y)+G(-y,-x)) - \mathrm{E}(y)(G(x,y)+F(-y,-x)).$$

Moreover Proposition 30, permits the fact to add λx to the symmetrised solution $F(x,y)+G(-y,-x)$. It is clear that the solution verifies the symmetry $F_1(-y,-x) = G_1(x,y)$, which completes the proof. □

7. Solution of the homogeneous equation

We are interested in finding all solutions of equation (1) in the free Lie algebra generated by two indeterminates x and y. And therefore this section is devoted to solving the homogeneous equation in order to set all the solutions of equation (1). That is to say solving equation

$$\mathrm{E}(-x)F(x,y) = \mathrm{E}(y)G(x,y), \tag{9}$$

for Lie series $F(x,y)$ and $G(x,y)$.

LEMMA 49. — *The pair of Lie series* $(P(x,y), Q(x,y))$ *is a solution of the equation*

$$\mathrm{ad}(x)P(x,y) + \mathrm{ad}(y)Q(x,y) = 0, \tag{10}$$

if and only if there exists a non-commutative series $p(x,y) \in \mathrm{Ker}\,\gamma$ *such that*

$$P(x,y) = \gamma\big(\big(p(x,y)\big)_x\big) \quad \textit{and} \quad Q(x,y) = \gamma\big(\big(p(x,y)\big)_y\big),$$

where $(p(x,y))_x$ *(resp.* $(p(x,y))_y$*) denotes the* x*-part (resp.* y*-part) of* $p(x,y)$.

Proof. — Remark that the adjunction $\mathrm{ad}(z)$, for $z \in V$, is a Lie homomorphism of degree 1 and the Dynkin idempotent γ is a degree preserving map. We can restrict the above equation to homogeneous polynomials $P(x,y)$ and $Q(x,y)$ of degree n.

Let $P(x,y)$ and $Q(x,y)$ be homogeneous Lie polynomials of degree n, defined as

$$P(x,y) := \gamma\big(\big(p(x,y)\big)_x\big), \quad Q(x,y) := \gamma\big(\big(p(x,y)\big)_y\big)$$

for a certain $p(x,y) \in \operatorname{Ker}\gamma_{n+1}$. Firstly, we verify that they satisfy equation (10). Replacing P and Q in equation (10) gives

$$\begin{aligned}
&\mathrm{ad}(x)\gamma\big(\big(p(x,y)\big)_x\big) + \mathrm{ad}(y)\gamma\big(\big(p(x,y)\big)_y\big) \\
&\quad = \frac{n+1}{n}\gamma(x(p(x,y))_x + y(p(x,y))_y) && \text{(Proposition 8)} \\
&\quad = \frac{n+1}{n}\gamma\big(p(x,y)\big) && \text{(Definition 5)} \\
&\quad = 0,
\end{aligned}$$

as $p(x,y) \in \operatorname{Ker}\gamma_{n+1}$. Therefore the two defined Lie polynomials verify equation (10).

Conversely, let $P(x,y), Q(x,y)$ be homogeneous Lie polynomials of degree n. Suppose that the pair (P,Q) verifies equation

$$\mathrm{ad}(x)P(x,y) + \mathrm{ad}(y)Q(x,y) = 0.$$

As $P(x,y)$ and $Q(x,y)$ are Lie polynomials they verify the following properties $\gamma(P(x,y)) = P(x,y)$ and $\gamma(Q(x,y)) = Q(x,y)$ respectively. So verifying the above equation is equivalent to verify

$$\mathrm{ad}(x)\gamma\big(P(x,y)\big) + \mathrm{ad}(y)\gamma\big(Q(x,y)\big) = 0. \tag{11}$$

