ANNALES

DE

L'INSTITUT FOURIER

Mihai MIHĂILESCU & Vicenţiu RĂDULESCU

Neumann problems associated to nonhomogeneous differential operators in Orlicz–Sobolev spaces

Tome 58, nº 6 (2008), p. 2087-2111.

<http://aif.cedram.org/item?id=AIF_2008__58_6_2087_0>

Ann. Inst. Fourier, Grenoble
58, 6 (2008) 2087-2111

NEUMANN PROBLEMS ASSOCIATED TO NONHOMOGENEOUS DIFFERENTIAL OPERATORS IN ORLICZ–SOBOLEV SPACES

by Mihai MIHĂILESCU & Vicențiu RĂDULESCU (*)

Ce travail est dédié au Professeur Philippe G. Ciarlet à l'occasion de son 70e anniversaire

Abstract. — We study a nonlinear Neumann boundary value problem associated to a nonhomogeneous differential operator. Taking into account the competition between the nonlinearity and the bifurcation parameter, we establish sufficient conditions for the existence of nontrivial solutions in a related Orlicz–Sobolev space.

Résumé. — On étudie un problème aux limites de Neumann associé à un opérateur différentiel non homogène. En tenant compte de la compétition entre le taux de croissance de la nonlinéarité et les valeurs du paramètre de bifurcation, on établit des conditions suffisantes pour l'existence de solutions non triviales dans un certain espace fonctionnel du type Orlicz–Sobolev.

1. Introduction and preliminary results

This paper is motivated by phenomena which are described by nonhomogeneous Neumann problems of the type

$$\begin{cases} -\mathrm{div}(a(x,|\nabla u|)\nabla u) + a(x,|u|)u = \lambda\, g(x,u), & \text{for } x \in \Omega \\ \dfrac{\partial u}{\partial \nu}(x) = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 3)$ with smooth boundary $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$. In (1.1) there are also involved the functions $a(x,t)$, $g(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ which will be specified later and the constant $\lambda > 0$.

Keywords: Nonhomogeneous differential operator, nonlinear partial differential equation, Neumann boundary value problem, Orlicz–Sobolev space.
Math. classification: 35D05, 35J60, 35J70, 58E05, 68T40, 76A02.
(*) The authors have been supported by Grant CNCSIS PNII–79/2007 *"Procese Neliniare Degenerate și Singulare"*.

In the particular case when in (1.1) we have $a(x,t) = t^{p(x)-2}$, with $p(x)$ a continuous function on $\overline{\Omega}$, we deal with problems involving variable growth conditions. The study of such problems has been stimulated by recent advances in elasticity (see [34, 35]), fluid dynamics (see [32, 31, 8, 17]), calculus of variations and differential equations with $p(x)$-growth conditions (see [1, 20, 21, 22, 24, 23, 25, 34, 35]).

Another recent application which uses operators as those described above can be found in the framework of image processing. In that context we refer to the study of Chen, Levine and Rao [4]. In [4] the authors study a functional with variable exponent, $1 < p(x) < 2$, which provides a model for image restoration. The diffusion resulting from the proposed model is a combination of Gaussian smoothing and regularization based on Total Variation. More exactly, the following adaptive model was proposed

$$\min_{I=u+v,\ u\in \mathrm{BV}\cap L^2(\Omega)} \int_\Omega \varphi(x,\nabla u)\,dx + \lambda\cdot \|u\|^2_{L^2(\Omega)}\,, \tag{1.2}$$

where $\Omega\subset\mathbb{R}^2$ is an open domain,

$$\varphi(x,r) = \begin{cases} \frac{1}{p(x)}|r|^{p(x)}, & \text{for } |r|\leqslant\beta\\ |r| - \frac{\beta\cdot p(x)-\beta^{p(x)}}{p(x)}, & \text{for } |r|>\beta, \end{cases}$$

where $\beta>0$ is fixed and $1<\alpha\leqslant p(x)\leqslant 2$. The function $p(x)$ involved here depends on the location x in the model. For instance it can be used

$$p(x) = 1 + \frac{1}{1+k|\nabla G_\sigma * I|^2}\,,$$

where $G_\sigma(x) = \frac{1}{\sigma}\exp(-|x|^2/(4\sigma^2))$ is the Gaussian filter and $k>0$ and $\sigma>0$ are fixed parameters (according to the notation in [4]). For problem (1.2) Chen, Levine and Rao establish the existence and uniqueness of the solution and the long-time behavior of the associated flow of the proposed model. The effectiveness of the model in image restoration is illustrated by some experimental results included in the paper.

We point out that the model proposed by Chen, Levine and Rao in problem (1.2) is linked with the energy which can be associated with problem (1.1) by taking $\varphi(x,\nabla u) = a(x,|\nabla u|)\nabla u$. Furthermore, the operators which will be involved in problem (1.1) can be more general than those presented in the above quoted model. That fact is due to the replacement of $|t|^{p(x)-2}t$ by more general functions $\varphi(x,t) = a(x,|t|)t$. Such functions will demand some new setting spaces for the associated energy, the *generalized Orlicz-Sobolev spaces* $L^\Phi(\Omega)$, where $\Phi(x,t) = \int_0^t \varphi(x,s)\,ds$. Such spaces originated with Nakano [28] and were developed by Musielak and Orlicz

[27, 26] ($f \in L^\Phi(\Omega)$ if and only if $\int \Phi(x, |f(x)|)\, dx < \infty$). Many properties of Sobolev spaces have been extended to Orlicz-Sobolev spaces, mainly by Dankert [7], Donaldson and Trudinger [10], and O'Neill [29] (see also Adams [2] for an excellent account of those works). Orlicz-Sobolev spaces have been used in the last decades to model various phenomena. Chen, Levine and Rao [4] proposed a framework for image restoration based on a variable exponent Laplacian. A second application which uses variable exponent type Laplace operators is modelling electrorheological fluids [1, 32]. According to Diening [9], we are strongly convinced that these more general spaces will become increasingly important in modelling modern materials.

In this paper we assume that the function $a(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ in (1.1) is such that $\varphi(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$,

$$\varphi(x,t) = \begin{cases} a(x,|t|)t, & \text{for} \quad t \neq 0 \\ 0, & \text{for} \quad t = 0\,, \end{cases}$$

and satisfies (φ) for all $x \in \Omega$, $\varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$; and $\Phi(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$,

$$\Phi(x,t) = \int_0^t \varphi(x,s)\, ds, \quad \forall\, x \in \overline{\Omega},\ t \geqslant 0\,,$$

belongs to *class* Φ (see [27], p. 33), i.e. Φ satisfies the following conditions (Φ_1) for all $x \in \Omega$, $\Phi(x,\cdot) : [0,\infty) \to \mathbb{R}$ is a nondecreasing continuous function, with $\Phi(x,0) = 0$ and $\Phi(x,t) > 0$ whenever $t > 0$; $\lim_{t\to\infty} \Phi(x,t) = \infty$; ($\Phi_2$) for every $t \geqslant 0$, $\Phi(\cdot,t) : \Omega \to \mathbb{R}$ is a measurable function.

Remark 1.1. — Since $\varphi(x,\cdot)$ satisfies condition (φ) we deduce that $\Phi(x,\cdot)$ is convex and increasing from $\mathbb{R}^+$ to $\mathbb{R}^+$.

For the function Φ introduced above we define the *generalized Orlicz class*,

$$K_\Phi(\Omega) = \{u : \Omega \to \mathbb{R},\ \text{measurable;}\ \int_\Omega \Phi(x, |u(x)|)\, dx < \infty\}$$

and the *generalized Orlicz space*,

$$L^\Phi(\Omega) = \{u : \Omega \to \mathbb{R},\ \text{measurable;}\ \lim_{\lambda\to 0^+} \int_\Omega \Phi(x, \lambda|u(x)|)\, dx = 0\}\,.$$

The space $L^\Phi(\Omega)$ is a Banach space endowed with the *Luxemburg norm*

$$|u|_\Phi = \inf\left\{\mu > 0;\ \int_\Omega \Phi\left(x, \frac{|u(x)|}{\mu}\right)\, dx \leqslant 1\right\}$$

or the equivalent norm (the *Orlicz norm*)

$$|u|_{(\Phi)} = \sup\left\{\left|\int_\Omega uv\,dx\right|;\ v\in L^{\overline{\Phi}}(\Omega),\ \int_\Omega \overline{\Phi}(x,|v(x)|)\,dx \leqslant 1\right\},$$

where $\overline{\Phi}$ denotes the *conjugate Young* function of Φ, that is,

$$\overline{\Phi}(x,t) = \sup_{s>0}\{ts - \Phi(x,s);\ s\in\mathbb{R}\},\quad \forall\, x\in\overline{\Omega},\ t\geqslant 0\,.$$

Furthermore, for Φ and $\overline{\Phi}$ conjugate Young functions, the Hölder type inequality holds true

$$\left|\int_\Omega uv\,dx\right| \leqslant C\cdot|u|_\Phi\cdot|v|_{\overline{\Phi}},\quad \forall\, u\in L^\Phi(\Omega),\ v\in L^{\overline{\Phi}}(\Omega)\,, \tag{1.3}$$

where C is a positive constant (see [27], Theorem 13.13).

