ANNALES

DE

L'INSTITUT FOURIER

Florent BALACHEFF & Daniel MASSART

Stable norms of non-orientable surfaces

STABLE NORMS OF NON-ORIENTABLE SURFACES

by Florent BALACHEFF & Daniel MASSART (*)

————

ABSTRACT. — We study the stable norm on the first homology of a closed non-orientable surface equipped with a Riemannian metric. We prove that in every conformal class there exists a metric whose stable norm is polyhedral. Furthermore the stable norm is never strictly convex if the first Betti number of the surface is greater than two.

RÉSUMÉ. — Nous étudions la norme stable sur le premier groupe d'homologie d'une surface fermée et non-orientable munie d'une métrique riemannienne. Nous montrons qu'il existe dans chaque classe conforme une métrique dont la norme stable est polyèdrale. De plus, la norme stable est strictement convexe dès que le premier nombre de Betti est au moins trois.

1. Introduction

Given a smooth compact Riemannian manifold (M, g) with first Betti number $b_1(M) > 0$, the stable norm $\| \ \|$ on $H_1(M, \mathbb{R})$ is defined in [10] (see also [9]) as

$$
\begin{array}{rcl}
H_1(M, \mathbb{R}) & \longrightarrow & \mathbb{R} \\
h & \longmapsto & \|h\| := \inf\left\{\sum_{i=1}^{n} |r_i| l_g(\gamma_i)\right\}
\end{array}
$$

where

- l_g denotes the length with respect to g
- the r_i are real numbers
- the γ_i are Lipschitz 1-cycles
- $h = \sum_{i=1}^{n} r_i [\gamma_i]$.

Note that since we want to minimize the length we may assume from the start that the γ_i are closed geodesics which minimize the length in their free homotopy class.

In general the infimum may not be achieved. It is remarkable that when the dimension of M is two, it is achieved for every integer homology class (Proposition 5.6). When the infimum is actually a minimum, we may wonder whether the minimizing cycles are connected. Note that every component $\gamma_i, i = 1 \dots n$ of a minimizing cycle $\sum_{i=1}^{n} r_i \gamma_i$ is itself minimizing in its own homology class. A minimizing cycle whose connected components have distinct homology classes yields a flat region in the unit sphere $\mathcal{S}_1$ of the stable norm, containing the convex hull of $\{[\gamma_i]/l_g(\gamma_i)\}_{i=1}^{n}$. So we may ask how often does such a flat appear, how many components do the minimizing cycles have and what is the dimension of the corresponding flat (that is the dimension of the affine subspace it spans). In this paper we give some answers when M is a closed non-orientable surface. Our first result is similar to Theorem 7 of [14] which addresses the orientable case. We denote by $[x]$ the integer part of a real number x.

THEOREM 1.1. — *Let M be a closed non-orientable surface endowed with a Riemannian metric. Then every connected minimizing cycle is a component of a minimizing cycle with at least $[(b_1(M) + 1)/2]$ homologically independent components, and at most $2b_1(M) - 1$ non pairwise homologically proportional components.*

So the stable norm is never strictly convex for $b_1(M) > 2$. The difference with Theorem 7 of [14] is that the dimension of the corresponding flat may be more than $[(b_1(M) + 1)/2] - 1$. Observe that if $b_1(M) = 2$ the stable norm may be strictly convex. For instance, take a hyperbolic punctured torus, cut off a sufficiently thin neighborhood of the cusp, and cap off with a projective plane.

Let $\pi \colon M_o \longrightarrow M$ be the orientation cover of a non-orientable surface M. A simple closed curve γ of M is said to be *of type I* (resp. *of type II*) if its inverse image $\pi^{-1}(\gamma)$ consists of either one curve or two homologous curves (resp. two non-homologous curves). Remark that one-sided simple closed curves on M (curves whose tubular neighborhood is homeomorphic to a Mőbius strip) are of type I while two-sided simple closed curves on M (curves whose tubular neighborhood is homeomorphic to an annulus) may be of type I or II. The following theorem states that the local geometry of the unit sphere $\mathcal{S}_1$ is special near homology classes whose minimizing cycles consist of curves of type I. Specifically, the intersection of the unit ball with a neighborhood of such a class is a cone.

THEOREM 1.2. — *Let M be a closed non-orientable surface endowed with a Riemannian metric. Let h_0 be an integer homology class all of whose*

minimizing cycles consist of geodesics of type I. Then for all $h \in H_1(M, \mathbb{R})$, there exists $s(h_0, h) > 0$ such that the subset of the unit sphere $\mathcal{S}_1$

$$\left\{ \frac{h_0 + sh}{||h_0 + sh||} \ : \ s \in [0, s(h_0, h)] \right\}$$

is a straight segment.

For manifolds in dimensions greater than 2 little is known about minimizing cycles. For flat tori they exist and are connected in every integer homology class (or multiple thereof). Other homology classes do not have minimizing cycles. Furthermore the stable norm of a flat torus is Euclidean. Apart from [16] which deals with hyperbolic metrics on a punctured torus, the only other examples ([1], [2]) where the stable norm is actually computed have very few connected minimizing cycles: the unit ball of the stable norm is a polyhedron. So there is only a finite number of connected minimizing cycles, corresponding to the vertices of the polyhedron. In every homology class there is a minimizing cycle which is a linear combination of the connected ones. All such examples assume $\dim M \geqslant 3$; if $\dim M = 2$ and M is orientable, [14] rules out the unit ball being a polyhedron. The situation is different when $\dim M = 2$ and M is not orientable:

THEOREM 1.3. — *Let M be a closed non-orientable surface. Then in every conformal class there exists a metric whose stable norm has a polyhedron as its unit ball.*

Now we briefly describe the contents of the paper. Section 2 contains basic facts about non-orientable surfaces. In Sections 3 and 4 we have gathered prerequisites about minimizing measures (in the sense of [15]) and stable norms. Some material from [13] and [14] has been included, either because it was not published, or because we found the exposition to be wanting. In Section 5 we prove the technical lemmas we need for our main theorems. Some consequences are derived, among which Lemma 5.5 and Proposition 5.6. In the last section we prove our main theorems.

2. Preliminaries: non-orientable surfaces

Let (M, g) be a closed non-orientable Riemannian surface.

2.1. First homology group

By the classical Surface Classification Theorem, any closed orientable surface is a connected sum of tori, any closed non-orientable surface is a

connected sum of tori and projective planes. In fact, since the connected sum of three projective planes is homeomorphic to the connected sum of a torus and a projective plane, any non-orientable surface is a connected sum of tori and one or two projective planes. Recall that the connected sum of two projective planes is the Klein bottle $\mathbb{K}$.

Denote by Σ_k an orientable surface of genus k (that is, $\Sigma_k \cong \sharp^k \mathbb{T}^2$). We have

$$H_1(\Sigma_k \sharp \mathbb{K}, \mathbb{R}) \cong H_1(\Sigma_k, \mathbb{R}) \oplus H_1(\mathbb{K}, \mathbb{R}) \cong \mathbb{R}^{2k} \oplus \mathbb{R}$$

whence the first Betti number $b_1(\Sigma_k \sharp \mathbb{K})$ of $\Sigma_k \sharp \mathbb{K}$ is $2k + 1$. Likewise,

$$H_1(\Sigma_k \sharp \mathbb{R}P^2, \mathbb{R}) \cong H_1(\Sigma_k, \mathbb{R}) \cong \mathbb{R}^{2k}$$

and $b_1(\Sigma_k \sharp \mathbb{R}P^2) = 2k$.

Similarly, we have

$$H_1(\Sigma_k \sharp \mathbb{R}P^2, \mathbb{Z}) \cong H_1(\Sigma_k, \mathbb{Z}) \oplus H_1(\mathbb{R}P^2, \mathbb{Z}) \cong \mathbb{Z}^{2k} \oplus \mathbb{Z}/2\mathbb{Z}$$

and

$$H_1(\Sigma_k \sharp \mathbb{K}, \mathbb{Z}) \cong H_1(\Sigma_k, \mathbb{Z}) \oplus H_1(\mathbb{K}, \mathbb{Z}) \cong \mathbb{Z}^{2k} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

For any manifold M, the torsion-free part of $H_1(M, \mathbb{Z})$ embeds as a lattice Λ in $H_1(M, \mathbb{R})$. We say

- an element of $H_1(M, \mathbb{R})$ is *integer* if it belongs to Λ
- a subspace of $H_1(M, \mathbb{R})$ is *integer* if it is generated by integer classes
- an element h of $H_1(M, \mathbb{R})$ is *rational* if rh belongs to Λ for some real number r.

2.2. Orientation cover

Let $\pi\colon M_o \longrightarrow M$ be the orientation cover of M. Then M_o is an orientable surface endowed with a fixed-point free, orientation-reversing involution I. Let I_* be the involution of $H_1(M_o, \mathbb{R})$ induced by I, and let E_1 (resp. E_{-1}) be the eigenspace of I_* for the eigenvalue 1 (resp.-1). First observe that

PROPOSITION 2.1. — E_1 and E_{-1} are Lagrangian for the (symplectic) intersection form Int on $H_1(M_o, \mathbb{R})$.

Proof. — Take $x, y \in E_1$ (resp. E_{-1}). We have $\text{Int}(I_*(x), I_*(y)) = \text{Int}(x, y)$ but on the other hand, since I reverses the orientation of M_o, $\text{Int}(I_*(x), I_*(y)) = -\text{Int}(x, y)$ whence $\text{Int}(x, y) = 0$, which proves that E_1 (resp. E_{-1}) is isotropic. In particular $\dim E_1 \leqslant 2^{-1} b_1(M_o)$ and $\dim E_{-1} \leqslant 2^{-1} b_1(M_o)$. Now since I_* is a linear involution, $\dim E_1 + \dim E_{-1} = b_1(M_o)$ whence

$\dim E_1 = \dim E_{-1} = 2^{-1} b_1(M_o)$ that is, E_1 (resp. E_{-1}) is Lagrangian for the symplectic form Int. $\qquad\square$

Furthermore

PROPOSITION 2.2. — $\ker \pi_* = E_{-1}$.

Proof. — Let γ be a 1-cycle in M_o such that $\pi_*([\gamma]) = 0$. That is, $\pi(\gamma)$ bounds a 2-chain C in M. Then $\pi^{-1}(\pi(\gamma))$ bounds the 2-chain $\pi^{-1}(C)$ in M_o. But $\pi^{-1}(\pi(\gamma)) = \gamma \cup I(\gamma)$, so $[\gamma] + [I(\gamma)] = 0$. Conversely, if γ is a 1-cycle in M_o such that $[\gamma] + [I(\gamma)] = 0$, then γ and $I(\gamma)$ together bound a two-chain C in M_o, so $\pi(\gamma) = \pi(I(\gamma))$ bounds the two-chain $\pi(C)$ in M, thus $[\pi(\gamma)] = 0$ in $H_1(M, \mathbb{R})$. $\qquad\square$

Consequently π_* identifies $H_1(M, \mathbb{R})$ with E_1.

3. Preliminaries: minimizing measures and stable norm

The material of this section is taken from [13] and was not published. Most of the ideas therein were presented to the second author by Albert Fathi.

