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An arithmetic Riemann-Roch theorem in higher degrees

AN ARITHMETIC RIEMANN-ROCH THEOREM IN HIGHER DEGREES

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ABSTRACT. — We prove an analog in Arakelov geometry of the Grothendieck-Riemann-Roch theorem.

RÉSUMÉ. — Nous démontrons un analogue du théorème de Grothendieck-Riemann-Roch en géométrie d'Arakelov.

1. Introduction

Recall that the Grothendieck-Riemann-Roch theorem (see for instance [14, par. 20.1]) says that, if Y and B are regular schemes which are quasi-projective and flat over the spectrum S of a Dedekind domain and $g : Y \to B$ is a flat and projective S-morphism, then the diagram

$$
\begin{array}{ccc}
K_0(Y) & \xrightarrow{\mathrm{Td}(g)\cdot\mathrm{ch}} & \mathrm{CH}^\cdot(Y)_{\mathbb{Q}} \\
\Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle g_*} \qquad \text{(GRR)} \\
K_0(B) & \xrightarrow{\quad\mathrm{ch}\quad} & \mathrm{CH}^\cdot(B)_{\mathbb{Q}}
\end{array}
$$

commutes. Here $K_0(Y)$ (resp. $K_0(B)$) is the Grothendieck group of locally free sheaves on Y (resp. on B). The group $\mathrm{CH}^\cdot(Y)$ (resp. $\mathrm{CH}^\cdot(B)$) is the Chow group of cycles modulo rational equivalence on Y (resp. B). The symbol g_* refers to the push-forward map in the corresponding theory. The symbol ch refers to the Chern character, which on each regular and quasi-projective S-scheme is a ring morphism from the Grothendieck group to the Chow group tensored with $\mathbb{Q}$. The element $\mathrm{Td}(g)$ is the Todd class of

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the virtual relative tangent bundle. In words, the Grothendieck-Riemann-Roch theorem implies that the Chern character does not commute with the push-forward maps but that this commutation can be obtained after multiplication of the Chern character with the Todd class of the virtual relative tangent bundle.

All the objects mentionned in the previous paragraph have extensions to Arakelov theory. Arakelov theory is an extension of scheme-theoretic geometry over the integers, where everything in sight is equipped with an analytic datum on the complex points of the scheme. This means that there will be "forgetful" maps from the Arakelov-theoretic Grothendieck groups, Chow groups etc. to the corresponding classical objects. We refer to [21] and [24] for an introduction to this subject, which originated in Arakelov's paper [1] and was later further developped by several people.

Now keep the same hypotheses as in the diagram (GRR) but suppose in addition that $S = \operatorname{Spec} \mathbb{Z}$ and that g is smooth over $\mathbb{Q}$. We shall show that there exists a commutative diagram

$$
\begin{array}{ccc}
\widehat{K}_0(Y) & \xrightarrow{\widehat{\operatorname{Td}}(g)\cdot(1-R(Tg_{\mathbb{C}}))\cdot\widehat{\operatorname{ch}}} & \widehat{\operatorname{CH}}^{\cdot}(Y)_{\mathbb{Q}} \\
g_* \downarrow & & \downarrow g_* \\
\widehat{K}_0(B) & \xrightarrow{\widehat{\operatorname{ch}}} & \widehat{\operatorname{CH}}^{\cdot}(B)_{\mathbb{Q}}
\end{array}
\qquad \text{(ARR)}
$$

where the objects with hats ($\widehat{\cdot}$) are the extensions of the corresponding objects to Arakelov theory. The class $1 - R(Tg_{\mathbb{C}})$ is an exotic cohomology class which has no classical analog. This diagram fits in a three-dimensional commutative diagram

where the various forgetful arrows $\twoheadrightarrow$ are surjective and their kernels are spaces of differential forms on the complex points of the corresponding schemes. The assertion that the diagram (ARR) commutes shall henceforth be referred to as the *arithmetic Riemann-Roch theorem*. The precise statement is given in Theorem 3.2 below.

Our proof of the arithmetic Riemann-Roch theorem combines the classical technique of proof of the Grothendieck-Riemann-Roch theorem with deep results of Bismut and his coworkers in local index theory.

The history of the previous work on this theorem and its variants is as follows. In [12] Faltings proved a variant of the theorem for surfaces. In [19, Th. 7, ii)], Gillet and Soulé proved a degree one version of the theorem (see after Theorem 3.2 for a precise statement). In his book [13], Faltings outlined an approach to the proof of Theorem 3.2, which is not based on Bismut's work. In [23, par. 8], Rössler proved a variant of Theorem 3.2, where the Chow groups are replaced by graded $\widehat{K}_0$-groups (in the spirit of [3]). This variant is a formal consequence of the Riemann-Roch theorem for the Adams operations acting on arithmetic K_0-groups, which is the main result of [23]. Finally, in his unpublished thesis [25], Zha obtained a general arithmetic Riemann-Roch theorem which does not involve analytic torsion.

A variant of Theorem 3.2 was conjectured in [15, Conjecture 3.3]. That conjecture is a variant of Theorem 3.2 in the sense that a definition of the push-forward map is used there which may differ from the one used here. A precise comparison has yet to be made. In the present setting all the morphisms are local complete intersections, because all the schemes are assumed to be regular; it is an open problem is to allow singularities at finite places and/or to allow more general morphisms. This problem is solved in degree 1 in [19, Th. 7, i)]. Note that arithmetic Chow theory (like ordinary Chow theory) is not defined outside the category of regular schemes and to tackle the problem of a Riemann-Roch theorem for singular schemes, one has to first extend that theory.

The structure of the article is the following. In the second section, we recall the definitions of the Grothendieck and Chow groups in Arakelov theory and some of their basic properties. In the third section, we formulate the arithmetic Riemann-Roch theorem. In the fourth section, we give a proof of the latter theorem. See the beginning of that section for a description of the structure of the proof.

2. Arithmetic Grothendieck and Chow groups

In this section, we shall define extensions of the classical Grothendieck and Chow groups to the framework of Arakelov theory.

Let X be a regular scheme, which is quasi-projective and flat over $\mathbb{Z}$. We shall call such a scheme an *arithmetic variety* (this definition is more restrictive than the definition given in [16, sec. 3.2]). Complex conjugation induces an antiholomorphic automorphism F_∞ on the manifold of complex points $X(\mathbb{C})$ of X. We shall write $A^{p,p}(X)$ for the set of real differential forms ω of type p, p on $X(\mathbb{C})$, which satisfy the equation $F_\infty^* \omega = (-1)^p \omega$ and we shall write $Z^{p,p}(X) \subseteq A^{p,p}(X)$ for the kernel of the operation $d = \partial + \overline{\partial}$. We also define $\widetilde{A}(X) := \bigoplus_{p \geqslant 0}(A^{p,p}(X)/(\mathrm{Im}\ \partial + \mathrm{Im}\ \overline{\partial}))$ and $Z(X) := \bigoplus_{p \geqslant 0} Z^{p,p}(X)$. A *hermitian bundle* $\overline{E} = (E, h^E)$ is a vector bundle E on X, endowed with a hermitian metric h^E, which is invariant under F_∞, on the holomorphic bundle $E_\mathbb{C}$ on $X(\mathbb{C})$ associated to E. We denote by $\mathrm{ch}(\overline{E})$ (resp. $\mathrm{Td}(\overline{E})$) the representative of the Chern character (resp. Todd class) of $E_\mathbb{C}$ associated by the formulae of Chern-Weil to the hermitian connection of type $(1,0)$ defined by h^E. Let

$$\mathcal{E} : 0 \to E' \to E \to E'' \to 0$$

be an exact sequence of vector bundles on X. We shall write $\overline{\mathcal{E}}$ for the sequence $\mathcal{E}$ and hermitian metrics on $E'_\mathbb{C}$, $E_\mathbb{C}$ and $E''_\mathbb{C}$ (invariant under F_∞). To $\overline{\mathcal{E}}$ is associated a secondary, or Bott-Chern class $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) \in \widetilde{A}(X)$ (resp. $\widetilde{\mathrm{Td}}(\overline{\mathcal{E}})$). This secondary class satisfies the equation

$$\frac{i}{2\pi} \partial \overline{\partial}(\widetilde{\mathrm{ch}}(\overline{\mathcal{E}})) = \mathrm{ch}(\overline{E}' \oplus \overline{E}'') - \mathrm{ch}(\overline{E})$$