In this paper we assume that there exist two positive constants φ_0 and φ^0 such that

$$1<\varphi_0\leqslant \frac{t\varphi(x,t)}{\Phi(x,t)}\leqslant \varphi^0<\infty,\quad \forall\, x\in\overline{\Omega},\ t\geqslant 0\,. \tag{1.4}$$

The above relation implies that Φ satisfies the Δ_2-*condition* (see Proposition 2.3), i.e.

$$\Phi(x,2t)\leqslant K\cdot\Phi(x,t),\quad \forall\, x\in\overline{\Omega},\ t\geqslant 0\,, \tag{1.5}$$

where K is a positive constant. Relation (1.5) and Theorem 8.13 in [27] imply that $L^\Phi(\Omega) = K_\Phi(\Omega)$.

Furthermore, we assume that Φ satisfies the following condition

$$\text{for each } x\in\overline{\Omega},\ \text{the function } [0,\infty)\ni t\to\Phi(x,\sqrt{t}) \text{ is convex}\,. \tag{1.6}$$

Relation (1.6) assures that $L^\Phi(\Omega)$ is an uniformly convex space and thus, a reflexive space (see Proposition 2.2).

On the other hand, we point out that assuming that Φ and Ψ belong to class Φ and

$$\Psi(x,t)\leqslant K_1\cdot\Phi(x,K_2\cdot t)+h(x),\quad \forall\, x\in\overline{\Omega},\ t\geqslant 0\,, \tag{1.7}$$

where $h\in L^1(\Omega)$, $h(x)\geqslant 0$ a.e. $x\in\Omega$ and K_1, K_2 are positive constants, then by Theorem 8.5 in [27] we have that there exists the continuous embedding $L^\Phi(\Omega)\subset L^\Psi(\Omega)$.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^\Phi(\Omega)$ space, which is the mapping $\rho_\Phi : L^\Phi(\Omega)\to\mathbb{R}$ defined by

$$\rho_\Phi(u) = \int_\Omega \Phi(x,|u(x)|)\,dx.$$

If (u_n), $u \in L^\Phi(\Omega)$ then the following relations hold true

$$|u|_\Phi > 1 \quad \Rightarrow \quad |u|_\Phi^{\varphi_0} \leqslant \rho_\Phi(u) \leqslant |u|_\Phi^{\varphi^0}\,, \tag{1.8}$$

$$|u|_\Phi < 1 \quad \Rightarrow \quad |u|_\Phi^{\varphi^0} \leqslant \rho_\Phi(u) \leqslant |u|_\Phi^{\varphi_0}\,, \tag{1.9}$$

$$|u_n - u|_\Phi \to 0 \quad \Leftrightarrow \quad \rho_\Phi(u_n - u) \to 0\,, \tag{1.10}$$

$$|u_n|_\Phi \to \infty \quad \Leftrightarrow \quad \rho_\Phi(u_n) \to \infty\,. \tag{1.11}$$

Next, we define the *generalized Orlicz-Sobolev space*

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^\Phi(\Omega);\ \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega),\ i = 1, ..., N \right\}.$$

On $W^{1,\Phi}(\Omega)$ we define the equivalent norms

$$\|u\|_{1,\Phi} = |\ |\nabla u|\ |_\Phi + |u|_\Phi$$
$$\|u\|_{2,\Phi} = \max\{|\ |\nabla u|\ |_\Phi, |u|_\Phi\}$$
$$\|u\| = \inf\left\{ \mu > 0;\ \int_\Omega \left[\Phi\left(x, \frac{|u(x)|}{\mu}\right) + \Phi\left(x, \frac{|\nabla u(x)|}{\mu}\right)\right]\ dx \leqslant 1 \right\}\,,$$

(see Proposition 2.4).

The generalized Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ endowed with one of the above norms is a reflexive Banach space.

Finally, we point out that assuming that Φ and Ψ belong to class Φ, satisfying relation (1.7) and $\inf_{x\in\Omega} \Phi(x,1) > 0$, $\inf_{x\in\Omega} \Psi(x,1) > 0$ then there exists the continuous embedding $W^{1,\Phi}(\Omega) \subset W^{1,\Psi}(\Omega)$.

We refer to Orlicz [30], Nakano [28], Musielak [27], Musielak and Orlicz [26], Diening [9] for further properties of generalized Lebesgue-Sobolev spaces.

Remark 1.2. — a) Assuming $\Phi(x,t) = \Phi(t)$, i.e. Φ is independent of variable x, we say that L^Φ and $W^{1,\Phi}$ are *Orlicz spaces*, respectively *Orlicz-Sobolev spaces* (see [2, 5, 6, 30]).

b) Assuming $\Phi(x,t) = |t|^{p(x)}$ with $p(x) \in C(\overline{\Omega})$, $p(x) > 1$ for all $x \in \overline{\Omega}$ we denote L^Φ by $L^{p(x)}$ and $W^{1,\Phi}$ by $W^{1,p(x)}$ and we refer to them as *variable exponents Lebesgue spaces*, respectively *variable exponents Sobolev spaces* (see [11, 12, 13, 16, 15, 18, 22, 24, 27, 26, 28])

c) Our framework enables us to work with spaces which are more general than those described in a) and b) (see the examples at the end of this paper).

2. Auxiliary results regarding generalized Orlicz-Sobolev spaces

In this section we point out certain useful results regarding the generalized Orlicz-Sobolev spaces.

Proposition 2.1. — *Assume condition (1.4) is satisfied. Then the following relations hold true*

$$|u|_\Phi^{\varphi_0} \leqslant \rho_\Phi(u) \leqslant |u|_\Phi^{\varphi^0}, \quad \forall\, u \in L^\Phi(\Omega) \text{ with } |u|_\Phi > 1\,, \tag{2.1}$$

$$|u|_\Phi^{\varphi^0} \leqslant \rho_\Phi(u) \leqslant |u|_\Phi^{\varphi_0}, \quad \forall\, u \in L^\Phi(\Omega) \text{ with } |u|_\Phi < 1\,. \tag{2.2}$$

Proof. — First, we show that $\rho_\Phi(u) \leqslant |u|_\Phi^{\varphi^0}$ for all $u \in L^\Phi(\Omega)$ with $|u|_\Phi > 1$. □

Indeed, since $\varphi^0 \geqslant (t\varphi(x,t))/\Phi(x,t)$ for all $x \in \overline{\Omega}$ and all $t \geqslant 0$ it follows that letting $\sigma > 1$ we have

$$\log(\Phi(x,\sigma\cdot t)) - \log(\Phi(x,t)) = \int_t^{\sigma\cdot t} \frac{\varphi(x,s)}{\Phi(x,s)}\, ds \leqslant \int_t^{\sigma\cdot t} \frac{\varphi^0}{s}\, ds = \log(\sigma^{\varphi^0})\,.$$

Thus, we deduce

$$\Phi(x,\sigma\cdot t) \leqslant \sigma^{\varphi^0}\cdot\Phi(x,t), \quad \forall\, x\in\overline{\Omega},\ t>0,\ \sigma>1\,. \tag{2.3}$$

Let now $u \in L^\Phi(\Omega)$ with $|u|_\Phi > 1$. Using the definition of the Luxemburg norm and relation (2.3) we deduce

$$\begin{aligned}\int_\Omega \Phi(x,|u(x)|)\, dx &= \int_\Omega \Phi\left(x, |u|_\Phi\cdot\frac{|u(x)|}{|u|_\Phi}\right)\, dx\\ &\leqslant |u|_\Phi^{\varphi^0}\cdot\int_\Omega \Phi\left(x,\frac{|u(x)|}{|u|_\Phi}\right)\, dx\\ &\leqslant |u|_\Phi^{\varphi^0}\,.\end{aligned}$$

Now, we show that $\rho_\Phi(u) \geqslant |u|_\Phi^{\varphi_0}$ for all $u \in L^\Phi(\Omega)$ with $|u|_\Phi > 1$.

Since $\varphi_0 \leqslant (t\varphi(x,t))/\Phi(x,t)$ for all $x \in \overline{\Omega}$ and all $t \geqslant 0$, similar techniques as those used in the proof of relation (2.3) imply

$$\Phi(x,\sigma\cdot t) \geqslant \sigma^{\varphi_0}\cdot\Phi(x,t), \quad \forall\, x\in\overline{\Omega},\ t>0,\ \sigma>1\,. \tag{2.4}$$

Let $u \in L^\Phi(\Omega)$ with $|u|_\Phi > 1$. We consider $\beta \in (1, |u|_\Phi)$. Since $\beta < |u|_\Phi$ it follows that $\int_\Omega \Phi\left(x, \frac{|u(x)|}{\beta}\right)\, dx > 1$ otherwise we will obtain a contradiction with the definition of the Luxemburg norm. The above considerations implies

$$\int_\Omega \Phi(x,|u(x)|)\,dx = \int_\Omega \Phi\left(x,\beta\cdot\frac{|u(x)|}{\beta}\right)\,dx \geqslant \beta^{\varphi_0}\cdot\int_\Omega \Phi\left(x,\frac{|u(x)|}{\beta}\right)\,dx \geqslant \beta^{\varphi_0};$$

Letting $\beta \nearrow |u|_\Phi$ we deduce that relation (2.1) holds true.