We introduce an alternative definition of the stable norm. It relies on invariant measures of the geodesic flow and is inspired by Mather's theory for Lagrangian systems. Then minimizing objects, in the form of measures (or asymptotic cycles as in [18]), exist in every homology class. The question of whether a minimizing cycle exists becomes "are minimizing measures supported on closed geodesics ?".

Let (M, g) be a compact Riemannian manifold of any dimension with first Betti number $b_1(M) > 0$. Denote by

- $T^1 M$ the unit tangent bundle of (M, g)
- p the canonical projection $T^1 M \longrightarrow M$
- ϕ_t the geodesic flow in $T^1 M$.

3.1. Minimizing measures

Define $\mathcal{M}$ as the set of all probability measures on $T^1 M$, endowed with the weak* topology. Then $\mathcal{M}$ is compact and metrizable [6, 13.4.2]. Moreover it embeds homeomorphically as a convex subset of the dual to the vector space $C^0(T^1 M)$ of continuous functions on $T^1 M$. Let $\mathcal{M}_g$ be the subset of $\mathcal{M}$ that consists of ϕ_t-invariant measures. Then $\mathcal{M}_g$ is closed in

$\mathcal{M}$, hence compact, and convex. Fix an element μ of $\mathcal{M}_g$. By [15], for any C^1 function f on M, we have

$$\int_{T^1 M} df(x).v \, d\mu(x, v) = 0.$$

Thus, if ω is a smooth closed one-form on M, the integral

$$\int_{T^1 M} \omega_x(v) d\mu(x, v)$$

only depends on the cohomology class of ω. By duality this endows μ with a homology class: $[\mu]$ is the unique element of $H_1(M, \mathbb{R})$ such that

$$\langle [\mu], [\omega] \rangle = \int_{T^1 M} \omega d\mu$$

for any smooth closed one-form ω on M. The map

$$[.]: \mathcal{M}_g \longrightarrow H_1(M, \mathbb{R})$$
$$\mu \longmapsto [\mu]$$

is continuous and affine, so the image $\mathcal{B}_1$ of $\mathcal{M}_g$ in $H_1(M, \mathbb{R})$ is compact and convex.

PROPOSITION 3.1. — $\mathcal{B}_1$ is the unit ball of the stable norm.

Proof. — We first prove that $\mathcal{B}_1$ is the unit ball of some norm N.
Denote by

$$\mathcal{I}: T^1 M \longrightarrow T^1 M$$
$$(x, v) \longmapsto (x, -v)$$

the canonical involution of $T^1 M$. We have, for any (x, v) in $T^1 M$,

$$\phi_t(x, -v) = \phi_{-t}(x, v)$$

so if μ is in $\mathcal{M}_g$, then $\mathcal{I}_* \mu$ is again in $\mathcal{M}_g$. Let ω be a smooth closed one-form on M. We have

$$\begin{aligned}
\langle [\mathcal{I}_* \mu], [\omega] \rangle &= \int_{T^1 M} \omega_x(v) d\mathcal{I}_* \mu(x, v) \\
&= \int_{T^1 M} \omega_x(-v) d\mu(x, v) \\
&= -\int_{T^1 M} \omega_x(v) d\mu(x, v) \\
&= -\langle [\mu], [\omega] \rangle
\end{aligned}$$

whence

$$[\mathcal{I}_* \mu] = -[\mu],$$

so $\mathcal{B}_1$ is centrally symmetric.

Now let us show that $\mathcal{B}_1$ contains the origin in its interior. Fix a basis $h_1, \ldots h_n$ of $H_1(M, \mathbb{R})$ such that $h_1, \ldots h_n$ are integer elements of $H_1(M, \mathbb{R})$. Let $\gamma_1, \ldots \gamma_n$ be closed geodesics parameterized by arc length such that $[\gamma_i] = h_i$, $i = 1, \ldots n$ and let $\mu_1, \ldots \mu_n$ be the probability measures defined by

$$\int_{T^1 M} f(x, v) d\mu_i(x, v) := \frac{1}{l_g(\gamma_i)} \int_0^{l_g(\gamma_i)} f(\gamma_i(t), \dot{\gamma}_i(t)) \, dt, \ i = 1, \ldots n$$

where $f \in C^0(T^1 M)$. We have, for any smooth closed one-form ω on M

$$\langle [\mu_i], [\omega] \rangle = \frac{1}{l_g(\gamma_i)} \langle [\gamma_i], [\omega] \rangle, \ i = 1, \ldots n$$

whence

$$[\mu_i] = \frac{1}{l_g(\gamma_i)} [\gamma_i], \ i = 1, \ldots n.$$

Therefore $\mathcal{B}_1$ contains the points $\pm l_g(\gamma_i)^{-1} [\gamma_i]$, $i = 1, \ldots n$, so it contains their convex hull, which contains the origin in its interior because $[\gamma_i]$, $i = 1, \ldots n$ generate $H_1(M, \mathbb{R})$.

So $\mathcal{B}_1$ is the unit ball for some norm N in $H_1(M, \mathbb{R})$, which justifies the notation $\mathcal{B}_1$.

Let us show this norm N is none other than the stable norm. First we show $\| \ \| \geqslant N$. Take h in $H_1(M, \mathbb{R})$ and $\epsilon > 0$. Let $\sum_i r_i \gamma_i$ be a cycle such that the γ_i are closed geodesics, $[\sum_i r_i \gamma_i] = h$, and $\sum_i |r_i| l_g(\gamma_i) \leqslant \|h\| + \epsilon$. After changing the orientations of the γ_i if need be, we may assume that the r_i are non-negative. Then the formula

$$\int_{T^1 M} f(x, v) d\mu(x, v) := \frac{\sum_i r_i \int_0^{l_g(\gamma_i)} f(\gamma_i(t), \dot{\gamma}_i(t)) \, dt}{\sum_i r_i l_g(\gamma_i)}$$

defines an element of $\mathcal{M}_g$, with associated homology class

$$[\mu] = \frac{[\sum_i r_i \gamma_i]}{\sum_i r_i l_g(\gamma_i)}$$

By definition we have $N([\mu]) \leqslant 1$, whence, since N is a norm

$$N\left(\left[\sum_i r_i \gamma_i \right] \right) \leqslant \sum_i |r_i| l_g(\gamma_i) \leqslant \|h\| + \epsilon.$$

Since ϵ is arbitrary, we conclude that $\| \ \| \geqslant N$.

Now let us show that $\| \ \| \leqslant N$. It suffices to show that for any $\mu \in \mathcal{M}_g$, we have $\|[\mu]\| \leqslant 1$. Here we use the dual stable norm (see [10, 4.35]). A norm $\| \ \|_0$ is defined on the space of C^1 closed one-forms on M by

$$\|\omega\|_0 := \max \left\{ \omega_x(v) : (x, v) \in T^1 M \right\}.$$

This norm induces a norm on $H^1(M, \mathbb{R})$: $\forall c \in H^1(M, \mathbb{R})$,

$$\|c\|_0 := \inf\left\{\|\omega\|_0 : \ [\omega] = c\right\}.$$

LEMMA 3.2 (Gromov [10]). — *The norm* $\| \ \|_0$ *on* $H^1(M, \mathbb{R})$ *is dual to the stable norm, that is, for any* $h \in H_1(M, \mathbb{R})$,

$$\|h\| = \max\left\{\langle c, h\rangle : \ c \in H^1(M, \mathbb{R}), \ \|c\|_0 \leqslant 1\right\}.$$

In view of the above lemma, what we need to show is that for any $c \in H^1(M, \mathbb{R})$ such that $\|c\|_0 \leqslant 1$, we have $\langle c, [\mu]\rangle \leqslant 1$. As $\|c\|_0 \leqslant 1$, for all $\epsilon > 0$ there exists a closed one-form ω such that $[\omega] = c$ and $|\omega_x(v)| \leqslant 1 + \epsilon$ for all $(x, v) \in T^1 M$. By the Ergodic Decomposition Theorem [11, Theorem 6.4, p. 170] we have

$$\int_{T^1 M} \omega d\mu = \int_{T^1 M} \left\{\int_{T^1 M} \omega d\mu_{x,v}\right\} d\mu(x, v)$$

where, for μ-almost every (x, v),

$$\int_{T^1 M} \omega d\mu_{x,v} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \omega(\phi_t(x, v)) dt.$$

Since $\phi_t(x, v)$ is in $T^1 M$ for all t, the above expression is bounded from above by $1 + \epsilon$, which proves that $\langle [\omega], [\mu]\rangle \leqslant 1 + \epsilon$. Thus $\langle c, [\mu]\rangle \leqslant 1$ for any $c \in H^1(M, \mathbb{R})$ such that $\|c\|_0 \leqslant 1$ so $\| \ \| \leqslant N$.

Finally $\| \ \| = N$ and $\mathcal{B}_1 = \{h \in H_1(M, \mathbb{R}) : \|h\| \leqslant 1\}$. □

We say an element μ of $\mathcal{M}_g$ is *minimizing* if and only if its homology class lies on the boundary $\mathcal{S}_1$ of $\mathcal{B}_1$, that is, if there exists a cohomology class c such that $\langle c, [\mu]\rangle = 1$ and $\langle c, h\rangle \leqslant 1$ for all $h \in \mathcal{B}_1$.

3.2. Link with Mather's theory

In this paragraph we prove that the minimizing measures just defined are minimizing in the sense of Mather [15], which allows us to use Mather's Graph Theorem.

Consider the set $\mathcal{M}'_g$ of all compactly supported, ϕ_t-invariant probability measures on the tangent bundle TM of M and not just $T^1 M$ (here ϕ_t denotes the geodesic flow in TM). We can define the homology class of an element of $\mathcal{M}'_g$ just like we do for an element of $\mathcal{M}_g$. Mather's β-function is defined in [15] as

$$\begin{aligned} \beta \colon H_1(M, \mathbb{R}) &\longrightarrow \mathbb{R} \\ h &\longmapsto \min\left\{\int_{TM} \tfrac{1}{2}\| \cdot \|_g^2 d\mu : \ \mu \in \mathcal{M}'_g, \ [\mu] = h\right\} \end{aligned}$$

where $\|(x, v)\|_g^2 := g_x(v, v)$ for all $(x, v) \in TM$.

The measures achieving the minimum for some h are called h-*minimizing.* Next we show that this definition of minimizing agrees with ours.

PROPOSITION 3.3. — *A minimizing measure $\mu \in \mathcal{M}_g$ is $[\mu]$-minimizing. Conversely an h-minimizing measure in $\mathcal{M}'_g$ with $h \in \mathcal{S}_1$ is in $\mathcal{M}_g$; in particular, it is minimizing. Furthermore, $2\beta = \| \ \|^2$.*

Proof. — Let us begin by showing that β is quadratic (i.e. 2-homogeneous). Take $h \in H_1(M, \mathbb{R})$, μ an h-minimizing measure, λ a real number. The formula

$$f \longmapsto \int_{TM} f(x, \lambda v) d\mu(x, v)$$

defines a probability measure on TM, whose homology class is λh. Therefore we have $\beta(\lambda h) \leqslant \lambda^2 \beta(h)$ and likewise,

$$\beta(h) = \beta(\frac{1}{\lambda}\lambda h) \leqslant \frac{1}{\lambda^2}\beta(\lambda h)$$

whence $\beta(\lambda h) = \lambda^2 \beta(h)$.