The linearity of γ and Proposition 8 lead to

$$\gamma\big(xP(x,y) + yQ(x,y)\big) = 0.$$

And so $xP(x,y) + yQ(x,y)$ is a polynomial of degree $n+1$ which lies in the kernel of the Dynkin idempotent. We denote $p(x,y)$ this element of $\operatorname{Ker}\gamma_{n+1}$, where its x-part is $P(x,y) = (p(x,y))_x$ and its y-part is $Q(x,y) = (p(x,y))_y$. Recall that $P(x,y)$ and $Q(x,y)$ are Lie series verifying $P(x,y) = \gamma(P(x,y)) = \gamma\big(\big(p(x,y)\big)_x\big)$ (resp. $Q(x,y) = \gamma(Q(x,y)) = \gamma\big(\big(p(x,y)\big)_y\big)$). Therefore $P(x,y) = \gamma\big(\big(p(x,y)\big)_x\big)$ and $Q(x,y) = \gamma\big(\big(p(x,y)\big)_y\big)$.

The proof is completed. □

Example 50. — M. Vergne found that the polynomial

$$P(x,y) = [x,[y,[x,[x,y]]]] - 2[y,[x,[x,[x,y]]] - [y,[y,[y,[y,x]] \neq 0$$

verifies equation (10):

$$\mathrm{ad}(x)P(x,y)+\mathrm{ad}(y)P(-y,-x)=0,$$

(private communication). Under Lemma 49 we should be able to prove that the polynomial $p(x,y):=xP(x,y)+yP(-y,-x)$ is in the kernel of the Dynkin idempotent. This is true as there exists a polynomial

$$\begin{aligned} q(x,y) = {} & 2xxxxyy - 8xxxyxy + xxxyyx + 12xxyxxy \\ & - 4xxyxyx + xxyyxx - 2xxyyyy - 8xyxxxxy \\ & + 6xyxxyx - 4xyxyxx + xyyxxx + 8xyxyyy \\ & - 12xyyxyy + 8xyyyxy - xyyyyx \\ & + yxxxxy - yxxyyy + 4yxyxyy - 6yxyyxy \\ & - yyxxyy + 4yyxyxy - yyyxxy \end{aligned}$$

such that $p(x,y)=q(x,y)-\gamma(q(x,y))$ which is in the kernel of the Dynkin idempotent.

Proposition 51. — *The pair $(F(x,y),G(x,y))$ is a solution of the homogeneous equation* (9) *if and only if there exists an element $p(x,y)$ in the kernel the Dynkin idempotent such that $F(x,y)$ is equal to*

$$F(x,y):=\mathrm{Ber}(-x)\gamma\big(\big(p(x,y)\big)_x\big)+\lambda_1 x$$

and $G(x,y)$ is equal to

$$G(x,y):=\mathrm{Ber}(y)\gamma\big(\big(p(x,y)\big)_y\big)+\lambda_2 y,$$

where $(p(x,y))_x$ (resp. $(p(x,y))_y$) denotes the x-part (resp. y-part) of $p(x,y)$ and $\lambda_1,\lambda_2\in\mathbb{K}$.

Proof. — First, we verify that the Lie series $F(x,y)$ and $G(x,y)$ are solutions of equation (9). As the kernel of the Dynkin idempotent can be described homogeneously we can restrict ourselves to elements of the kernel of $\gamma_n : V^{\otimes n}\to V^{\otimes n}$.

Let $p_n(x,y)\in\operatorname{Ker}\gamma_n$. Define Lie polynomials $F(x,y)$ and $G(x,y)$ as

$$F(x,y):=\mathrm{Ber}(-x)\gamma\big(\big(p_n(x,y)\big)_x\big)\quad\text{and}\quad G(x,y):=\mathrm{Ber}(y)\gamma\big(\big(p_n(x,y)\big)_y\big).$$

Then the equation $\mathrm{E}(-x)F(x,y)-\mathrm{E}(y)(G(x,y))$ becomes by definition of $F(x,y)$ and $G(x,y)$ and Proposition 25:

$$\mathrm{ad}(-x)\gamma\big(\big(p_n(x,y)\big)_x\big)-\mathrm{ad}(y)\gamma\big(\big(p_n(x,y)\big)_y\big).$$

Lemma 49 assures that the pair $(F(x,y),G(x,y))$ is solution of the homogeneous equation.