Next, we show that $\rho_\Phi(u) \leqslant |u|_\Phi^{\varphi_0}$ for all $u \in L^\Phi(\Omega)$ with $|u|_\Phi < 1$. It is easy to show (see the proof of relations (2.3) and (2.4)) that

$$\Phi(x,t) \leqslant \tau^{\varphi_0}\cdot\Phi(x,t/\tau), \quad \forall\, x\in\overline{\Omega},\ t>0,\ \tau\in(0,1)\,. \tag{2.5}$$

Let $u \in L^\Phi(\Omega)$ with $|u|_\Phi < 1$. The definition of the Luxemburg norm and relation (2.5) imply

$$\begin{aligned}\int_\Omega \Phi(x,|u(x)|)\,dx &= \int_\Omega \Phi\left(x,|u|_\Phi\cdot\frac{|u(x)|}{|u|_\Phi}\right)\,dx\\ &\leqslant |u|_\Phi^{\varphi_0}\cdot\int_\Omega \Phi\left(x,\frac{|u(x)|}{|u|_\Phi}\right)\,dx\\ &\leqslant |u|_\Phi^{\varphi_0}\,.\end{aligned}$$

Finally, we show that $\rho_\Phi(u) \geqslant |u|_\Phi^{\varphi^0}$ for all $u \in L^\Phi(\Omega)$ with $|u|_\Phi < 1$.

As in the proof of (2.3) we deduce

$$\Phi(x,t) \geqslant \tau^{\varphi^0}\cdot\Phi(x,t/\tau), \quad \forall\, x\in\overline{\Omega},\ t>0,\ \tau\in(0,1)\,. \tag{2.6}$$

Let $u \in L^\Phi(\Omega)$ with $|u|_\Phi < 1$ and $\xi \in (0,|u|_\Phi)$. By (2.6) we find

$$\int_\Omega \Phi(x,|u(x)|)\,dx \geqslant \xi^{\varphi^0}\cdot\int_\Omega \Phi\left(x,\frac{|u(x)|}{\xi}\right)\,dx\,. \tag{2.7}$$

Define $v(x) = u(x)/\xi$, for all $x \in \Omega$. We have $|v|_\Phi = |u|_\Phi/\xi > 1$. Using relation (2.1) we find

$$\int_\Omega \Phi(x,|v(x)|)\,dx \geqslant |v|_\Phi^{\varphi_0} > 1\,. \tag{2.8}$$

By (2.7) and (2.8) we obtain

$$\int_\Omega \Phi(x,|u(x)|)\,dx \geqslant \xi^{\varphi^0}, \quad \forall\,\xi\in(0,|u|_\Phi)\,.$$

Letting $\xi \nearrow |u|_\Phi$ we deduce that relation (2.2) holds true. The proof of Proposition 2.1 is complete. □

Proposition 2.2. — *Assume Φ satisfies conditions (1.5) and (1.6). Then the space $L^\Phi(\Omega)$ is uniformly convex.*

Proof. — From the above hypotheses we deduce that we can apply Lemma 2.1 in [19] in order to deduce

$$\frac{1}{2}[\Phi(x,|t|)+\Phi(x,|s|)] \geqslant \Phi\left(x, \frac{|t+s|}{2}\right)+\Phi\left(x, \frac{|t-s|}{2}\right), \quad ;\forall\, x\in\Omega,\ t,\ s\in\mathbb{R}\,.$$

The above inequality yields

$$\frac{1}{2}[\rho_\Phi(u)+\rho_\Phi(v)] \geqslant \rho_\Phi\left(\frac{u+v}{2}\right)+\rho_\Phi\left(\frac{u-v}{2}\right), \quad ;\ \forall\, u,v\in L^\Phi(\Omega)\,. \tag{2.9}$$

Assume that $|u|_\Phi<1$ and $|v|_\Phi<1$ and $|u-v|_\Phi>\epsilon$ (with $\epsilon\in(0,1/K)$). Then we have

$$\begin{aligned} \rho_\Phi(u-v) &\geqslant |u-v|_\Phi^{\varphi_0} \quad \text{if } |u-v|_\Phi>1\\ \rho_\Phi(u-v) &\geqslant |u-v|_\Phi^{\varphi^0} \quad \text{if } |u-v|_\Phi<1\,, \end{aligned}$$

and

$$\rho_\Phi(u)<1, \quad \rho_\Phi(v)<1\,.$$

The above information and relation (1.4) yield

$$\rho_\Phi\left(\frac{u-v}{2}\right) \geqslant \frac{1}{K}\cdot\rho_\Phi(u-v) \geqslant \begin{cases} \dfrac{1}{K}\cdot\epsilon^{\varphi_0}, & \text{if} \quad |u-v|_\Phi>1\\ \dfrac{1}{K}\cdot\epsilon^{\varphi^0}, & \text{if} \quad |u-v|_\Phi<1\,. \end{cases}$$

By (2.9) and the above inequality we have

$$\rho_\Phi\left(\frac{u+v}{2}\right) < \begin{cases} 1-\dfrac{1}{K}\cdot\epsilon^{\varphi_0}, & \text{if} \quad |u-v|_\Phi>1\\ 1-\dfrac{1}{K}\cdot\epsilon^{\varphi^0}, & \text{if} \quad |u-v|_\Phi<1\,. \end{cases} \tag{2.10}$$

On the other hand, we have

$$\rho_\Phi\left(\frac{u+v}{2}\right) \geqslant \begin{cases} \left|\dfrac{u+v}{2}\right|_\Phi^{\varphi_0}, & \text{if} \quad \left|\dfrac{u+v}{2}\right|_\Phi>1\\ \left|\dfrac{u+v}{2}\right|_\Phi^{\varphi^0}, & \text{if} \quad \left|\dfrac{u+v}{2}\right|_\Phi<1\,. \end{cases} \tag{2.11}$$

Relations (2.10) and (2.11) show that there exists $\delta>0$ such that

$$\left|\frac{u+v}{2}\right|_\Phi<1-\delta\,.$$

Thus, we proved that $L^\Phi(\Omega)$ is an uniformly convex space. The proof of Proposition 2.2 is complete. ☐

Remark 3. Condition (1.6) (via relation (2.9)) also implies the fact that for every $x \in \Omega$ fixed, the function $\Phi(x, \cdot)$ is convex from $\mathbb{R}^+$ to $\mathbb{R}^+$.

PROPOSITION 2.3. — *Condition (1.4) implies condition (1.5).*

Proof. — Since relation (1.4) holds true by Proposition 2.1 it follows that condition (2.3) works. We deduce that

$$\Phi(x, 2 \cdot t) \leqslant 2^{\varphi^0} \cdot \Phi(x, t), \quad \forall\, x \in \Omega,\ t > 0\,.$$

Thus, relation (1.5) holds true with $K = 2^{\varphi^0}$. The proof of Proposition 2.3 is complete. □

PROPOSITION 2.4. — *On $W^{1,\Phi}(\Omega)$ the following norms*

$$\|u\|_{1,\Phi} = |\ |\nabla u|\ |_\Phi + |u|_\Phi\,,$$

$$\|u\|_{2,\Phi} = \max\{|\ |\nabla u|\ |_\Phi, |u|_\Phi\}\,,$$

$$\|u\| = \inf\left\{\mu > 0;\ \int_\Omega \left[\Phi\left(x, \frac{|u(x)|}{\mu}\right) + \Phi\left(x, \frac{|\nabla u(x)|}{\mu}\right)\right]\, dx \leqslant 1\right\}\,,$$

are equivalent.

Proof. — First, we point out that $\|\ \|_{1,\Phi}$ and $\|\ \|_{2,\Phi}$ are equivalent, since

$$2 \cdot \|u\|_{2,\Phi} \geqslant \|u\|_{1,\Phi} \geqslant \|u\|_{2,\Phi}, \quad \forall\, u \in W^{1,\Phi}(\Omega)\,. \tag{2.12}$$

Next, we remark that

$$\int_\Omega \Phi\left(x, \frac{|u(x)|}{|u|_\Phi}\right)\, dx \leqslant 1 \quad \text{and} \quad \int_\Omega \Phi\left(x, \frac{|\nabla u(x)|}{|\ |\nabla u|\ |_\Phi}\right)\, dx \leqslant 1\,,$$

and

$$\int_\Omega \left[\Phi\left(x, \frac{|u(x)|}{\|u\|}\right) + \Phi\left(x, \frac{|\nabla u(x)|}{\|u\|}\right)\right]\, dx \leqslant 1\,.$$

Using the above relations we obtain

$$\int_\Omega \Phi\left(x, \frac{|u(x)|}{\|u\|}\right)\, dx \leqslant 1 \quad \text{and} \quad \int_\Omega \Phi\left(x, \frac{|\nabla u(x)|}{\|u\|}\right)\, dx \leqslant 1.$$

Taking into account the way in which $|\ |_\Phi$ is defined we find

$$2\|u\| \geqslant (|u|_\Phi + |\ |\nabla u|\ |_\Phi) = \|u\|_{1,\Phi}, \quad \forall\, u \in W^{1,\Phi}(\Omega)\,. \tag{2.13}$$

On the other hand, by relation (2.4) we deduce that

$$\Phi(x, 2 \cdot t) \geqslant 2 \cdot \Phi(x, t), \quad \forall\, x \in \Omega,\ t > 0\,.$$

Thus, we deduce that

$$2 \cdot \Phi\left(x, \frac{|u(x)|}{2 \cdot \|u\|_{2,\Phi}}\right) \leqslant \Phi\left(x, \frac{|u(x)|}{\|u\|_{2,\Phi}}\right), \quad \forall\, u \in W^{1,\Phi}(\Omega),\ x \in \Omega$$

and

$$2\cdot\Phi\left(x,\frac{|\nabla u(x)|}{2\cdot\|u\|_{2,\Phi}}\right)\leqslant\Phi\left(x,\frac{|\nabla u(x)|}{\|u\|_{2,\Phi}}\right),\quad\forall\,u\in W^{1,\Phi}(\Omega),\ x\in\Omega\,.$$