Now, since 2β and $\| \ \|^2$ are both quadratic, proving that

$$\mathcal{B}_1 = \left\{ h \in H_1(M, \mathbb{R}) : \ \beta(h) \leqslant \frac{1}{2} \right\}$$

suffices to prove that $2\beta = \| \ \|^2$. Note that

$$\mathcal{B}_1 \subset \left\{ h \in H_1(M, \mathbb{R}) : \ \beta(h) \leqslant \frac{1}{2} \right\}$$

for if μ is an element of $\mathcal{M}_g$, we can view it as a measure on TM supported on T^1M, thus

$$\int_{TM} \frac{1}{2}\| \cdot \|_g^2 d\mu = \int_{TM} \frac{1}{2}d\mu = \frac{1}{2}$$

whence $\beta([\mu]) \leqslant 1/2$.

Conversely, let $h \in H_1(M, \mathbb{R})$ be such that $\beta(h) = 1/2$, and let μ be an h-minimizing measure. Then by [4] the support of μ is contained in the energy level one half, that is, T^1M. Thus $[\mu] \in \mathcal{B}_1$, and

$$\mathcal{B}_1 = \left\{ h \in H_1(M, \mathbb{R}) : \ \beta(h) \leqslant \frac{1}{2} \right\} \text{ whence } \beta = \frac{1}{2}\| \ \|^2.$$

Moreover, since $\beta(h) = 1/2$, h lies on the boundary of $\mathcal{B}_1$, hence μ is minimizing in our sense. Now we would like to prove that a minimizing measure in our sense minimizes in the sense of Mather. Let $\mu \in \mathcal{M}_g$ be such that $[\mu] \in \mathcal{S}_1$. Then as we have just seen $\beta([\mu]) = 1/2$ so

$$\int_{TM} \frac{1}{2}\| \cdot \|_g^2 d\mu \geqslant \frac{1}{2}.$$

On the other hand since μ is supported in T^1M, we have

$$\int_{TM} \frac{1}{2} \| \cdot \|_g^2 d\mu \leqslant \frac{1}{2}$$

whence

$$\int_{TM} \frac{1}{2} \| \cdot \|_g^2 d\mu = \frac{1}{2} = \beta([\mu])$$

that is, μ is minimizing in the sense of Mather. $\qquad\square$

The main payoff for our efforts is that we may use Mather's Graph Theorem. Let c be a cohomology class such that $\langle c, h \rangle \leqslant 1$ for all $h \in \mathcal{B}_1$ and $\langle c, h_0 \rangle = 1$ for some $h_0 \in \mathcal{B}_1$ (thus $\|c\|_0 = 1$). We say a measure $\mu \in \mathcal{M}_g$ is *c-minimizing* if $\langle c, [\mu] \rangle = 1$. Let $\mathcal{M}_c \subset T^1M$ be the union of the supports of all c-minimizing measures. Recall that p denote the canonical projection $T^1M \longrightarrow M$.

THEOREM 3.4 (Mather [15]). — *The restriction of p to $\mathcal{M}_c$ is injective, and its inverse is Lipschitz.*

The reason why this is called the Graph Theorem is that it says the support of a minimizing measure is a Lipschitz graph over its projection to M. Recall (see [3] for an introduction to measured geodesic laminations) that a geodesic lamination is a subset S of a Riemannian manifold, with the following property : there exists a compact subset $\tilde{S}$ of T^1M, such that

- $\tilde{S}$ is invariant by the geodesic flow
- the canonical projection p, restricted to $\tilde{S}$, is injective
- $p(\tilde{S}) = S$.

A measured geodesic lamination is a geodesic lamination S, with the additional data of a transverse measure μ, invariant by the geodesic flow and whose support is $\tilde{S}$. So the Graph Theorem implies that minimizing measures can be identified with measured geodesic laminations.

4. Flats of the unit ball

Let (M, g) be a closed Riemannian manifold of any dimension. We call

- supporting subspace to the unit ball of the stable norm, any affine subspace of $H_1(M, \mathbb{R})$ that meets the unit sphere but not the open unit ball
- flat of the unit ball, the intersection of the unit sphere with a supporting subspace

- dimension of a flat, the dimension of the affine subspace it generates in $H_1(M, \mathbb{R})$
- interior of a flat, its interior in the affine subspace it generates.

As a trivial example, all points of the unit sphere are zero-dimensional flats. If c is a cohomology class of dual stable norm one, that is, $\langle c, h \rangle \leqslant 1$ for all $h \in \mathcal{B}_1$ and $\langle c, h_0 \rangle = 1$ for some $h_0 \in \mathcal{B}_1$, then

$$\{h \in H_1(M, \mathbb{R}) : \langle c, h \rangle = 1\}$$

is a supporting hyperplane to the unit ball of the stable norm, and

$$\{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}$$

is a flat, which may or may not be trivial. Note that by the Hahn-Banach Theorem, any supporting subspace is contained in a supporting hyperplane. So for any flat F, there exists $c \in H^1(M, \mathbb{R})$ such that

$$\langle c, h \rangle \leqslant 1, \forall h \in \mathcal{B}_1 \text{ and } F \subset \{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\} .$$

By [12], if a minimizing measure μ is ergodic, then $[\mu]$ is an extremal point of $\mathcal{B}_1$, hence it cannot be in the interior of any non-trivial flat. In particular non connected minimizing cycles, when they exist, are the simplest examples of non-trivial flats. Recall Proposition 4 of [14]:

LEMMA 4.1. — Let F_1 and F_2 be two flats of the unit ball, both containing a point h_0 such that h_0 is an interior point of F_1. Then there exists a flat F containing $F_1 \cup F_2$.

Proof. — Let $c \in H^1(M, \mathbb{R})$ be such that

$$\langle c, h \rangle \leqslant 1, \forall h \in \mathcal{B}_1 \text{ and } F_2 \subset \{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\} .$$

Then c restricted to the convex set F_1 has a maximum at the interior point h_0. Since c is linear, this implies that c is constant on F_1. Hence

$$\{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}$$

is a flat containing $F_1 \cup F_2$. □

LEMMA 4.2. — Let F_1 and F_2 be two flats of the unit ball, both containing a point h_0 in their interiors. Then there exists a flat F containing $F_1 \cup F_2$ such that h_0 is an interior point of F.

Proof. — Let

- V_i, $i = 1, 2$ be the underlying vector space of the affine space generated by F_i
- $V := V_1 + V_2$
- A be the affine subspace $h_0 + V$

- $F := A \cap \mathcal{B}_1$
- $c \in H^1(M, \mathbb{R})$ be given by the previous lemma such that

$$\langle c, h \rangle \leqslant 1 \ , \forall h \in \mathcal{B}_1 \text{ and } F_1, F_2 \subset \{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\} \,.$$

Since $F_1, F_2 \subset \{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}$ we have $A \subset \{h \in \mathcal{B}_1 : \langle c, h \rangle = 1\}$ so A is a supporting subspace whence F is a flat. Moreover, since F is convex and contains F_1 and F_2, it contains the convex hull C of F_1 and F_2. Now, since h_0 is interior to both F_1 and F_2, there exist open neighborhoods of zero U_1, U_2 in V_1, V_2 respectively such that $h_0 + U_i \subset F_i$, $i = 1, 2$. So the convex hull of $h_0 + U_1$ and $h_0 + U_2$ is open in A, and contained in C, hence in F. Thus h_0 is an interior point of F.   □

The preceding lemma means that for any homology class h, there exists an unique maximal flat containing h in its interior.

Orientable surfaces.

Assume, for the remainder of this section, that M is an orientable surface. If F is a flat of the unit ball of the stable norm, $h_1, h_2 \in F$ and μ_1, μ_2 are minimizing measures such that $[\mu_i] = h_i$, $i = 1, 2$, then by Mather's Graph Theorem the supports of μ_1 and μ_2 do not intersect transversally so $\text{Int}(h_1, h_2) = 0$. Thus the vector space generated by F in $H_1(M, \mathbb{R})$ is isotropic with respect to the symplectic intersection form. In particular its dimension is $\leqslant 2^{-1} b_1(M)$ so $\dim F \leqslant 2^{-1} b_1(M) - 1$. This was first observed by M.J. Carneiro [4]. In [14] this upper bound is proved to be optimal, so non-trivial flats always exists for orientable surfaces of genus $\geqslant 2$ (see Theorem 6.1 and its corollary).

It is proved in [14, Proposition 6] that a flat containing a rational point in its interior is a finite polyhedron with at most $3(2^{-1} b_1(M) - 1)$ vertices. Furthermore the vertices are rational homology classes which have connected minimizing cycles.

5. Technical lemmas

5.1. Key lemma, one-sided case

After writing up this lemma we came across reference [17], where a similar result is proved in a topological setting. Lemma 5.1 and its orientable companion Lemma 5.2 are improved versions of lemmas 15, 16, 17 of [14]. The purpose of Lemma 5.1 was to mimic the approach of [14]. Later on, inspired by [8] we realized it is simpler to use the orientation cover.

Let γ_1 be a closed, one-sided, simple geodesic on a closed non-orientable surface M endowed with a Riemannian metric.

LEMMA 5.1. — *There exists a neighborhood V_1 of $(\gamma_1, \dot{\gamma}_1)$ in $T^1 M$ such that, for any simple geodesic γ, if $(\gamma, \dot{\gamma})$ enters (resp. leaves) V_1 then γ is forever trapped in $p(V_1)$ in the future (resp. past), that is*

$$\exists t_1 \in \mathbb{R}, \ \forall t \geqslant (resp. \leqslant) t_1, \ \gamma(t) \in p(V_1).$$

Proof. — Let U_1 be a neighborhood of γ_1 in M homeomorphic to a Mőbius strip. Let $P := \gamma_1(0)$ be a point of γ_1, and let δ be an smooth open arc transverse at P to γ_1, such that $U_1 \setminus \delta$ is simply connected. Let V_1 be the neighborhood of $(\gamma_1, \dot{\gamma}_1)$ in $T^1 M$ defined by

(1) $p(V_1) = U_1$

(2) $\forall (x, v) \in V_1, p(\phi_t(x, v))$ intersects δ transversally at least three times $t_1 < t_2 < t_3$ (the points $p(\phi_{t_i}(x, v)), i = 1, 2, 3$ may coincide if $p(\phi_t(x, v))$ is a closed geodesic)

(3) we would like a condition along the lines of "the geodesics that enter V always cross δ in the same direction as γ_1". This must be done with some care because M is not orientable. So we choose a smooth vector field X in U_1, transverse to δ, which has γ_1 as a trajectory and such that every other trajectory is closed and homotopic to the boundary of U_1, that is, bounds a Mőbius strip containing γ_1. We require that

$$\forall (x, v) \in V_1, g(X(x), v) > 0.$$

These conditions define an open set of $T^1 M$ because δ is an open arc and we demand that the intersections be transverse.

Now consider a simple geodesic γ such that for some $t \in \mathbb{R}, (\gamma(t), \dot{\gamma}(t)) \in V_1$. Let $t_1 < t_2 < t_3$ be such that $\gamma(t_1), \gamma(t_2), \gamma(t_3)$ are the transverse intersection points of γ and δ given by the definition of V_1.

