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SINGULAR HECKE ALGEBRAS, MARKOV TRACES, AND HOMFLY-TYPE INVARIANTS

by Luis PARIS & Loïc RABENDA

ABSTRACT. — We define the singular Hecke algebra $\mathcal{H}(SB_n)$ as the quotient of the singular braid monoid algebra $\mathbb{C}(q)[SB_n]$ by the Hecke relations $\sigma_k^2 = (q-1)\sigma_k + q$, $1 \leqslant k \leqslant n-1$. We define the notion of Markov trace in this context, fixing the number d of singular points, and we prove that a Markov trace determines an invariant on the links with d singular points which satisfies some skein relation. Let TR_d denote the set of Markov traces with d singular points. This is a $\mathbb{C}(q,z)$-vector space. Our main result is that TR_d is of dimension $d+1$. This result is completed with an explicit construction of a basis of TR_d. Thanks to this result, we define a universal Markov trace and a universal HOMFLY-type invariant on singular links. This invariant is the unique invariant which satisfies some skein relation and some desingularization relation.

RÉSUMÉ. — On définit l'algèbre de Hecke singulière $\mathcal{H}(SB_n)$ comme le quotient de l'algèbre de monoïde $\mathbb{C}(q)[SB_n]$ par les relations de Hecke $\sigma_k^2 = (q-1)\sigma_k + q$, $1 \leqslant k \leqslant n-1$. On définit la notion de trace de Markov dans ce cadre, en fixant le nombre d de points singuliers, et on démontre qu'une trace de Markov détermine un invariant sur les entrelacs à d points singuliers qui vérifie une relation d'écheveau. Soit TR_d l'ensemble des traces de Markov à d points singuliers fixés. C'est un espace vectoriel sur $\mathbb{C}(q,z)$. Notre résultat principal est que TR_d est de dimension $d+1$. Ce résultat est complété par une construction explicite d'une base de TR_d. Grâce à ces résultats, nous définissons une trace de Markov universelle et un invariant universel de type HOMFLY sur les entrelacs singuliers. Cet invariant est l'unique invariant qui vérifie une certaine relation d'écheveau et une certaine relation de désingularisation.

1. Introduction

The *Hecke algebra* $\mathcal{H}(B_n)$ of the symmetric group is a one parameter deformation of the symmetric group algebra studied in representation theory as well as in knot theory. Let $\mathbb{K} = \mathbb{C}(q)$ be the field of rational functions on

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a variable q, and let B_n denote the braid group on n strands. Then $\mathcal{H}(B_n)$ is the quotient of the group algebra $\mathbb{K}[B_n]$ by the relations

$$\sigma_k^2 = (q-1)\sigma_k + q\,, \quad 1 \leqslant k \leqslant n-1\,,$$

where $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators of B_n.

Let z be a new variable. A *Markov trace* on the tower of algebras $\{\mathcal{H}(B_n)\}_{n=1}^{+\infty}$ is defined to be a collection of $\mathbb{K}$-linear maps

$$\mathrm{tr}_n : \mathcal{H}(B_n) \to \mathbb{K}(z)\,, \quad n \geqslant 1\,,$$

such that

- $\mathrm{tr}_n(\alpha\beta) = \mathrm{tr}_n(\beta\alpha)$ for all $\alpha, \beta \in B_n$ and all $n \geqslant 1$;
- $\mathrm{tr}_{n+1}(\beta) = \mathrm{tr}_n(\beta)$ for all $\beta \in B_n \subset B_{n+1}$ and all $n \geqslant 1$;
- $\mathrm{tr}_{n+1}(\beta\sigma_n) = z \cdot \mathrm{tr}_n(\beta)$ for all $\beta \in B_n$ and all $n \geqslant 1$.

Let $\mathcal{L}_0$ denote the set of (isotopy classes of) links in $\mathbb{R}^3$. According to Jones [7], a Markov trace $T = \{\mathrm{tr}_n\}_{n=1}^{+\infty}$ on $\{\mathcal{H}(B_n)\}_{n=1}^{+\infty}$ determines a link invariant $I_T : \mathcal{L}_0 \to \mathbb{K}(\sqrt{y})$, where $y = \frac{z-q+1}{qz}$. On the other hand, by a result of Ocneanu (see [7], [3]), there exists a unique Markov trace which takes the value 1 on the identity. In particular, the set of Markov traces form a one dimensional $\mathbb{K}(z)$-vector space spanned by the Ocneanu trace.

Let A be an abelian group, let $I : \mathcal{L}_0 \to A$ be an invariant, and let $t, x \in A$. We say that I satisfies the (t, x) *skein relation* if

$$t^{-1} \cdot I(L_+) - t \cdot I(L_-) = x \cdot I(L_0)\,,$$

for all links $L_+, L_-, L_0 \in \mathcal{L}_0$ that have the same link diagram except in a neighborhood of a crossing where they are as in Figure 1.1. It is well-known that there exists a unique invariant $I : \mathcal{L}_0 \to \mathbb{C}[t^{\pm 1}, x^{\pm 1}]$ which satisfies the (t, x) skein relation and which takes the value 1 on the trivial knot. This invariant is equal to I_T (up to a change of variables), where T is the Ocneanu trace, and it is called the *HOMFLY polynomial* (see [3], [6], [7], [10]).

Figure 1.1. The links L_+, L_-, and L_0.

Our goal in this paper is to extend these constructions to the singular braids and links.

Let SB_n denote the monoid of singular braids on n strands. After some preliminaries on singular links and braids in Section 2, we develop the study of singular Hecke algebras, Markov traces, and related singular link invariants in Section 3. We define the *singular Hecke algebra* in a naïve way, as the quotient of the singular braid monoid algebra $\mathbb{K}[SB_n]$ by the Hecke relations

$$\sigma_k^2 = (q-1)\sigma_k + q\,, \quad 1 \leqslant k \leqslant n-1\,.$$

For $d \geqslant 0$, let S_dB_n denote the set of braids with d singular points. The algebra $\mathcal{H}(SB_n)$ has a natural graduation

$$\mathcal{H}(SB_n) = \bigoplus_{d=0}^{+\infty} \mathcal{H}(S_dB_n)\,,$$

where $\mathcal{H}(S_dB_n)$ is the linear subspace of $\mathcal{H}(SB_n)$ spanned by S_dB_n. The algebra $\mathcal{H}(SB_n)$ itself is of infinite dimension, but we show that each subspace $\mathcal{H}(S_dB_n)$ of the graduation is of finite dimension over $\mathbb{K}$ (see Proposition 3.1).

A Markov trace on the sequence $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$ is defined in the same way as a Markov trace on $\{\mathcal{H}(B_n)\}_{n=1}^{+\infty}$. Let $\mathcal{L}_d$ denote the set of (isotopy classes of) links with d singular points. We prove that a Markov trace T on $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$ determines an invariant $I_T : \mathcal{L}_d \to \mathbb{K}(\sqrt{y})$ (see Proposition 3.3), and that this invariant satisfies the (t,x) skein relation for $t = \sqrt{q}\sqrt{y}$ and $x = \sqrt{q} - \frac{1}{\sqrt{q}}$ (see Proposition 3.4). Conversely, any invariant $I : \mathcal{L}_d \to \mathbb{C}(\sqrt{q},\sqrt{y})$ which satisfies the (t,x) skein relation is of the form $I = I_T$, where T is a Markov trace on $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$ (with coefficients in $\mathbb{C}(\sqrt{q},\sqrt{y})$).

Section 4 contains the main result of the paper. Let TR_d denote the set of traces on $\mathcal{H}(S_dB_n)$. This is a $\mathbb{K}(z)$-vector space. We prove that the dimension of TR_d is $d+1$, and construct an explicit basis of TR_d (see Theorem 4.7).

Let $\mathcal{L}$ denote the set of all (isotopy classes of) singular links. Thanks to Section 4, we define in Section 5 a universal trace and a universal HOMFLY-type invariant $\hat{I} : \mathcal{L} \to \mathbb{C}(\sqrt{q},\sqrt{y})[X,Y]$, where y, X, Y are variables. We prove that $\hat{I}$ distinguishes two singular links $L, L' \in \mathcal{L}_d$ (where d is fixed) if and only if there exists an invariant $I : \mathcal{L}_d \to \mathbb{C}(\sqrt{q},\sqrt{y})$ which satisfies the (t,x) skein relation for $t = \sqrt{q}\sqrt{y}$ and $x = \sqrt{q} - \frac{1}{\sqrt{q}}$, and which distinguishes

L and L' (see Theorem 5.3). We also prove that $\hat{I}$ is the unique invariant with values in $\mathbb{C}[t^{\pm 1}, x^{\pm 1}, X, Y]$, which satisfies the (t, x) skein relation and some desingularization relation, and which takes the value 1 on the trivial knot (see Proposition 5.4 and Theorem 5.5).

Our invariant $\hat{I}$ is more or less equivalent to the invariant of Kauffman and Vogel defined in [9]. More precisely, the invariant of Kauffman and Vogel is a specialization of our invariant (see Lemma 5.6), but this specialization does not make much difference. Nevertheless, their approach is different from ours in the sense that they use singular Reidemeister moves to prove that their invariant is an invariant. They define some "generalized Hecke algebras" and define a Markov trace on this family of generalized Hecke algebras from which they can recover their invariant, but they definition of generalized Hecke algebra involves many relations besides the Hecke ones that are not natural in the context of an algebraic study.

Note that the notion of Markov traces to study singular braids and links is also present in [1], but the considered algebra in this paper is a one parameter deformation of $\mathbb{C}[B_n]$, which is specially adapted to study Vassiliev invariants, but which has nothing to do with the Hecke relations.

2. Singular links and braids

Let $n \geqslant 1$, and let $\mathbb{S}_1, \ldots, \mathbb{S}_n$ be n copies of the circle $\mathbb{S}^1$. A *singular link on n components* is defined to be a smooth immersion $L : \mathbb{S}_1 \sqcup \cdots \sqcup \mathbb{S}_n \to \mathbb{R}^3$ whose image has finitely many singularities (called *singular points*) that are all ordinary double points. In this context, the admissible isotopies preserve the local structures. Moreover, the circle $\mathbb{S}^1$ as well as the links are always assumed to be oriented.

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n punctures in $\mathbb{R}^2$ (except mention of the contrary, we will always assume $P_k = (k, 0)$ for all $1 \leqslant k \leqslant n$). A *singular braid on n strands based at $\mathcal{P}$* is defined to be a n-tuple $\beta = (b_1, \ldots, b_n)$ of smooth paths, $b_k : [0, 1] \to \mathbb{R}^2 \times [0, 1]$, such that:

- There exists a permutation $\chi \in \mathrm{Sym}_n$ such that $b_k(0) = (P_k, 0)$ and $b_k(1) = (P_{\chi(k)}, 1)$ for all $1 \leqslant k \leqslant n$.
- $\beta_k(t)$ runs monotonically on the third coordinate for all $1 \leqslant k \leqslant n$.
- The image of $\beta_1 \cup \cdots \cup \beta_n$ has finitely many singularities (called *singular points*) that are all ordinary double points.

The isotopy classes of singular braids form a monoid (and not a group) called the *singular braid monoid on n strands* and denoted by SB_n. The monoid operation is the concatenation.