It follows that

$$\int_\Omega\left[\Phi\left(x,\frac{|u(x)|}{2\|u\|_{2,\Phi}}\right)+\Phi\left(x,\frac{|\nabla u(x)|}{2\|u\|_{2,\Phi}}\right)\right] dx\leqslant\frac{1}{2}\left\{\int_\Omega\left[\Phi\left(x,\frac{|u(x)|}{\|u\|_{2,\Phi}}\right)+\Phi\left(x,\frac{|\nabla u(x)|}{\|u\|_{2,\Phi}}\right)\right]\,dx\right\}. \tag{2.14}$$

But, since

$$\|u\|_{2,\Phi}\geqslant|u|_\Phi\quad\text{and}\quad\|u\|_{2,\Phi}\geqslant|\,|\nabla u|\,|_\Phi,\quad\forall\,u\in W^{1,\Phi}(\Omega)\,,$$

we obtain

$$\frac{|u(x)|}{|u|_\Phi}\geqslant\frac{|u(x)|}{\|u\|_{2,\Phi}}\quad\text{and}\quad\frac{|\nabla u(x)|}{|\,|\nabla u|\,|_\Phi}\geqslant\frac{|\nabla u(x)|}{\|u\|_{2,\Phi}},\quad\forall\,u\in W^{1,\Phi}(\Omega),\ x\in\Omega\,. \tag{2.15}$$

Taking into account that Φ is increasing by (2.14) and (2.15) we deduce that

$$\int_\Omega\left[\Phi\left(x,\frac{|u(x)|}{2\|u\|_{2,\Phi}}\right)+\Phi\left(x,\frac{|\nabla u(x)|}{2\|u\|_{2,\Phi}}\right)\right] dx\leqslant\frac{1}{2}\left\{\int_\Omega\left[\Phi\left(x,\frac{|u(x)|}{|u|_\Phi}\right)+\Phi\left(x,\frac{|\nabla u(x)|}{|\,|\nabla u\,|_\Phi}\right)\right]\,dx\right\}\leqslant 1\,,$$

for all $u\in W^{1,\Phi}(\Omega)$.

We conclude that

$$2\cdot\|u\|_{1,\Phi}\geqslant 2\cdot\|u\|_{2,\Phi}\geqslant\|u\|,\quad\forall\,u\in W^{1,\Phi}(\Omega)\,. \tag{2.16}$$

By relations (2.12), (2.13) and (2.16) we deduce that Proposition 2.4 holds true. □

Proposition 2.5. — *The following relations hold true*

$$\int_\Omega[\Phi(x,|u(x)|)+\Phi(x,|\nabla u(x)|)]\,dx\geqslant\|u\|^{\varphi_0},\qquad\forall\,u\in W^{1,\Phi}(\Omega)\text{ with }\|u\|>1; \tag{2.17}$$

$$\int_\Omega[\Phi(x,|u(x)|)+\Phi(x,|\nabla u(x)|)]\,dx\geqslant\|u\|^{\varphi^0},\qquad\forall\,u\in W^{1,\Phi}(\Omega)\text{ with }\|u\|<1. \tag{2.18}$$

Proof. — First, assume that $\|u\| > 1$. Let $\beta \in (1, \|u\|)$. By relation (2.4) we have

$$\int_\Omega [\Phi(x, |u(x)|) + \Phi(x, |\nabla u(x)|)]\, dx \geqslant \beta^{\varphi_0} \cdot \int_\Omega \left[\Phi\left(x, \frac{|u(x)|}{\beta}\right) + \Phi\left(x, \frac{|\nabla u(x)|}{\beta}\right)\right].$$

Since $\beta < \|u\|$ we find

$$\int_\Omega \left[\Phi\left(x, \frac{|u(x)|}{\beta}\right) + \Phi\left(x, \frac{|\nabla u(x)|}{\beta}\right)\right] > 1\,.$$

Thus, we find

$$\int_\Omega [\Phi(x, |u(x)|) + \Phi(x, |\nabla u(x)|)]\, dx \geqslant \beta^{\varphi_0}\,.$$

Letting $\beta \nearrow \|u\|$ we deduce that (2.17) holds true.

Next, assume $\|u\| < 1$. Let $\xi \in (0, \|u\|)$. By relation (2.6) we obtain

$$\int_\Omega [\Phi(x, |u(x)|) + \Phi(x, |\nabla u(x)|)]\, dx \geqslant \xi^{\varphi^0} \cdot \int_\Omega \left[\Phi\left(x, \frac{|u(x)|}{\xi}\right) + \Phi\left(x, \frac{|\nabla u(x)|}{\xi}\right)\right] dx. \tag{2.19}$$

Defining $v(x) = u(x)/\xi$, for all $x \in \Omega$, we have $\|v\| = \|u\|/\xi > 1$. Using relation (2.17) we find

$$\int_\Omega [\Phi(x, |v(x)|) + \Phi(x, |\nabla v(x)|)]\, dx \geqslant \|v\|^{\varphi_0} > 1. \tag{2.20}$$

Relations (2.19) and (2.20) show that

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\, dx \geqslant \xi^{p^0}.$$

Letting $\xi \nearrow \|u\|$ in the above inequality we obtain that relation (2.18) holds true. The proof of Proposition 2.5 is complete. □

3. Main results

In this paper we study problem (1.1) in the particular case when Φ satisfies

$$M \cdot |t|^{p(x)} \leqslant \Phi(x, t), \quad \forall\, x \in \overline{\Omega},\ t \geqslant 0\,, \tag{3.1}$$

where $p(x) \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$ and $M > 0$ is a constant.

Remark 3.1. — In the following, for each continuous function $s:\overline{\Omega}\to(1,\infty)$ we will use the following notations:

$$s^- = \inf_{x\in\Omega} s(x), \qquad s^+ = \inf_{x\in\Omega} s(x)\,.$$

We point out that s^- and s^+ correspond to the constants ϕ_0 and ϕ^0 defined previously, for the particular function $\Phi(x,t)=|t|^{s(x)}$ when $\phi(x,t)=|t|^{s(x)-2}t$. By relation (3.1) we deduce that $W^{1,\Phi}(\Omega)$ is continuously embedded in $W^{1,p(x)}(\Omega)$ (see relation (1.7) with $\Psi(x,t)=|t|^{p(x)}$). On the other hand, it is known (see [18, 16, 22]) that $W^{1,p(x)}(\Omega)$ is compactly embedded in $L^{r(x)}(\Omega)$ for any $r(x)\in C(\overline{\Omega})$ with $1<r^-\leqslant r^+<\frac{Np^-}{N-p^-}$. Thus, we deduce that $W^{1,\Phi}(\Omega)$ is compactly embedded in $L^{r(x)}(\Omega)$ for any $r(x)\in C(\overline{\Omega})$ with $1<r(x)<\frac{Np^-}{N-p^-}$ for all $x\in\overline{\Omega}$.

On the other hand, we assume that the function g from problem (1.1) satisfies the hypotheses

$$|g(x,t)|\leqslant C_0\cdot|t|^{q(x)-1},\quad \forall\, x\in\Omega,\ t\in\mathbb{R} \tag{3.2}$$

and

$$C_1\cdot|t|^{q(x)}\leqslant G(x,t):=\int_0^t g(x,s)\,ds\leqslant C_2\cdot|t|^{q(x)},\quad \forall\, x\in\Omega,\ t\in\mathbb{R}\,, \tag{3.3}$$

where C_0, C_1 and C_2 are positive constants and $q(x)\in C(\overline{\Omega})$ satisfies $1<q(x)<\frac{Np^-}{N-p^-}$ for all $x\in\overline{\Omega}$.

Examples. — *We point out certain examples of functions g and G which satisfy hypotheses (3.2) and (3.3).*

(1) *$g(x,t)=q(x)\cdot|t|^{q(x)-2}t$ and $G(x,t)=|t|^{q(x)}$, where $q(x)\in C(\overline{\Omega})$ satisfies $2\leqslant q(x)<\frac{Np^-}{N-p^-}$ for all $x\in\overline{\Omega}$;*

(2) *$g(x,t)=q(x)\cdot|t|^{q(x)-2}t+(q(x)-2)\cdot[\log(1+t^2)]\cdot|t|^{q(x)-4}t+\frac{t}{1+t^2}|t|^{q(x)-2}$ and $G(x,t)=|t|^{q(x)}+\log(1+t^2)\cdot|t|^{q(x)-2}$, where $q(x)\in C(\overline{\Omega})$ satisfies $4\leqslant q(x)<\frac{Np^-}{N-p^-}$ for all $x\in\overline{\Omega}$;*

(3) *$g(x,t)=q(x)\cdot|t|^{q(x)-2}t+(q(x)-1)\cdot\sin(\sin(t))\cdot|t|^{q(x)-3}t+\cos(\sin(t))\cdot\cos(t)\cdot|t|^{q(x)-1}$ and $G(x,t)=|t|^{q(x)}+\sin(\sin(t))\cdot|t|^{q(x)-1}$, where $q(x)\in C(\overline{\Omega})$ satisfies $3\leqslant q(x)<\frac{Np^-}{N-p^-}$ for all $x\in\overline{\Omega}$.*

We say that $u\in W^{1,\Phi}(\Omega)$ is a *weak solution* of problem (1.1) if

$$\int_\Omega a(x,|\nabla u|)\nabla u\nabla v\,dx+\int_\Omega a(x,|u|)uv\,dx-\lambda\int_\Omega g(x,u)v\,dx=0,$$

for all $v\in W^{1,\Phi}(\Omega)$.