5.1.1. First case

Assume $\gamma(t_3)$ is farther away from P than $\gamma(t_1)$ with respect to the distance on δ induced by the metric of M. The domain U_1' bounded by $\gamma([t_1, t_3])$ and the subsegment of δ joining $\gamma(t_1)$ with $\gamma(t_3)$ is homeomorphic to a Mőbius strip. The geodesic γ does not self-intersect, hence it can only cut the boundary of U_1' along δ. By Condition 3 of the definition of V_1, γ can only intersect δ from left to right as pictured in Figure 5.1, that is, outwards of U_1'. Therefore γ is trapped in U_1' in the past.
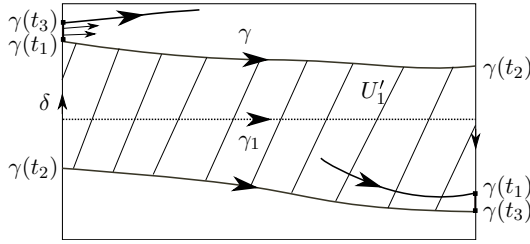
Figure 5.1

5.1.2. Second case

Assume $\gamma(t_1)$ is farther away from P than $\gamma(t_3)$ with respect to the distance on δ induced by the metric of M. We can prove as in the first case that γ is trapped in U_1' in the future.

5.1.3. Third case

Assume $\gamma(t_1) = \gamma(t_3)$. Then, since γ has no self-intersections, it must be a closed geodesic, hence is trapped in U_1 in both past and future. □

5.2. Key lemma, two-sided case

Let γ_2 be a closed, two-sided, simple geodesic without conjugate points on a closed (possibly non-orientable) surface M endowed with a Riemannian metric. Again, we shall only use the orientable case in the sequel, but the non-orientable case comes for free. Let U_2 be a neighborhood of γ_2 in M homeomorphic to an annulus. Choose a symplectic form ω in U_2, yielding a local orientation of U_2.

LEMMA 5.2. — *There exists a neighborhood V_2 of $(\gamma_2, \dot{\gamma}_2)$ in $T^1 M$ such that any simple geodesic γ, if $(\gamma, \dot{\gamma})$ enters (resp. leaves) V_2 then either γ intersects γ_2 or γ is forever trapped in $p(V_2)$ in the future (resp. past), that is*

$$\exists t_2 \in \mathbb{R}, \ \forall t \geqslant (\text{resp.} \leqslant) t_2, \ \gamma(t) \in p(V_2).$$

Moreover, all intersections with γ_2, if any, have the same sign with respect to ω.

Proof. — Let $P := \gamma_2(0)$ be a point of γ_2, and let δ be an smooth open arc transverse at P to γ_2, such that $U_2 \setminus \delta$ is simply connected. Assume δ is oriented so that $\omega(P)(\dot{\gamma}_2, \dot{\delta}) > 0$. Let V_2 be the neighborhood of $(\gamma_2, \dot{\gamma}_2)$ in $T^1 M$ defined by

(1) $p(V_2) = U_2$

(2) $\forall (x, v) \in V_2$, $p(\phi_t(x, v))$ intersects δ transversally at least twice before intersecting γ_2, if it intersects γ_2 at all; and if it does, it must intersect δ transversally at least twice more before either leaving U_2 or meeting γ_2 again

(3) the geodesics that enter V_2 always crosses δ in the same direction as γ_2, that is,

$$\forall x \in \delta, \ \forall v \in T_x^1 M \text{ such that } (x, v) \in V_2, \ \omega(x)(v, \dot{\delta}(x)) > 0.$$

These conditions define an open set of $T^1 M$ because δ is an open arc, γ_2 has no conjugate points and we demand that the intersections be transverse.

Now consider a simple geodesic γ such that for some $t \in \mathbb{R}$, $(\gamma(t), \dot{\gamma}(t)) \in V_2$. Let $t_1 < t_2$ be such that $\gamma(t_1), \gamma(t_2)$ are the first two transverse intersection points of γ and δ given by the definition of V_2.

5.2.1. First case

Assume $\gamma(t_2)$ is farther away from P than $\gamma(t_1)$ with respect to the distance on δ induced by the metric of M. The domain U_2' bounded by $\gamma([t_1, t_2])$ and the subarc of δ joining $\gamma(t_1)$ with $\gamma(t_2)$ on one side, and by γ_2 on the other side is homeomorphic to an annulus. The geodesic γ is simple so it cannot self-intersect, hence it can only cut the boundary of U_2' along δ or γ_2. By Condition 3 of the definition of V_2, γ can only intersect δ from left to right as pictured in Figure 5.2, that is, outwards of U_2'. Therefore γ either intersects γ_2 or is trapped in U_2' in the past.
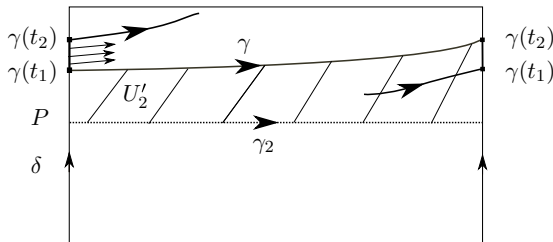

Figure 5.2

5.2.2. Second case

Assume $\gamma(t_1)$ is farther away from P than $\gamma(t_2)$ with respect to the distance on δ induced by the metric of M. Likewise we prove that γ either intersects γ_2 or is trapped in U_2' in the future.

5.2.3. Third case

Assume $\gamma(t_1) = \gamma(t_2)$. Then, since γ doesn't self-intersect, it must be a closed geodesic, and the conclusion readily follows.

We still have to prove the statement about the sign of the intersections. Assume γ cuts γ_2 once with positive sign, that is, downwards in Figure 5.3. Assume for convenience that the intersection point is $\gamma(0)$. Let $t_1 < t_2 < 0 < t_3 < t_4$ be such that $\gamma(t_1), \gamma(t_2)$ are the last two transverse intersection points of γ and δ before γ meets γ_2, and $\gamma(t_3), \gamma(t_4)$ are the first two transverse intersection points of γ and δ after γ meets γ_2. The domain U_2'' bounded by $\gamma([t_1, t_2])$ and the subarc of δ joining $\gamma(t_1)$ with $\gamma(t_2)$ on one side, and by $\gamma([t_3, t_4])$ and the subarc of δ joining $\gamma(t_3)$ with $\gamma(t_4)$ on the other side, is homeomorphic to an annulus and contains γ_2 in its interior.
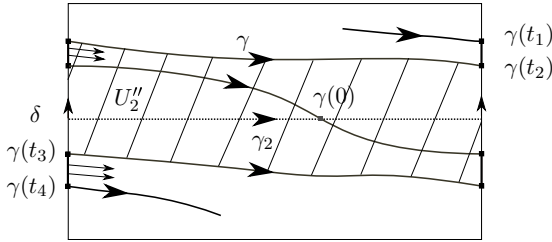

Figure 5.3

The geodesic γ does not self intersect so it cannot enter U_2'' through segments of γ. It only intersects δ from left to right, that is, either between $\gamma(t_3)$ and $\gamma(t_4)$ and outwards of U_2'', or between $\gamma(t_1)$ and $\gamma(t_2)$ and inwards of U_2''. So it can only enter U_2'' through δ between $\gamma(t_1)$ and $\gamma(t_2)$, that is, from above in Figure 5.3. Therefore it always cut γ_2 with positive sign.   □

5.3. Consequences of the key lemmas

To facilitate the exposition, rather than loading up the sentences with "resp." we have split the next proposition in two, one part for the one-sided case and the other for the two-sided case.

PROPOSITION 5.3. — *Let γ_1 be a closed, simple, one-sided geodesic on a closed non-orientable surface M endowed with a Riemannian metric. There exists a neighborhood V_1 of $(\gamma_1, \dot\gamma_1)$ in $T^1 M$ such that for any simple geodesic γ, if $(\gamma, \dot\gamma)$ enters (resp. leaves) V_1, then*

- γ is a closed geodesic homotopic to γ_1 or $\gamma_1.\gamma_1$, this being the product in $\pi_1(M)$
- or γ is positively (resp. negatively) asymptotic to a closed without conjugate points geodesic homotopic to γ_1 or $\gamma_1.\gamma_1$.

PROPOSITION 5.4. — Let γ_2 be a closed, simple, two-sided geodesic on a closed (possibly non-orientable) surface M endowed with a Riemannian metric. There exists a neighborhood V_2 of $(\gamma_2, \dot{\gamma}_2)$ in $T^1 M$ such that for any simple geodesic γ, if γ enters (resp. leaves) V_2, then

- either γ is a closed geodesic homotopic to γ_2
- or γ is asymptotic to a closed geodesic homotopic to γ_2
- or γ intersects γ_2, and all intersections have the same sign with respect to some orientation of $p(V_2)$.

Proof. — Let us prove Proposition 5.4. Let V_1 be a neighborhood of $(\gamma_1, \dot{\gamma}_1)$ in $T^1 M$ given by Lemma 5.2, and small enough so it does not contain any contractible closed geodesic. By contractible we mean null-homotopic. Let γ be a simple geodesic such that $(\gamma, \dot{\gamma})$ enters V_1. Let t_1 be such that $(\gamma, \dot{\gamma})([t_1, +\infty[) \subset V_1$. Extend $\dot{\gamma}([t_1, +\infty[)$ to a smooth vector field in the annulus $p(V_1)$. Since the annulus may be embedded in the two-sphere, the Poincaré-Bendixon Theorem applies. So γ is either a fixed point, a cycle of fixed points and heteroclinic orbits, or a closed orbit, or asymptotic to one of the preceding. Four out of six cases are impossible here because γ is a geodesic so its velocity is constant, hence cannot go to zero. Moreover, a closed orbit of a vector field must be a simple closed curve, and a non-contractible simple closed curve in an annulus is homotopic to the boundary of the annulus. This proves Proposition 5.4. The proof of Proposition 5.3 is identical, mutatis mutandis. $\qquad\square$

5.4. Geodesics asymptotic to closed geodesics

LEMMA 5.5. — Let M be a closed (possibly non-orientable) surface with a Riemannian metric. If a geodesic γ is asymptotic to a simple closed geodesic, then $(\gamma, \dot{\gamma})$ is not in the support of any minimizing measure.

Proof. — Let

- γ_0 be a simple closed geodesic
- γ be a geodesic asymptotic to γ_0
- V_0 be a neighborhood of $(\gamma_0, \dot{\gamma}_0)$ in $T^1 M$ given by Lemma 5.1 or 5.2 depending on whether γ_0 is one-sided or two-sided, and such that $(\gamma(0), \dot{\gamma}(0)) \notin V_0$

- V be a neighborhood of $(\gamma(0), \dot{\gamma}(0))$ such that V is disjoint from V_0 but for some t, $\phi_t(V) \subset V_0$
- μ be a minimizing measure.