THEOREM 2.1 (Baez [1], Birman [2]). — *The monoid SB_n has a monoid presentation with generators*

$$\sigma_1, \dots, \sigma_{n-1}, \sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}, \tau_1, \dots, \tau_{n-1}\,,$$

and relations

$$\begin{aligned}
\sigma_k\sigma_k^{-1} = \sigma_k^{-1}\sigma_k = 1 \qquad & \text{for } 1 \leqslant k \leqslant n-1\,,\\
\sigma_k\tau_k = \tau_k\sigma_k \qquad & \text{for } 1 \leqslant k \leqslant n-1\,,\\
\sigma_k\sigma_l\sigma_k = \sigma_l\sigma_k\sigma_l \qquad & \text{if } |k-l| = 1\,,\\
\sigma_k\sigma_l\tau_k = \tau_l\sigma_k\sigma_l \qquad & \text{if } |k-l| = 1\,,\\
\sigma_k\sigma_l = \sigma_l\sigma_k \qquad & \text{if } |k-l| \geqslant 2\,,\\
\sigma_k\tau_l = \tau_l\sigma_k \qquad & if |k-l| \geqslant 2\,,\\
\tau_k\tau_l = \tau_l\tau_k \qquad & \text{if } |k-l| \geqslant 2\,.
\end{aligned}$$

The braid σ_k in the above presentation is the standard k-th generator of the braid group B_n (see Figure 2.1). The braid τ_k is a singular braid with a single singular point which involves the k-th strand and the $(k+1)$-th strand (see Figure 2.1).

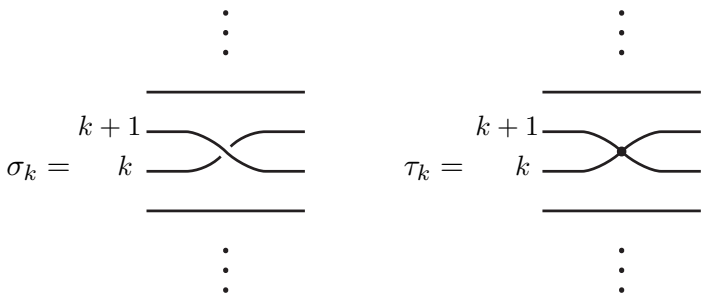

Figure 2.1. Generators of SB_n.

From a singular braid $\beta = (b_1, \dots, b_n)$ one can construct a singular link, called the *closure* of β and denoted by $\hat{\beta}$, connecting $(P_k, 0)$ with $(P_k, 1)$ for all $1 \leqslant k \leqslant n$ as in Figure 2.2.

THEOREM 2.2 (Birman [2]). — *Every singular link is a closed singular braid.*

Now, consider the set $\sqcup_{n=1}^{+\infty} SB_n$ of all singular braids. We may often use the notation (β, n) to denote a braid $\beta \in SB_n$ in case we want to emphasize the number n of strands.

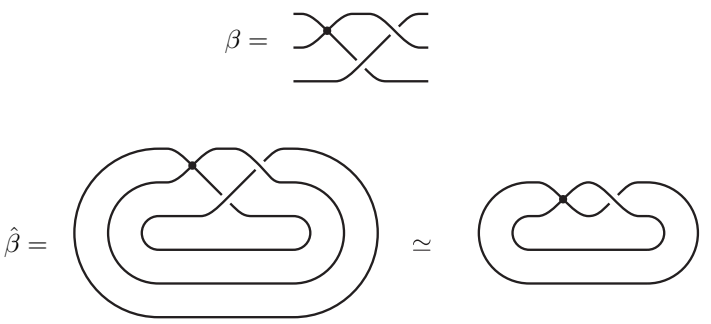

Figure 2.2. A closed singular braid.

We say that two singular braids (α, n) and (β, m) are connected by a *Markov move* if either

- $n = m$, $\alpha = \gamma_1\gamma_2$, and $\beta = \gamma_2\gamma_1$, for some $\gamma_1, \gamma_2 \in SB_n$; or
- $n = m + 1$ and $\alpha = \beta\sigma_m^{\pm 1}$; or
- $m = n + 1$ and $\beta = \alpha\sigma_n^{\pm 1}$.

THEOREM 2.3 (Gemein [5]). — *Let (α, n) and (β, m) be two singular braids. Then $\hat{\alpha}$ and $\hat{\beta}$ are isotopic if and only if (α, n) and (β, m) are connected by a finite sequence of Markov moves.*

3. Singular Hecke algebras, Markov traces, and singular link invariants

Recall that $\mathbb{K} = \mathbb{C}(q)$ denotes the field of rational functions on a variable q. We define the *singular Hecke algebra* $\mathcal{H}(SB_n)$ to be the quotient of the monoid algebra $\mathbb{K}[SB_n]$ by the relations

$$\sigma_k^2 = (q-1)\sigma_k + q\,, \quad 1 \leqslant k \leqslant n-1\,. \tag{3.1}$$

For $d \geqslant 0$, we denote by S_dB_n the set of (isotopy classes of) singular braids on n strands with d singular points, and by $\mathbb{K}[S_dB_n]$ the $\mathbb{K}$-linear subspace of $\mathbb{K}[SB_n]$ spanned by S_dB_n. Note that $S_0B_n = B_n$ is the braid group, and $\mathbb{K}[S_0B_n] = \mathbb{K}[B_n]$ is the group algebra of B_n. We have the graduation

$$\mathbb{K}[SB_n] = \bigoplus_{d=0}^{+\infty} \mathbb{K}[S_dB_n]\,.$$

The relations (3.1) that define the singular Hecke algebra involve only elements of degree zero, thus the graduation of $\mathbb{K}[SB_n]$ induces a graduation

of $\mathcal{H}(SB_n)$:

$$\mathcal{H}(SB_n) = \bigoplus_{d=0}^{+\infty} \mathcal{H}(S_d B_n)\,,$$

where $\mathcal{H}(S_d B_n)$ is the $\mathbb{K}$-linear subspace of $\mathcal{H}(SB_n)$ spanned by $S_d B_n$.

It is known that $\mathcal{H}(B_n)$ has dimension $n!$, and has a basis $\mathcal{B}_n$ which can be described as follows. For $n \geqslant 2$ we set

$$\mathcal{U}_n = \{1, \sigma_{n-1}, \sigma_{n-1}\sigma_{n-2}, \ldots, \sigma_{n-1}\sigma_{n-2}\cdots\sigma_2\sigma_1\}\,.$$

Then $\mathcal{B}_n$ is defined by induction on n by

$$\mathcal{B}_1 = \{1\}\,, \quad \mathcal{B}_n = \{\beta u\ ;\ \beta \in \mathcal{B}_{n-1} \text{ and } u \in \mathcal{U}_n\} \quad \text{if } n \geqslant 2\,.$$

The singular Hecke algebra $\mathcal{H}(SB_n)$ is not of finite dimension, but each subspace $\mathcal{H}(S_d B_n)$ of the graduation is of finite dimension. Indeed:

PROPOSITION 3.1. — *Let $d \geqslant 0$, and let $n \geqslant 2$. Let $\mathcal{C}_{d,n}$ denote the set of singular braids of the form $\tau_{i_1}\cdots\tau_{i_d}\beta$, where $1 \leqslant i_j \leqslant n-1$ for $1 \leqslant j \leqslant d$, and $\beta \in \mathcal{B}_n$. Then $\mathcal{C}_{d,n}$ spans $\mathcal{H}(S_d B_n)$.*

Proof. — Observe that the Hecke relation (3.1) implies that

$$\sigma_k^{-1} = q^{-1}\sigma_k - q^{-1}(q-1)\,, \quad \text{for all } 1 \leqslant k \leqslant n-1\,.$$

Let $i, j \in \{1, \ldots, n-1\}$ such that $|i-j| = 1$, and let $a \geqslant 1$. We calculate $\sigma_i^2\tau_j^a$ in two ways. Firstly,

$$\begin{aligned}
\sigma_i^2\tau_j^a &= \sigma_i\sigma_j^{-1}\sigma_j\sigma_i\tau_j^a \\
&= \sigma_i\sigma_j^{-1}\tau_i^a\sigma_j\sigma_i \\
&= q^{-1}\sigma_i\sigma_j\tau_i^a\sigma_j\sigma_i - q^{-1}(q-1)\sigma_i\tau_i^a\sigma_j\sigma_i \\
&= q^{-1}\tau_j^a\sigma_i\sigma_j^2\sigma_i - q^{-1}(q-1)\tau_i^a\sigma_i\sigma_j\sigma_i \\
&= q^{-1}(q-1)\tau_j^a\sigma_i\sigma_j\sigma_i + (q-1)\tau_j^a\sigma_i + q\tau_j^a - q^{-1}(q-1)\tau_i^a\sigma_i\sigma_j\sigma_i
\end{aligned}$$

Secondly,

$$\sigma_i^2\tau_j^a = (q-1)\sigma_i\tau_j^a + q\tau_j^a\,.$$

These two equalities imply

$$\sigma_i\tau_j^a = q^{-1}\tau_j^a\sigma_i\sigma_j\sigma_i - q^{-1}\tau_i^a\sigma_i\sigma_j\sigma_i + \tau_j^a\sigma_i\,. \tag{3.2}$$

On the other hand, by Theorem 2.1, if $i, j \in \{1, \ldots, n-1\}$ are such that $|i-j| \neq 1$, then

$$\sigma_i\tau_j^a = \tau_j^a\sigma_i\,. \tag{3.3}$$

Equalities (3.2) and (3.3) show that every element of $\mathcal{H}(S_d B_n)$ is a linear combination of elements of the form $\tau_{i_1}\cdots\tau_{i_d}\omega$, where $1 \leqslant i_j \leqslant n-1$ for $1 \leqslant j \leqslant d$, and $\omega \in \mathcal{H}(B_n)$. Now, since $\mathcal{B}_n$ is a basis of $\mathcal{H}(B_n)$, we conclude

that every element of $\mathcal{H}(S_dB_n)$ is a linear combination of elements of the form $\tau_{i_1}\cdots\tau_{i_d}\beta$, where $1\leqslant i_j\leqslant n-1$ for $1\leqslant j\leqslant d$, and $\beta\in\mathcal{B}_n$. □

However, $\mathcal{C}_{d,n}$ is not a basis of $\mathcal{H}(S_dB_n)$ in general. Indeed:

LEMMA 3.2. — *Let $i,j\in\{1,\dots,n-1\}$ such that $|i-j|=1$, and let $a\geqslant 1$. Then*

$$\begin{aligned}(3.4)\quad &\tau_i^a(\sigma_i\sigma_j+\sigma_j\sigma_i-(q-1)\sigma_i-(q-1)\sigma_j+(q^2-q+1))\\ &\qquad=\tau_j^a(\sigma_i\sigma_j+\sigma_j\sigma_i-(q-1)\sigma_i-(q-1)\sigma_j+(q^2-q+1))\,.\end{aligned}$$

Proof. — Recall the equality (3.2) in the proof of Proposition 3.1:

$$\sigma_i\tau_j^a=q^{-1}\tau_j^a\sigma_i\sigma_j\sigma_i-q^{-1}\tau_i^a\sigma_i\sigma_j\sigma_i+\tau_j^a\sigma_i\,.$$

We multiply this equality on the right hand side by $\sigma_i^{-1}\sigma_j^{-1}$ and we get

$$\begin{aligned}&\sigma_i\tau_j^a\sigma_i^{-1}\sigma_j^{-1}=q^{-1}\tau_j^a\sigma_i-q^{-1}\tau_i^a\sigma_i+\tau_j^a\sigma_j^{-1}\\ \Leftrightarrow\quad &\sigma_i\sigma_i^{-1}\sigma_j^{-1}\tau_i^a=q^{-1}\tau_j^a\sigma_i-q^{-1}\tau_i^a\sigma_i+q^{-1}\tau_j^a\sigma_j-q^{-1}(q-1)\tau_j^a\\ \Leftrightarrow\quad &q^{-1}\sigma_j\tau_i^a-q^{-1}(q-1)\tau_i^a=q^{-1}\tau_j^a\sigma_i-q^{-1}\tau_i^a\sigma_i+q^{-1}\tau_j^a\sigma_j-q^{-1}(q-1)\tau_j^a\end{aligned}$$

thus

$$(3.5)\qquad \sigma_j\tau_i^a=\tau_j^a(\sigma_i+\sigma_j-(q-1))-\tau_i^a(\sigma_i-(q-1))\,.$$

Now, we apply twice (3.5) to $\sigma_i\sigma_j\tau_i^a$ and obtain

$$\begin{aligned}\sigma_i\sigma_j\tau_i^a&=\sigma_i\tau_j^a(\sigma_i+\sigma_j-(q-1))-\sigma_i\tau_i^a(\sigma_i-(q-1))\\ &=\tau_i^a(\sigma_i+\sigma_j-(q-1))^2-\tau_j^a(\sigma_j-(q-1))(\sigma_i+\sigma_j-(q-1))\\ &\qquad-\tau_i^a\sigma_i(\sigma_i-(q-1))\\ &=\tau_i^a(\sigma_i\sigma_j+\sigma_j\sigma_i-(q-1)\sigma_i-(q-1)\sigma_j+(q^2-q+1))\\ &\qquad-\tau_j^a(\sigma_j\sigma_i-(q-1)\sigma_i-(q-1)\sigma_j+(q^2-q+1))\,.\end{aligned}$$

Since $\sigma_i\sigma_j\tau_i^a=\tau_j^a\sigma_i\sigma_j$, it follows that

$$\begin{aligned}&\tau_i^a(\sigma_i\sigma_j+\sigma_j\sigma_i-(q-1)\sigma_i-(q-1)\sigma_j+(q^2-q+1))\\ &\qquad=\tau_j^a(\sigma_i\sigma_j+\sigma_j\sigma_i-(q-1)\sigma_i-(q-1)\sigma_j+(q^2-q+1))\,.\end{aligned}$$

□

Remark. — We do not know the dimension of $\mathcal{H}(S_dB_n)$ if $d\geqslant 1$ and $n\geqslant 3$.