The main results of this paper are given by the following theorems.

THEOREM 3.2. — *Assume φ and Φ verify conditions (φ), (Φ_1), (Φ_2), (1.4), (1.6) and (3.1) and the functions g and G satisfy conditions (3.2) and (3.3). Furthermore, we assume that $q^- < \varphi_0$. Then there exists $\lambda_\star > 0$ such that for any $\lambda \in (0, \lambda_\star)$ problem (1.1) has a nontrivial weak solution.*

THEOREM 3.3. — *Assume φ and Φ verify conditions (φ), (Φ_1), (Φ_2), (1.4), (1.6) and (3.1) and the functions g and G satisfy conditions (3.2) and (3.3). Furthermore, we assume that $q^+ < \varphi_0$. Then there exists $\lambda_\star > 0$ and $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda_\star) \cup (\lambda^\star, \infty)$ problem (1.1) has a nontrivial weak solution.*

4. Proof of the main results

Let E denote the generalized Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$.

For each $\lambda > 0$ we define the energy functional $J_\lambda : E \to \mathbb{R}$ by

$$J_\lambda(u) = \int_\Omega [\Phi(x, |\nabla u|) + \Phi(x, |u|)]\, dx - \lambda \int_\Omega G(x, u)\, dx, \quad \forall\, u \in E\,.$$

We first establish some basic properties of J_λ.

PROPOSITION 4.1. — *For each $\lambda > 0$ the functional J_λ is well-defined on E and $J_\lambda \in C^1(E, \mathbb{R})$ with the derivative given by*

$$\langle J_\lambda^{'}(u), v\rangle = \int_\Omega a(x, |\nabla u|)\nabla u \cdot \nabla v\, dx + \int_\Omega a(x, |u|)uv\, dx - \lambda \int_\Omega g(x, u)v\, dx\,,$$

for all u, $v \in E$.

To prove Proposition 4.1 we define the functional $\Lambda : E \to \mathbb{R}$ by

$$\Lambda(u) = \int_\Omega [\Phi(x, |\nabla u|) + \Phi(x, |u|)]\, dx, \quad \forall\, u \in E\,.$$

LEMMA 4.2. — *The functional Λ is well defined on E and $\Lambda \in C^1(E, \mathbb{R})$ with*

$$\langle \Lambda^{'}(u), v\rangle = \int_\Omega a(x, |\nabla u|)\nabla u \cdot \nabla v\, dx + \int_\Omega a(x, |u|)uv\, dx\,,$$

for all u, $v \in E$.

Proof. — Clearly, Λ is well defined on E.

Existence of the Gâteaux derivative. — Let u, $v \in E$. Fix $x \in \Omega$ and $0 < |r| < 1$. Then, by the mean value theorem, there exists $\nu, \theta \in [0,1]$ such that

$$\begin{aligned}(4.1)\qquad &|\Phi(x, |\nabla u(x) + r\nabla v(x)|) - \Phi(x, |\nabla u(x)|)|/|r| \\ &\qquad = |\varphi(x, |(1-\nu)|\nabla u(x) + r\nabla v(x)| + \nu|\nabla u(x)|)| \cdot \\ &\qquad\quad ||\nabla u(x) + r\nabla v(x)| - |\nabla u(x)||\end{aligned}$$

and

$$\begin{aligned}(4.2)\qquad &|\Phi(x, |u(x) + rv(x)|) - \Phi(x, |u(x)|)|/|r| \\ &\qquad = |\varphi(x, |(1-\theta)|u(x) + rv(x)| + \theta|u(x)|)| \cdot \\ &\qquad\quad ||u(x) + rv(x)| - |u(x)||\,.\end{aligned}$$

Next, we claim that $\varphi(x, |u(x)|) \in L^{\overline{\Phi}}(\Omega)$ provided that $u \in L^{\Phi}(\Omega)$, where $\overline{\Phi}$ is the conjugate Young function of Φ.

Indeed, we know that

$$\overline{\Phi}(x,t) = \sup_{s>0}\{ts - \Phi(x,s);\ s \in \mathbb{R}\}, \quad \forall\, x \in \overline{\Omega},\ t \geqslant 0$$

or

$$\overline{\Phi}(x,t) = \int_0^t \overline{\varphi}(x,s)\, ds, \quad \forall\, x \in \overline{\Omega},\ t \geqslant 0\,,$$

where $\overline{\varphi}(x,t) = \sup\limits_{\varphi(x,s) \leqslant t} s$, for all $x \in \overline{\Omega}$ and $t \geqslant 0$.

On the other hand, by relation (φ) we know that for all $x \in \Omega$, $\varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and thus, an increasing homeomorphism from $\mathbb{R}^+$ onto $\mathbb{R}^+$. It follows that for each $x \in \overline{\Omega}$ we can denote by $\varphi^{-1}(x,t)$ the inverse function of $\varphi(x,t)$ relative to variable t. Thus, we deduce that $\varphi(x,s) \leqslant t$ if and only if $s \leqslant \varphi^{-1}(x,t)$. Taking into account the above pieces of information we deduce that $\overline{\varphi}(x,t) = \varphi^{-1}(x,t)$. Consequently we find

$$\overline{\Phi}(x,t) = \int_0^t \varphi^{-1}(x,s)\, ds, \quad \forall\, x \in \overline{\Omega},\ t \geqslant 0\,.$$

Next, since

$$\overline{\Phi}(x, \varphi^{-1}(x,s)) = \int_0^{\varphi^{-1}(x,s)} \varphi(x,\theta)\, d\theta, \quad \forall\, x \in \overline{\Omega},\ s \geqslant 0\,,$$

taking $\varphi(x,\theta) = r$ we find

$$\overline{\Phi}(x, \varphi^{-1}(x,s)) = \int_0^s r \cdot (\varphi^{-1}(x,r))'_r\, dr = s \cdot \varphi^{-1}(x,s) - \overline{\Phi}(x,s), \qquad \forall\, x \in \overline{\Omega},\ s \geqslant 0\,.$$

The above relation implies

$$\overline{\Phi}(x,s)\leqslant s\cdot\varphi^{-1}(x,s),\quad \forall\, x\in\overline{\Omega},\ s\geqslant 0\,.$$

Taking into the above inequality $s=\varphi(x,t)$ we find

$$\overline{\Phi}(x,\varphi(x,t))\leqslant t\cdot\varphi(x,t),\quad \forall\, x\in\overline{\Omega},\ t\geqslant 0\,.$$

The last inequality and relation (1.4) yield

$$\overline{\Phi}(x,\varphi(x,t))\leqslant \varphi^{0}\cdot\Phi(x,t),\quad \forall\, x\in\overline{\Omega},\ t\geqslant 0\,.$$

Thus, we deduce that for any $u\in L^{\Phi}(\Omega)$ we have $\varphi(x,|u(x)|)\in L^{\overline{\Phi}}(\Omega)$ and the claim is verified. By applying relations (4.1), (4.2), the above claim and (1.3) we infer that

$$\begin{gathered}|\Phi(x,|\nabla u(x)+r\nabla v(x)|)+\Phi(x,|u(x)+rv(x)|)-\Phi(x,|\nabla u(x)|)-\\ \Phi(x,|u(x)|)||/|r|\leqslant|\varphi(x,|(1-\nu)|\nabla u(x)+r\nabla v(x)|+\\ \nu|\nabla u(x)|)|\cdot||\nabla u(x)+r\nabla v(x)|-|\nabla u(x)||+\\ |\varphi(x,|(1-\theta)|u(x)+rv(x)|+\theta|\nabla u(x)|)|\cdot||u(x)+rv(x)|-|u(x)||\in L^{1}(\Omega),\end{gathered}$$

for all u, $v\in E$, $x\in\overline{\Omega}$ and $|r|\in(0,1)$. It follows from the Lebesgue theorem that

$$\langle\Lambda^{'}(u),v\rangle=\int_{\Omega}a(x,|\nabla u|)\nabla u\cdot\nabla v\;dx+\int_{\Omega}a(x,|u|)uv\;dx\,.$$

Continuity of the Gâteaux derivative. — Assume $u_n\to u$ in E. The above claim and the Lebesgue theorem imply

$$a(x,|\nabla u_n|)\nabla u_n\to a(x,|\nabla u|)\nabla u,\quad\text{in}\quad (L^{\overline{\Phi}}(\Omega))^N$$

and

$$a(x,|u_n|)u_n\to a(x,|u|)u,\quad\text{in}\quad L^{\overline{\Phi}}(\Omega)\,.$$

Those facts and (1.3) imply

$$\begin{aligned}&|\langle\Lambda^{'}(u_n)-\Lambda^{'}(u),v\rangle|\leqslant|a(x,|\nabla u_n|)\\&\quad\nabla u_n-a(x,|\nabla u|)\nabla u|_{\overline{\Phi}}\cdot|\;|\nabla v|\;|_{\Phi}+|a(x,|u_n|)u_n-a(x,|u|)u|_{\overline{\Phi}}\cdot|v|_{\Phi}\,,\end{aligned}$$

for all $v\in E$, and so

$$\begin{aligned}&\|\Lambda^{'}(u_n)-\Lambda^{'}(u)\|\leqslant|a(x,|\nabla u_n|)\\&\nabla u_n-a(x,|\nabla u|)\nabla u|_{\overline{\Phi}}+|a(x,|u_n|)u_n-a(x,|u|)u|_{\overline{\Phi}}\to 0,\quad\text{as } n\to\infty\,.\end{aligned}$$

The proof of Lemma 4.2 is complete. □

Combining Lemma 4.2 and Remark 4 we infer that Proposition 4.1 holds true.