(resp.

$$\frac{i}{2\pi} \partial \overline{\partial}(\widetilde{\mathrm{Td}}(\overline{\mathcal{E}})) = \mathrm{Td}(\overline{E}' \oplus \overline{E}'') - \mathrm{Td}(\overline{E})\ ).$$

Here we write $\overline{E}' \oplus \overline{E}''$ for the hermitian bundle $(E' \oplus E'', h' \oplus h'')$, which is the orthogonal direct sum of the hermitian bundles $\overline{E}'$ and $\overline{E}''$. For the definition of the secondary classes, we refer to [6, Par. f)].

DEFINITION 2.1 ([17, section 6]). — *The arithmetic Grothendieck group* $\widehat{K}_0(X)$ *associated to* X *is the abelian group generated by* $\widetilde{A}(X)$ *and the isometry classes of hermitian bundles on* X, *with the following relations:*

 - $\widetilde{\mathrm{ch}}(\overline{\mathcal{E}}) = \overline{E}' - \overline{E} + \overline{E}''$ *for every exact sequence* $\overline{\mathcal{E}}$ *as above;*
 - $\eta = \eta' + \eta''$ *if* $\eta \in \widetilde{A}(X)$ *is the sum of two elements* η' *and* η''.

Notice that, by construction, there is an exact sequence of abelian groups

$$\widetilde{A}(X) \to \widehat{K}_0(X) \to K_0(X) \to 0$$

where the "forgetful" map $\widehat{K}_0(X) \to K_0(X)$ sends a hermitian bundle onto its underlying locally free sheaf and sends an element of $\widetilde{A}(X)$ to 0.

We shall now define a commutative ring structure on $\widehat{K}_0(X)$. To this end, let us consider the group $\Gamma(X) := Z(X) \oplus \widetilde{A}(X)$. We equip it with the $\mathbb{N}$-grading whose term of degree p is $Z^{p,p}(X) \oplus \widetilde{A}^{p-1,p-1}(X)$ if $p \geqslant 1$ and $Z^{0,0}(X)$ if $p = 0$. We define an $\mathbb{R}$-bilinear map $*$ from $\Gamma(X) \times \Gamma(X)$ to $\Gamma(X)$ via the formula

$$(\omega, \eta) * (\omega', \eta') = (\omega \wedge \omega', \omega \wedge \eta' + \eta \wedge \omega' + (\frac{i}{2\pi}\partial\bar{\partial}\eta) \wedge \eta').$$

This map endows $\Gamma(X)$ with the structure of a commutative graded $\mathbb{R}$-algebra (cf. [17, Lemma 7.3.1, p. 233]). Now let $\overline{E} + \eta$ and $\overline{E}' + \eta'$ be two generators of $\widehat{K}_0(X)$; we define

$$(\overline{E} + \eta) \otimes (\overline{E}' + \eta') := \overline{E} \otimes \overline{E}' + [(\mathrm{ch}(\overline{E}), \eta) * (\mathrm{ch}(\overline{E}'), \eta')].$$

Here [.] refers to the projection on the second component of $\Gamma(X)$. Gillet and Soulé have shown in [17, Th. 7.3.2] that $\otimes$ is compatible with the defining relations of $\widehat{K}_0(X)$ and defines a commutative ring structure on $\widehat{K}_0(X)$.

Now let Y be another arithmetic variety. Let $f : X \to Y$ be any morphism. If $(E, h^E) + \eta$ is a generator of $\widehat{K}_0(Y)$, we define

$$f^*((E, h^E) + \eta) = (f^*(E), f_{\mathbb{C}}^* h^E) + f_{\mathbb{C}}^*(\eta),$$

where $f_{\mathbb{C}}^*(\eta)$ is the pull-back of η by $f_{\mathbb{C}}$ as a differential form. It follows from the definitions that the just defined map f^* descends to a morphism of commutative rings

$$f^* : \widehat{K}_0(Y) \to \widehat{K}_0(X).$$

We shall call this morphism the *pull-back map* associated to f.

We now turn to arithmetic Chow groups. We shall write $D^{p,p}(X)$ for the space of real currents of type p, p on $X(\mathbb{C})$ on which F_∞^* acts by multiplication by $(-1)^p$. If Z is a p-cycle on X, a Green current g_Z for Z is an element of $D^{p,p}(X)$ which satisfies the equation

$$\frac{i}{2\pi}\partial\bar{\partial}g_Z + \delta_{Z(\mathbb{C})} = \omega_Z$$

where ω_Z is a differential form and $\delta_{Z(\mathbb{C})}$ is the Dirac current associated to $Z(\mathbb{C})$.

DEFINITION 2.2 ([16, section 3]). — *The arithmetic Chow group* $\widehat{\mathrm{CH}}^p(X)$ *is the abelian group generated by the ordered pairs* (Z, g_Z), *where* Z *is a* p-*cycle on* X *and* g_Z *is a Green current for* $Z(\mathbb{C})$, *with the following relations:*

- $(Z, g_Z) + (Z', g_{Z'}) = (Z + Z', g_Z + g_{Z'})$;
- $(\mathrm{div}(f), -\log |f|^2 + \partial u + \bar{\partial} v) = 0$

where f *is a non-zero rational function defined on a closed integral subscheme of codimension* $p - 1$ *in* X *and* u *(resp.* v*) is a complex current of type* $(p - 2, p - 1)$ *(resp.* $(p - 1, p - 2)$*).*

We shall write $\widehat{\mathrm{CH}}^{\cdot}(X)$ for the direct sum $\oplus_{p \geqslant 0} \widehat{\mathrm{CH}}^p(X)$. There is by construction a morphism of groups $\omega : \widehat{\mathrm{CH}}^{\cdot}(X) \to Z(X)$, given by the formula $\omega((Z, g_Z)) := \omega_Z$. As for the arithmetic Grothendieck group, there is a natural exact sequence

$$\widetilde{A}(X) \to \widehat{\mathrm{CH}}^{\cdot}(X) \to \mathrm{CH}^{\cdot}(X) \to 0$$

where the "forgetful" map $\widehat{\mathrm{CH}}^{\cdot}(X) \to \mathrm{CH}^{\cdot}(X)$ sends a pair (Z, g_Z) (as above) on Z.

The maps $\widetilde{A}(X) \to \widehat{\mathrm{CH}}^{\cdot}(X)$ and $\widetilde{A}(X) \to \widehat{\mathrm{CH}}^{\cdot}(X)$ are usually both denoted by the letter a. To lighten formulae, we shall usually drop that letter in our computations. This is in the spirit of ordinary K_0-theory, where the brackets $[\cdot]$ (which map an object into the Grothendieck group) are often dropped in computations.