We turn now to the definition of a Markov trace, but, before, we make the following remark.

Remark. — The basis $\mathcal{B}_n$ of $\mathcal{H}(B_n)$ can be viewed as a subset of $\mathcal{B}_{n+1}$. This implies that the natural embedding $B_n \hookrightarrow B_{n+1}$ leads to an injective homomorphism $\mathcal{H}(B_n) \hookrightarrow \mathcal{H}(B_{n+1})$. In the case of singular Hecke algebras, the natural embedding $SB_n \hookrightarrow SB_{n+1}$ also leads to a homomorphism $\iota_n : \mathcal{H}(SB_n) \to \mathcal{H}(SB_{n+1})$, but we do not know whether this homomorphism is injective.

Let z be a new variable. Let $d \geqslant 0$. A *Markov trace* on the sequence $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$ is defined to be a collection of $\mathbb{K}$-linear maps

$$\mathrm{tr}_n^d : \mathcal{H}(S_dB_n) \to \mathbb{K}(z)\,, \quad n \geqslant 1\,,$$

such that

- $\mathrm{tr}_n^d(\alpha\beta) = \mathrm{tr}_n^d(\beta\alpha)$ for all singular braids $\alpha \in S_kB_n$ and $\beta \in S_lB_n$ such that $k + l = d$, and all $n \geqslant 1$;
- $\mathrm{tr}_{n+1}^d \circ \iota_n = \mathrm{tr}_n^d$ for all $n \geqslant 1$;
- $\mathrm{tr}_{n+1}^d(\iota_n(\beta)\sigma_n) = z \cdot \mathrm{tr}_n^d(\beta)$ for all $\beta \in S_dB_n$ and all $n \geqslant 1$.

Define a *Markov trace* on the sequence $\{\mathcal{H}(SB_n)\}_{n=1}^{+\infty}$ to be a collection of $\mathbb{K}$-linear maps

$$\mathrm{tr}_n : \mathcal{H}(SB_n) \to \mathbb{K}(z)\,, \quad n \geqslant 1\,,$$

such that

- $\mathrm{tr}_n(\alpha\beta) = \mathrm{tr}_n(\beta\alpha)$ for all singular braids $\alpha, \beta \in SB_n$, and all $n \geqslant 1$;
- $\mathrm{tr}_{n+1} \circ \iota_n = \mathrm{tr}_n$ for all $n \geqslant 1$;
- $\mathrm{tr}_{n+1}(\iota_n(\beta)\sigma_n) = z \cdot \mathrm{tr}_n(\beta)$ for all $\beta \in SB_n$ and all $n \geqslant 1$.

Note that, if $T = \{\mathrm{tr}_n\}_{n=1}^{+\infty}$ is a Markov trace on $\{\mathcal{H}(SB_n)\}_{n=1}^{+\infty}$, then, for all $d \geqslant 0$, the collection $T^d = \{\mathrm{tr}_n^d = \mathrm{tr}_n|_{\mathcal{H}(S_dB_n)}\}_{n=1}^{+\infty}$ of restrictions is a Markov trace on $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$. Conversely, a collection $\{T^d\}_{d=0}^{+\infty}$, where T^d is a Markov trace on $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$ for all $d \geqslant 0$, determines a unique Markov trace on $\{\mathcal{H}(SB_n)\}_{n=1}^{+\infty}$. So, both definitions of Markov traces are more or less equivalent. Now, since the number d of singular points can be fixed in our study, we will mainly consider Markov traces with a fixed number of singular points in the remainder.

Remark. — We do not impose the condition $\mathrm{tr}_1(1) = 1$ in the above definitions because this condition has no real meaning in the context of singular braids. Moreover, without this condition, the Markov traces on $\{\mathcal{H}(SB_n)\}_{n=1}^{+\infty}$ (or on $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$) form a $\mathbb{K}(z)$-vector space. This will be of importance in the remainder.

Let $\mathcal{L}_d$ denote the set of (isotopy classes of) singular links with d singular points. We fix a Markov trace $T = \{tr_n^d\}_{n=1}^{+\infty}$ on $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$, and turn

to define an invariant $I_T : \mathcal{L}_d \to \mathbb{K}(\sqrt{y})$. We follow the same strategy as Jones in [7].

Let $\pi : SB_n \to \mathcal{H}(SB_n)$ denote the natural map, and let $\varepsilon : SB_n \to \mathbb{Z}$ be the homomorphism defined by

$$\varepsilon(\sigma_i) = 1\,,\ \varepsilon(\sigma_i^{-1}) = -1\,,\ \varepsilon(\tau_i) = 0\,,\ \text{for } 1 \leqslant i \leqslant n-1\,.$$

We consider the following change of variables:

$$z = \frac{q-1}{1-qy} \quad \Leftrightarrow \quad y = \frac{z-q+1}{qz}\,.$$

For a braid $\beta \in S_dB_n$ we set

$$I_T(\beta) = \left(\frac{q-1}{1-qy}\right)^{-n+1} \cdot (\sqrt{y})^{\varepsilon(\beta)-n+1} \cdot \mathrm{tr}_n^d(\pi(\beta))\,.$$

This is an element of $\mathbb{K}(\sqrt{y})$.

PROPOSITION 3.3. — *Let (α, n) and (β, m) be two singular braids with d singular points. If $\hat{\alpha}$ is isotopic to $\hat{\beta}$, then $I_T(\alpha) = I_T(\beta)$.*

Proof. — This is standard application of Theorem 2.3. □

For $L \in \mathcal{L}_d$, we choose a singular braid (β, n) such that $\hat{\beta} = L$, and we set $I_T(L) = I_T(\beta)$. By Proposition 3.3, $I_T(L)$ is a well-defined invariant.

Let A be an abelian group, let $I : \mathcal{L}_d \to A$ be an invariant, and let $t, x \in A$. We say that I satisfies the (t, x) *skein relation* if

$$t^{-1} \cdot I(L_+) - t \cdot I(L_-) = x \cdot I(L_0)\,,$$

for all singular links $L_+, L_-, L_0 \in \mathcal{L}_d$ that have the same link diagram except in a neighborhood of a crossing where they are as in Figure 3.1.

Figure 3.1. The singular links L_+, L_-, and L_0.

Now, we set

$$t = \sqrt{y} \cdot \sqrt{q}\,, \quad x = \sqrt{q} - \frac{1}{\sqrt{q}}\,,$$

and we define $\widetilde{\mathrm{tr}}_n^d : S_dB_n \to \mathbb{C}(\sqrt{q}, \sqrt{y})$ by

$$\widetilde{\mathrm{tr}}_n^d(\beta) = (\sqrt{q})^{-\varepsilon(\beta)} \cdot \mathrm{tr}_n^d(\pi(\beta))\,.$$

With these new notations $I_T(\beta)$ can be written

$$I_T(\beta) = \left(\frac{1-t^2}{tx}\right)^{n-1} \cdot t^{\varepsilon(\beta)} \cdot \widetilde{\mathrm{tr}}_n^d(\beta)\,.$$

PROPOSITION 3.4. — *The invariant $I_T : \mathcal{L}_d \to \mathbb{C}(\sqrt{q}, \sqrt{y})$ satisfies the (t, x) skein relation.*

Proof. — Let $L_+, L_-, L_0 \in \mathcal{L}_d$ be three singular links that have the same singular link diagram except in a neighborhood of a non-singular crossing where they are as in Figure 3.1. A careful reading of Birman's proof of Theorem 2.3 shows that there exist a singular braid (β, n) with d singular points, and an index $1 \leqslant i \leqslant n-1$, such that $L_+ = \widehat{\beta\sigma_i}$, $L_- = \widehat{\beta\sigma_i^{-1}}$, and $L_0 = \hat{\beta}$. On the other hand, the Hecke relation (3.1) implies

$$\widetilde{\mathrm{tr}}_n^d(\beta\sigma_i) = x\widetilde{\mathrm{tr}}_n^d(\beta) + \widetilde{\mathrm{tr}}_n^d(\beta\sigma_i^{-1})\,.$$

Hence,

$$\begin{aligned} t^{-1}I_T(L_+) - tI_T(L_-) &= \left(\frac{1-t^2}{tx}\right)^{n-1} t^{\varepsilon(\beta)}(\widetilde{\mathrm{tr}}_n^d(\beta\sigma_i) - \widetilde{\mathrm{tr}}_n^d(\beta\sigma_i^{-1})) \\ &= x\left(\frac{1-t^2}{tx}\right)^{n-1} t^{\varepsilon(\beta)}\widetilde{\mathrm{tr}}_n^d(\beta) \\ &= I_T(L_0)\,. \end{aligned}$$

□

Define a *Markov trace* on the sequence $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$ *with coefficients in* $\mathbb{C}(\sqrt{q}, \sqrt{y})$ to be a collection of $\mathbb{K}$-linear maps

$$\mathrm{tr}_n^d : \mathcal{H}(S_dB_n) \to \mathbb{C}(\sqrt{q}, \sqrt{y})\,, \quad n \geqslant 1\,,$$

such that

- $\mathrm{tr}_n^d(\alpha\beta) = \mathrm{tr}_n^d(\beta\alpha)$ for all singular braids $\alpha \in S_kB_n$ and $\beta \in S_lB_n$ such that $k + l = d$, and all $n \geqslant 1$;
- $\mathrm{tr}_{n+1}^d \circ \iota_n = \mathrm{tr}_n^d$ for all $n \geqslant 1$;
- $\mathrm{tr}_{n+1}^d(\iota_n(\beta)\sigma_n) = z \cdot \mathrm{tr}_n^d(\beta)$ for all $\beta \in S_dB_n$ and all $n \geqslant 1$.

Using the same trick as above, a Markov trace T on the sequence $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$ with coefficients in $\mathbb{C}(\sqrt{q}, \sqrt{y})$ defines an invariant $I_T : \mathcal{L}_d \to \mathbb{C}(\sqrt{q}, \sqrt{y})$ which satisfies the (t, x) skein relation for $t = \sqrt{y}\sqrt{q}$ and $x = \sqrt{q} - \frac{1}{\sqrt{q}}$. Now, the reverse of Proposition 3.4 is true in the following sense.

PROPOSITION 3.5. — *Let $I : \mathcal{L}_d \to \mathbb{C}(\sqrt{q}, \sqrt{y})$ be an invariant which satisfies the (t, x) skein relation for $t = \sqrt{y}\sqrt{q}$ and $x = \sqrt{q} - \frac{1}{\sqrt{q}}$. Then there*

exists a Markov trace T on $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$ with coefficients in $\mathbb{C}(\sqrt{q},\sqrt{y})$ such that $I = I_T$.