Assume $(\gamma, \dot{\gamma})$ is contained in the support of μ. Then since $\operatorname{supp}(\mu)$ is closed and invariant under the geodesic flow, it contains the α-and ω-limit sets of $(\gamma, \dot{\gamma})$, in particular it contains $(\gamma_0, \dot{\gamma}_0)$. Moreover, $\mu(V) > 0$. By the Ergodic Decomposition Theorem [11, Theorem 6.4, p. 170], we have

$$\mu(V) = \int_{T^1 M} \mu_{x,v}(V) d\mu(x, v)$$

where, for μ-almost every x, v,

$$\mu_{x,v}(V) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \chi_V(\phi_t(x, v)) dt$$

where χ_V denote the characteristic function of V. Thus for some (x, v) we have $\mu_{x,v}(V) > 0$. So for some t in $\mathbb{R}$, $\phi_t(x, v) \in V$. By our hypothesis on V, this implies that for some other t we have $\phi_t(x, v) \in V_0$. But since $(\gamma_0, \dot{\gamma}_0)$ is contained in the support of μ, by Mather's Graph Theorem $p(\phi_t(x, v))$ cannot intersect γ_0. Thus by Proposition 5.3 or 5.4 the geodesic $p(\phi_t(x, v))$ is asymptotic to a geodesic homotopic to γ_0 whose lift to $T^1 M$ is contained in V_0. Therefore $\phi_t(x, v)$ never comes back to V, whence

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \chi_V(\phi_t(x, v)) dt = 0$$

which contradicts the fact that $\mu_{x,v}(V) > 0$. $\qquad\square$

5.5. Minimizing measures with rational homology classes

The proposition below was stated as Lemma 2.1.6 in [13] and Proposition 5 of [14]. The proofs given there were mere sketches, the reader will find a complete proof below. After writing the proof below, we realized another complete proof was already published as Proposition 2.1 of [5].

PROPOSITION 5.6. — *Let M be a closed (possibly non-orientable) surface with a Riemannian metric. If h is a rational homology class and μ is an h-minimizing measure, then the support of μ consists of periodic orbits.*

First we need

LEMMA 5.7. — *Let M be a closed non-orientable surface with a Riemannian metric g and $\pi : (M_o, \tilde{g}) \to (M, g)$ denote the Riemannian orientation cover.*

If $\mu \in \mathcal{M}_g$ is a c-*minimizing measure where* c *is a cohomology class with* $\|c\|_0 = 1$, *then there exists* $\nu \in \mathcal{M}_{\tilde{g}}$ *such that* $\pi_*(\nu) = \mu$, ν *is* I_*-*invariant and* $\pi^*(c)$-*minimizing.*

Proof of the lemma. — Let us assume for the time being that μ is ergodic. That is, there exists $(x, v) \in T^1 M$ such that for any continuous function F on $T^1 M$,

$$\int_{T^1 M} F d\mu = \lim_{T \to +\infty} \frac{1}{T} \int_0^T F(\phi_t(x, v)) dt$$

We lift the orbit $\phi_t(x, v)$ to an orbit $\tilde{\phi}_t(x, v)$ of the geodesic flow of $(M_o, \tilde{g})$. Let ν_T be the probability measure on $T^1 M_o$ defined by

$$\int_{T^1 M_o} F d\nu_T = \frac{1}{T} \int_0^T F(\tilde{\phi}_t(x, v)) dt$$

for any continuous function F on $T^1 M_o$. Since the set of probability measures on $T^1 M_o$ is compact for the weak* topology, there exists a sequence $T_n \to +\infty$ such that ν_{T_n} converges to some measure ν. Then ν is invariant by the geodesic flow on M_o, that is, $\nu \in \mathcal{M}_{\tilde{g}}$. Moreover $\pi_*(\nu) = \mu$ since, for any continuous function F on $T^1 M$

$$
\begin{aligned}
\int_{T^1 M} F d\pi_*(\nu) &= \int_{T^1 M_o} F \circ \pi \, d\nu = \lim_{n \to +\infty} \frac{1}{T_n} \int_0^{T_n} F(\pi(\tilde{\phi}_t(x, v))) dt \\
&= \lim_{n \to +\infty} \frac{1}{T_n} \int_0^{T_n} F(\phi_t(x, v)) dt = \int_{T^1 M} F d\mu.
\end{aligned}
$$

Furthermore, since μ is a c-minimizing measure, for all $\epsilon > 0$ there exists a closed one-form ω such that $[\omega] = c$, $|\omega_x(v)| \leqslant 1 + \epsilon$ for all $x \in M$, $v \in T_x^1 M$, and $\int \omega d\mu = 1$. Set $\tilde{\omega} := \pi^*(\omega)$, then $\tilde{c} = [\tilde{\omega}] = \pi^*(c)$ and $|\tilde{\omega}_x(v)| \leqslant 1 + \epsilon$ for all $x \in M_o$, $v \in T_x^1 M_o$, and $\int \tilde{\omega} d\nu = 1$. So ν is $\pi^*(c)$-minimizing. Notice that $\tilde{\omega}$ is I^*-invariant, so $I_*\nu$ is also $\pi^*(c)$-minimizing. Then so is $2^{-1}(\nu + I_*\nu)$, which is I_*-invariant. This proves the ergodic case of the lemma as $\pi_*(I_*\nu) = \mu$.

Now consider the restriction of π_* to the compact convex set

$$\{\mu \in \mathcal{M}_{\tilde{g}} : I_*(\mu) = \mu\}$$

with image in $\mathcal{M}_g$. This restriction is affine and surjective onto the extremal points of $\mathcal{M}_g$, hence surjective onto $\mathcal{M}_g$. $\square$

Proof of the proposition. — First let us address the case when M is orientable.

Let h be a rational homology class and μ be an h-minimizing measure. Then $\text{Int}(h, H_1(M, \mathbb{Z}))$ is a discrete subgroup of $\mathbb{R}$. Assume the projection

$p(\mathrm{supp}\mu)$ of the support of μ to M contains a non-closed geodesic γ. Since M is compact γ has a limit point, say x_γ in M. Let t_n be an increasing sequence of real numbers such that $\gamma(t_n) \longrightarrow x_\gamma$ when $n \longrightarrow \infty$. Denote by γ_n the closed curve obtained by closing up $\gamma([t_n, t_{n+1}])$ with a geodesic segment δ_n of length $d(\gamma(t_n), \gamma(t_{n+1}))$. Such a segment is unique for n large enough because $d(\gamma(t_n), \gamma(t_{n+1}))$ tends to zero. We claim that $\mathrm{Int}(h, [\gamma_n])$ is not zero for n large enough, and tends to zero, which contradicts the discreteness of $\mathrm{Int}(h, H_1(M, \mathbb{Z}))$.

By Mather's Graph Theorem [15], for any x in $p(\mathrm{supp}\mu)$, there exists a unique geodesic, denoted γ_x, which is the projection of an orbit in $\mathrm{supp}\mu$ and such that $\gamma_x(0) = x$. In order to make the notation less cumbersome we denote by γ_γ the orbit γ_x with $x = x_\gamma$. Call

$$R_n := \{\gamma_x(t) : x \in p(\mathrm{supp}\mu) \cap \delta_n, \ t \in [0,1]\}.$$

This is a closed subset of M.

First let us show that

(5.1) $$p_*\mu(R_n) \longrightarrow 0$$

Denote by χ_n the characteristic function of R_n. The sequence of functions χ_n converges pointwise to the characteristic function of $\gamma_\gamma([0,1])$, so

$$p_*\mu(R_n) \longrightarrow p_*\mu\left(\gamma_\gamma([0,1])\right).$$

Now the latter cannot be positive unless the geodesic γ_γ is closed, for otherwise, since μ is invariant by the geodesic flow, the total mass of γ_γ would be infinite, contradicting the fact that μ is a probability measure. Assume γ_γ is closed. It is two-sided because we are assuming M to be orientable for the time being. Since $\mathrm{supp}\mu$ is closed, x_γ is in $\mathrm{supp}\mu$. Since $\mathrm{supp}\mu$ is invariant by the geodesic flow γ_γ is contained in $\mathrm{supp}\mu$. Therefore by Mather's Graph Theorem γ_γ and γ do not intersect. Thus by Proposition 5.4, γ is asymptotic to a closed geodesic, hence cannot be in the support of a minimizing measure by Lemma 5.5. This proves Equation (5.1). Moreover, since γ is in the support of μ,

(5.2) $$p_*\mu(R_n) > 0.$$

Next we evaluate $\mathrm{Int}(h, [\gamma_n])$ and find it equals $p_*\mu(R_n)$, which together with the previous paragraph proves the proposition.

First note that by the Ergodic Decomposition Theorem [11, Theorem 6.4, p. 170],

(5.3) $$p_*\mu(R_n) = \int_M \left\{ \int \chi_n d\mu_x \right\} dp_*\mu(x)$$

where, for $p_*\mu$-almost every x in M

$$\int \chi_n d\mu_x \;=\; \lim_{T\to+\infty} \frac{1}{T} \int_0^T \chi_n(\gamma_x(t))dt$$

$$= \lim_{T\to+\infty} \frac{1}{T} \sharp\,\{t \in [0,T] \,:\, \gamma_x(t) \in \delta_n\}$$

by the definition of R_n, denoting $\sharp$ the cardinal of a set.

For x in $p(\mathrm{supp}\mu)\cap\delta_n$, let $\gamma_{x,T}$ be a closed curve obtained by closing up $\gamma_x([0,T])$ with a geodesic segment $\delta_{x,T}$ of length $\leqslant \mathrm{diam}M$. By Birkhoff's Ergodic Theorem, for $p_*\mu$-almost every x, for any closed one-form ω on M,

$$\int \omega d\mu_x = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \omega_{\gamma_x(t)}(\dot\gamma_x(t))dt = \lim_{T\to+\infty} \frac{1}{T} \langle[\omega]\,,[\gamma_{x,T}]\rangle.$$

Thus, for $p_*\mu$-almost every x,

$$[\mu_x] \;=\; \lim_{T\to+\infty} \frac{1}{T}\,[\gamma_{x,T}]\,.$$

Since the dimension of $H_1(M,\mathbb{R})$ is finite, the bilinear form $\mathrm{Int}(.,.)$ is continuous so for $p_*\mu$-almost every x,

$$\mathrm{Int}([\mu_x]\,,[\gamma_n]) = \lim_{T\to+\infty} \frac{1}{T}\mathrm{Int}([\gamma_{x,T}]\,,[\gamma_n]).$$

Observe that since both γ and γ_x are in the support of μ, by the Graph Theorem they cannot intersect transversally. So the transverse intersections of $\gamma_{x,T}$ and γ_n, if any, occur along δ_n or $\delta_{x,T}$. Note that for fixed n the number $n_{x,T}$ of intersections (counted with sign) of $\delta_{x,T}$ with γ_n is bounded independantly of T because the length of $\delta_{x,T}$ is bounded independantly of T.

Furthermore, by the Graph Theorem, all intersections of $\gamma_x([0,T])$ with δ_n have the same sign if δ_n is small enough. This is where we need the orientability assumption.

By smoothing the corners one can make $\gamma_{x,T}$ and γ_n of class C^1 without modifying their transverse intersections. The curve obtained in this way are transverse unless $\gamma = \gamma_x$. In the latter case one moves γ_x slightly away from γ without modifying the transverse intersections of $\gamma_{x,T}$ and γ_n. Since all intersections of $\gamma_x([0,T])$ with δ_n have the same sign, we get

(5.4) $\qquad \mathrm{Int}([\gamma_{x,T}]\,,[\gamma_n]) = \sharp\,\{t \in [0,T] \,:\, \gamma_x(t) \in \delta_n\} + n_{x,T}$

whence, since $n_{x,T}$ is bounded independantly of T,

$$\mathrm{Int}([\mu_x]\,,[\gamma_n]) = \lim_{T\to+\infty} \frac{1}{T}\sharp\,\{t \in [0,T] \,:\, \gamma_x(t) \in \delta_n\}$$

so, using Equation (5.3),

$$\text{Int}(h, [\gamma_n]) = p_*\mu(R_n)$$

which finishes the proof of the orientable case of the proposition.