Let $F(x,y)$ and $G(x,y)$ be a Lie series which are solutions of equation (9). By Proposition 26, there exists two Lie series $P(x,y)$ and $Q(x,y)$ such that $\mathrm{E}(-x)F(x,y) = \mathrm{ad}(-x)P(x,y)$ and $\mathrm{E}(y)G(x,y) = \mathrm{ad}(-x)Q(x,y)$ respectively. As $(F(x,y), G(x,y))$ is a solution of equation (9) the pair $(P(x,y), Q(x,y))$ is a solution of the equation

$$\mathrm{ad}(x)P(x,y) + \mathrm{ad}(y)Q(x,y) = 0.$$

And we note

$$p(x,y) := xP(x,y) + yQ(x,y).$$

Therefore Lemma 49 gives that $P(x,y) = \gamma\big(\big(p(x,y)\big)_x\big)$ and $Q(x,y) = \gamma\big(\big(p(x,y)\big)_y\big)$. As the pair $(F(x,y), P(x,y))$ verify equation (5) we can apply Lemma 29 to obtain for a certain $\lambda_1 \in \mathbb{K}$

$$F(x,y) = \mathrm{Ber}(-x)P(x,y) + \lambda_1 x.$$

An analogue to Lemma 29 (changing $\mathrm{E}(-x)$ in $\mathrm{E}(y)$) gives that

$$G(x,y) = \mathrm{Ber}(y)Q(x,y) + \lambda_2 y,$$

for a certain $\lambda_1 \in \mathbb{K}$. This ends the proof. □

We would like to have a more explicit formula for the Lie series solution of the homogeneous equation (9). So we construct maps $\Psi_x : T(V) \to T(V)$ which will simplify the span of the vector space of solutions.

Definition 52. — We denote by $\Psi_x : T(V) \to T(V)$ the map defined by

$$p(x,y) \longmapsto \gamma\big(\big(p(x,y) - \gamma\big(p(x,y)\big)\big)_x\big).$$

This map can be explicited in terms of permutations as follows.

PROPOSITION 53. — *The map $\Psi_x : T(V) \to T(V)$ is induced by*

$$x_1 \cdots x_n \longmapsto \frac{1}{n-1}\,\Big((-1)^{n-2}\delta_{x,x_j}\Big(\sum_{\substack{1\leqslant k\leqslant n-2\\ \tau\in D_{\{1,\ldots,k\}}\subset S_{n-1}}} (n-1)x_{\tau(2)}\cdots x_{\tau(n)}\Big)$$
$$+\sum_{\substack{1\leqslant k\leqslant n-2\\ 1\leqslant j\leqslant n-2}}\ \sum_{\substack{\sigma\in D_{\{1,\ldots,k\}}\subset S_n\\ \omega\in D_{\{1,\ldots,j\}}\subset S_{n-1}}} (-1)^{k+j-1}\delta_{x,x_{\sigma(n)}}\big(x_{\omega(\sigma(n-1))}\cdots x_{\omega(\sigma(1))}\big)\Big),$$

where the map $\delta_{x,x_j} : T(V) \to \mathbb{K}$ is the map induced by

$$x_1\cdots x_n \longmapsto \begin{cases} 1 & \text{if } x_j = x,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. — Define the non-commutative polynomial

$$p_n(x_1,\ldots,x_n) := n\big(x_1\cdots x_n - \gamma(x_1\cdots x_n)\big)$$

which can be explicited by Proposition 12 as

$$p_n(x_1,\ldots,x_n) := (n-1)x_1\cdots x_n + \sum_{k=1}^{n-2}\sum_{\sigma\in D_{\{1,\ldots,k\}}} (-1)^{n+k-1}(x_n\cdots x_1)^{\sigma},$$

for $x_i \in V$ and $n \geqslant 2$.