Proof. — Recall that $\pi : SB_n \to \mathcal{H}(SB_n)$ denotes the natural map, and that $\varepsilon : SB_n \to \mathbb{Z}$ is the homomorphism defined by $\varepsilon(\sigma_i) = 1$, $\varepsilon(\sigma_i^{-1}) = -1$, and $\varepsilon(\tau_i) = 0$, for $1 \leqslant i \leqslant n-1$.

Let $\widetilde{\mathrm{tr}}_n^d : S_dB_n \to \mathbb{C}(\sqrt{q},\sqrt{y})$ be the map defined by

$$\widetilde{tr}_n^d(\beta) = t^{-\varepsilon(\beta)} \cdot \left(\frac{1-t^2}{tx}\right)^{1-n} \cdot I(\hat{\beta})\,.$$

and let $\mathrm{tr}_n^u : \mathbb{K}[S_dB_n] \to \mathbb{C}(\sqrt{q},\sqrt{y})$ be the $\mathbb{K}$-linear map defined by

$$\mathrm{tr}_n^u(\beta) = (\sqrt{q})^{\varepsilon(\beta)} \cdot \widetilde{\mathrm{tr}}_n^d(\beta)\,, \quad \text{for } \beta \in S_dB_n\,.$$

Thanks to the (t,x) skein relation, it is easily checked that $\mathrm{tr}_n^u : \mathbb{K}[S_dB_n] \to \mathbb{C}(\sqrt{q},\sqrt{y})$ induces a $\mathbb{K}$-linear map $\mathrm{tr}_n^d : \mathcal{H}(S_dB_n) \to \mathbb{C}(\sqrt{q},\sqrt{y})$, and that $T = \{\mathrm{tr}_n^d\}_{n=1}^{+\infty}$ is a Markov trace on $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$ with coefficients in $\mathbb{C}(\sqrt{q},\sqrt{y})$. On the other hand, a direct calculation shows that $I = I_T$. □

4. The space of traces

For $d \geqslant 0$, we denote by TR_d the set of all traces on $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$. This is a $\mathbb{K}(z)$-vector space. Note also that the space of all traces on $\{\mathcal{H}(SB_n)\}_{n=1}^{+\infty}$ is the completion of $\mathrm{TR} = \oplus_{d=0}^{+\infty}\mathrm{TR}_d$. We start our analysis recalling the following.

Theorem 4.1 (Ocneanu [7], [3]). — *There exists a unique trace $T_0^0 = \{\mathrm{tr}_n^0\}_{n=1}^{+\infty}$ on $\{\mathcal{H}(B_n)\}_{n=1}^{+\infty}$ such that $\mathrm{tr}_1^0(1) = 1$.*

Corollary 4.2. — *TR_0 is a one-dimensional $\mathbb{K}(z)$-vector space spanned by T_0^0.*

The above trace T_0^0 is called the *Ocneanu trace*. It will be a master piece in our study.

In this section we prove that TR_d is of dimension $d+1$ and construct an explicit basis $\{T_0^d, T_1^d, \ldots, T_d^d\}$ of TR_d.

We start with the definition of the Markov traces T_k^d, $0 \leqslant k \leqslant d$.

Let

$$g_0^d, g_1^d : \mathcal{H}(S_{d+1}B_n) \to \mathcal{H}(S_dB_n)$$

be the $\mathbb{K}$-linear maps defined as follows. Let $\beta \in S_{d+1}B_n$. Write β in the form

$$\beta = \alpha_0 \tau_{i_1} \alpha_1 \cdots \tau_{i_d} \alpha_d \tau_{i_{d+1}} \alpha_{d+1} \,,$$

where $1 \leqslant i_j \leqslant n-1$ for $1 \leqslant j \leqslant d+1$, and $\alpha_j \in B_n$ for $0 \leqslant j \leqslant d+1$. Then

$$g_0^d(\beta) = \sum_{j=1}^{d+1} \alpha_0 \tau_{i_1} \alpha_1 \cdots \tau_{i_{j-1}} \alpha_{j-1} \cdot \alpha_j \cdot \tau_{i_{j+1}} \alpha_{j+1} \cdots \tau_{i_{d+1}} \alpha_{d+1} \,,$$

$$g_1^d(\beta) = \sum_{j=1}^{d+1} \alpha_0 \tau_{i_1} \alpha_1 \cdots \tau_{i_{j-1}} \alpha_{j-1} \cdot \sigma_{i_j} \alpha_j \cdot \tau_{i_{j+1}} \alpha_{j+1} \cdots \tau_{i_{d+1}} \alpha_{d+1} \,.$$

It is easily seen from the presentation of SB_n given in Theorem 2.1 that g_0^d and g_1^d are well-defined.

Let

$$\Phi_0^d, \Phi_1^d : \mathrm{TR}_d \to \mathrm{TR}_{d+1}$$

be the $\mathbb{K}(z)$-linear maps defined as follows. Let $T = \{\mathrm{tr}_n^d\}_{n=1}^{+\infty}$ be an element of TR_d. Then, for $\omega \in \mathcal{H}(S_{d+1}B_n)$, we set

$$\Phi_0^d(T)(\omega) = \mathrm{tr}_n^d(g_0^d(w)) \,, \quad \Phi_1^d(T)(\omega) = \mathrm{tr}_n^d(g_1^d(w)) \,.$$

It is easily checked that $\Phi_0^d \circ \Phi_1^{d-1} = \Phi_1^d \circ \Phi_0^{d-1}$ for all $d \geqslant 1$.

Now, we define T_k^d by induction on d. According to the previous notation, T_0^0 is the Ocneanu trace of Theorem 4.1. If $d \geqslant 1$, then

$$T_k^d = \begin{cases} \Phi_0^{d-1}(T_k^{d-1}) & \text{if } k \leqslant d-1 \,, \\ \Phi_1^{d-1}(T_{d-1}^{d-1}) & \text{if } k = d \,. \end{cases}$$

Note that we also have $T_k^d = \Phi_1^{d-1}(T_{k-1}^{d-1})$ for all $1 \leqslant k \leqslant d-1$.

THEOREM 4.3. — *Let $d \geqslant 0$. Then $\{T_0^d, T_1^d, \ldots, T_d^d\}$ is a linearly independent family of TR_d.*

The following lemmas 4.4 to 4.6 are preliminaries to the proof of Theorem 4.3.

The submonoid of B_n generated (as a monoid) by $\sigma_1, \ldots, \sigma_{n-1}$ is called the *positive braid monoid* and is denoted by B_n^+. By [4], it has a monoid presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\begin{aligned} \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j \qquad \text{if } |i-j| = 1 \,, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \qquad \text{if } |i-j| \geqslant 2 \,. \end{aligned}$$

LEMMA 4.4. — *Let $n \geqslant 1$, and let $\beta \in B_n^+$. Then $T_0^0(\beta) \in \mathbb{Z}[q, z]$.*

Proof. — Let U_n denote the $\mathbb{Z}[q]$-submodule of $\mathcal{H}(B_n)$ spanned by B_n^+. We prove by induction on $n \geqslant 2$ that U_n is actually spanned as a $\mathbb{Z}[q]$-module by $B_{n-1}^+ \cup \{\alpha\sigma_{n-1}\alpha'; \alpha, \alpha' \in B_{n-1}^+\}$.

Suppose $n = 2$. Then U_2 is spanned as a $\mathbb{Z}[q]$-module by $\{\sigma_1^a; a \geqslant 0\}$. Now, the Hecke relation (3.1) implies that

$$\sigma_1^a = (q-1)\sigma_1^{a-1} + q\sigma_1^{a-2}\,, \quad \text{for all } a \geqslant 2\,,$$

thus U_2 is spanned by $\{1, \sigma_1\}$.

Suppose $n \geqslant 3$. Let V_n be the $\mathbb{Z}[q]$-submodule spanned by $B_{n-1}^+ \cup \{\alpha\sigma_{n-1}\alpha'; \alpha, \alpha' \in B_{n-1}^+\}$. Let $\beta \in B_n^+$. We write β in the form

$$\beta = \beta_0\sigma_{n-1}\beta_1 \cdots \sigma_{n-1}\beta_l\,,$$

where $\beta_0, \beta_1, \ldots, \beta_l \in B_{n-1}^+$, and prove that $\beta \in V_n$ by induction on l. The cases $l = 0$ and $l = 1$ are obvious. So, we can suppose that $l \geqslant 2$. By induction (on n), we can assume that either $\beta_1 \in B_{n-2}^+$, or $\beta_1 = \beta_1'\sigma_{n-2}\beta_1''$ for some $\beta_1', \beta_1'' \in B_{n-2}^+$. If $\beta_1 \in B_{n-2}^+$, then

$$\begin{aligned}\beta &= \beta_0\beta_1\sigma_{n-1}^2\beta_2\sigma_{n-1}\beta_3 \cdots \sigma_{n-1}\beta_l \\ &= (q-1)\cdot\beta_0\beta_1\sigma_{n-1}\beta_2\sigma_{n-1}\beta_3 \cdots \sigma_{n-1}\beta_l \\ &\quad + q\cdot\beta_0\beta_1\beta_2\sigma_{n-1}\beta_3 \cdots \sigma_{n-1}\beta_l\,,\end{aligned}$$

thus, by induction (on l), we have $\beta \in V_n$. If $\beta_1 = \beta_1'\sigma_{n-2}\beta_1''$ for some $\beta_1', \beta_1'' \in B_{n-2}^+$, then

$$\begin{aligned}\beta &= \beta_0\beta_1'\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}\beta_1''\beta_2\sigma_{n-1}\beta_3 \cdots \sigma_{n-1}\beta_l \\ &= (\beta_0\beta_1'\sigma_{n-2})\sigma_{n-1}(\sigma_{n-2}\beta_1''\beta_2)\sigma_{n-1}\beta_3 \cdots \sigma_{n-1}\beta_l\,,\end{aligned}$$

thus, by induction (on l), we have $\beta \in V_n$.

Now, we take $\beta \in B_n^+$ and turn to prove that $T_0^0(\beta) \in \mathbb{Z}[q, z]$. We argue by induction on n.

Suppose $n \geqslant 2$. By the above observation, we can assume that either $\beta \in B_{n-1}^+$, or $\beta = \alpha\sigma_{n-1}\alpha'$ for some $\alpha, \alpha' \in B_{n-1}^+$. If $\beta \in B_{n-1}^+$, then, by induction, $T_0^0(\beta) \in \mathbb{Z}[q, z]$. If $\beta = \alpha\sigma_{n-1}\alpha'$ for some $\alpha, \alpha' \in B_{n-1}^+$, then, by induction, $T_0^0(\beta) = z \cdot T_0^0(\alpha\alpha') \in \mathbb{Z}[q, z]$. □

LEMMA 4.5. — *Let $1 \leqslant a \leqslant n-1$, and let $\alpha, \alpha' \in \langle\sigma_{a+1}, \ldots, \sigma_{n-1}\rangle^+$, where $\langle\sigma_{a+1}, \ldots, \sigma_{n-1}\rangle^+$ denotes the submonoid generated by $\sigma_{a+1}, \ldots, \sigma_{n-1}$. Then*

$$\begin{aligned}T_0^0(\alpha\sigma_a\alpha')|_{z=0} &= 0\,, \\ T_0^0(\alpha\sigma_a^2\alpha')|_{z=0} &= q \cdot T_0^0(\alpha\alpha')|_{z=0}\,.\end{aligned}$$

Proof. — The first equality is a consequence of the following one

$$T_0^0(\alpha\sigma_a\alpha') = z \cdot T_0^0(\alpha\alpha')$$

whose proof is left to the reader. The second equality follows from the first one and the Hecke relation (3.1). □

The following lemma is a direct consequence of the previous one.