Lemma 4.3. — *The functional Λ is weakly lower semi-continuous.*

Proof. — By Corollary III.8 in [3], it is enough to show that Λ is lower semi-continuous. For this purpose, we fix $u \in E$ and $\epsilon > 0$. Since Λ is convex (because Φ is convex) we deduce that for any $v \in E$ the following inequality holds true

$$\Lambda(v) \geqslant \Lambda(u) + \langle \Lambda^{'}(u), v - u \rangle \,,$$

or

$$\begin{aligned}\Lambda(v) &\geqslant \Lambda(u) - \int_\Omega [a(x,|\nabla u|)|\nabla u| \cdot |\nabla v - \nabla u| + a(x,|u|)|u| \cdot |v-u|]\,dx \\ &= \Lambda(u) - \int_\Omega [\varphi(x,|\nabla u|)|\nabla v - \nabla u| + \varphi(x,|u|)|v-u|]\,dx\,.\end{aligned}$$

But, by the claim proved in Proposition 4.1 we know that for any $u \in L^\Phi(\Omega)$ we have $\varphi(x,|u|)$, $\varphi(x,|\nabla u|) \in L^{\overline{\Phi}}(\Omega)$. Thus, by relation (1.3) we find

$$\begin{aligned}\Lambda(v) &\geqslant \Lambda(u) - C \cdot [|\varphi(x,|\nabla u|)|_{\overline{\Phi}} \cdot |\,|\nabla v - \nabla u|\,|_\Phi + |\varphi(x,|u|)|_{\overline{\Phi}} \cdot |v-u|_\Phi] \\ &\geqslant \Lambda(u) - C^{'} \cdot \|u - v\| \\ &\geqslant \Lambda(u) - \epsilon\,,\end{aligned}$$

for all $v \in E$ with $\|v - u\| < \delta = \epsilon / C^{'}$, where C and $C^{'}$ are positive constants. The proof of Lemma 4.3 is complete. □

Proposition 4.4. — *The functional J_λ is weakly lower semi-continuous.*

Proof. — Using Lemma 4.3 we have that Λ is weakly lower semi-continuous. We show that J_λ is weakly lower semi-continuous. Let $\{u_n\} \subset E$ be a sequence which converges weakly to u in E. By Lemma 4.3 we deduce

$$\Lambda(u) \leqslant \liminf_{n\to\infty} \Lambda(u_n)\,.$$

On the other hand, Remark 4 and conditions (3.2) and (3.3) imply

$$\lim_{n\to\infty} \int_\Omega G(x,u_n)\,dx = \int_\Omega G(x,u)\,dx\,.$$

Thus, we find

$$J_\lambda(u) \leqslant \liminf_{n\to\infty} J_\lambda(u_n)\,.$$

Therefore, J_λ is weakly lower semi-continuous and Proposition 4.4 is verified. □

Proposition 4.5. — *Assume that the sequence $\{u_n\}$ converges weakly to u in E and*

$$\limsup_{n\to\infty} \langle \Lambda^{'}(u_n), u_n - u \rangle \leqslant 0.$$

Then $\{u_n\}$ converges strongly to u in E.

Proof. — Since $\{u_n\}$ converges weakly to u in E it follows that $\{\|u_n\|\}$ is a bounded sequence of real numbers. That fact and Proposition 2.4 imply that $\{|u_n|_\Phi\}$ and $\{|\,|\nabla u_n|\,|_\Phi\}$ are bounded sequences of real numbers. That information and relations (1.8) and (1.9) yield that the sequence $\{\Lambda(u_n)\}$ is bounded. Then, up to a subsequence, we deduce that $\Lambda(u_n) \to c$.

By Lemma 4.3 we obtain

$$\Lambda(u) \leqslant \liminf_{n\to\infty} \Lambda(u_n) = c\,.$$

On the other hand, since Λ is convex, we have

$$\Lambda(u) \geqslant \Lambda(u_n) + \langle \Lambda^{'}(u_n), u - u_n\rangle\,.$$

Using the above hypothesis we conclude that $\Lambda(u) = c$. Taking into account that $\{(u_n + u)/2\}$ converges weakly to u in E and using Lemma 4.3 we find

$$c = \Lambda(u) \leqslant \Lambda\left(\frac{u_n + u}{2}\right)\,. \tag{4.3}$$

We assume by contradiction that $\{u_n\}$ does not converge to u in E or $\{(u_n - u)/2\}$ does not converge to 0 in E. It follows that there exist $\epsilon > 0$ and a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that

$$\left\|\frac{u_{n_m} - u}{2}\right\| \geqslant \epsilon, \quad \forall m\,. \tag{4.4}$$

Furthermore, relations (2.17), (2.18) and (4.4) imply that there exists $\epsilon_1 > 0$ such that

$$\Lambda\left(\frac{u_{n_m} - u}{2}\right) \geqslant \epsilon_1, \quad \forall m\,. \tag{4.5}$$

On the other hand, relations (2.9) and (4.5) yield

$$\frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(u_{n_m}) - \Lambda\left(\frac{u_{n_m} + u}{2}\right) \geqslant \Lambda\left(\frac{u_{n_m} - u}{2}\right) \geqslant \epsilon_1, \quad \forall m\,.$$

Letting $m \to \infty$ in the above inequality we obtain

$$c - \epsilon_1 \geqslant \limsup_{m\to\infty} \Lambda\left(\frac{u_{n_m} + u}{2}\right)\,,$$

and that is a contradiction with (4.3). We conclude that $\{u_n\}$ converges strongly to u in E and Proposition 4.5 is proved. □

LEMMA 4.6. — *Assume the hypotheses of Theorem 3.2 are fulfilled. Then there exists $\lambda_\star > 0$ such that for any $\lambda \in (0, \lambda_\star)$ there exist ρ, $\alpha > 0$ such that $J_\lambda(u) \geqslant \alpha > 0$ for any $u \in E$ with $\|u\| = \rho$.*

Proof. — By Remark 4 and conditions (3.2) and (3.3) it follows that E is continuously embedded in $L^{q(x)}(\Omega)$. So, there exists a positive constant c_1 such that

$$|u|_{q(x)} \leqslant c_1 \cdot \|u\|, \quad \forall\, u \in E\,. \tag{4.6}$$

where by $|\cdot|_{q(x)}$ we denoted the norm on $L^{q(x)}(\Omega)$.

We fix $\rho \in (0,1)$ such that $\rho < 1/c_1$. Then relation (4.6) implies

$$|u|_{q(x)} < 1, \quad \forall\, u \in E, \text{ with } \|u\| = \rho\,.$$

Furthermore, relation (1.9) applied to $\Phi(x,t) = |t|^{q(x)}$ yields

$$\int_\Omega |u|^{q(x)}\, dx \leqslant |u|_{q(x)}^{q^-}, \quad \forall\, u \in E, \text{ with } \|u\| = \rho\,. \tag{4.7}$$

Relations (4.6) and (4.7) imply

$$\int_\Omega |u|^{q(x)}\, dx \leqslant c_1^{q^-} \|u\|^{q^-}, \quad \forall\, u \in E, \text{ with } \|u\| = \rho\,. \tag{4.8}$$

Taking into account relations (2.18), (4.8) and (3.3) we deduce that for any $u \in E$ with $\|u\| = \rho$ the following inequalities hold true

$$J_\lambda(u) \geqslant \|u\|^{\varphi^0} - \lambda \cdot C_2 \cdot c_1^{q^-} \cdot \|u\|^{q^-} = \rho^{q^-}(\rho^{\varphi^0 - q^-} - \lambda \cdot C_2 \cdot c_1^{q^-})\,.$$

By the above inequality we remark that if we define

$$\lambda_\star = \frac{\rho^{\varphi^0 - q^-}}{2 \cdot C_2 \cdot c_1^{q^-}}\,, \tag{4.9}$$

then for any $\lambda \in (0, \lambda_\star)$ and any $u \in E$ with $\|u\| = \rho$ there exists $\alpha = \frac{\rho^{\varphi^0}}{2} > 0$ such that

$$J_\lambda(u) \geqslant \alpha > 0\,.$$

The proof of Lemma 4.6 is complete. □

Lemma 4.7. — *Assume the hypotheses of Theorem 3.2 are fulfilled. Then there exists $\theta \in E$ such that $\theta \geqslant 0$, $\theta \neq 0$ and $J_\lambda(t\theta) < 0$, for $t > 0$ small enough.*

Proof. — By the hypotheses of Theorem 3.2 we have $q^- < \varphi_0$. Let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < \varphi_0$. On the other hand, since $q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_0 \subset\subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. Thus, we conclude that $q(x) \leqslant q^- + \epsilon_0 < \varphi_0$ for all $x \in \Omega_0$.

Let $\theta \in C_0^\infty(\Omega) \subset E$ be such that $\operatorname{supp}(\theta) \supset \overline{\Omega}_0$, $\theta(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \leqslant \theta \leqslant 1$ in Ω.