Let $x_1\cdots x_n$ be a monomial of degree n in $V^{\otimes n}$. Then, the x-part of this monomial is the monomial $x_2\cdots x_n$ if $x_1 = x$. This can be sum up as $(x_1\cdots x_n)_x = \delta_{x,x_j}x_2\cdots x_n$. Therefore the x-part of $p_n(x_1,\ldots,x_n)$ is

$$\begin{aligned}\big(p_n(x_1,\ldots,x_n)\big)_x &= (n-1)\delta_{x,x_j}x_2\cdots x_n\\ &\quad+\sum_{k=1}^{n-2}\sum_{\sigma\in D_{\{1,\ldots k\}}}(-1)^{n+k-1}\delta_{x,x_{\sigma(n)}}\big(x_{\sigma(n-1)}\cdots x_{\sigma(1)}\big).\end{aligned}$$

By the explicit formula of Proposition 11

$$\gamma_n(x_1,\ldots,x_n) = \frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(-1)^k\sum_{\sigma\in D_{\{1,\ldots,k\}}}(x_n\cdots x_1)^{\sigma},$$

taking the Dynkin idempotent of $(p_n(x_1,\ldots,x_n))_x$ gives the following:

$$\begin{aligned}&\gamma\Big(\big(p_n(x_1,\ldots,x_n)\big)_x\Big)\\ &\quad= (n-1)\delta_{x,x_j}\gamma(x_2,\ldots,x_n)\\ &\qquad\quad+\sum_{\substack{1\leqslant k\leqslant n-2\\ \sigma\in D_{\{1,\ldots,k\}\subset S_n}}}(-1)^{n+k-1}\delta_{x,x_{\sigma(n)}}\gamma\big(x_{\sigma(n-1)},\ldots,x_{\sigma(1)}\big)\\ &\quad= (-1)^{n-2}\delta_{x,x_j}\Big(\sum_{\substack{1\leqslant k\leqslant n-2\\ \tau\in D_{\{1,\ldots,k\}}\subset S_{n-1}}} x_{\tau(2)}\cdots x_{\tau(n)}\Big)\\ &\qquad\quad+(-1)^{n-2}\sum_{\substack{1\leqslant k\leqslant n-2\\ \sigma\in D_{\{1,\ldots,k\}}}}\delta_{x,x_{\sigma(n)}}\Big(\sum_{\substack{1\leqslant j\leqslant n-2\\ \omega\in D_{\{1,\ldots,j\}\subset S_{n-1}}}}\\ &\qquad\qquad\qquad\qquad\qquad\frac{(-1)^{n+k+j-1}}{n-1}x_{\omega(\sigma(n-1))}\cdots x_{\omega(\sigma(1))}\Big)\\ &\quad= n\Psi_x(x_1,\ldots,x_n).\end{aligned}$$

□

Proposition 54. — *Let V be the vector space spanned by the indeterminates x and y. Let Ψ_x, Ψ_y be the maps defined in Definition 52. Let*

$(F(x,y),G(x,y)) \in \mathrm{Lie}(V)^2$ *be a solution of the homogeneous equation* (9). *Then there exists a series* $m(x,y) \in T(V)$ *such that*

$$F(x,y) = \mathrm{Ber}(-x)\Psi_x\big(m(x,y)\big), \quad G(x,y) = \mathrm{Ber}(y)\Psi_y\big(m(x,y)\big).$$

Proof. — Recall that the kernel of the Dynkin idempotent is generated by 1 and elements

$$p_n(x_1,\ldots,x_n) := (n-1)x_1\cdots x_n + \sum_{k=1}^{n-2}\sum_{\sigma\in D_{\{1,\ldots k\}}}(-1)^{n+k-1}(x_n\cdots x_1)^\sigma,$$

for $x_i \in V$ and $n \geqslant 2$. Then, Proposition 51 and Proposition 12 complete the proof. □