LEMMA 4.6. — *Let $0 \leqslant a, b \leqslant n-1$, and let $i_1, \dots, i_a \in \{1, \dots, n-1\}$ such that $i_1 < i_2 < \cdots < i_a$. Let*

$$\gamma = \sigma_{i_a} \cdots \sigma_{i_2}\sigma_{i_1}\sigma_1\sigma_2 \cdots \sigma_b\,.$$

Then

$$T_0^0(\gamma)|_{z=0} = \begin{cases} q^a & \text{if } a = b,\ i_1 = 1,\ \dots,\ i_a = a\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Proof of Theorem 4.3. For $0 \leqslant b \leqslant d$, we set

$$\gamma_b^d = \tau_d \cdots \tau_2\tau_1\sigma_1\sigma_2 \cdots \sigma_b\,.$$

A direct calculation shows that

$$T_a^d(\gamma_b^d) = (d-a)!\,a! \sum_{1 \leqslant i_1 < \cdots < i_a \leqslant n-1} T_0^0(\sigma_{i_a} \cdots \sigma_{i_2}\sigma_{i_1}\sigma_1\sigma_2 \cdots \sigma_b)\,.$$

By Lemma 4.4, we have $T_a^d(\gamma_b^d) \in \mathbb{Z}[q, z]$, and, by Lemma 4.6,

$$T_a^d(\gamma_b^d)|_{z=0} = \begin{cases} (d-a)!\,a!\,q^a & \text{if } a = b\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This implies that $T_0^d, T_1^d, \dots, T_d^d$ are linearly independent. □

THEOREM 4.7. — *Let $d \geqslant 0$. Then TR_d is a $\mathbb{K}(z)$-vector space of dimension $d+1$. In particular, $\{T_0^d, T_1^d, \dots, T_d^d\}$ is a basis of TR_d.*

The main ingredient in the proof of Theorem 4.7 are the relations in $\mathcal{H}(SB_n)$ that will be proved in the following lemmas 4.8 to 4.11. We will prove Theorem 4.7 after these lemmas.

LEMMA 4.8. — *Let $i, j \in \{1, \dots, n-1\}$ such that $|i-j| = 1$, and let $a \geqslant 1$. Set*

$$B_{i\,j} = \sigma_i + \sigma_j - (q-1)\,.$$

Then

$$\sigma_i\tau_j^a = q^{-1}\tau_j^a\sigma_i\sigma_j\sigma_i - q^{-1}\tau_i^a\sigma_i\sigma_j\sigma_i + \tau_j^a\sigma_i\,; \tag{4.1}$$

$$\sigma_j\tau_i^a = \tau_j^a(\sigma_i+\sigma_j-(q-1)) - \tau_i^a(\sigma_i-(q-1))\,; \tag{4.2}$$

$$B_{i\,j}\tau_i^a = \tau_j^a B_{i\,j}\,; \tag{4.3}$$

$$\begin{aligned}&\tau_i^a(\sigma_i\sigma_j+\sigma_j\sigma_i-(q-1)\sigma_i-(q-1)\sigma_j+(q^2-q+1))\\ =\;&\tau_j^a(\sigma_i\sigma_j+\sigma_j\sigma_i-(q-1)\sigma_i-(q-1)\sigma_j+(q^2-q+1))\,.\end{aligned} \tag{4.4}$$

Proof. — The equalities (4.1), (4.2), and (4.4) are proved in Section 3 (see (3.2), (3.5), (3.4)). Since τ_i^a commutes with $\sigma_i-(q-1)$, the equality (4.3) is a direct consequence of (4.2). □

LEMMA 4.9. — *Let $i,j\in\{1,\dots,n-1\}$ such that $|i-j|=1$. Then $B_{i\,j}$ is invertible in $\mathcal{H}(B_n)$.*

Proof. — A direct calculation shows that

$$-q^{-1}(q+1)^{-2}(q(q-1)-2q\sigma_i-2q\sigma_j-(q-1)\sigma_i\sigma_j-(q-1)\sigma_j\sigma_i+2\sigma_i\sigma_j\sigma_i)$$

is the inverse of $B_{i\,j}$. □

LEMMA 4.10. — *Let $i,j\in\{1,\dots,n-1\}$ such that $|i-j|\geqslant 2$, and let $a\geqslant 1$. Let*

$$C_{i\,j} = 2\sigma_i\sigma_j-(q-1)\sigma_i-(q-1)\sigma_j+q^2+1\,.$$

Then

$$\sigma_i\tau_j^a = \tau_j^a\sigma_i\,; \tag{4.5}$$

$$(\sigma_i-\sigma_j)^2 = (q+1)^2 - C_{i\,j}\,; \tag{4.6}$$

$$\tau_i^a C_{i\,j} = \tau_j^a C_{i\,j}\,. \tag{4.7}$$

Proof. — The equality (4.5) is a straightforward consequence of Theorem 2.1, and (4.6) can be easily proved with a direct calculation. So, it remains to prove (4.7).

First, we study the case where $i=1$ and $j=3$. We apply twice (4.2) to $\sigma_1\sigma_3\tau_2^a$ and obtain

$$\begin{aligned}\sigma_1\sigma_3\tau_2^a &= \sigma_1\tau_3^a(\sigma_2+\sigma_3-(q-1)) - \sigma_1\tau_2^a(\sigma_2-(q-1))\\ &= \tau_3^a\sigma_1(\sigma_2+\sigma_3-(q-1)) - \tau_1^a(\sigma_1+\sigma_2-(q-1))(\sigma_2-(q-1))\\ &\qquad +\tau_2^a(\sigma_2-(q-1))^2\\ &= \tau_3^a(\sigma_1\sigma_2+\sigma_1\sigma_3-(q-1)\sigma_1) - \tau_1^a\Big(\sigma_1\sigma_2-(q-1)\sigma_1-(q-1)\sigma_2\\ &\qquad +(q^2-q+1)\Big) + \tau_2^a(\sigma_2-(q-1))^2\,.\end{aligned}$$

Similarly,

$$\sigma_3\sigma_1\tau_2^a = \tau_1^a(\sigma_3\sigma_2 + \sigma_1\sigma_3 - (q-1)\sigma_3) - \tau_3^a(\sigma_3\sigma_2 - (q-1)\sigma_2 - (q-1)\sigma_3 + (q^2-q+1)) + \tau_2^a(\sigma_2 - (q-1))^2 .$$

Since $\sigma_1\sigma_3\tau_2^a = \sigma_3\sigma_1\tau_2^a$, it follows that

$$\begin{aligned} &\tau_1^a\Big(\sigma_1\sigma_2 + \sigma_3\sigma_2 + \sigma_1\sigma_3 - (q-1)\sigma_1 - (q-1)\sigma_2 - (q-1)\sigma_3 + (q^2-q+1)\Big) \\ &= \tau_3^a\Big(\sigma_1\sigma_2 + \sigma_3\sigma_2 + \sigma_1\sigma_3 - (q-1)\sigma_1 - (q-1)\sigma_2 - (q-1)\sigma_3 + (q^2-q+1)\Big) . \end{aligned} \tag{4.8}$$

Set

$$\omega_0 = \sigma_1\sigma_2 + \sigma_3\sigma_2 + \sigma_1\sigma_3 - (q-1)\sigma_1 - (q-1)\sigma_2 - (q-1)\sigma_3 + (q^2-q+1) .$$

By (4.8), we have $\tau_1^a\omega_0 = \tau_3^a\omega_0$. A direct calculation shows that

$$C_{1\,3} = q^{-2}(q\sigma_1\omega_0 + q\omega_0\sigma_3 + (q-1)\sigma_1\omega_0\sigma_3 - \sigma_1\omega_0\sigma_3\sigma_2)(\sigma_1 - (q-1)) .$$

Since τ_1 and τ_3 commute with σ_1, it follows that

$$\tau_1^a C_{1\,3} = \tau_3^a C_{1\,3} . \tag{4.9}$$

Now, suppose that $1 \leqslant i < j-1 \leqslant n-2$. Set

$$B_{j\,3} = B_{j\,j-1}\cdots B_{5\,4}B_{4\,3} , \quad B_{i\,1} = B_{i\,i-1}\cdots B_{3\,2}B_{2\,1} , \quad \delta_{i\,j} = B_{i\,1}B_{j\,3} .$$

By Lemma 4.9, $\delta_{i\,j}$ is invertible, and by (4.3), we have

$$\delta_{i\,j}\sigma_1\delta_{i\,j}^{-1} = \sigma_i , \quad \delta_{i\,j}\tau_1\delta_{i\,j}^{-1} = \tau_i , \quad \delta_{i\,j}\sigma_3\delta_{i\,j}^{-1} = \sigma_j , \quad \delta_{i\,j}\tau_3\delta_{i\,j}^{-1} = \tau_j ,$$

thus, by (4.9),

$$\tau_i^a C_{i\,j} = \delta_{i\,j}\tau_1^a C_{1\,3}\delta_{i\,j}^{-1} = \delta_{i\,j}\tau_3^a C_{1\,3}\delta_{i\,j}^{-1} = \tau_j^a C_{i\,j} .$$

□

LEMMA 4.11. — *Let $a, b \geqslant 1$. Then*

$$\tau_1^a\tau_3^b(\sigma_3 - \sigma_1) = (\tau_2^b\tau_1^a + \tau_2^a\tau_3^b)(\sigma_3 - \sigma_1) + \tau_2^{a+b}(B_{1\,2} - B_{2\,3}) . \tag{4.10}$$

Proof. — Applying twice (4.2) to $\sigma_2\tau_1^a\tau_3^b$ we obtain

$$\begin{aligned} \sigma_2\tau_1^a\tau_3^b &= \tau_2^a(\sigma_1 + \sigma_2 - (q-1))\tau_3^b - \tau_1^a(\sigma_1 - (q-1))\tau_3^b \\ &= \tau_2^a\tau_3^b(\sigma_1 - (q-1)) + \tau_2^a\sigma_2\tau_3^b - \tau_1^a\tau_3^b(\sigma_1 - (q-1)) \\ &= \tau_2^a\tau_3^b(\sigma_1 - (q-1)) + \tau_2^{a+b}(\sigma_2 + \sigma_3 - (q-1)) \\ &\qquad - \tau_2^a\tau_3^b(\sigma_3 - (q-1)) - \tau_1^a\tau_3^b(\sigma_1 - (q-1)) \\ &= \tau_2^a\tau_3^b(\sigma_1 - \sigma_3) + \tau_2^{a+b}B_{2\,3} - \tau_1^a\tau_3^b(\sigma_1 - (q-1)) . \end{aligned}$$

Similarly,

$$\sigma_2\tau_3^b\tau_1^a = \tau_2^b\tau_1^a(\sigma_3-\sigma_1) + \tau_2^{a+b}B_{1\,2} - \tau_1^a\tau_3^b(\sigma_3-(q-1))\,.$$

Since $\sigma_2\tau_1^a\tau_3^b = \sigma_2\tau_3^b\tau_1^a$, it follows that

$$\tau_1^a\tau_3^b(\sigma_3-\sigma_1) = (\tau_2^b\tau_1^a+\tau_2^a\tau_3^b)(\sigma_3-\sigma_1) + \tau_2^{a+b}(B_{1\,2}-B_{2\,3})\,.$$

□

Proof of Theorem 4.7. — We fix once for all the number $d \geqslant 1$ of singular points. We set

$$\widetilde{\mathrm{TR}}_d = \bigoplus_{n=2}^{+\infty} \big(\mathbb{K}(z)\otimes\mathcal{H}(S_dB_n)\big)\,.$$

Then TR_d can and will be viewed as the quotient of $\widetilde{TR}_d$ by the following relations:

- $(\alpha\beta,n) = (\beta\alpha,n)$ for all $\alpha\in S_kB_n$ and $\beta\in S_lB_n$ such that $k+l=d$, and all $n\geqslant 2$;
- $(\beta,n+1) = (\beta,n)$ for all $\beta\in S_dB_n$ and all $n\geqslant 2$;
- $(\beta\sigma_n,n+1) = z\cdot(\beta,n)$ for all $\beta\in S_dB_n$ and all $n\geqslant 2$.