Taking into account all the above pieces of information and relations (2.5) and (3.3) we have

$$\begin{aligned}
&J_\lambda(t\cdot\theta)\\
&=\int_\Omega[\Phi(x,t|\nabla\theta(x)|)+\Phi(x,t|\theta(x)|)]\,dx-\lambda\int_\Omega G(x,t\cdot\theta(x))\,dx\\
&\leqslant t^{\varphi_0}\cdot\int_\Omega[\Phi(x,|\nabla\theta(x)|)+\Phi(x,|\theta(x)|)]\,dx-\lambda\cdot C_1\cdot\int_\Omega t^{q(x)}|\theta|^{q(x)}\,dx\\
&\leqslant t^{\varphi_0}\cdot\Lambda(\theta)-\lambda\cdot C_1\cdot\int_{\Omega_0} t^{q(x)}|\theta|^{q(x)}\,dx\\
&\leqslant t^{\varphi_0}\cdot\Lambda(\theta)-\lambda\cdot C_1\cdot t^{q^-+\epsilon_0}\cdot\int_{\Omega_0}|\theta|^{q(x)}\,dx\,,
\end{aligned}$$

for any $t\in(0,1)$, where by $|\Omega_0|$ we denoted the Lebesgue measure of Ω_0. Therefore

$$J_\lambda(t\cdot\theta)<0$$

for $t<\delta^{1/(\varphi_0-q^--\epsilon_0)}$ with

$$0<\delta<\min\left\{1,\frac{\lambda\cdot C_1\cdot\int_{\Omega_0}|\theta|^{q(x)}\,dx}{\Lambda(\theta)}\right\}.$$

Finally, we point out that $\Lambda(\theta)>0$. Indeed, it is clear that

$$0<\int_{\Omega_0}|\theta|^{q(x)}\,dx\leqslant\int_\Omega|\theta|^{q(x)}\,dx\int_\Omega|\theta|^{q^-}\,dx\leqslant c_1^{q^-}\|u\|^{q^-}.$$

Thus, we infer that $\|\theta\|>0$. That fact and relations (2.17) and (2.18) imply that $\Lambda(\theta)>0$. The proof of Lemma 4.7 is complete. □

Proof of Theorem 3.2. — Let $\lambda_\star>0$ be defined as in (4.9) and $\lambda\in(0,\lambda_\star)$. By Lemma 4.6 it follows that on the boundary of the ball centered in the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)}J_\lambda>0.$$

On the other hand, by Lemma 4.7, there exists $\theta\in E$ such that $J_\lambda(t\cdot\theta)<0$ for all $t>0$ small enough. Moreover, relations (2.18), (4.8) and (3.3) imply that for any $u\in B_\rho(0)$ we have

$$J_\lambda(u)\geqslant\|u\|^{\varphi^0}-\lambda\cdot C_2\cdot c_1^{q^-}\|u\|^{q^-}.$$

It follows that

$$-\infty<\underline{c}:=\inf_{\overline{B_\rho(0)}}J_\lambda<0\,.$$

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland's variational principle [14] to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{aligned} J_\lambda(u_\epsilon) &< \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \\ J_\lambda(u_\epsilon) &< J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|, \quad u \neq u_\epsilon. \end{aligned}$$

Since

$$J_\lambda(u_\epsilon) \leqslant \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \leqslant \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda\,,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \geqslant 0$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_\lambda(u_\epsilon + t \cdot v) - J_\lambda(u_\epsilon)}{t} + \epsilon \cdot \|v\| \geqslant 0.$$

Letting $t \to 0$ it follows that $\langle J_\lambda'(u_\epsilon), v\rangle + \epsilon \cdot \|v\| > 0$ and we infer that $\|J_\lambda'(u_\epsilon)\| \leqslant \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \to \underline{c} \quad \text{and} \quad J_\lambda'(w_n) \to 0. \tag{4.10}$$

It is clear that $\{w_n\}$ is bounded in E. Thus, there exists $w \in E$ such that, up to a subsequence, $\{w_n\}$ converges weakly to w in E. Since, by Remark 4, E is compactly embedded in $L^{q(x)}(\Omega)$ it follows that $\{w_n\}$ converges strongly to w in $L^{q(x)}(\Omega)$. The above information combined with relation (3.2) and Hölder's inequality implies

$$\begin{aligned} &\left|\int_\Omega g(x, w_n) \cdot (w_n - w)\, dx\right| \\ &\leqslant C_0 \cdot \int_\Omega |w_n|^{q(x)-1} |w_n - w|\, dx \\ &\leqslant C_0 \cdot \big|\, |w_n|^{q(x)-1}\, \big|_{\frac{q(x)}{q(x)-1}} \cdot |w_n - w|_{q(x)} \to 0, \quad \text{as } n \to \infty\,. \end{aligned} \tag{4.11}$$

On the other hand, by (4.10) we have

$$\lim_{n\to\infty} \langle J_\lambda'(w_n), w_n - w\rangle = 0\,. \tag{4.12}$$

Relations (4.11) and (4.12) imply

$$\lim_{n\to\infty} \langle \Lambda'(w_n), w_n - w\rangle = 0\,.$$

Thus, by Proposition 4.5 we find that $\{w_n\}$ converges strongly to w in E. So, by (4.10),

$$J_\lambda(w) = \underline{c} < 0 \quad \text{and} \quad J_\lambda^{'}(w) = 0\,.$$

We conclude that w is a nontrivial weak solution for problem (1.1) for any $\lambda \in (0, \lambda_\star)$. The proof of Theorem 3.2 is complete. □

Lemma 4.8. — *Assume the hypotheses of Theorem 3.3 are fulfilled. Then for any $\lambda > 0$ the functional J_λ is coercive.*

Proof. — For each $u \in E$ with $\|u\| > 1$ and $\lambda > 0$ relations (2.17), (3.2) and Remark 4 imply

$$\begin{aligned} J_\lambda(u) &\geqslant \|u\|^{\varphi_0} - \lambda \cdot C_2 \cdot \int_\Omega |u|^{q(x)}\,dx \\ &\geqslant \|u\|^{\varphi_0} - \lambda \cdot C_2 \cdot \left[\int_\Omega |u|^{q^-}\,dx + \int_\Omega |u|^{q^+}\,dx\right] \\ &\geqslant \|u\|^{\varphi_0} - \lambda \cdot C_3 \cdot [\|u\|^{q^-} + \|u\|^{q^+}]\,, \end{aligned}$$

where C_3 is a positive constant. Since $q^+ < \varphi_0$ the above inequality implies that $J_\lambda(u) \to \infty$ as $\|u\| \to \infty$, that is, J_λ is coercive. The proof of Lemma 4.8 is complete. □

Proof of Theorem 3.3. — Since $q^+ < \varphi_0$ it follows that $q^- < \varphi_0$ and thus, by Theorem 3.2 there exists $\lambda_\star > 0$ such that for any $\lambda \in (0, \lambda_\star)$ problem (1.1) has a nontrivial weak solution.

Next, by Lemma 4.8 and Proposition 4.4 we infer that J_λ is coercive and weakly lower semi-continuous in E, for all $\lambda > 0$. Then Theorem 1.2 in [33] implies that there exists $u_\lambda \in E$ a global minimizer of I_λ and thus a weak solution of problem (1.1).

We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0 > 1$ be a fixed real and $u_0(x) = t_0$, for all $x \in \Omega$ we have $u_0 \in E$ and

$$\begin{aligned} J_\lambda(u_0) &= \Lambda(u_0) - \lambda \int_\Omega G(x, u_0)\,dx \\ &\leqslant \int_\Omega \Phi(x, t_0)\,dx - \lambda \cdot C_1 \cdot \int_\Omega |t_0|^{q(x)}\,dx \\ &\leqslant L - \lambda \cdot C_1 \cdot t_0^{q^+} \cdot |\Omega_1|\,, \end{aligned}$$

where L is a positive constant. Thus, there exists $\lambda^\star > 0$ such that $J_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^\star, \infty)$. It follows that $J_\lambda(u_\lambda) < 0$ for any $\lambda \geqslant \lambda^\star$ and thus u_λ is a nontrivial weak solution of problem (1.1) for λ large enough. The proof of Theorem 3.3 is complete. □

5. Examples

In this section we point out certain examples of functions $\varphi(x,t)$ and $\Phi(x,t)$ for which the results of this paper can be applied.

I) We can take

$$\varphi(x,t)=p(x)|t|^{p(x)-2}t \quad \text{and} \quad \Phi(x,t)=|t|^{p(x)}\,,$$

with $p(x)\in C(\overline{\Omega})$ satisfying $2\leqslant p(x)<N$, for all $x\in\overline{\Omega}$. It is easy to verify that φ and Φ satisfy conditions (φ), (Φ_1), (Φ_2), (1.4), (1.6) and (3.1) since in this case we can take $\varphi_0=p^-$ and $\varphi^0=p^+$.

II) We can take

$$\varphi(x,t)=p(x)\frac{|t|^{p(x)-2}t}{\log(1+|t|)}$$

and

$$\Phi(x,t)=\frac{|t|^{p(x)}}{\log(1+|t|)}+\int_0^{|t|}\frac{s^{p(x)}}{(1+s)(\log(1+s))^2}\,ds\,,$$

with $p(x)\in C(\overline{\Omega})$ satisfying $3\leqslant p(x)<N$, for all $x\in\overline{\Omega}$.

It is easy to see that relations (φ), (Φ_1) and (Φ_2) are verified.

For each $x\in\overline{\Omega}$ fixed by Example 3 on p. 243 in [6] we have

$$p(x)-1\leqslant\frac{t\cdot\varphi(x,t)}{\Phi(x,t)}\leqslant p(x),\quad \forall\, t\geqslant 0\,.$$

Thus, relation (1.4) holds true with $\varphi_0=p^--1$ and $\varphi^0=p^+$.