Example 55. — With the notations of Example 50 we know that

$$\big(\mathrm{Ber}(-x)P(x,y), \mathrm{Ber}(y)P(-y,-x)\big)$$

is a solution of the homogeneous equation (9). This solution can be explicited as in the above proposition since

$$P(x,y) = \gamma\big(P(x,y)\big) = \gamma\Big(\big(q(x,y) - \gamma\big(q(x,y)\big)\big)_x\Big) = \Psi_x\big(q(x,y)\big).$$

We can now state the theorem giving all solutions of equation (1) in the free Lie algebra generated by the two non-commutative indeterminates x and y:

THEOREM 56. — *Let* V *be the* $\mathbb{K}$*-vector space spanned by the indeterminates* x *and* y. *Let* $(F_0(x,y),F_0(-y,-x))$ *be the particular symmetric solution of equation* (1) *constructed in Definition* 39. *Let* Ψ_x,Ψ_y *be the maps defined in Definition 52.*

Let the pair $(F(x,y),G(x,y)) \in \mathrm{Lie}(V)^2$ *of Lie series be a solution of equation* (1). *Then there exists polynomial* $p(x,y) \in T(V)$ *such that*

$$F(x,y) = F_0(x,y) + \sum_{m\geqslant 1}\frac{B_m}{m!}(-1)^m\big(\mathrm{ad}(x)\big)^m\Psi_x\big(p(x,y)\big),$$

$$G(x,y) = F_0(-y,-x) + \sum_{m\geqslant 1}\frac{B_m}{m!}\big(\mathrm{ad}(y)\big)^m\Psi_y\big(p(x,y)\big).$$

Conversely, the pair $(F(x,y),G(x,y))$ *is a solution of equation* (1).

Proof. — It is clear by Proposition 51 and by Theorem 42 that the pair $(F(x,y),G(x,y))$ is solution of equation (1). Conversely, let $(F(x,y), G(x,y))$ be a solution of equation (1). As $(F_0(x,y),F_0(-y,-x))$ is also a solution, their difference is solution of the homogeneous equation (9). And Proposition 51 ends the proof. □

8. Another decription of all the solutions of equation (1)

In [3] and in [7] there is another description of the kernel of the Dynkin idempotent recalled as Proposition 13. The same proofs as before brings up another formulation of Theorem 56. In order to simplify the statement of this theorem we introduce a few notations.

Definition 57. — Let $p(x,y)$ be a non-commutative polynomial in indeterminates x and y. We define:

$$A(p)(x,y) := \gamma\big(p(x,y)\big)p(x,y) - \gamma\big(p(-y,-x)\big)p(-y,-x),$$
$$E(p)(x,y) := \gamma\big(p(x,y)\big)p(x,y).$$

Moreover we define $A(p)_n$ (resp. $E(p)_n$) as the homogeneous part of $A(p)$ (resp. $E(p)$) of degree n, as a Lie polynomial can uniquely be seen as a non-commutative polynomial.

Proposition 58. — *Let V be the vector space spanned by the indeterminates x and y. Let $(F_0(x,y), F_0(-y,-x))$ be the particular symmetric solution of equation* (1) *constructed in Definition* 39. *Let the pair $(F(x,y), G(x,y)) \in \mathrm{Lie}(V)^2$ of Lie series in indeterminates x and y.*

If $(F(x,y), G(x,y))$ is a solution of equation (1) *then, there exists λ_1, λ_2 in $\mathbb{K}$ and a finite set of indices I indexing a finite family of non-commutative polynomials $p^i(x,y) \in T(V)$ and a finite family of scalars $\mu_i \in \mathbb{K}$ such that*

$$F(x,y) := F_0(x,y) + \sum_{i\in I} \mathrm{Ber}(-x)\gamma\Big(\sum_{n\geqslant 0} \frac{n}{n+1}\mu_i\big(E(p^i)_n(x,y)\big)_x\Big) + \lambda_1 x,$$

$$G(x,y) := F_0(-y,-x) + \sum_{i\in I} \mathrm{Ber}(y)\gamma\Big(\sum_{n\geqslant 0} \frac{n}{n+1}\mu_i\big(E(p^i)_n(x,y)\big)_y\Big) + \lambda_2 y,$$

with the notations of Definition 57. *Conversely, the pair $(F(x,y), G(x,y))$ is a solution of equation* (1).