For $\omega\in\mathbb{K}(z)\otimes\mathcal{H}(S_dB_n)$, we will denote by $[\omega]$ the element of TR_d represented by ω.

We already know that $\dim TR_d \geqslant d+1$ (see Theorem 4.3). So, in order to prove Theorem 4.7, it suffices to show that TR_d is spanned by $d+1$ elements.

Recall the basis $\mathcal{B}_n$ of $\mathcal{H}(B_n)$ described in Section 3. For $n\geqslant 2$ we set

$$\mathcal{U}_n = \{1,\sigma_{n-1},\sigma_{n-1}\sigma_{n-2},\dots,\sigma_{n-1}\cdots\sigma_2\sigma_1\}\,.$$

Then $\mathcal{B}_n$ is defined by induction on n by

$$\mathcal{B}_1=\{1\}\,,\quad \mathcal{B}_n=\{\beta u;\beta\in\mathcal{B}_{n-1}\text{ and }u\in\mathcal{U}_n\}\quad\text{if } n\geqslant 2\,.$$

Let $\mathcal{C}_n$ be the set of elements of TR_d of the form $[\tau_{i_1}\cdots\tau_{i_d}\beta]$, where $1\leqslant i_j\leqslant n-1$ for $1\leqslant j\leqslant d$, and $\beta\in\mathcal{B}_n$. Set $\mathcal{C}_\infty=\cup_{n=2}^{+\infty}\mathcal{C}_n$. By Proposition 3.1, $\mathcal{C}_\infty$ spans TR_d.

Let $\omega\in\mathcal{C}_n$. Let $1\leqslant l\leqslant d$. If ω can be written in the form $\omega=[\tau_{i_1}^{a_1}\cdots\tau_{i_l}^{a_l}\beta]$, where $1\leqslant i_j\leqslant n-1$ and $a_j\geqslant 1$ for $1\leqslant j\leqslant l$, $a_1+\cdots+a_l=d$, and $\beta\in\mathcal{B}_n$, then we say that ω *has a syllable length less or equal to* l, and we write $\mathrm{Syl}(\omega)\leqslant l$. We set

$$\mathcal{D}_{l,n}=\{\omega\in\mathcal{C}_n;\mathrm{Syl}(\omega)\leqslant l\}\,,\quad\text{and }\mathcal{D}_{l,\infty}=\cup_{n=2}^{+\infty}\mathcal{D}_{l,n}\,.$$

Note that $\mathcal{D}_{d,\infty}=\mathcal{C}_\infty$ spans TR_d.

For $\mathcal{X} \subset \mathrm{TR}_d$, we denote by $\mathrm{Span}(\mathcal{X})$ the $\mathbb{K}(z)$-linear subspace spanned by $\mathcal{X}$. The first step in the proof of Theorem 4.7 will consist on proving that $\mathrm{Span}(\mathcal{D}_{l,\infty}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty})$ for all $l \geqslant 3$ (see Claims 1 to 4). Since $\mathrm{TR}_d = \mathrm{Span}(\mathcal{D}_{d,\infty})$, it will follow that $\mathrm{TR}_d = \mathrm{Span}(\mathcal{D}_{2,\infty})$. The second step will consist on proving that there exists a subset $\mathcal{F}_3 \subset \mathcal{D}_{2,3}$ with $d+1$ elements such that $\mathrm{Span}(\mathcal{F}_3) = \mathrm{Span}(\mathcal{D}_{2,\infty}) = \mathrm{TR}_d$ (see Claims 5 to 7).

Let $1 \leqslant l \leqslant d$, and let $1 \leqslant r \leqslant l$. Set $\varepsilon = 2$ if r is even, and $\varepsilon = 1$ if r is odd. Then we denote by $\mathcal{E}_{r,l,n}$ the set of elements of $\mathcal{D}_{l,n}$ of the form

$$\omega = [\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_\varepsilon^{a_r}\tau_{i_{r+1}}^{a_{r+1}}\cdots\tau_{i_l}^{a_l}\beta]\,,$$

where $1 \leqslant i_j \leqslant n-1$ for $r+1 \leqslant j \leqslant l$, and $\beta \in \mathcal{B}_n$. We set $\mathcal{E}_{r,l,\infty} = \cup_{n=2}^{+\infty}\mathcal{E}_{r,l,n}$.

Claim 1. — *Let $2 \leqslant l \leqslant d$. Then*

$$\mathrm{Span}(\mathcal{D}_{l,\infty}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{2,l,\infty})\,. \tag{4.11}$$

Proof. — For $k \geqslant 2$, we denote by $\mathcal{E}'_{1,l,n}(k)$ the set of elements $\omega \in \mathcal{D}_{l,n}$ of the form $\omega = [\tau_{i_1}^{a_1}\tau_{i_2}^{a_2}\cdots\tau_{i_l}^{a_l}\beta]$, where $1 \leqslant i_1 \leqslant k$, $1 \leqslant i_j \leqslant n-1$ for $2 \leqslant j \leqslant l$, and $\beta \in \mathcal{B}_n$. We set $\mathcal{E}'_{1,l,\infty}(k) = \cup_{n=2}^{+\infty}\mathcal{E}'_{1,l,n}(k)$. Note that $\mathcal{E}'_{1,l,\infty}(1) = \mathcal{E}_{1,l,\infty}$, and $\mathcal{D}_{l,n} = \mathcal{D}_{l-1,n} \cup \mathcal{E}'_{1,l,n}(n-1)$ for all $n \geqslant 2$.

We prove that

$$\mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}'_{1,l,\infty}(k)) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}'_{1,l,\infty}(k-1)) \tag{4.12}$$

for all $k \geqslant 2$. This implies that

$$\mathrm{Span}(\mathcal{D}_{l,\infty}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{1,l,\infty})\,. \tag{4.13}$$

Let $\omega \in \mathcal{E}'_{1,l,\infty}(k)$ be of the form $\omega = [\tau_k^{a_1}\tau_{i_2}^{a_2}\cdots\tau_{i_l}^{a_l}\beta]$. By (4.1), we have

$$\begin{aligned}
\omega &= [\tau_k^{a_1}\sigma_k\sigma_{k-1}\sigma_k\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_k^{-1}\tau_{i_2}^{a_2}\cdots\tau_{i_l}^{a_l}\beta] \\
&= [\tau_{k-1}^{a_1}\tau_{i_2}^{a_2}\cdots\tau_{i_l}^{a_l}\beta] - q[\sigma_k\tau_{k-1}^{a_1}\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_k^{-1}\tau_{i_2}^{a_2}\cdots\tau_{i_l}^{a_l}\beta] \\
&\qquad\qquad + q[\tau_{k-1}^{a_1}\sigma_{k-1}^{-1}\sigma_k^{-1}\tau_{i_2}^{a_2}\cdots\tau_{i_l}^{a_l}\beta] \\
&= [\tau_{k-1}^{a_1}\tau_{i_2}^{a_2}\cdots\tau_{i_l}^{a_l}\beta] - q[\tau_{k-1}^{a_1}\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_k^{-1}\tau_{i_2}^{a_2}\cdots\tau_{i_l}^{a_l}\beta\sigma_k] \\
&\qquad\qquad + q[\tau_{k-1}^{a_1}\sigma_{k-1}^{-1}\sigma_k^{-1}\tau_{i_2}^{a_2}\cdots\tau_{i_l}^{a_l}\beta]\,.
\end{aligned}$$

It is easily checked by means of (4.2) and (4.5) that this element belongs to $\mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}'_{1,l,\infty}(k-1))$.

Now, for $k \geqslant 2$, we denote by $\mathcal{E}'_{2,l,n}(k)$ the set of elements $\omega \in \mathcal{D}_{l,n}$ of the form $\omega = [\tau_1^{a_1}\tau_{i_2}^{a_2}\tau_{i_3}^{a_3}\cdots\tau_{i_l}^{a_l}\beta]$, where $2 \leqslant i_2 \leqslant k$, $1 \leqslant i_j \leqslant n-1$ for $3 \leqslant j \leqslant l$, and $\beta \in \mathcal{B}_n$. We set $\mathcal{E}'_{2,l,\infty}(k) = \cup_{n=2}^{+\infty}\mathcal{E}'_{2,l,n}(k)$. Note that $\mathcal{E}'_{2,l,\infty}(2) = \mathcal{E}_{2,l,\infty}$, and $\mathcal{E}'_{2,l,n}(n-1) = \mathcal{E}_{1,l,n}$ for all $n \geqslant 2$.

Using the same arguments as in the proof of (4.12), one can easily show that

$$\mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}'_{2,l,\infty}(k)) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}'_{2,l,\infty}(k-1)) \tag{4.14}$$

for all $k \geqslant 3$. (Here we also need to use the fact that σ_k commutes with τ_1.) It follows that

$$\mathrm{Span}(\mathcal{D}_{l,\infty}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{2,l,\infty})\,.$$

□

Claim 2. — *Let $l \geqslant 3$, and let $2 \leqslant r \leqslant l-1$. Then*

$$\mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{r,l,\infty}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{r+1,l,\infty})\,. \tag{4.15}$$

Proof. — Set $\varepsilon = 2$ if r is even, and $\varepsilon = 1$ if r is odd. For $k \geqslant 3$, we denote by $\mathcal{E}'_{r+1,l,n}(k)$ the set of elements $\omega \in \mathcal{D}_{l,n}$ of the form $\omega = [\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_\varepsilon^{a_r}\tau_{i_{r+1}}^{a_{r+1}}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta]$, where $1 \leqslant i_{r+1} \leqslant k$, $1 \leqslant i_j \leqslant n-1$ for $r+2 \leqslant j \leqslant l$, and $\beta \in \mathcal{B}_n$. We set $\mathcal{E}'_{r+1,l,\infty}(k) = \cup_{n=2}^{+\infty}\mathcal{E}'_{r+1,l,n}(k)$. Note that $\mathcal{E}'_{r+1,l,n}(n-1) = \mathcal{E}_{r,l,n}$ for all $n \geqslant 2$.

Using the same arguments as in the proof of (4.12), one can easily show that

$$\mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}'_{r+1,l,\infty}(k)) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}'_{r+1,l,\infty}(k-1)) \tag{4.16}$$

for all $k \geqslant 4$. (Here we also need to use the fact that σ_k commutes with τ_1 and τ_2.) This implies that

$$\mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{r,l,\infty}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}'_{r+1,l,\infty}(3))\,. \tag{4.17}$$

Let $\omega \in \mathcal{E}'_{r+1,l,\infty}(3)$ be an element of the form

$$\omega = [\tau_1^{a_1}\tau_2^{a_2}\cdots\tau_\varepsilon^{a_r}\tau_3^{a_{r+1}}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta].$$

Now, in order to prove Claim 2, it suffices to show that such an element belongs to $\mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{r+1,l,\infty})$.