The group $\widehat{\mathrm{CH}}(X)_{\mathbb{Q}}$ is equipped with a $\mathbb{Z}$-bilinear pairing $\cdot$, such that

$$(Z, g_Z) \cdot (Z', g_{Z'}) = (Z \cap Z', g_Z \wedge \delta_{Z'(\mathbb{C})} + \omega_Z \wedge g_{Z'})$$

if Z, Z' are integral and meet properly in X; the multiplicity of each component in $Z \cap Z'$ is given by Serre's Tor formula. See [16, Th. 4.2.3] for the definition of the pairing in general. It is proven in [16] and [20] that this pairing makes the group $\widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}}$ into a commutative $\mathbb{N}$-graded ring (the reference [20] fills a gap in [16]). Let now $f : X \to Y$ be a morphism of arithmetic varieties. We can associate to f a *pull-back map*

$$f^* : \widehat{\mathrm{CH}}^{\cdot}(Y)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}}.$$

which is a morphism of $\mathbb{N}$-graded rings. We shall describe this map under the hypothesis that f is smooth over $\mathbb{Q}$ and flat. Under this hypothesis, let Z be a p-cycle on Y and let g_Z be a Green current for Z. Write f^*Z for the pull-back of Z to X and f^*g_Z for the pull-back of g_Z to $X(\mathbb{C})$ as a current (which exists because $f_{\mathbb{C}}$ is smooth). The rule which associates the pair (f^*Z, f^*g_Z) to the pair (Z, g_Z) descends to a morphism of abelian

groups $\widehat{\mathrm{CH}}^p(Y) \to \widehat{\mathrm{CH}}^p(X)$. The induced morphism $\widehat{\mathrm{CH}}^p(Y)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$ is the pull-back map. See [16, sec. 4.4].

There is a unique ring morphism

$$\widehat{\mathrm{ch}} : \widehat{K}_0(X) \to \widehat{\mathrm{CH}}^{\cdot}(X)_{\mathbb{Q}}$$

commuting with pull-back maps and such that

(ch-1): the formula $\widehat{\mathrm{ch}}(\eta) = (0, \eta)$ holds, if $\eta \in \widetilde{A}(X)$;

(ch-2): the formula $\widehat{\mathrm{ch}}(\overline{L}) = \exp(\widehat{\mathrm{c}}_1(\overline{L}))$ holds, if $\overline{L} = (L, h^L)$ is a hermitian line bundle on X;

(ch-3): the formula $\omega(\widehat{\mathrm{ch}}(\overline{E})) = \mathrm{ch}(\overline{E})$ holds for any hermitian vector bundle $\overline{E}$ on X.

Here the first Chern class $\widehat{\mathrm{c}}_1(\overline{L}) \in \widehat{\mathrm{CH}}^1(X)$ of a hermitian line bundle $\overline{L}$ is defined as the class of $(\mathrm{div}\,(s), -\log\,h^L(s, s))$ for any choice of a rational section s of L over X. The fact that $-\log\,h^L(s, s)$ is a Green current for $\mathrm{div}\,(s)$ is implied by the Poincaré-Lelong formula (see [22]). The morphism $\widehat{\mathrm{ch}}$ is called the *arithmetic Chern character* and is compatible with the traditional Chern character $K_0(X) \to \mathrm{CH}^{\cdot}(X)_{\mathbb{Q}}$ via the forgetful maps. See [17, sec. 7.2] for a proof of the existence and unicity of $\widehat{\mathrm{ch}}$.

The Todd class $\widehat{\mathrm{Td}}(\overline{E})$ of a hermitian vector bundle $\overline{E}$ is defined similarly. It commutes with pull back maps and is multiplicative :

$$\widehat{\mathrm{Td}}(\overline{E}' \oplus \overline{E}'') = \widehat{\mathrm{Td}}(\overline{E}')\widehat{\mathrm{Td}}(\overline{E}'')\,.$$

If $\overline{L}$ is a hermitian line bundle, the formula

$$\widehat{\mathrm{Td}}(\overline{L}) = \mathrm{td}(\widehat{\mathrm{c}}_1(\overline{L}))$$

holds, where $\mathrm{td}(x)$ is the formal power series $\mathrm{td}(x) = \frac{xe^x}{e^x-1} \in \mathbb{Q}[[x]]$. The arithmetic Todd class is compatible with the usual Todd class via the forgetful maps. Furthermore, it has the following properties:

(Td-1): if $\overline{E}$ is a hermitian bundle on X, then

$$\omega(\widehat{\mathrm{Td}}(\overline{E})) = \mathrm{Td}(\overline{E});$$

(Td-2): if $\overline{\mathcal{E}}$ is an exact sequence of hermitian bundles on X as in Definition 2.1, then

$$\widehat{\mathrm{Td}}(\overline{E}' \oplus \overline{E}'') - \widehat{\mathrm{Td}}(\overline{E}) = \widetilde{\mathrm{Td}}(\overline{\mathcal{E}}).$$

For $\widehat{\mathrm{Td}}$ to be uniquely defined, one still needs to give an expression of the Todd class of the tensor product $\overline{E} \otimes \overline{L}$ of a hermitian vector bundle with a hermitian line bundle. See [17, Th. 4.1, Th 4.8, par. 4.9] for a proof.

3. The statement

Recall that an *arithmetic variety* denotes a regular scheme, which is quasi-projective and flat over $\mathbb{Z}$. Let $g : Y \to B$ be a projective, flat morphism of arithmetic varieties, which is smooth over $\mathbb{Q}$ (abbreviated p.f.s.r.).

We shall first define a push-forward map $g_* : \widehat{K}_0(Y) \to \widehat{K}_0(B)$. To this end, fix a conjugation invariant Kähler metric h_Y on $Y(\mathbb{C})$. Denote by ω_Y the corresponding Kähler form, given by the formula

$$\omega_Y = i \sum_{\alpha,\beta} h_Y \left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta} \right) dz_\alpha \, d\bar{z}_\beta$$

for any choice (z_α) of local holomorphic coordinates. Let (E, h^E) be a hermitian bundle on Y, such that E is g-acyclic. This means that $R^k g_* E = 0$ if $k > 0$ or equivalently that $H^k(Y_{\bar{b}}, E_{\bar{b}}) = 0$ if $k > 0$, for any geometric point $\bar{b} \to B$. The sheaf of modules $R^0 g_* E$ is then locally free by the semi-continuity theorem. Furthermore, in the holomorphic category, the natural map

$$R^0 g_{\mathbb{C}*}(E_\mathbb{C})_b \to H^0(Y(\mathbb{C})_b, E(\mathbb{C})|_{Y(\mathbb{C})_b})$$

is then an isomorphism for every point $b \in B(\mathbb{C})$. Here $Y(\mathbb{C})_b$ denotes the (analytic) fiber of the morphism $g_\mathbb{C}$ above b. For every $b \in B(\mathbb{C})$, we endow $H^0(Y(\mathbb{C})_b, E(\mathbb{C})|_{Y(\mathbb{C})_b})$ with the hermitian metric given by the formula

$$\langle s, t \rangle_{L^2} := \frac{1}{(2\pi)^{d_b}} \int_{Y(\mathbb{C})_b} h^E(s, t) \frac{\omega_Y^{d_b}}{d_b!}$$