We can also produce all symmetric solutions of equation (1) as follows.

Proposition 59. — *Let V be the vector space spanned by the indeterminates x and y. Let $(F_0(x,y), F_0(-y,-x))$ be the particular symmetric solution of equation* (1) *constructed in Definition* 39. *Let $F(x,y) \in \mathrm{Lie}(V)$ be a Lie series in indeterminates x and y.*

If $(F(x,y), F(-y,-x))$ is a symmetric solution of equation (1) *then, there exists $\lambda_1 \in \mathbb{K}$ and a finite set of indices I indexing a finite family of non-commutative polynomials $p^i(x,y) \in T(V)$ and a finite family of scalars*

$\mu_i \in \mathbb{K}$ *such that*

$$F(x,y) := F_0(x,y) + \sum_{i\in I} \mathrm{Ber}(-x)\gamma\Big(\sum_{n\geqslant 0} \frac{n}{n+1}\mu_i\big(A(p^i)_n(x,y)\big)_x\Big) + \lambda_1 x,$$

with notations of Definition 57*. Conversely, the pair* $(F(x,y), F(-y,-x))$ *is a symmetric solution of equation* (1)*.*

The proof is analogous to the one used for Proposition 51. It is to be noted that instead of elements of the kernel of the Dynkin idempotent we need the anti-symmetric elements of the kernel of the Dynkin idempotent which is spanned by:

Proposition 60. — *The anti-symmetric elements of the kernel of the Dynkin idempotent are spanned by the elements*

$$A(p)(x,y) = \gamma\big(p(x,y)\big)p(x,y) - \gamma\big(p(-y,-x)\big)p(-y,-x),$$

for $p(x,y) \in T(V)$*.*

Proof. — Let $p(x,y)$ denote a non-commutative polynomial. By Proposition 13 it is clear that the elements spanned by $A(p)(x,y)$ are anti-symmetric elements of the kernel of the Dynkin idempotent.

Conversely, let $q(x,y)$ be an anti-symmetric element of the kernel of the Dynkin idempotent. By Proposition 13, there exist a finite family $p^i(x,y) \in T(V)$ and $\lambda_i \in \mathbb{K}$ such that

$$q(x,y) = \sum_{i\geqslant 0} \lambda_i \gamma\big(p^i(x,y)\big)p^i(x,y).$$

By the anti-symmetry property of $q(x,y)$ this sum can also be rewritten as

$$q(x,y) = \frac{1}{2}\sum_{i\geqslant 0} \lambda_i\Big(\gamma\big(p^i(x,y)\big)p^i(x,y) - \gamma\big(p^i(-y,-x)\big)p^i(-y,-x)\Big).$$

Therefore anti-symmetric elements of the kernel of the Dynkin idempotent are spanned by elements $\gamma\big(p(x,y)\big)p(x,y) - \gamma(p(-y,-x))p(-y,-x)$, where $p(x,y) \in T(V)$, which ends the proof. □

9. Multilinearised Kashiwara-Vergne conjecture

Remark that the previous method can be extended in order to find all the solutions in $\mathrm{Lie}(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n)$ of the first equation of the following multilinear version of Kashiwara-Vergne conjecture.