Assume that r is odd. So,

$$\omega = [\tau_1^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r}\tau_3^{a_{r+1}}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta]\,.$$

Let

$$\begin{aligned}\omega_1 &= [\tau_1^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r}\tau_3^{a_{r+1}}(\sigma_3-\sigma_1)^2\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta]\,,\\ \omega_2 &= [\tau_1^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r}\tau_3^{a_{r+1}}C_{1\,3}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta]\,.\end{aligned}$$

By Lemma 4.11, we have

$$\begin{aligned}\omega_1 &= [\tau_1^{a_1}\cdots\tau_2^{a_{r-1}+a_r}\tau_3^{a_{r+1}}(\sigma_3-\sigma_1)^2\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta]\\ &\quad+[\tau_1^{a_1}\cdots\tau_2^{a_{r-1}+a_{r+1}}\tau_1^{a_r}(\sigma_3-\sigma_1)^2\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta]\\ &\quad+[\tau_1^{a_1}\cdots\tau_2^{a_{r-1}+a_r+a_{r+1}}(B_{1\,2}-B_{2\,3})(\sigma_3-\sigma_1)\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta]\\ &\in \operatorname{Span}(\mathcal{D}_{l-1,\infty})\subset\operatorname{Span}(\mathcal{D}_{l-1,\infty}\cup\mathcal{E}_{r+1,l,\infty})\,.\end{aligned}$$

On the other hand, by Lemma 4.10,

$$\begin{aligned}\omega_2 &= [\tau_1^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r+a_{r+1}}C_{1\,3}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta]\\ &\in \operatorname{Span}(\mathcal{D}_{l-1,\infty})\subset\operatorname{Span}(\mathcal{D}_{l-1,\infty}\cup\mathcal{E}_{r+1,l,\infty})\,.\end{aligned}$$

Hence, by Lemma 4.10,

$$\omega=(q+1)^{-2}(\omega_1+\omega_2)\in\operatorname{Span}(\mathcal{D}_{l-1,\infty}\cup\mathcal{E}_{r+1,l,\infty})\,.$$

Now, assume that r is even. So

$$\omega=[\tau_1^{a_1}\cdots\tau_1^{a_{r-1}}\tau_2^{a_r}\tau_3^{a_{r+1}}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta]\,.$$

Let

$$\begin{aligned}\omega_1 &= [\tau_1^{a_1}\cdots\tau_1^{a_{r-1}}\tau_2^{a_r}\tau_1^{a_{r+1}}B_{1\,2}B_{2\,3}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-2}]\,,\\ \omega_2 &= [\tau_2^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r}\tau_3^{a_{r+1}}(\sigma_1-\sigma_3)\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\,.\end{aligned}$$

Obviously, $\omega_1\in\operatorname{Span}(\mathcal{D}_{l-1,\infty}\cup\mathcal{E}_{r+1,l,\infty})$. On the other hand, by Lemma 4.11,

$$\begin{aligned}\omega_2 &= [\tau_2^{a_1}\cdots\tau_2^{a_{r-1}+a_r}\tau_3^{a_{r+1}}(\sigma_1-\sigma_3)\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\\ &\quad+[\tau_2^{a_1}\cdots\tau_2^{a_{r-1}+a_{r+1}}\tau_1^{a_r}(\sigma_1-\sigma_3)\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\\ &\quad+[\tau_2^{a_1}\cdots\tau_2^{a_{r-1}+a_r+a_{r+1}}(B_{2\,3}-B_{1\,2})\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\\ &\in \operatorname{Span}(\mathcal{D}_{l-1,\infty})\subset\operatorname{Span}(\mathcal{D}_{l-1,\infty}\cup\mathcal{E}_{r+1,l,\infty})\,.\end{aligned}$$

Hence, by Lemma 4.8,

$$\begin{aligned}\omega &= [B_{1\,2}\tau_1^{a_1}\cdots\tau_1^{a_{r-1}}\tau_2^{a_r}\tau_3^{a_{r+1}}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\\ &= [\tau_2^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r}B_{1\,2}\tau_3^{a_{r+1}}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\\ &= [\tau_2^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r}\tau_3^{a_{r+1}}(\sigma_1-(q-1))\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\\ &\quad+[\tau_2^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r}\sigma_2\tau_3^{a_{r+1}}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\\ &= [\tau_2^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r}\tau_3^{a_{r+1}}(\sigma_1-(q-1))\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\\ &\quad+[\tau_2^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r}\tau_2^{a_{r+1}}B_{2\,3}\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\\ &\quad-[\tau_2^{a_1}\cdots\tau_2^{a_{r-1}}\tau_1^{a_r}\tau_3^{a_{r+1}}(\sigma_3-(q-1))\tau_{i_{r+2}}^{a_{r+2}}\cdots\tau_{i_l}^{a_l}\beta B_{1\,2}^{-1}]\\ &= \omega_1+\omega_2\in\operatorname{Span}(\mathcal{D}_{l-1,\infty}\cup\mathcal{E}_{r+1,l,\infty})\,.\end{aligned}$$

□

At this point, thanks to Claims 1 and 2, we have proved that

$$\mathrm{Span}(\mathcal{D}_{l,\infty}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{l,l,\infty})\,, \tag{4.18}$$

for all $l \geqslant 3$.

Claim 3. — *Let $l \geqslant 3$. Then*

$$\mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{l,l,\infty}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{l,l,3})\,. \tag{4.19}$$

Proof. — It suffices to show that

$$\mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{l,l,n}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{l,l,n-1})$$

for all $n \geqslant 4$.

Set $\varepsilon = 2$ if l is even, and $\varepsilon = 1$ if l is odd. Let $\omega \in \mathcal{E}_{l,l,n}$ be an element of the form $\omega = [\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_\varepsilon^{a_l}\beta]$, where $\beta \in \mathcal{B}_n$. By construction, either $\beta \in \mathcal{B}_{n-1}$, or $\beta = \alpha_1\sigma_{n-1}\alpha_2$ for some $\alpha_1, \alpha_2 \in B_{n-1}$. If $\beta \in \mathcal{B}_{n-1}$, then $\omega \in \mathcal{E}_{l,l,n-1}$. If $\beta = \alpha_1\sigma_{n-1}\alpha_2$ for some $\alpha_1, \alpha_2 \in B_{n-1}$, then

$$\omega = z[\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\cdots\tau_\varepsilon^{a_l}\alpha_1\alpha_2] \in \mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{l,l,n-1})\,.$$

□

Claim 4. — *Let $l \geqslant 3$. Then*

$$\mathrm{Span}(\mathcal{D}_{l-1,\infty} \cup \mathcal{E}_{l,l,3}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty})\,. \tag{4.20}$$

Proof. — Let

$$\delta_0 = (z^2 - (q-1)z - q)^{-1}(z - (q-1) + \sigma_1)\,.$$

A direct calculation shows that we have

$$\delta_0(z - \sigma_1) = (z - \sigma_1)\delta_0 = 1$$

in $\mathbb{K}(z) \otimes \mathcal{H}(B_2)$.

We set $\varepsilon = 1$ if l is odd, and $\varepsilon = 2$ if l is even. Let $\omega \in \mathcal{E}_{l,l,3}$. We write $\omega = [\tau_1^{a_1}\tau_2^{a_2}\cdots\tau_\varepsilon^{a_l}\beta]$, where $\beta \in \mathcal{B}_3$. Set

$$\begin{aligned}
\omega_1 &= [\tau_1^{a_1}\tau_3^{a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\,,\\
\omega_2 &= [\tau_1^{a_1+a_3}\tau_3^{a_2}(\sigma_3 - \sigma_1)\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\,,\\
\omega_3 &= [\tau_1^{a_1+a_2}B_{1\,2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\,.
\end{aligned}$$

Obviously, $\omega_2, \omega_3 \in \mathrm{Span}(\mathcal{D}_{l-1,\infty})$. On the other hand, by Lemma 4.11,

$$\begin{aligned}[(\sigma_3-\sigma_1)^2\tau_1^{a_1}\tau_3^{a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0] &= [(\sigma_3-\sigma_1)^2\tau_1^{a_1}\tau_2^{a_2+a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &\quad+[(\sigma_3-\sigma_1)^2\tau_3^{a_2}\tau_2^{a_1+a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &\quad+[(\sigma_3-\sigma_1)(B_{1\,2}-B_{2\,3})\tau_2^{a_1+a_2+a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &\in \mathrm{Span}(\mathcal{D}_{l-1,\infty})\,.\end{aligned}$$

Moreover, by Lemma 4.10,

$$\begin{aligned}[C_{1\,3}\tau_1^{a_1}\tau_3^{a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0] &= [C_{1\,3}\tau_1^{a_1+a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &\in \mathrm{Span}(\mathcal{D}_{l-1,\infty})\,.\end{aligned}$$

Hence, by Lemma 4.10,

$$\begin{aligned}\omega_1 &= (q+1)^{-2}[(\sigma_3-\sigma_1)^2\tau_1^{a_1}\tau_3^{a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &\quad+(q+1)^{-2}[C_{1\,3}\tau_1^{a_1}\tau_3^{a_2}\tau_2^{a_3}B_{1\,2}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &\in \mathrm{Span}(\mathcal{D}_{l-1,\infty})\,.\end{aligned}$$

Finally, by (4.2),

$$\begin{aligned}\omega &= [(z-\sigma_1)\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &= [(\sigma_3-\sigma_1)\tau_1^{a_1}\tau_2^{a_2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &= [\tau_1^{a_1}\sigma_3\tau_2^{a_2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]-[\tau_1^{a_1}\sigma_1\tau_2^{a_2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &= [\tau_1^{a_1}\tau_3^{a_2}\sigma_2\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]+[\tau_1^{a_1+a_3}\tau_3^{a_2}(\sigma_3-(q-1))\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &\qquad-[\tau_1^{a_1+a_2}B_{1\,2}\tau_1^{a_3}\tau_2^{a_4}\cdots\tau_\varepsilon^{a_l}\beta\delta_0]\\ &= \omega_1+\omega_2-\omega_3\in\mathrm{Span}(\mathcal{D}_{l-1,\infty})\,.\end{aligned}$$

□

At this point we have proved that

$$\mathrm{Span}(\mathcal{D}_{l,\infty}) = \mathrm{Span}(\mathcal{D}_{l-1,\infty})$$

for all $l \geqslant 3$. This implies that

$$\mathrm{TR}_d = \mathrm{Span}(\mathcal{D}_{d,\infty}) = \mathrm{Span}(\mathcal{D}_{2,\infty})\,. \tag{4.21}$$

Now, let

$$\begin{aligned}\mathcal{F}_1 = \Big\{[\tau_1^d],[\tau_1^d\sigma_1]\Big\}\cup\Big\{&[\tau_1^a\tau_2^b],[\tau_1^a\tau_2^b\sigma_1],[\tau_1^a\tau_2^b\sigma_2],[\tau_1^a\tau_2^b\sigma_1\sigma_2],[\tau_1^a\tau_2^b\sigma_2\sigma_1],\\ &[\tau_1^a\tau_2^b\sigma_1\sigma_2\sigma_1]\,;\,a,b\geqslant 1 \text{ and } a+b=d\Big\}\,.\end{aligned}$$

Claim 5. — $\mathrm{TR}_d = \mathrm{Span}(\mathcal{F}_1)$.

Proof. — One can easily prove using the same arguments as in the proof of Claim 1 that

$$\mathrm{Span}(\mathcal{D}_{2,\infty}) = \mathrm{Span}(\mathcal{E}_{1,1,\infty} \cup \mathcal{E}_{2,2,\infty})\,.$$

On the other hand, using the same arguments as in the proof of Claim 3, it is easily seen that

$$\mathrm{Span}(\mathcal{E}_{1,1,\infty} \cup \mathcal{E}_{2,2,\infty}) = \mathrm{Span}(\mathcal{F}_1)$$

□

Now, let

$$\mathcal{F}_2 = \{[\tau_1^d], [\tau_1^d\sigma_1]\} \cup \{[\tau_1^a\tau_2^b], [\tau_1^a\tau_2^b\sigma_1]\,;\, a, b \geqslant 1 \text{ and } a+b=d\}\,.$$

Claim 6. — $\mathrm{TR}_d = \mathrm{Span}(\mathcal{F}_2)$.