where $d_b := \dim(Y(\mathbb{C})_b)$. It can be shown that these metrics depend on b in a C^∞ manner (see [2, p. 278]) and thus define a hermitian metric on $(R^0 g_* E)_\mathbb{C}$. We shall write $g_* h^E$ for this hermitian metric; it is called the L^2-metric (obtained from $g_\mathbb{C}$, h^E and h_Y). Apart from that, we shall write $T(h_Y, h^E)$ for the higher analytic torsion form determined by (E, h^E), $g_\mathbb{C}$ and h_Y. The higher analytic torsion form is an element of $\widetilde{A}(B)$, which satisfies the equality

$$\frac{i}{2\pi} \partial\bar{\partial} T(h_Y, h^E) = \mathrm{ch}((R^0 g_* E, g_* h^E)) - \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{Td}(\overline{Tg}_\mathbb{C})\mathrm{ch}(\overline{E}),$$

where $\overline{Tg}_\mathbb{C}$ is the tangent bundle relatively to $g_\mathbb{C}$, endowed with the hermitian metric induced by h_Y. For the definition of $T(h_Y, h^E)$ and for the proof of the last equality, we refer to [10]. In [23, Prop. 3.1], it is shown that there is a unique group morphism

$$g_* : \widehat{K}_0(Y) \to \widehat{K}_0(B)$$

such that

$$g_*((E, h^E) + \eta) = (R^0 g_* E, g_* h^E) - T(h_Y, h^E) + \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{Td}(\overline{Tg_{\mathbb{C}}})\eta,$$

where $\eta \in \widetilde{A}(Y)$ and (E, h^E) is a hermitian bundle as above on Y. We shall call the morphism g_* the *push-forward map* associated to g and h_Y.

There is also a *push-forward map*

$$g_* : \widehat{\mathrm{CH}}^{\cdot}(Y) \to \widehat{\mathrm{CH}}^{\cdot}(B).$$

This map is uniquely characterised by the fact that it is a group morphism and by the fact that

$$g_*((Z, g_Z)) = (\deg(Z/g(Z))g(Z), \int_{Y(\mathbb{C})/B(\mathbb{C})} g_Z)$$

for every integral closed subscheme Z of Y and for every Green current g_Z of Z. Here $\deg(Z/g(Z))$ is the degree of the corresponding extension of function fields if $\dim(g(Z)) = \dim(Z)$ and $\deg(Z/g(Z)) = 0$ otherwise.

Notice that this push-forward map does not depend on the choice of a Kähler metric on $Y(\mathbb{C})$, unlike the push-forward map for arithmetic Grothendieck groups.

Let now

$$Y \xrightarrow{\;i\;} P$$

with g to B and f from P to B
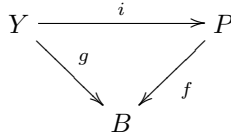
be a factorisation of g into a closed immersion i and a projective smooth morphism f. Let N be the normal bundle of the immersion i. Let

$$\mathcal{N} : 0 \to Tg_{\mathbb{C}} \to Tf_{\mathbb{C}} \to N_{\mathbb{C}} \to 0$$

be the exact sequence associated to $i_{\mathbb{C}}$. Endow as before $Tg_{\mathbb{C}}$ with the metric induced by h_Y. Endow $Tf_{\mathbb{C}}$ with some (not necessarily Kähler) hermitian metric extending the metric on $Tg_{\mathbb{C}}$ and endow $N_{\mathbb{C}}$ with the resulting quotient metric. These choices being made, we define

$$\widehat{\mathrm{Td}}(g) = \widehat{\mathrm{Td}}(g, h_Y) := \widehat{\mathrm{Td}}(i^* \overline{Tf}) \cdot \widehat{\mathrm{Td}}^{-1}(\overline{N}) + \widetilde{\mathrm{Td}}(\overline{\mathcal{N}})\mathrm{Td}(\overline{N})^{-1} \in \widehat{\mathrm{CH}}^{\cdot}(Y)_{\mathbb{Q}}.$$

It is shown in [19, Prop. 1, par. 2.6.2] that the element $\widehat{\mathrm{Td}}(g)$ depends only on g and on the restriction of h_Y to $Tg_{\mathbb{C}}$.

Before we state the Riemann-Roch theorem, we still have to define a characteristic class.

DEFINITION 3.1 ([15, 1.2.3, p. 25]). — *The R-genus is the unique addi-tive characteristic class defined for a line bundle L by the formula*

$$R(L) = \sum_{m \text{ odd}, \geqslant 1} (2\zeta'(-m) + \zeta(-m)(1 + \frac{1}{2} + \cdots + \frac{1}{m}))c_1(L)^m/m!$$

where $\zeta(s)$ is the Riemann zeta function.

In the definition 3.1, it is understood as usual that R is defined for any C^∞-vector bundle on a C^∞-manifold and that it has values in ordinary (de Rham) cohomology with complex coefficients. This being said, let

$$H_{\text{red}}^{\cdot}(Y) := \sum_{p \geqslant 0} Z^{p,p}(Y)/(A^{p,p}(Y) \cap \text{Im } d).$$

By construction, there is an inclusion $H_{\text{red}}^{\cdot}(Y) \subset H^{\cdot}(Y(\mathbb{C}), \mathbb{C})$. If E is a vector bundle on Y, then $R(E_\mathbb{C})$ can be computed via the formulae of Chern-Weil using a connection of type $(1,0)$. The local curvature matrices associated to such connections are of type $(1,1)$; this shows that $R(E_\mathbb{C}) \in H_{\text{red}}^{\cdot}(Y)$. On the other hand, also by construction, there is a natural map $H_{\text{red}}^{\cdot}(Y) \to \widetilde{A}(Y)$. Hence we may (and shall) consider that $R(E_\mathbb{C}) \in \widetilde{A}(Y)$. Similar remarks apply to any other characteristic class.

THEOREM 3.2 (arithmetic Riemann-Roch theorem). — *Let $y \in \widehat{K}_0(Y)$. The equality*

$$\widehat{\text{ch}}(g_*(y)) = g_*(\widehat{\text{Td}}(g) \cdot (1 - a(R(Tg_\mathbb{C}))) \cdot \widehat{\text{ch}}(y))$$

holds in $\widehat{\text{CH}}^{\cdot}(B)_\mathbb{Q}$.

In [19], it is proved that the equality in Theorem 3.2 holds after projection of both sides of the equality on $\widehat{\text{CH}}^1(B)_\mathbb{Q}$.

4. The Proof

In this section, we shall prove Theorem 3.2. The structure of the proof is as follows. In the first subsection, we prove various properties of the (putatively non vanishing) difference between the two sides of the asserted equality. Let us call this difference the error term. We first prove that the error term is independent of all the involved hermitian metrics (Lemmata 4.4, 4.5 and 4.6), using Bismut-Koehler's anomaly formulae for the analytic torsion form. We then proceed to prove that it is invariant under immersions (Theorem 4.7). The proof of this fact relies on two difficult results, which are proved elsewhere: the arithmetic Riemann-Roch for closed immersions

(Theorem 4.1), which is a generalisation of Arakelov's adjunction formula and Bismut's immersion formula (Theorem 4.2). This last result is the most difficult part of the proof of the arithmetic Riemann-Roch theorem and is of a purely analytic nature. In the third section, we show that the error term vanishes in the special case of relative projective spaces; this is is shown to be either the consequence of the article [19], where the arithmetic Riemann-Roch theorem is proved in degree 1 or of the more recent article by Bost [11], where explicit resolutions of the diagonal are used. Finally, in the third subsection, we show that the error term always vanishes. This is achieved by reduction to the case of relative projective spaces, using the invariance of the error term under immersions.