Assume now that M is not orientable. Let μ be a minimizing measure such that $[\mu] = rh$ with $h \in \Lambda$ and $r \in \mathbb{R}$. Let ν be an I_*-invariant minimizing measure given by Lemma 5.7. Let $c_1, \ldots c_b$ be an integer basis of $H^1(M, \mathbb{R})$ and let $\omega_1, \ldots \omega_b$ be closed one-forms such that $[\omega_i] = c_i$, $i = 1 \ldots b$. Then $\int \omega_i d\mu \in r\mathbb{Z}$ for $i = 1 \ldots b$.

Let $\tilde{\omega}_1, \ldots \tilde{\omega}_b$ be the lifts of $\omega_1, \ldots \omega_b$ to M_o. They are integer one-forms and $[\tilde{\omega}_1], \ldots [\tilde{\omega}_b]$ is a basis of $E_1 = \{c \in H^1(M_o, \mathbb{R}) : I^*c = c\}$. Moreover,

$$\int \tilde{\omega}_i d\nu = \int \omega_i d\mu \in r\mathbb{Z}, \quad i = 1, \ldots, b.$$

Let us take an integer basis $c_{b+1}, \ldots c_{2b}$ of

$$E_{-1} = \{c \in H^1(M_o, \mathbb{R}) : I^*c = -c\}$$

and closed one-forms $\tilde{\omega}_1, \ldots \tilde{\omega}_b$ such that $[\tilde{\omega}_i] = c_i$ for $i = b+1, \ldots 2b$. Since ν is I_*-invariant we have

$$\int \tilde{\omega}_i d\nu = 0, \quad i = b+1, \ldots, 2b.$$

Let $x_1, \ldots x_{2b}$ be the coordinates of $[\nu]$ in the basis of $H_1(M_o, \mathbb{R})$ dual to the integer basis $[\tilde{\omega}_1], \ldots [\tilde{\omega}_{2b}]$ of $H^1(M_o, \mathbb{Z})$. We have just seen that $x_1, \ldots x_{2b}$ are all in $r\mathbb{Z}$, so $[\nu]$ is rational. Thus, using the orientable case of the proposition, we conclude that ν, hence μ, is supported on periodic orbits. $\qquad\square$

6. Proofs of the main theorems

6.1. Local results - Orientable case

Let h be a rational homology class of a closed (possibly non-orientable) Riemannian surface M. Then by Proposition 5.6 any h-minimizing measure is supported on periodic orbits. Call $\mathcal{P}_h$ the union of the projections on M of the supports of all h-minimizing measures. By Mather's Graph Theorem $\mathcal{P}_h$ is a union of pairwise disjoint closed geodesics. Denote by $\mathcal{VP}_h$ the vector subspace of $H_1(M, \mathbb{R})$ generated by all homology classes of geodesics contained in $\mathcal{P}_h$. Note that the convex hull of all homology classes of curves in $\mathcal{P}_h$ is contained in a flat of the unit ball containing h in its interior.

The following theorem proved in [14] describes the local geometry of the unit ball of the stable norm near a rational homology class in the orientable case.

THEOREM 6.1. — [14] *Let M be a closed orientable surface endowed with a Riemannian metric g. Let h_0 be a rational point of $\mathcal{S}_1$. For all $h \in \mathcal{VP}_{h_0}^{\perp}$, there exists $s(h_0, h) > 0$ such that the subset of the unit sphere $\mathcal{S}_1$*

$$\left\{ \frac{h_0 + sh}{||h_0 + sh||} \, : \, s \in [0, s(h_0, h)] \right\}$$

is a straight segment.

Proof. — For any $n \in \mathbb{N}^*$ let us denote

$$h_n := \frac{h_0 + \frac{1}{n}h}{||h_0 + \frac{1}{n}h||}.$$

Let

- μ_n be an h_n-minimizing measure
- μ_0 be a limit point, in the weak-$*$ topology, of the sequence μ_n.

Then μ_0 is an h_0-minimizing measure. By Proposition 5.6, μ_0 is supported on periodic orbits $\gamma_i, i \in I$ where I is some set, not necessarily finite. Note that for all $i \in I$ the class $[\gamma_i]$ belongs to $\mathcal{VP}_{h_0}$. For each $i \in I$ let V_i be the neighborhood of $(\gamma_i, \dot{\gamma}_i)$ given by Proposition 5.4. Let V be the union over $i \in I$ of the V_i. First let us prove that $V \cap \mathrm{supp}(\mu_n)$ is ϕ_t-invariant and consists of periodic orbits homotopic to some or all of the γ_i. Indeed by Proposition 5.4 a minimizing geodesic that enters V is either

- asymptotic to one of the γ_i, which is ruled out by Lemma 5.5
- or homotopic to one of the γ_i
- or cuts one of the γ_i with constant sign, which is ruled out by hypothesis.

Suppose $\mu_n(V) \neq 0$. For any measurable subset A of T^1M, set

$$\alpha_n(A) \;\; := \;\; \frac{\mu_n(A \cap V)}{\mu_n(V)}$$

$$\beta_n(A) \;\; := \;\; \frac{\mu_n(A \setminus V)}{\mu_n(T^1M \setminus V)}$$

$$\lambda_n \;\; := \;\; \mu_n(V).$$

Then α_n and β_n are two probability measures on T^1M. They are invariant by the geodesic flow because $V \cap \mathrm{supp}(\mu_n)$, as well as its complement in $\mathrm{supp}(\mu_n)$, is ϕ_t-invariant. In case $\mu_n(V) = 0$, set $\alpha_n := \mu_n$ and $\lambda_n := 1$. Since the support of α_n consists of periodic orbits homotopic to some or

all of the γ_i, the homology class of α_n is contained in the convex hull of the $[\gamma_i]/l_g(\gamma_i)$. Note that since the support of μ_0 consists of all of the γ_i, the homology class of μ_0 is contained in the relative interior of the convex hull of the $[\gamma_i]/l_g(\gamma_i)$.

We have
$$\mu_n = \lambda_n \alpha_n + (1 - \lambda_n)\beta_n$$
and λ_n tends to 1 as n tends to infinity, so the homology class of α_n converges to h_0. Therefore, when n is large enough, the homology class of α_n is contained in the relative interior of the convex hull of the $[\gamma_i]/l_g(\gamma_i)$. Thus any supporting cohomology class c to $\mathcal{S}_1$ at $[\alpha_n]$, i.e. such that $\langle c, [\alpha_n]\rangle = 1$ and $\langle c, h\rangle \leqslant 1$ for all $h \in \mathcal{B}_1$, is also a supporting cohomology class to $\mathcal{S}_1$ at h_0. In other words, any flat of $\mathcal{S}_1$ that contains $[\alpha_n]$ also contains h_0.

Let c be a supporting cohomology class to $\mathcal{S}_1$ at h_N. We have $\langle c, h_N\rangle = 1$ and $|\langle c, h\rangle| \leqslant 1, \forall h \in \mathcal{S}_1$. Therefore
$$\lambda_N \langle c, [\alpha_N]\rangle + (1 - \lambda_N)\langle c, [\beta_N]\rangle = 1.$$
Since $\langle c, [\alpha_N]\rangle \leqslant 1$, $\langle c, [\beta_N]\rangle \leqslant 1$, $\lambda_N \in [0, 1]$, this implies
$$\langle c, [\alpha_N]\rangle = \langle c, h_N\rangle = 1$$
that is, $[\alpha_N]$ and h_N are in the same flat of $\mathcal{S}_1$, whence h_0 and h_N are in the same flat of $\mathcal{S}_1$. □

Recall from [14] the orientable analogue of the first part of Theorem 1.1:

COROLLARY 6.2. — *Let M be a closed orientable surface endowed with a Riemannian metric g. Then every rational homology class contained in $\mathcal{S}_1$ lies in a flat of $\mathcal{S}_1$ of dimension at least $b_1(M)/2 - 1$.*

Proof. — Let h_0 be a rational point of $\mathcal{S}_1(M, g)$. Set
$$p := \dim \mathcal{VP}_{h_0}$$
and assume $p < b_1(M)/2$. Choose curves γ_i in $\mathcal{P}_{h_0}$ for $i = 1, \ldots, p$, such that $\{[\gamma_i] \mid i = 1, \ldots, p\}$ generate $\mathcal{VP}_h$. Since $p < b_1(M)/2$, there exists $h \in H_1(M, \mathbb{R})$ such that
$$h \notin \mathcal{VP}_{h_0} \text{ and } \text{Int}(h, [\gamma_i]) = 0, \forall i = 1, \ldots, p.$$
By Theorem 6.1 there exists $s > 0$ such that
$$F_1 := \left\{ \frac{h_0 + sh}{\|h_0 + sh\|} : s \in [0, s(h_0, h)] \right\}$$
is a straight segment contained in $\mathcal{S}_1(M, g)$. On the other hand, the convex hull of $[\gamma_i]$ for $i = 1, \ldots p + q$ is contained in a flat F_0 of $\mathcal{S}_1(M, g)$ of

dimension p that contains h_0 in its interior. From Lemma 4.1 we deduce that there exists a flat containing F_0 and F_1. The dimension of said flat is greater than $p = \dim F_0$ because $h \notin \mathcal{VP}_{h_0}$. $\qquad\square$

6.2. Local results - Non-orientable case

In this section, we suppose that M is a closed non-orientable surface endowed with a Riemannian metric g and prove Theorems 1.1 and 1.2. The proofs combine basic facts about the orientation cover of a non-orientable surface and Theorem 6.1.

PROPOSITION 6.3. — *Let M be a closed non-orientable surface endowed with a Riemannian metric g and $\pi : (M_o, \tilde{g}) \to (M, g)$ its orientation cover. Then*

$$\pi_* \mathcal{B}_1(M_o, \tilde{g}) = \mathcal{B}_1(M, g).$$

and furthermore the vector space E_1 endowed with the restriction of the stable norm of $(M_o, \tilde{g})$ is isometric to $H_1(M, \mathbb{R})$ endowed with the stable norm of (M, g).