Conjecture 61. — *For any Lie algebra $\mathfrak{g}$ of finite dimension, we can find series $F_1, \dots, F_n$ such that they satisfy*

$$\begin{aligned}(12)\quad & x_1 + \dots + x_n - \log(\mathrm{e}^{x_n} \cdots \mathrm{e}^{x_1}) \\ &= (1 - \mathrm{e}^{-\operatorname{ad} x_1}) F_1(x_1, \dots, x_n) + (1 - \mathrm{e}^{\operatorname{ad} x_2}) F_2(x_1, \dots, x_n) \\ &\qquad + \dots + (1 - \mathrm{e}^{(-1)^n \operatorname{ad} x_n}) F_n(x_1, \dots, x_n),\end{aligned}$$

(13) $F_1, \dots, F_n$ *give $\mathfrak{g}$-valued convergent power series on* $(x_1, \dots, x_n) \in \mathfrak{g}^{\times n}$,

$$\begin{aligned}(14)\quad & \operatorname{tr}(\operatorname{ad} x_1 \circ \partial_{x_1} F_1; \mathfrak{g}) + \dots + \operatorname{tr}(\operatorname{ad} x_n \circ \partial_{x_n} F_n; \mathfrak{g}) \\ &= \frac{1}{n} \operatorname{tr}\Big(\frac{\operatorname{ad} x_1}{\mathrm{e}^{\operatorname{ad} x_1} - 1} + \dots + \frac{\operatorname{ad} x_n}{\mathrm{e}^{\operatorname{ad} x_n} - 1} - \frac{\operatorname{ad} \Phi(x_1, \dots, x_n)}{\mathrm{e}^{\operatorname{ad} \Phi(x_1, \dots, x_n)} - 1} - n; \mathfrak{g} \Big).\end{aligned}$$

Here $\Phi(x_1, \dots, x_n) = \log(\mathrm{e}^{x_1} \cdots \mathrm{e}^{x_n})$ and $\partial_{x_i} F_i$ is the $\mathbf{End}(\mathfrak{g})$-valued real analytic function defined by

$$\mathfrak{g} \ni a \longmapsto \frac{\mathrm{d}}{\mathrm{d}t} F_i(x_1, \dots, x_{i-1}, x_i + ta, x_{i+1}, \dots, x_n)_{|t=0}$$

and tr *denotes the trace of an endomorphism of $\mathfrak{g}$.*

We are able to construct explicitly a particular solution $F_{1,0}, \dots, F_{n,0}$ of equation (12).

Theorem 62. — *Let $\mathbb{K}$ be a characteristic zero field, and V the vector space $V = \mathbb{K}x_1 \oplus \dots \oplus \mathbb{K}x_n$. Let $F_{i,0}(x, y)$ be the Lie series defined below:*

$$\begin{aligned} a_i(x_1, \dots, x_n) := \sum_{m \geqslant 1} \frac{m}{m+1} \sum_{\substack{i_1 + \dots + i_n = m \\ i_1, \dots, i_n \geqslant 1}} \frac{1}{i_1!} \cdots \frac{1}{(i_k+1)!} \cdots \frac{1}{i_n!} \\ \gamma\Big(\big(e_n(\underbrace{x_n, \dots, x_n}_{i_n}, \dots, \underbrace{x_1, \dots, x_1}_{i_1})\big)_{x_i}\Big).\end{aligned}$$

$$F_{i,0}(x_1, \dots, x_n) := -\sum_{m \geqslant 0} \frac{B_m}{m!} (-1)^m (\operatorname{ad} x_i)^m a_i(x_1, \dots, x_n).$$

On $\mathrm{Lie}(V)$, *the Lie series $F_{1,0}(x_1, \dots, x_n), \dots, F_{n,0}(x_1, \dots, x_n)$ verify equation* (12) *of the multilinear Kashiwara-Vergne conjecture:*

$$\begin{aligned} x_1 + \dots + x_n - \log(\mathrm{e}^{x_n} \cdots \mathrm{e}^{x_1}) = (1 - \mathrm{e}^{-\operatorname{ad} x_1}) F_1(x_1, \dots, x_n) \\ + (1 - \mathrm{e}^{\operatorname{ad} x_2}) F_2(x_1, \dots, x_n) + \dots + (1 - \mathrm{e}^{(-1)^n \operatorname{ad} x_n}) F_n(x_1, \dots, x_n)\end{aligned}$$

Remark that we can also give an analogous to Proposition 40 where the solutions will be unique up to $(\lambda_1 x_1, \dots, \lambda_n x_n)$ for $\lambda_i \in \mathbb{K}$.