4.1. Properties of the error term

Before beginning with the study of the properties of the error term of the arithmetic Riemann-Roch theorem, we shall recall a few results on direct images in arithmetic Chow and K_0-theory.

Let $i : Y \hookrightarrow P$ be a closed immersion of arithmetic varieties. Let η be a locally free sheaf on Y and let

$$\Xi : \ 0 \to \xi_m \to \xi_{m-1} \to \ldots \xi_0 \to i_*\eta \to 0$$

be a resolution of $i_*\eta$ by locally free sheaves on P. Denote by N be the normal bundle of the immersion i. Let $F := \oplus_{l=0}^m H^l(i^*\Xi)$. There is a canonical isomorphism of graded bundles $F \simeq \oplus_{l=0}^{rk(N)} \Lambda^l(N^\vee) \otimes \eta$ (see for instance [3, Lemme 2.4 and Prop. 2.5, i'), exposé VII]). Both of the latter graded bundles carry natural metrics, if N, η and the ξ_i are endowed with metrics. We shall say that hermitian metrics on the bundles ξ_i satisfy Bismut's assumption (A) with respect to the hermitian metrics on N and η if the isomorphism $i^*F \simeq \oplus_{i=0}^{rk(N)} \Lambda^i(N^\vee) \otimes \eta$ also identifies the metrics. It is proved in [4] that if metrics on N and η are given, there always exist metrics on the ξ_i such that this assumption is satisfied. We now equip N and η with arbitrary hermitian metrics and we suppose that the ξ_i are endowed with hermitian metrics such that Bismut's condition (A) is satisfied with respect to the metric on N and η. The *singular Bott-Chern current* of $\overline{\Xi}$ is an element $T(h^{\xi\cdot})$ of $\oplus_{p\geqslant 0} D^{p,p}(X)$ satisfying the equation

$$\frac{i}{2\pi}\partial\overline{\partial}T(h^{\xi\cdot}) = i_*(\mathrm{Td}^{-1}(\overline{N})\mathrm{ch}(\overline{\eta})) - \sum_{i=0}^m (-1)^i \mathrm{ch}(\overline{\xi}_i)$$

(see [4, Th. 2.5, p. 266]). Here i_* refers to the pushforward of currents.

We now suppose given a commutative diagram

where g is a p.f.s.r. morphism and f is projective and smooth. Endow $P(\mathbb{C})$ with a Kähler metric h_P and $Y(\mathbb{C})$ with the restricted metric h_Y. As before Definition 3.1, consider the sequence

$$\mathcal{N} : 0 \to Tg_{\mathbb{C}} \to Tf_{\mathbb{C}} \to N_{\mathbb{C}} \to 0.$$

Endow $Tf_{\mathbb{C}}$ (resp. $Tg_{\mathbb{C}}$) with the metric induced from h_P (resp. h_Y). With these conventions, we shall suppose from now on that the metric on $N_{\mathbb{C}}$ is the quotient metric induced from the map $Tf_{\mathbb{C}} \to N_{\mathbb{C}}$ in the sequence $\mathcal{N}$. The following result is proved in [9, Th. 4.13].

THEOREM 4.1 (arithmetic Riemann-Roch theorem for closed immersions).
Let $\alpha \in \widehat{CH}^{\cdot}(P)$. The current

$$\int_{P(\mathbb{C})/B(\mathbb{C})} \omega(\alpha) T(h^\xi)$$

is then a differential form and the equality

$$f_*(\alpha \cdot \widehat{ch}(\xi.)) = g_*(i^*(\alpha) \cdot \widehat{Td}^{-1}(\overline{N}) \cdot \widehat{ch}(\overline{\eta})) - \int_{P(\mathbb{C})/B(\mathbb{C})} \omega(\alpha) T(h^\xi)$$

is satisfied in $\widehat{CH}^{\cdot}(B)_{\mathbb{Q}}$.

Notice that if one applies the forgetful map to both sides of the last equality, one obtains a consequence of the Grothendieck-Riemann-Roch theorem for closed immersions.

Suppose from now on that the ξ_i are f-acyclic and that η is g-acyclic. The next theorem is a $\widehat{K}_0$-theoretic translation of a difficult result of Bismut, often called Bismut's immersion theorem. The translation is made in [23, Th. 6.6]. Bismut's immersion theorem is proved in [5].

THEOREM 4.2. — *The equality*

$$g_*(\overline{\eta}) - \sum_{i=0}^{m} (-1)^i f_*(\overline{\xi}_i) =$$

$$= \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{ch}(\eta_{\mathbb{C}}) R(N_{\mathbb{C}}) \mathrm{Td}(Tg_{\mathbb{C}}) + \int_{P(\mathbb{C})/B(\mathbb{C})} T(h^{\xi \cdot}) \mathrm{Td}(\overline{Tf})$$

$$+ \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{ch}(\overline{\eta}) \widetilde{\mathrm{Td}}(\overline{\mathcal{N}}) \mathrm{Td}^{-1}(\overline{N})$$

holds in $\widehat{K}_0(B)$.

Notice that the complex $R^0 f_*(\Xi)$ is exact with our hypotheses. Hence we have $g_*(\eta) - \sum_{i=0}^{m} (-1)^i f_*(\xi_i) = 0$ in $K_0(B)$. This is the equality to which the Theorem 4.2 reduces after application of the forgetful maps. We shall also need the following theorem, which studies the dependence of the analytic torsion form on h_Y:

THEOREM 4.3. — *Let* h'_Y *be another Kähler metric on* $Y(\mathbb{C})$. *Let* h'^{Tg} *be the metric induced on* $Tg_{\mathbb{C}}$ *by* h'_Y. *The identity*

$$T(h'_Y, h^\eta) - T(h_Y, h^\eta) = \widetilde{\mathrm{ch}}(g_*^{h_Y} h^\eta, g_*^{h'_Y} h^\eta) - \int_{Y(\mathbb{C})/B(\mathbb{C})} \widetilde{\mathrm{Td}}(h^{Tg}, h'^{Tg}) \mathrm{ch}(\overline{\eta}).$$

holds in $\widetilde{A}(B)$.

Here $\widetilde{\mathrm{Td}}(h^{Tg}, h'^{Tg})$ refers to the Todd secondary class of the sequence

$$0 \to 0 \to Tg_{\mathbb{C}} \to Tg_{\mathbb{C}} \to 0,$$

where the first non-zero term is endowed with the metric h^{Tg} and the second non-zero term with the metric h'^{Tg}. The term $\widetilde{\mathrm{ch}}(g_*^{h_Y} h^\eta, g_*^{h'_Y} h^\eta)$ is the Bott-Chern secondary class of the sequence

$$0 \to 0 \to R^0 g_* \eta \to R^0 g_* \eta \to 0,$$

where the first non-zero term carries the metric obtain by integration along the fibers with the volume form coming from h_Y and the second non-zero term the metric obtain by integration along the fibers with the volume form coming from h'_Y. For the proof, we refer to [10, Th. 3.10, p. 670].