Next, Φ satisfies condition (3.1) since

$$\Phi(x,t)\geqslant t^{p(x)-1},\quad \forall\, x\in\overline{\Omega},\ t\geqslant 0\,.$$

Finally, we point out that trivial computations imply that $\frac{d^2(\Phi(x,\sqrt{t}))}{dt^2}\geqslant 0$ for all $x\in\overline{\Omega}$ and $t\geqslant 0$. Thus, relation (1.6) is satisfied.

III) We can take

$$\varphi(x,t)=p(x)\cdot\log(1+\alpha+|t|)\cdot|t|^{p(x)-1}t\,,$$

and

$$\Phi(x,t)=\log(1+\alpha+|t|)\cdot|t|^{p(x)}-\int_0^{|t|}\frac{s^{p(x)}}{1+\alpha+s}\,dx\,,$$

where $\alpha>0$ is a constant and $p(x)\in C(\overline{\Omega})$ satisfying $2\leqslant p(x)<N$, for all $x\in\overline{\Omega}$.

It is easy to see that relations (φ), (Φ_1) and (Φ_2) are verified.

Next, it is easy to remark that for each $x\in\overline{\Omega}$ fixed we have

$$p(x)\leqslant\frac{t\cdot\varphi(x,t)}{\Phi(x,t)},\quad \forall\, t\geqslant 0\,.$$

The above information shows that taking $\varphi_0 = p^-$ we have

$$1 < p^- \leqslant \frac{t \cdot \varphi(x,t)}{\Phi(x,t)}, \quad \forall\, x \in \overline{\Omega},\ t \geqslant 0\,.$$

On the other hand, some simple computations imply

$$\lim_{t\to\infty} \frac{t \cdot \varphi(x,t)}{\Phi(x,t)} = p(x), \quad \forall\, x \in \overline{\Omega}$$

and

$$\lim_{t\to 0} \frac{t \cdot \varphi(x,t)}{\Phi(x,t)} = p(x), \quad \forall\, x \in \overline{\Omega}\,.$$

Thus, defining $H(x,t) = \frac{t\cdot\varphi(x,t)}{\Phi(x,t)}$ we remark that $H(x,t)$ is continuous on $\overline{\Omega} \times [0,\infty)$ and $1 < p^- \leqslant \lim_{t\to 0} H(x,t) \leqslant p^+ < \infty$ and $1 < p^- \leqslant \lim_{t\to\infty} H(x,t) \leqslant p^+ < \infty$. It follows that

$$\varphi^0 = \sup_{t>0,\ x\in\overline{\Omega}} \frac{t \cdot \varphi(x,t)}{\Phi(x,t)} < \infty\,.$$

We conclude that relation (1.4) is satisfied.

On the other hand, since

$$\varphi(x,t) \geqslant p^- \cdot \log(1+\alpha) \cdot t^{p(x)-1}, \quad \forall\, x \in \overline{\Omega},\ t \geqslant 0\,,$$

it follows that

$$\Phi(x,t) \geqslant \frac{p^-}{p^+} \cdot (1+\alpha) \cdot t^{p(x)}, \quad \forall\, x \in \overline{\Omega},\ t \geqslant 0\,.$$

The above relation assures that relation (3.1) is verified.

Finally, we point out that trivial computations imply that $\frac{d^2(\Phi(x,\sqrt{t}))}{dt^2} \geqslant 0$ for all $x \in \overline{\Omega}$ and $t \geqslant 0$ and thus, relation (1.6) is satisfied.

BIBLIOGRAPHY

[1] E. Acerbi & G. Mingione, "Regularity results for a class of functionals with non-standard growth", *Arch. Ration. Mech. Anal.* **156** (2001), no. 2, p. 121-140.

[2] R. A. Adams, *Sobolev spaces*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Pure and Applied Mathematics, Vol. 65, xviii+268 pages.

[3] H. Brezis, *Analyse fonctionnelle*, Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree], Masson, Paris, 1983, Théorie et applications. [Theory and applications], xiv+234 pages.

[4] Y. Chen, S. Levine & M. Rao, "Variable exponent, linear growth functionals in image restoration", *SIAM J. Appl. Math.* **66** (2006), no. 4, p. 1383-1406 (electronic).

[5] P. Clément, M. García-Huidobro, R. Manásevich & K. Schmitt, "Mountain pass type solutions for quasilinear elliptic equations", *Calc. Var. Partial Differential Equations* **11** (2000), no. 1, p. 33-62.

[6] P. Clément, B. de Pagter, G. Sweers & F. de Thélin, "Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces", *Mediterr. J. Math.* **1** (2004), no. 3, p. 241-267.

[7] G. Dankert, "Sobolev Embedding Theorems in Orlicz Spaces", PhD Thesis, University of Köln, 1966, Ph. D. Thesis.

[8] L. Diening, "Theorical and numerical results for electrorheological fluids", PhD Thesis, University of Freiburg, 2002, Ph. D. Thesis.

[9] ———, "Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces", *Bull. Sci. Math.* **129** (2005), no. 8, p. 657-700.

[10] T. K. Donaldson & N. S. Trudinger, "Orlicz-Sobolev spaces and imbedding theorems", *J. Functional Analysis* **8** (1971), p. 52-75.

[11] D. E. Edmunds, J. Lang & A. Nekvinda, "On $L^{p(x)}$ norms", *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **455** (1999), no. 1981, p. 219-225.

[12] D. E. Edmunds & J. Rákosník, "Density of smooth functions in $W^{k,p(x)}(\Omega)$", *Proc. Roy. Soc. London Ser. A* **437** (1992), no. 1899, p. 229-236.

[13] ———, "Sobolev embeddings with variable exponent", *Studia Math.* **143** (2000), no. 3, p. 267-293.

[14] I. Ekeland, "On the variational principle", *J. Math. Anal. Appl.* **47** (1974), p. 324-353.

[15] X. Fan, J. Shen & D. Zhao, "Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$", *J. Math. Anal. Appl.* **262** (2001), no. 2, p. 749-760.

[16] X. Fan & D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$", *J. Math. Anal. Appl.* **263** (2001), no. 2, p. 424-446.

[17] T. C. Halsey, "Electrorheological Fluids", *Science* **258** (1992), no. 5083, p. 761 - 766.

[18] O. Kováčik & J. Rákosník, "On spaces $L^{p(x)}$ and $W^{k,p(x)}$", *Czechoslovak Math. J.* **41(116)** (1991), no. 4, p. 592-618.

[19] J. Lamperti, "On the isometries of certain function-spaces", *Pacific J. Math.* **8** (1958), p. 459-466.

[20] P. Marcellini, "Regularity and existence of solutions of elliptic equations with p, q-growth conditions", *J. Differential Equations* **90** (1991), no. 1, p. 1-30.

[21] M. Mihăilescu, P. Pucci & V. Rădulescu, "Nonhomogeneous boundary value problems in anisotropic Sobolev spaces", *C. R. Math. Acad. Sci. Paris* **345** (2007), no. 10, p. 561-566.

[22] M. Mihăilescu & V. Rădulescu, "A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids", *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **462** (2006), no. 2073, p. 2625-2641.

[23] ———, "Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz-Sobolev space setting", *J. Math. Anal. Appl.* **330** (2007), no. 1, p. 416-432.

[24] ———, "On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent", *Proc. Amer. Math. Soc.* **135** (2007), no. 9, p. 2929-2937 (electronic).

[25] ———, "Continuous spectrum for a class of nonhomogeneous differential operators", *Manuscripta Math.* **125** (2008), no. 2, p. 157-167.

[26] J. Musielak & W. Orlicz, "On modular spaces", *Studia Math.* **18** (1959), p. 49-65.

[27] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, vol. 1034, Springer-Verlag, Berlin, 1983, iii+222 pages.

[28] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen Co. Ltd., Tokyo, 1950, i+288 pages.

[29] R. O'Neill, "Fractional integration in Orlicz spaces", *Trans. Amer. Math. Soc.* **115** (1965), p. 300-328.

[30] W. Orlicz, "Über konjugierte Exponentenfolgen", *Studia Math.* **3** (1931), p. 200-211.

[31] K. R. Rajagopal & M. Růžička, "Mathematical modelling of electrorheological fluids", *Cont. Mech. Term.* **13** (2001), p. 59-78.

[32] M. Růžička, "Electrorheological fluids: modeling and mathematical theory", *Sūrikaisekikenkyūsho Kōkyūroku* (2000), no. 1146, p. 16-38, Mathematical analysis of liquids and gases (Japanese) (Kyoto, 1999).

[33] M. Struwe, *Variational methods*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 34, Springer-Verlag, Berlin, 1996, Applications to nonlinear partial differential equations and Hamiltonian systems, xvi+272 pages.

[34] V. V. Zhikov, "Averaging of functionals of the calculus of variations and elasticity theory", *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), no. 4, p. 675-710, 877.

[35] ———, "Meyer-type estimates for solving the nonlinear Stokes system", *Differ. Uravn.* **33** (1997), no. 1, p. 107-114, 143.

Mihai MIHĂILESCU
University of Craiova
Department of Mathematics
200585 Craiova (Romania)
Central European University
Department of Mathematics
1051 Budapest (Hungary)
mmihailes@yahoo.com

Vicențiu RĂDULESCU
Institute of Mathematics "Simion Stoilow" of the Romanian Academy
P.O. Box 1-764
014700 Bucharest (Romania)
University of Craiova
Department of Mathematics
200585 Craiova (Romania)
vicentiu.radulescu@math.cnrs.fr