We can state the theorem giving explicitly all the solutions of equation (12).

THEOREM 63. — *Let V be the vector space spanned by the indeterminates $x_1,\dots,x_n$. Let $F_{1,0}(x_1,\dots,x_n),\dots,F_{n,0}(x_1,\dots,x_n)$ be the particular symmetric solution of equation* (1) *constructed in Theorem* 62. *Let $\Psi_{x_i}: T(V)\to T(V)$ be the map of Definition* 52.

If the pair $(F_1(x_1,\dots,x_n),\dots,F_n(x_1,\dots,x_n))\in \mathrm{Lie}(V)^n$ of Lie series in indeterminates $x_1,\dots,x_n$ is a solution of equation (12), *then there exists a polynomial $m(x_1,\dots,x_n)$ such that*

$$F_i(x_1,\dots,x_n)=F_{i,0}(x_1,\dots,x_n)+\mathrm{Ber}\big((-1)^i x_i\big)\Psi_{x_i}(m(x_1,\dots,x_n)\big).$$

Conversely, the n-tuple $(F_1,\dots,F_n)$ is a solution of equation (12).

The above theorem has an analogous version with the description of the kernel of the Dynkin idempotent due to Patras and Reutenauer's:

PROPOSITION 64. — *Let V be the vector space spanned by the indeterminates $x_1,\dots,x_n$. Let $F_{1,0}(x_1,\dots,x_n),\dots,F_{n,0}(x_1,\dots,x_n)$ be the particular solution of equation* (12) *constructed in Theorem* 62. *Let the n-tuple $(F_1(x_1,\dots,x_n),\dots,F_n(x_1,\dots,x_n))\in \mathrm{Lie}(V)^n$ of Lie series in indeterminates $x_1,\dots,x_n$.*

If $(F_1,\dots,F_n)$ is a solution of the equation (1) *then, there exists $\lambda_1,\dots,\lambda_n$ in $\mathbb{K}$ and a finite set of indices J indexing a finite family of non-commutative polynomials $p^j(x_1,\dots,x_n)\in T(V)$ and a finite family of scalars $\mu_j\in\mathbb{K}$ such that*

$$\begin{aligned}F_i(x_1,\dots,x_n)&:=F_{i,0}(x_1,\dots,x_n)\\&\quad+\mathrm{Ber}\big((-1)^i x_i\big)\gamma\Big(\sum_{j\in J}\sum_{m\geqslant 0}\frac{m}{m+1}\mu_j(E(p^j)_n(x_1,\dots,x_n))_{x_i}\Big)+\lambda_i x_i,\end{aligned}$$

with the notations of Definition 57. *Conversely, the n-tuple is solution of equation* (12).

Indeed, there exists an analogous formula linking the multilinearized Baker-Campbell-Hausdorff formula defined as

$$\Phi(x_1,\dots,x_n):=\log(\mathrm{e}^{x_1}\cdots\mathrm{e}^{x_n})$$

and the Eulerian idempotent (cf. [6]). All the proofs will be analogous as the Eulerian and the Dynkin idempotent are defined on the tensor module $T(V)$ over any vector space V and can be particularised in the case where $V=\mathbb{K}x_1\oplus\cdots\oplus\mathbb{K}x_n$.

The case treated in this paper is the case

$$n=2,\quad F_1(x,y)=F(x,y)\quad\text{and}\quad F_2(x,y)=\sum_{n\geqslant 0}(-1)^n G_n(x,y).$$

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