Proof. — Let μ_o be an element of $\mathcal{M}_{\tilde{g}}$. Then $\pi_* \mu_o$ is an element of $\mathcal{M}_g$. So $\pi_* \mathcal{B}_1(M_o, \tilde{g}) \subset \mathcal{B}_1(M, g)$. Conversely, let μ be a minimizing measure of M. Let $\nu \in \mathcal{M}_{\tilde{g}}$ be given by Lemma 5.7. We have $\pi_*(\nu) = \mu$ so $\pi_*([\nu]) = [\mu]$. Thus π_* restricted to $E_1 \cap \mathcal{S}_1(M_o, \tilde{g})$ is surjective onto $\mathcal{S}_1(M, g)$. Since π_* is linear, it must then be surjective from $E_1 \cap \mathcal{B}_1(M_o, \tilde{g})$ onto $\mathcal{B}_1(M, g)$. Moreover, since the dimensions of E_1 and $H_1(M, \mathbb{R})$ are equal, π_* restricted to E_1 must be injective. So π_* restricted to E_1 is a linear isomorphism sending $E_1 \cap \mathcal{B}_1(M_o, \tilde{g})$ to $\mathcal{B}_1(M, g)$. $\qquad\square$

The purpose of the next proposition is to evaluate the maximal dimension of a flat containing a rational class h (not necessarily as an interior point), depending on the topological properties of h-minimizing curves. Recall that a simple closed curve γ of M is said *of type I* (resp. *of type II*) if its inverse image $\pi^{-1}(\gamma)$ consists of either one curve or two homologous curves (resp. two non-homologous curves). Let h be a rational point of $\mathcal{S}_1(M, g)$. Partition $\mathcal{P}_h$ in two subsets $\mathcal{P}_h^1$ and $\mathcal{P}_h^2$, the first set consisting only of curves of type I and the latter only of curves of type II. Let $\mathcal{VP}_h^2$ be the vector subspace of $H_1(M, \mathbb{R})$ generated by all homology classes of geodesics contained in $\mathcal{P}_h^2$. Let $\mathcal{VP}_h^1$ be such that $\mathcal{VP}_h^1$ is generated by homology classes of curves of type I and

$$\mathcal{VP}_h^2 \oplus \mathcal{VP}_h^1 = \mathcal{VP}_h.$$

PROPOSITION 6.4. — *Let M be a closed non-orientable surface endowed with a Riemannian metric g and let h_0 be a rational point of $\mathcal{S}_1(M,g)$. Set*

$$p := \dim \mathcal{VP}_{h_0}^1 \text{ and } q := \dim \mathcal{VP}_{h_0}^2$$

and assume $p+2q < b_1(M)$. Then there exists a flat of $\mathcal{B}_1(M,g)$ containing h_0, of dimension $> p + q$.

Proof. — Choose curves

- γ_i in $\mathcal{P}_{h_0}^1$ for $i = 1, \ldots, p$, such that $\{[\gamma_i] \mid i = 1, \ldots, p\}$ generate $\mathcal{VP}_h^1$
- γ_i in $\mathcal{P}_{h_0}^2$ for $i = p+1, \ldots, p+q$, such that $\{[\gamma_i] \mid i = p+1, \ldots, p+q\}$ generate $\mathcal{VP}_h^2$.

For $i = 1, \ldots p + q$ denote by μ_i the ϕ_t-invariant probability measure supported on γ_i. Let $c \in H^1(M, \mathbb{R})$ be such that h_0 is c-minimizing. Then each μ_i, and each convex combination thereof, is also c-minimizing. Let $\lambda_i \in]0,1[$, $i = 1, \ldots p + q$ be such that $\sum_i \lambda_i = 1$ and $\sum_i \lambda_i [\mu_i] = h_0$.

If $i \in \{1, \ldots p\}$, choose a closed geodesic $\tilde{\gamma}_i$ in M_o such that

- $\pi(\tilde{\gamma}_i) = \gamma_i$
- $[\tilde{\gamma}_i]$ lies in the eigenspace E_1 for the involution I.

If $i \in \{p+1, \ldots p+q\}$, choose two closed geodesics $\tilde{\gamma}_i$ and $\tilde{\gamma}_{i+q}$ in M_o such that

- $\pi(\tilde{\gamma}_i) = \gamma_i$
- $I(\tilde{\gamma}_i) = \tilde{\gamma}_{i+q}$
- $[\tilde{\gamma}_i] \neq [\tilde{\gamma}_{i+q}]$.

Define

- $\tilde{\mu}_i$ the $\tilde{\phi}_t$-invariant probability measure supported on $\tilde{\gamma}_i$ for $i = 1, \ldots, p + 2q$
- $\tilde{\lambda}_i := \lambda_i$ if $i = 1, \ldots, p$
- $\tilde{\lambda}_i := \lambda_i/2$ if $i = p+1, \ldots, p+q$
- $\tilde{\lambda}_{i+q} := \lambda_i/2$ if $i = p+q+1, \ldots, p+2q$
- $\tilde{\mu} := \sum_{i=1}^{p+2q} \tilde{\lambda}_i \tilde{\mu}_i$
- $\tilde{h}_0 = [\tilde{\mu}]$.

We have

- $I_*(\tilde{\mu}) = \tilde{\mu}$ whence $I_*(\tilde{h}_0) = \tilde{h}_0$
- $\pi_*(\tilde{\mu}) = \mu$ whence $\pi_*(\tilde{h}_0) = h_0$
- $\tilde{\mu}$ is $\pi^*(c)$-minimizing
- the vector space generated by $[\tilde{\gamma}_i]$ for $i = 1, \ldots p + 2q$ equals $\mathcal{VP}_{\tilde{h}_0}$.

The last equality holds because of Proposition 6.3. Call V the vector subspace of $H_1(M_o, \mathbb{R})$ generated by the integer classes $[\tilde{\gamma}_i]$ for $i = 1, \ldots, p+2q$. Note that $I(V) = V$, so $V = V_1 \oplus V_{-1}$ where $V_i = E_i \cap V$, $i = \pm 1$. Also, $V = \mathcal{VP}_{\tilde{h}_0}$. We have

$$V_1 = \text{Vect}\left(\{[\tilde{\gamma}_i] : i = 1, \ldots p\} \cup \{[\tilde{\gamma}_i] + [\tilde{\gamma}_{i+q}] : i = p+1, \ldots p+q\}\right)$$
$$V_{-1} = \text{Vect}\left(\{[\tilde{\gamma}_i] - [\tilde{\gamma}_{i+q}] : i = p+1, \ldots p+q\}\right)$$

We would like to use Theorem 6.1 with $\tilde{h}_0$ playing the part of h_0 and some h in $E_1 \cap V^\perp$ but not in V. Observe that

$$\dim V_{-1}^\perp = b_1(M_o) - q \text{ so}$$
$$\dim V_{-1}^\perp \cap E_1 \geqslant b_1(M_o) - q + b_1(M) - b_1(M_0)$$
$$= b_1(M) - q > p + q = \dim V_1$$

since we assume $b_1(M) > p + 2q$. So there exists $h \in V_{-1}^\perp \cap E_1$ such that $h \notin V_1$. Since $V_1 \subset E_1$, we have $E_1 = E_1^\perp \subset V_1^\perp$ thus $h \in V_1^\perp$ and

$$h \in V_1^\perp \cap V_{-1}^\perp = (V_1 \oplus V_{-1})^\perp = V^\perp.$$

So $h \in E_1 \cap V^\perp$. Furthermore $h \notin V$ since $h \in E_1$ and $h \notin V_1 = E_1 \cap V$.

By Theorem 6.1 there exists $s > 0$ such that

$$F_1 := \left\{ \frac{\tilde{h}_0 + sh}{||\tilde{h}_0 + sh||} : s \in [0, s(h_0, h)] \right\}$$

is a straight segment contained in $\mathcal{S}_1(M_o, \tilde{g}) \cap E_1 \cong \mathcal{S}_1(M, g)$. On the other hand, the convex hull of $[\gamma_i]$ for $i = 1, \ldots p+q$ is contained in a flat F_0 of $\mathcal{S}_1(M, g)$ that contains h_0 in its interior. From Lemma 4.1 we deduce that there exists a flat containing F_0 and F_1. The dimension of such a flat is greater than $p + q = \dim F_0$ because $h \notin V$. $\qquad\square$

Taking a rational h in the proof of Proposition 6.4, we deduce the first part of Theorem 1.1.

COROLLARY 6.5. — *Let M be a closed non-orientable surface endowed with a Riemannian metric. Then every connected minimizing cycle is a component of a minimizing cycle whose homology class lies in a flat of $\mathcal{S}_1$ of dimension at least $[(b_1(M) + 1)/2] - 1$.*

Let Γ be a minimizing cycle whose connected components are not pairwise proportional in homology.

PROPOSITION 6.6. — *There are at most $2b_1(M) - 1$ disjoint non pairwise homologically proportional simple closed curves on M.*

Proof. — The argument is classical. Let $\alpha_1, \ldots, \alpha_{p+q}$ be a maximal family of disjoint pairwise non homologically proportional simple closed curves of M. Suppose that p is the number of one-sided curves of this family and q the number of two-sided curves. By cutting M along these simple closed curves we obtain an union of $b_1(M) - 1$ pair of pants. So we must have $3(b_1(M) - 1) = p + 2q$. This implies $2(p + q) = 3.b_1(M) + p - 3$. As $p \leqslant b_1(M) + 1$, the assertion follows. □

So Γ has at most $2b_1(M) - 1$ components. The second part of theorem 1.1 is proved.

By specializing Proposition 6.4 to the case when $q = 0$, we now deduce Theorem 1.2 which describes the local geometry of the unit ball of the stable norm near a rational homology class for which the connected components of minimizing cycles are curves of type I.

COROLLARY 6.7. — *Let M be a closed non-orientable surface endowed with a Riemannian metric. Let h_0 be an integer homology class all of whose minimizing cycles consist of curves of type I. Then for all $h \in H_1(M, \mathbb{R})$, there exists $s(h_0, h) > 0$ such that the subset of the unit sphere $\mathcal{S}_1$*

$$\left\{ \frac{h_0 + sh}{\|h_0 + sh\|} \ : \ s \in [0, s(h_0, h)] \right\}$$

is a straight segment.

6.3. Global result - Proof of Theorem 1.3

We first prove the following theorem.

THEOREM 6.8. — *Let M be a closed (possibly non-orientable) surface endowed with a Riemannian metric g and $c_1, \ldots, c_l$ a family of disjoint smooth, simple, closed curves whose homology classes are not pairwise proportional (that is $[c_i] \notin \mathbb{R}[c_j]$ for $i \neq j$).*

For all sequence $\{r_i\}_{i=1}^l$ of positive real numbers, there exists a smooth metric g^* conformal to g such that the intersection of $\mathcal{S}_1(g^*)$ with the subspace spanned by the curves $[c_1], \ldots, [c_l]$ coincides with the polyhedron

$$Conv_s \left(\frac{[c_1]}{r_1}, \ldots, \frac{[c_l]}{r_l} \right)$$

where $Conv_s$ denote the convex hull of the symmetrization of a set.

Proof. — Let P denote the polyhedron generated as the convex hull

$$\text{Conv}_s \left(\frac{[c_1]}{r_1}, \ldots, \frac{[c_l]}{r_l} \right).$$

We can suppose that each curve $[c_i]$ corresponds to an exposed point of the polyhedron (if not we can discard this curve and the polyhedron P remains unchanged).