We are now ready to study the *error term*

$$\delta(y, g, h_Y) := \widehat{\mathrm{ch}}(g_*(y)) - g_*(\widehat{\mathrm{Td}}(g) \cdot (1 - R(Tg_{\mathbb{C}})) \cdot \widehat{\mathrm{ch}}(y))$$

of the arithmetic Riemann-Roch theorem. Notice that by construction

$$\delta(y' + y'', g, h_Y) = \delta(y', g, h_Y) + \delta(y'', g, h_Y)$$

for all $y', y'' \in \widehat{K}_0(Y)$.

LEMMA 4.4. — $\delta(y, g, h_Y) = 0$ if y is represented by a differential form.

Proof. — This follows directly from the definitions.   □

LEMMA 4.5. — Let $\overline{E}, \overline{E}'$ be hermitian vector bundles on Y such that $E \simeq E'$. Then we have $\delta(\overline{E}, g, h_Y) = \delta(\overline{E}', g, h_Y)$.

Proof. — We have $\delta(\overline{E}, g) - \delta(\overline{E}', g) = \delta(\overline{E} - \overline{E}', g)$ and from the definition of arithmetic K_0-theory, the element $\overline{E} - \overline{E}'$ is represented by a differential form. Hence we can apply the last lemma.   □

LEMMA 4.6. — Let h'_Y be another Kähler metric on Y. Then

$$\delta(y, g, h_Y) = \delta(y, g, h'_Y).$$

Proof. — We shall use Theorem 4.3. Using Lemma 4.4 and the fact that every locally free sheaf on Y has a finite resolution by g-acyclic locally free sheaves, we see that we may assume without loss of generality that $y = \overline{E} = (E, h^E)$, where $\overline{E}$ a hermitian bundle on Y such that E is g-acyclic. We now compute

$\delta(\overline{E}, g, h_Y) - \delta(\overline{E}, g, h'_Y)$

$\overset{(1)}{=} \widehat{\mathrm{ch}}((g_* E, g_*^{h_Y} h^E)) - T(h_Y, h^E) - g_*(\widehat{\mathrm{Td}}(g, h_Y) \cdot (1 - R(Tg_{\mathbb{C}}))\widehat{\mathrm{ch}}(\overline{E}))$

$- \widehat{\mathrm{ch}}((g_* E, g_*^{h'_Y} h^E)) + T(h'_Y, h^E) + g_*(\widehat{\mathrm{Td}}(g, h'_Y) \cdot (1 - R(Tg_{\mathbb{C}}))\widehat{\mathrm{ch}}(\overline{E}))$

$\overset{(2)}{=} -\widetilde{\mathrm{ch}}(g_*^{h_Y} h^E, g_*^{h'_Y} h^E) + T(h'_Y, h^E) - T(h_Y, h^E)$

$+ g_*(\widetilde{\mathrm{Td}}(h^{Tg}, h'^{Tg}) \cdot (1 - R(Tg_{\mathbb{C}}))\widehat{\mathrm{ch}}(\overline{E}))$

$\overset{(3)}{=} T(h'_Y, h^E) - T(h_Y, h^E)$

$- \left(\widetilde{\mathrm{ch}}(g_*^{h_Y} h^E, g_*^{h'_Y} h^E) - \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{ch}(\overline{E})\widetilde{\mathrm{Td}}(h^{Tg}, h'^{Tg}) \right) \overset{(4)}{=} 0.$

The equality (1) follows from the definitions. The equality (2) is justified by the property (ch-1) of the arithmetic Chern character and by [19, par. 2.6.2, Prop.1,(ii)], which implies that

$$\widehat{\mathrm{Td}}(g, h'_Y) - \widehat{\mathrm{Td}}(g, h_Y) = \widetilde{\mathrm{Td}}(h^{Tg}, h'^{Tg})$$

(notice that this follows from (Td-2) if g is smooth). From the definition of the ring structure of $\widehat{\mathrm{CH}}(X)_{\mathbb{Q}}$ (see after Definition 2.2), we obtain (3). The equality (4) is the content of Theorem 4.3.   □

In view of the Lemma 4.6, we shall from now on drop the reference to the Kähler metric and write $\delta(y, g)$ for the error term. Notice that the Lemmata 4.4 and 4.5 imply that $\delta(y, g)$ depends only on the image of y in $K_0(Y)$. This justifies writing $\delta(E, g)$ for $\delta(\overline{E}, g)$ if $\overline{E}$ is a hermitian bundle on Y.

The following theorem studies the compatibility of the error term with the immersion i and is the core of the proof of the arithmetic Riemann-Roch theorem.

THEOREM 4.7. — *The equality*

$$\sum_{i=0}^{m}(-1)^i\delta(\xi_i, f) = \delta(\eta, g)$$

holds.

Proof. — Using Theorem 4.2, we compute

$$\sum_{i=0}^{m}(-1)^i\delta(\xi_i, f) = \sum_{i=0}^{m}(-1)^i(\widehat{\mathrm{ch}}(f_*(\overline{\xi}_i)) - f_*(\widehat{\mathrm{Td}}(\overline{Tf}) \cdot (1 - R(Tf_{\mathbb{C}})) \cdot \widehat{\mathrm{ch}}(\overline{\xi}_i)))$$

$$= \widehat{\mathrm{ch}}(g_*(\overline{\eta})) - \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{ch}(\eta_{\mathbb{C}})R(N_{\mathbb{C}})\mathrm{Td}(Tg_{\mathbb{C}}) - \int_{P(\mathbb{C})/B(\mathbb{C})} T(h^{\xi\cdot})\mathrm{Td}(\overline{Tf})$$

$$- \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{ch}(\overline{\eta})\widetilde{\mathrm{Td}}(\overline{\mathcal{N}})\mathrm{Td}^{-1}(\overline{N})$$

$$- \sum_{i=0}^{m}(-1)^i f_*(\widehat{\mathrm{Td}}(\overline{Tf}) \cdot \widehat{\mathrm{ch}}(\overline{\xi}_i)) + \sum_{i=0}^{m}(-1)^i \int_{P(\mathbb{C})/B(\mathbb{C})} \mathrm{Td}(Tf_{\mathbb{C}})\mathrm{ch}(\xi_{i,\mathbb{C}})R(Tf_{\mathbb{C}}).$$

Now by the definition of the arithmetic tangent element, we have

$$\widetilde{\mathrm{Td}}(\overline{\mathcal{N}})\mathrm{Td}^{-1}(\overline{N}) + \widehat{\mathrm{Td}}(i^*\overline{Tf}) \cdot \widehat{\mathrm{Td}}^{-1}(\overline{N}) = \widehat{\mathrm{Td}}(g)$$

and hence

$$\int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{ch}(\overline{\eta})\widetilde{\mathrm{Td}}(\overline{\mathcal{N}})\mathrm{Td}^{-1}(\overline{N}) =$$

$$g_*(\widehat{\mathrm{Td}}(g) \cdot \widehat{\mathrm{ch}}(\overline{\eta})) - g_*(\widehat{\mathrm{ch}}(\overline{\eta}) \cdot \widehat{\mathrm{Td}}(i^*\overline{Tf}) \cdot \widehat{\mathrm{Td}}^{-1}(\overline{N})).$$

Hence, using Theorem 4.1 with $\alpha = \widehat{\mathrm{Td}}(\overline{Tf})$ and property (Td-1) of the arithmetic Todd class, we obtain that

$$\sum_{i=0}^{m}(-1)^{i}\delta(\xi_{i},f) = \widehat{\mathrm{ch}}(g_{*}(\overline{\eta})) - g_{*}(\widehat{\mathrm{Td}}(g) \cdot \widehat{\mathrm{ch}}(\overline{\eta}))$$

$$- \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{ch}(\eta_{\mathbb{C}})R(N_{\mathbb{C}})\mathrm{Td}(Tg_{\mathbb{C}}) +$$

$$\sum_{i=0}^{m}(-1)^{i}\int_{P(\mathbb{C})/B(\mathbb{C})} \mathrm{Td}(Tf_{\mathbb{C}})\mathrm{ch}(\xi_{i,\mathbb{C}})R(Tf_{\mathbb{C}}).$$

Furthermore, using the Grothendieck-Riemann-Roch theorem with values in singular cohomology, we compute that

$$\sum_{i=0}^{m}(-1)^{i}\mathrm{ch}(\xi_{i,\mathbb{C}}) = i_{*}(\mathrm{Td}^{-1}(N_{\mathbb{C}})\mathrm{ch}(\eta_{\mathbb{C}}))$$

where i_{*} is the direct image in singular cohomology. Hence, using the multiplicativity of the Todd class and the additivity of the R-genus,

$$\sum_{i=0}^{m}(-1)^{i}\int_{P(\mathbb{C})/B(\mathbb{C})} \mathrm{Td}(Tf_{\mathbb{C}})\mathrm{ch}(\xi_{i,\mathbb{C}})R(Tf_{\mathbb{C}})$$

$$- \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{ch}(\eta_{\mathbb{C}})R(N_{\mathbb{C}})\mathrm{Td}(Tg_{\mathbb{C}})$$

$$= \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{ch}(\eta_{\mathbb{C}})\mathrm{Td}(Tg_{\mathbb{C}})(R(Tf_{\mathbb{C}}) - R(N_{\mathbb{C}}))$$

$$= \int_{Y(\mathbb{C})/B(\mathbb{C})} \mathrm{ch}(\eta_{\mathbb{C}})\mathrm{Td}(Tg_{\mathbb{C}})R(Tg_{\mathbb{C}}).$$

Thus

$$\sum_{i=0}^{m}(-1)^{i}\delta(\xi_{i},f) = \widehat{\mathrm{ch}}(g_{*}(\overline{\eta})) - g_{*}(\widehat{\mathrm{Td}}(g) \cdot (1 - R(Tg_{\mathbb{C}})) \cdot \widehat{\mathrm{ch}}(\overline{\eta})) = \delta(\eta,g)$$

which was the claim to be proved. $\qquad\square$

4.2. The case of relative projective spaces

Suppose now that B is an arithmetic variety and that $Y = \mathbb{P}_{B}^{r} \simeq \mathbb{P}_{\mathbb{Z}}^{r} \times B$ is some relative projective space over B ($r \geqslant 0$). Let $g : Y \to B$ (resp. $p : Y \to \mathbb{P}_{\mathbb{Z}}^{r}$) be the natural projection. Endow $Y(\mathbb{C})$ with the product h_{Y} of the standard Fubini-Study Kähler metric on $\mathbb{P}^{r}(\mathbb{C})$ with a fixed (conjugation invariant) Kähler metric on $B(\mathbb{C})$. Let $\omega_{\mathbb{P}^{r}}$ and ω_{B} be the corresponding

Kähler forms on $\mathbb{P}^r(\mathbb{C})$ and $B(\mathbb{C})$. Let $k \in \mathbb{Z}$ and let $\overline{\mathcal{O}}(k)$ be the k-th tensor power of the tautological bundle on $\mathbb{P}^r_{\mathbb{Z}}$, endowed with the Fubini-Study metric. We shall write $\tau(\overline{\mathcal{O}}(k))$ for the analytic torsion form of $\overline{\mathcal{O}}(k)$ with respect to the map from $\mathbb{P}(\mathbb{C})$ to the point. The form $\tau(\overline{\mathcal{O}}(k))$ is in this case a real number (which coincides with the Ray-Singer torsion of $\overline{\mathcal{O}}(k)_{\mathbb{C}}$). We shall need the

LEMMA 4.8. — *Let $\overline{E}$ be a hermitian vector bundle on B. Let $\overline{V} := g^*(\overline{E}) \otimes p^*(\overline{\mathcal{O}}(k))$. Then the equality*

$$T(h_Y, h^V) = \mathrm{ch}(\overline{E})\tau(\overline{\mathcal{O}}(k))$$

holds for the analytic torsion form $T(h_Y, h^V)$ of $\overline{V}$ with respect to g and h_Y.

Proof. — See [23, Lemma 7.15]. In that reference, it is assumed that $k >> 0$ but this assumption is not used in the proof and is thus not necessary. □

We shall also need the following projection formulae.

PROPOSITION 4.9. — *Let $b \in \widehat{K}_0(B)$ and $a \in \widehat{K}_0(Y)$. Then the projection formula*

$$g_*(a \otimes g^*(b)) = g_*(a) \otimes b$$

holds in $\widehat{K}_0(B)$.
Similarly, let $b \in \widehat{\mathrm{CH}}^{\cdot}(B)_{\mathbb{Q}}$ and $a \in \widehat{\mathrm{CH}}^{\cdot}(Y)_{\mathbb{Q}}$. Then the projection formula

$$g_*(a \cdot g^*(b)) = g_*(a) \cdot b$$

holds in $\widehat{\mathrm{CH}}^{\cdot}(B)_{\mathbb{Q}}$.

Proof. — For the first formula, see [23, Prop. 7.16]. For the second one, see [16, Th. in par. 4.4.3]. □

PROPOSITION 4.10. — *We have $\delta(\mathcal{O}_Y, g)^{[0]} = \delta(\mathcal{O}_Y, g)^{[1]} = 0$.*

Proof. — To prove that $\delta(\mathcal{O}_Y, g)^{[0]} = 0$, notice that the forgetful map $\widehat{\mathrm{CH}}^0(B) \to \mathrm{CH}^0(B)$ is an isomorphism by construction. Hence the assertion that $\delta(\mathcal{O}_Y, g)^{[0]} = 0$ follows from the Grothendieck-Riemann-Roch theorem. The fact that $\delta(\mathcal{O}_Y, g)^{[1]} = 0$ is a special case of [19, Th. 7]. A different proof is given in [11, par. 4.2]. □

COROLLARY 4.11. — *We have $\delta(\mathcal{O}_Y, g) = 0$.*

Proof. — Endow $\mathcal{O}_Y$ with the trivial metric. Recall that

$$\delta(\mathcal{O}_Y, g) = \widehat{\mathrm{ch}}(g_*(\overline{\mathcal{O}}_Y)) - g_*(\widehat{\mathrm{Td}}(g)(1 - R(Tg_{\mathbb{C}}))).$$

We shall first show that $\widehat{\mathrm{ch}}(g_*(\overline{\mathcal{O}}_Y))$ has no components of degree > 1 in $\widehat{\mathrm{CH}}^{\cdot}(B)_{\mathbb{Q}}$.