LEMMA 6.9. — *There exists a smooth metric $\bar{g}$ conformal to g and an open neighborhood V_i of each c_i such that c_i is the unique closed $\bar{g}$-geodesic of V_i and $l_{\bar{g}}(c_i) = r_i$.*

Proof of Lemma 6.9. For ϵ small enough, the ϵ-tubular neighborhoods $U_\epsilon(c_i)$ are pairwise disjoints and the g-orthogonal projections $p_i : U_\epsilon(c_i) \to c_i$ are well defined. For each $x \in U_\epsilon(c_i)$, there exists two g-unitary vectors $\pm v(x) \in T_x M$ orthogonal to the fiber $p_i^{-1}(p_i(x))$. The function $\alpha_i : U_i \to]0, \infty[$ given by the formula

$$\alpha_i(x) = g(Dp_i(x)v(x), Dp_i(x)v(x))$$

is smooth and such that $\alpha_i \circ c_i = 1$. We define a new metric g' conformal to g by $\alpha_i g$ on $U_\epsilon(c_i)$ and by extending the local conformal factors $\{\alpha_i\}_{i=1}^k$ into a smooth positive function α on M. We claim that the projections $p_i : U_\epsilon(c_i) \to c_i$ do not increase the lengths with respect to g'. Indeed, take $x \in U_i$ and $w \in T_x U_i$. Write $w = \lambda v(x) + \mu v'$, with $v' \in T_x p_i^{-1}(p_i(x))$. Note that v and v' are g-orthogonal; since g' is conformal to g, v and v' are g'-orthogonal. Orthogonal projections do not increase distances, so

$$\lambda^2 g'(v(x), v(x)) = g'(\lambda v(x), \lambda v(x)) \leqslant g'(w, w).$$

Now

$$g'(Dp_i(x)w, Dp_i(x)w) = g'(Dp_i(x)\lambda v, Dp_i(x)\lambda v)$$

because p_i is the orthogonal projection to c_i, whence

$$\begin{aligned} g'(Dp_i(x)w, Dp_i(x)w) &= \lambda^2 \alpha_i(x) \\ &= \lambda^2 \alpha_i(x)g(v,v) \\ &= \lambda^2 g'(v,v) \leqslant g'(w,w) \end{aligned}$$

which proves the claim.

Choose a function $f \in \mathcal{C}^\infty(M)$ null on $\cup_{i=1}^k c_i$, positive elsewhere and such that $\Delta(f) < -\mathrm{Scal}_{g'}$ where $\mathrm{Scal}_{g'}$ denote the scalar curvature of g'. Let $g'' := \exp(f)g'$. We can easily verify that each projection $p_i : U_\epsilon(c_i) \to c_i$ now strictly contracts the lengths. So c_i is a g''-geodesic and the negativity of $\mathrm{Scal}_{g''} = \exp(f)\Delta(f) + \mathrm{Scal}_{g'}$ ensures that the orbit c_i of the geodesic flow associated to g'' is hyperbolic, hence isolated in a neighborhood V_i.

We extend the functions $\lambda_i = r_i/l_g(c_i)$ defined on each neighborhood V_i into a smooth function λ defined on the whole surface and set $\bar{g} := \lambda^2 g''$. The lemma is proved. $\qquad\square$

For any sequence $\bar{\epsilon}$ such that $\epsilon_i = 0, \pm 1$ we denote by $\gamma(\bar{\epsilon})$ the multicurve $\cup_{i=1}^l \epsilon_i \cdot c_i$ of $\cup_{i=1}^l V_i$ minimizing the length in the class $\sum_{i=1}^l \epsilon_i[c_i]$.

LEMMA 6.10. — *There exists a smooth metric g^* conformal to $\bar{g}$ such that for any sequence $\bar{\epsilon} := \{\epsilon_i\}_{i=1}^l$ with $\epsilon_i = 0, \pm 1$,*

$$|| \sum_{i=1}^l \epsilon_i[c_i]||_s^{g^*} = l_{g^*}(\gamma(\bar{\epsilon})).$$

Proof of Lemma 6.10. Set

$$\delta(\bar{\epsilon}) := l_{\bar{g}}(\gamma(\bar{\epsilon})) - || \sum_{i=1}^l \epsilon_i[c_i]||_s^{\bar{g}}.$$

The set $\Gamma(\bar{\epsilon})$ of unions of closed geodesics different from $\gamma(\bar{\epsilon})$ homologous to $\sum_{i=1}^l \epsilon_i[c_i]$ such that their length is bounded from above by $l_{\bar{g}}(\gamma(\bar{\epsilon}))$ is compact. It is clear that no multicurve γ in $\Gamma(\bar{\epsilon})$ can be totally contained in $\cup_{i=1}^l V_i$. So, if $\delta(\bar{\epsilon}) > 0$, the infimum $t(\bar{\epsilon})$ of time t such that there exists a multicurve $\gamma \in \Gamma(\bar{\epsilon})$ spending a time t outside $\cup_{i=1}^l V_i$ is achieved and not zero.

Since there is but a finite number of sequences $\bar{\epsilon}$ we may choose β so big that for all $\bar{\epsilon}$ with $\delta(\bar{\epsilon}) > 0$

$$\beta > \log\left[1 + \frac{\delta(\bar{\epsilon})}{t(\bar{\epsilon})}\right].$$

Now we choose a function $f' \in \mathcal{C}^\infty(M)$ null on $\cup_{i=1}^l c_i$, positive elsewhere and such that $f' > \beta$ outside $\cup_{i=1}^l V_i$. For any multicurve γ spending some time t outside $\cup_{i=1}^l V_i$, we have

$$l_{\exp(f')\bar{g}}(\gamma) > (\exp(\beta) - 1)t + l_{\bar{g}}(\gamma).$$

Now let $g^* = \exp(f')\bar{g}$. All the lengths except those of the c_i's increase for g^* so for all $\bar{\epsilon}$ such that $\delta(\bar{\epsilon}) > 0$ and for all multicurve γ in the class $\sum_{i=1}^{l} \epsilon_i [c_i]$,

$$l_{g^*}(\gamma) \geqslant l_{g^*}(\gamma(\bar{\epsilon})).$$

Recall that $l_{g^*}(c_i) = r_i$ as the length of each c_i does not increase. Thus each exposed point of P belongs to the unit sphere $\mathcal{S}_1(g^*)$ of the stable norm. Furthermore by lemma 6.10 each face of P contains an interior point that belongs to $\mathcal{S}_1(g^*)$. This proves the theorem by convexity of the unit sphere of the stable norm. $\qquad\square$
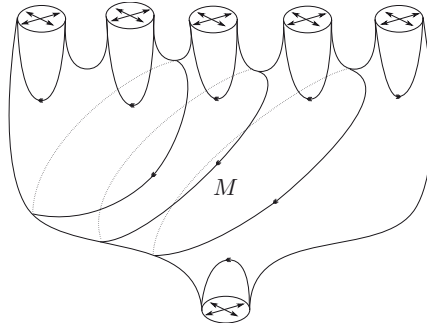

Figure 6.1

COROLLARY 6.11. — *Let M be a closed orientable surface endowed with a Riemannian metric g. For each isotropic subspace L of $H_1(M, \mathbb{R})$ (with respect to Int), there exists a metric g^* conformal to g such that the restriction of $\mathcal{S}_1(g^*)$ to L is a polyhedra with rational vertices.*

Proof. — There exists a system of disjoint, smooth, simple and non-pairwise homotopic closed curves $c_1, \ldots, c_l$ that span L. We apply Theorem 6.8 to obtain the claim. $\qquad\square$

Remark. — It is a classical result that such a system has cardinality at most $(3/2)b_1(M) - 3$ (same argument as in the proof of Proposition 6.6), thus this bounds the number of vertices of the polyhedra obtained that way by $3b_1(M) - 6$.

We now deduce, as a corollary of Theorem 6.8, Theorem 1.3 as stated in the introduction:

COROLLARY 6.12. — *Let M be a closed non-orientable surface endowed with a Riemannian metric g. There exists a metric g^* conformal to g such that $\mathcal{S}_1(g^*)$ is a polyhedra with rational vertices.*

Proof. — There exists a system $c_1, \ldots, c_l$ of smooth, simple, closed curves such that $c_i \cap c_j = \emptyset$, $[c_i] \notin \mathbb{R}[c_j]$ for $i \neq j$ and $H_1(M, \mathbb{R}) = \mathrm{Vect}([c_1], \ldots, [c_l])$ (see figure 6.1 for an example of such a system with $l = 2b_1(M) - 1$). We apply Theorem 6.8 to obtain the claim. □

Remark. — Such a system has cardinality at most $2b_1(M) - 1$ (Proposition 6.6), thus this bounds the number of vertices of the polyhedra obtained that way by $4b_1(M) - 2$.

BIBLIOGRAPHY

[1] I. BABENKO & F. BALACHEFF, "Sur la forme de la boule unité de la norme stable unidimensionnelle", *Manuscripta Math.* **119** (2006), no. 3, p. 347-358.

[2] V. BANGERT, "Minimal geodesics", *Ergodic Theory Dynam. Systems* **10** (1990), no. 2, p. 263-286.

[3] F. BONAHON, "Geodesic laminations on surfaces", *Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998)* (2001), p. 1-37, Contemp. Math., 269, Amer. Math. Soc., Providence, RI (2001).

[4] M. J. D. CARNEIRO, "On minimizing measures of the action of autonomous Lagrangians", *Nonlinearity* **8** (1995), no. 6, p. 1077-1085.

[5] G. CONTRERAS, L. MACARINI & G. P. PATERNAIN, "Periodic orbits for exact magnetic flows on surfaces", *Int. Math. Res. Not.* (2004), no. 8, p. 361-387.

[6] J. DIEUDONNÉ, *Eléments d'Analyse*, Cahiers Scientifiques, vol. 2, Fasc. XXXI Gauthier-Villars, 1968.

[7] H. FARKAS & I. KRA, *Riemann surfaces*, Second ed., Graduate Texts in Mathematics, vol. 71, Springer-Verlag, New York, 1992.

[8] A. FATHI, "Sur la convergence du semi-groupe de Lax-Oleinik", *C. R. Acad. Sci. Paris Sér. I Math.* **327** (1998), no. 3, p. 267-270.

[9] H. FEDERER, "Real flat chains, cochains and variational problems", *Indiana Univ. Math. J.* **24** (1974), p. 351-407.

[10] M. GROMOV, *Structures métriques pour les variétés riemanniennes*, Edited by J. Lafontaine and P. Pansu. Textes Mathématiques, 1. CEDIC, Paris, 1981.

[11] R. MAÑÉ, *Introdução à teoria ergódica*, Projeto Euclides, vol. 14, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1983.

[12] ———, "On the minimizing measures of Lagrangian dynamical systems", *Nonlinearity* **5** (1992), no. 3, p. 623-638.

[13] D. MASSART, "Norme stable des surfaces", Thèse de doctorat. Ecole Normale Supérieure de Lyon, 1996.

[14] ———, "Stable norms of surfaces: local structure of the unit ball of rational directions", *Geom. Funct. Anal.* **7** (1997), no. 6, p. 996-1010.

[15] J. N. MATHER, "Action minimizing invariant measures for positive definite Lagrangian systems", *Math. Z.* **207** (1991), p. 169-207.

[16] G. MCSHANE & I. RIVIN, "Simple curves on hyperbolic tori", *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), no. 12, p. 1523-1528.

[17] M. SCHARLEMANN, "The complex of curves on nonorientable surfaces", *J. London Math. Soc.* **25** (1982), no. 1, p. 171-184.

[18] S. SCHWARTZMAN, "Asymptotic cycles", *Ann. of Math.* **66** (1957), p. 270-284.

Florent BALACHEFF
Université de Neuchâtel
Institut de mathématiques
Rue Émile Argand 11
CP 158
2009 Neuchâtel (Switzerland)
Florent.Balacheff@unine.ch

Daniel MASSART
Université Montpellier II
Institut de Mathématiques et de Modélisation de
Montpellier
UMR 5149
Case Courier 051
Place Eugène Bataillon
34095 Montpellier Cedex 5 (France)
massart@math.univ-montp2.fr