Notice that for every $l > 0$, we have $R^l g_*(\mathcal{O}_Y) = 0$ and that the isomorphism $R^0 g_*(\mathcal{O}_Y) \simeq \mathcal{O}_B$ given by adjunction is an isomorphism. We shall compute the L^2-norm of the section 1, which trivialises $R^0 g_*(\mathcal{O}_Y)$. For $b \in B(\mathbb{C})$, we compute

$$\langle 1_b, 1_b \rangle_{L^2} = \frac{1}{(2\pi)^r r!} \int_{\mathbb{P}^r(\mathbb{C})} i_b^*(g^*(\omega_B) + p^*(\omega_{\mathbb{P}^r}))^r = \frac{1}{(2\pi)^r r!} \int_{\mathbb{P}^r(\mathbb{C})} \omega_{\mathbb{P}^r}^r$$

where $i_b : \mathbb{P}^r(\mathbb{C}) \hookrightarrow Y$ is the embedding of the fiber of the map $g_{\mathbb{C}}$ above b. The first equality holds by definition. The second one is justified by the binomial formula and by the fact that $i_b^* \cdot g^*(\omega_B) = 0$, since the image of $g \circ i_b$ is the point b. We thus see that $\langle 1_b, 1_b \rangle_{L^2}$ is independent of $b \in B(\mathbb{C})$. This shows that $R^0 g_*(\mathcal{O}_Y)$ endowed with its L^2-metric is the trivial bundle endowed with a constant metric. Aside from that, by Lemma 4.8 we have $T(h_Y, h^{\mathcal{O}_Y}) = \tau(\overline{\mathcal{O}}_{\mathbb{P}^r(\mathbb{C})})$. Hence the differential form $T(h_Y, h^{\mathcal{O}_Y})$ is a constant function on $B(\mathbb{C})$. Now, using the definition of the push-forward map in arithmetic K_0-theory (see Section 3), we compute that

$$\widehat{\mathrm{ch}}(g_*(\overline{\mathcal{O}}_Y)) = \widehat{\mathrm{ch}}((R^0 g_*(\mathcal{O}_Y), g_* h^{\mathcal{O}_Y}) - \tau(\overline{\mathcal{O}}_{\mathbb{P}^r(\mathbb{C})}).$$

Using the property (ch-2) of the Chern character, we see that

$$\widehat{\mathrm{ch}}((R^0 g_*(\mathcal{O}_Y), g_* h^{\mathcal{O}_Y}))$$

has no components of degree > 1. We can thus conclude that $\widehat{\mathrm{ch}}(g_*(\overline{\mathcal{O}}_Y))$ has no components of degree > 1.

We shall now show that $g_*(\widehat{\mathrm{Td}}(g)(1 - R(Tg_{\mathbb{C}})))$ has no components of degree > 1. This will conclude the proof. By construction, we have

$$g_*(\widehat{\mathrm{Td}}(g) \cdot (1 - R(Tg_{\mathbb{C}}))) = g_*\big(p^*(\widehat{\mathrm{Td}}(\overline{T\mathbb{P}^r_{\mathbb{Z}}}) \cdot (1 - R(T\mathbb{P}^r_{\mathbb{C}})))\big).$$

Now, since $\mathbb{P}^r_{\mathbb{Z}}$ has dimension $r + 1$, the element $p^*(\widehat{\mathrm{Td}}(\overline{T\mathbb{P}^r_{\mathbb{Z}}})(1 - R(T\mathbb{P}^r_{\mathbb{C}})))$ has no component of degree $> r + 1$. Hence the element

$$g_*\big(p^*(\widehat{\mathrm{Td}}(\overline{T\mathbb{P}^r_{\mathbb{Z}}}) \cdot (1 - R(T\mathbb{P}^r_{\mathbb{C}})))\big)$$

has no component of degree > 1.

We thus see that $\delta(\mathcal{O}_Y, g)$ has no components of degree > 1. We can now conclude the proof using the last Proposition. $\square$

COROLLARY 4.12. — *We have $\delta(y, g) = 0$ for all $y \in \widehat{K}_0(Y)$.*

Proof. — We know that $K_0(Y)$ is generated by elements of the form $g^*(E) \otimes p^*(\mathcal{O}(k))$, where E is a vector bundle on B and $k \geqslant 0$ (see [3, exp. VI]). In view of Lemma 4.4 and Lemma 4.5, we are thus reduced to prove that $\delta(g^*(E) \otimes p^*(\mathcal{O}(k)), g) = 0$. Now in view of Proposition 4.9 and the fact that the arithmetic Chern character is multiplicative and commutes with pull-backs, we have

$$\delta(g^*(E) \otimes p^*(\mathcal{O}(k)), g) = \delta(p^*(\mathcal{O}(k)), g) \cdot \widehat{\mathrm{ch}}((E, h^E)).$$

for any hermitian metric h^E on E. We are thus reduced to prove that $\delta(p^*(\mathcal{O}(k)), g) = 0$ (for $k \geqslant 0$). In order to emphasize the fact that g depends only on B and r, we shall write $\delta(p^*(\mathcal{O}(k)), B, r)$ for $\delta(p^*(\mathcal{O}(k)), g)$ until the end of the proof. Now note that $\delta(p^*(\mathcal{O}(k)), B, r) = 0$ if $r = 0$. Furthermore, $\delta(p^*(\mathcal{O}(k)), B, r) = 0$ if $k = 0$ by Corollary 4.11. By induction, we may thus assume that $k, r > 0$ and that $\delta(p^*(\mathcal{O}(k')), B, r') = 0$ for all $k', r' \in \mathbb{N}$ such that $k'^2 + r'^2 < k^2 + r^2$. Now recall that there is an exact sequence of coherent sheaves

$$0 \to \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^r_{\mathbb{Z}}} \to j_* \mathcal{O}_{\mathbb{P}^{r-1}_{\mathbb{Z}}} \to 0$$

where j is the immersion of $\mathbb{P}^{r-1}_{\mathbb{Z}}$ into $\mathbb{P}^r_{\mathbb{Z}}$ as the hyperplane at ∞. If we tensor this sequence with $\mathcal{O}(k)$, we obtain the sequence
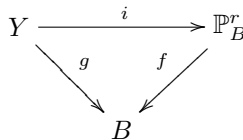
$$0 \to \mathcal{O}(k-1) \to \mathcal{O}(k) \to j_* \mathcal{O}_{\mathbb{P}^{r-1}_{\mathbb{Z}}}(k) \to 0.$$

If we apply Theorem 4.7 to this sequence, we see that the equalities $\delta(p^*(\mathcal{O}(k-1)), B, r) = 0$ and $\delta(p^*(\mathcal{O}(k)), B, r-1) = 0$ together imply the equality $\delta(p^*(\mathcal{O}(k)), B, r) = 0$. The first two equalities hold by induction, so this concludes the proof. $\qquad\square$

4.3. The general case

To conclude the proof of the arithmetic Riemann-Roch theorem, we consider again the case of a general p.f.s.r morphism of arithmetic varieties $g : Y \to B$. We want to prove that $\delta(y, g) = 0$ for every $y \in K_0(Y)$. Since $K_0(Y)$ is generated by g-acyclic bundles, we may assume that $y = \eta$, where η is a g-acyclic vector bundle.

Now notice that, by assumption, there is an $r \in \mathbb{N}$ and a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ \ i\ \ } & \mathbb{P}^r_B \\ & \searrow{\scriptstyle g} \quad {\scriptstyle f}\swarrow & \\ & B & \end{array}$$

where f is the natural projection and i is a closed immersion. Choose a resolution

$$0 \to \xi_m \to \xi_{m-1} \to \ldots \xi_0 \to i_*\eta \to 0$$

of $i_*\eta$ by f-acyclic locally free sheaves ξ_i on $\mathbb{P}_B^r$. Corollary 4.12 implies that $\delta(\xi_i, f) = 0$. From this and Theorem 4.7 we deduce that $\delta(\eta, g) = 0$ and this concludes the proof of the arithmetic Riemann-Roch theorem.

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