Proof. — Let $a, b \geqslant 1$ such that $a+b=d$. Then

$$\begin{aligned}[\tau_1^a\tau_2^b\sigma_1\sigma_2\sigma_1] = [\tau_1^a\sigma_1\sigma_2\sigma_1\tau_1^b] = [\tau_1^{a+b}\sigma_1\sigma_2\sigma_1] = z[\tau_1^{a+b}\sigma_1^2] \\ = z(q-1)[\tau_1^d\sigma_1] + zq[\tau_1^d] \in \mathrm{Span}(\mathcal{F}_2)\,.\end{aligned}$$

$$[\tau_1^a\tau_2^b\sigma_1\sigma_2] = [\tau_1^a\sigma_1\sigma_2\tau_1^b] = [\tau_1^{a+b}\sigma_1\sigma_2] = z[\tau_1^d\sigma_1] \in \mathrm{Span}(\mathcal{F}_2)\,.$$

By (4.3) we have

$$\begin{aligned}\left[\tau_1^a\tau_2^bB_{1\,2}\right] = [\tau_1^aB_{1\,2}\tau_1^b] = [\tau_1^{a+b}(\sigma_1+\sigma_2-(q-1))] \\ = [\tau_1^d(\sigma_1+z-(q-1))] \in \mathrm{Span}(\mathcal{F}_2)\,.\end{aligned}$$

On the other hand,

$$[\tau_1^a\tau_2^b\sigma_2] = [\tau_1^a\tau_2^bB_{1\,2}] - [\tau_1^a\tau_2^b\sigma_1] + (q-1)[\tau_1^a\tau_2^b]\,,$$

thus $[\tau_1^a\tau_2^b\sigma_2] \in \mathrm{Span}(\mathcal{F}_2)$. By (4.4) we have

$$\begin{aligned}&[\tau_1^a\tau_2^b(\sigma_1\sigma_2+\sigma_2\sigma_1-(q-1)\sigma_1-(q-1)\sigma_2+(q^2-q+1))] \\ &\quad= [\tau_1^{a+b}(\sigma_1\sigma_2+\sigma_2\sigma_1-(q-1)\sigma_1-(q-1)\sigma_2+(q^2-q+1))] \\ &\quad= [\tau_1^d(2z\sigma_1-(q-1)\sigma_1-(q-1)z+(q^2-q+1))] \\ &\quad\in \mathrm{Span}(\mathcal{F}_2)\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}[\tau_1^a\tau_2^b\sigma_2\sigma_1] = \Big[\tau_1^a\tau_2^b\Big(\sigma_1\sigma_2+\sigma_2\sigma_1-(q-1)\sigma_1-(q-1)\sigma_2 \\ +(q^2-q+1)\Big)\Big] - [\tau_1^a\tau_2^b\sigma_1\sigma_2] + (q-1)[\tau_1^a\tau_2^b\sigma_1] + (q-1)[\tau_1^a\tau_2^b\sigma_2] \\ -(q^2-q+1)[\tau_1^a\tau_2^b]\,,\end{aligned}$$

thus $[\tau_1^a\tau_2^b\sigma_2\sigma_1] \in \mathrm{Span}(\mathcal{F}_2)$. □

Let

$$\mathcal{F}_3 = \{[\tau_1^d], [\tau_1^d\sigma_1]\} \cup \{[\tau_1^a\tau_2^b]\,;\, a \geqslant b \geqslant 1 \text{ and } a+b=d\} \\ \cup \{[\tau_1^a\tau_2^b\sigma_1]\,;\, a > b \geqslant 1 \text{ and } a+b=d\}\,.$$

Note that $|\mathcal{F}_3| = d+1$, thus the following finishes the proof of Theorem 4.7.

Claim 7. — $\mathrm{TR}_d = \mathrm{Span}(\mathcal{F}_3)$.

Proof. — Let $a, b \geqslant 1$ such that $a < b$ and $a+b=d$.

$$[\tau_1^a\tau_2^b] = [\tau_2^b\tau_1^a] = [B_{1\,2}\tau_2^b\tau_1^aB_{1\,2}^{-1}] = [\tau_1^b\tau_2^a] \in \mathrm{Span}(\mathcal{F}_3)\,.$$

By Lemma 4.8,

$$[\tau_1^b\tau_2^aB_{1\,2}] = [\tau_1^bB_{1\,2}\tau_1^a] = [\tau_1^{a+b}(\sigma_1+\sigma_2-(q-1))] \\ = [\tau_1^d(\sigma_1+z-(q-1))] \in \mathrm{Span}(\mathcal{F}_3)\,.$$

Moreover,

$$[\tau_1^b\tau_2^a\sigma_2] = [\tau_1^b\tau_2^aB_{1\,2}] - [\tau_1^b\tau_2^a\sigma_1] + (q-1)[\tau_1^b\tau_2^a]\,,$$

thus $[\tau_1^b\tau_2^a\sigma_2] \in \mathrm{Span}(\mathcal{F}_3)$. It follows that

$$[\tau_1^a\tau_2^b\sigma_1] = [\sigma_1\tau_1^a\tau_2^b] = [\tau_1^a\sigma_1\tau_2^b] = [\tau_2^b\tau_1^a\sigma_1] = [B_{1\,2}\tau_2^b\tau_1^a\sigma_1B_{1\,2}^{-1}] \\ = [\tau_1^b\tau_2^a\sigma_2] \in \mathrm{Span}(\mathcal{F}_3)\,.$$

Now, assume that d is even, and let $a=b=\frac{d}{2}$. We have

$$[\tau_1^a\tau_2^a\sigma_1] = [\sigma_1\tau_1^a\tau_2^a] = [\tau_1^a\sigma_1\tau_2^a] = [\tau_2^a\tau_1^a\sigma_1] = [B_{1\,2}\tau_2^a\tau_1^a\sigma_1B_{1\,2}^{-1}] = [\tau_1^a\tau_2^a\sigma_2]\,.$$

Moreover,

$$[\tau_1^a\tau_2^aB_{1\,2}] = [\tau_1^aB_{1\,2}\tau_1^a] = [\tau_1^d(\sigma_1+\sigma_2-(q-1))] \\ = [\tau_1^d(\sigma_1+z-(q-1))] \in \mathrm{Span}(\mathcal{F}_3)\,.$$

Thus

$$[\tau_1^a\tau_2^a\sigma_1] = \frac{1}{2}([\tau_1^a\tau_2^a\sigma_1] + [\tau_1^a\tau_2^a\sigma_2]) \\ = \frac{1}{2}[\tau_1^a\tau_2^aB_{1\,2}] + \frac{1}{2}(q-1)[\tau_1^a\tau_2^a] \in \mathrm{Span}(\mathcal{F}_3)\,.$$

□

5. Universal Markov trace and universal HOMFLY-type invariant

Let X, Y be two new variables. We define the *universal Markov trace* as the collection $\hat{T} = \{\widehat{\mathrm{tr}}_n\}_{n=1}^{+\infty}$ of $\mathbb{K}$-linear maps

$$\widehat{\mathrm{tr}}_n : \mathcal{H}(SB_n) \to \mathbb{C}(\sqrt{q}, z)[X, Y]\,, \quad n \geqslant 1,$$

defined as follows. Let $d \geqslant 0$, and let $\omega \in \mathcal{H}(S_d B_n)$. Then

$$\widehat{\mathrm{tr}}_n(\omega) = \sum_{k=0}^{d} \frac{\sqrt{q}^k}{(d-k)!\,k!} X^k \cdot Y^{d-k} \cdot T_k^d(\omega)\,,$$

where $\{T_0^d, T_1^d, \ldots, T_d^d\}$ is the $\mathbb{K}(z)$-basis of TR_d constructed in Section 4.

Proposition 5.1. —

(1) *We have* $\widehat{\mathrm{tr}}_n(\alpha\beta) = \widehat{\mathrm{tr}}_n(\beta\alpha)$ *for all* $\alpha, \beta \in SB_n$, *and all* $n \geqslant 1$.
(2) *Let* $\iota_n : \mathcal{H}(SB_n) \to \mathcal{H}(SB_{n+1})$ *be the morphism induced by the inclusion* $SB_n \hookrightarrow SB_{n+1}$. *Then* $\widehat{\mathrm{tr}}_{n+1} \circ \iota_n = \widehat{\mathrm{tr}}_n$ *for all* $n \geqslant 1$.
(3) $\widehat{\mathrm{tr}}_{n+1}(\iota_n(\omega)\sigma_n) = z \cdot \widehat{\mathrm{tr}}_n(\omega)$ *for all* $\omega \in \mathcal{H}(SB_n)$, *and all* $n \geqslant 1$.
(4) *We have*

$$\widehat{\mathrm{tr}}_n(\tau_i\omega) = X\sqrt{q} \cdot \widehat{\mathrm{tr}_n}(\sigma_i\omega) + Y \cdot \widehat{\mathrm{tr}}_n(\omega)$$

for all $\omega \in \mathcal{H}(SB_n)$, *all* $n \geqslant 2$, *and all* $1 \leqslant i \leqslant n-1$.

Proof. — Parts (1), (2), and (3) follow from the definition of a Markov trace (see Section 3), and from the fact that $T_0^d, T_1^d, \ldots, T_d^d$ are Markov traces for all $d \geqslant 0$.

We turn now to prove (4). Let $\beta \in S_d B_n$. We write β in the form

$$\beta = \alpha_0 \tau_{i_1} \alpha_1 \cdots \tau_{i_d} \alpha_d\,,$$

where $1 \leqslant i_j \leqslant n-1$ for $1 \leqslant j \leqslant d$, and $\alpha_j \in B_n$ for $0 \leqslant j \leqslant d$. For $S \subset \{1, \ldots, d\}$ we set

$$\beta(S) = \alpha_0 u_1 \alpha_1 \cdots u_d \alpha_d\,,$$

where $u_j = \sigma_{i_j}$ if $j \in S$, and $u_j = 1$ if $j \notin S$. It is easily checked that, for $0 \leqslant k \leqslant d$, $T_k^d(\beta)$ is given by the formula

$$T_k^d(\beta) = k!\,(d-k)! \sum_{\substack{S \subset \{1,\ldots,d\} \\ |S| = k}} T_0^0(\beta(S))\,. \tag{5.1}$$

This implies that

$$\widehat{\mathrm{tr}}_n(\beta) = \sum_{S \subset \{1,\ldots,d\}} \sqrt{q}^{|S|} X^{|S|} Y^{d-|S|} \cdot T_0^0(\beta(S))\,. \tag{5.2}$$

Now, from (5.2), it follows that

$$\begin{aligned}\widehat{\mathrm{tr}}_n(\tau_i\beta) &= X\sqrt{q}\sum_{S\subset\{1,\dots,d\}}\sqrt{q}^{|S|}X^{|S|}Y^{d-|S|}\cdot T_0^0(\sigma_i\beta(S))\\ &\quad + Y\sum_{S\subset\{1,\dots,d\}}\sqrt{q}^{|S|}X^{|S|}Y^{d-|S|}\cdot T_0^0(\beta(S))\\ &= X\sqrt{q}\cdot\widehat{\mathrm{tr}}_n(\sigma_i\beta)+Y\cdot\widehat{\mathrm{tr}}_n(\beta)\,.\end{aligned}$$

□

Recall from Section 3 that $\pi : SB_n \to \mathcal{H}(SB_n)$ denotes the natural map, and that $\varepsilon : SB_n \to \mathbb{Z}$ is the homomorphism defined by

$$\varepsilon(\sigma_i)=1\,,\quad \varepsilon(\sigma_i^{-1})=-1\,,\quad \varepsilon(\tau_i)=0\,,\quad \text{for } 1\leqslant i\leqslant n-1\,.$$

We consider the following change of variables:

$$z=\frac{q-1}{1-qy}\quad\Leftrightarrow\quad y=\frac{z-q+1}{qz}\,.$$

For $\beta\in SB_n$, we set

$$\hat{I}(\beta)=\left(\frac{q-1}{1-qy}\right)^{-n+1}\cdot(\sqrt{y})^{\varepsilon(\beta)-n+1}\cdot\widehat{\mathrm{tr}}_n(\pi(\beta))\,.$$

This is an element of $\mathbb{C}(\sqrt{q},\sqrt{y})[X,Y]$.

The following can be proved in the same way as Proposition 3.3.

PROPOSITION 5.2. — *Let (α,n) and (β,m) be two singular braids. If $\hat{\alpha}$ is isotopic to $\hat{\beta}$, then $\hat{I}(\alpha)=\hat{I}(\beta)$.*

Let $\mathcal{L}$ denote the set of (isotopy classes of) singular links. For $L\in\mathcal{L}$, we choose a singular braid (β,n) such that $\hat{\beta}=L$, and we set $\hat{I}(L)=\hat{I}(\beta)$. By Proposition 5.2, the map $\hat{I}:\mathcal{L}\to\mathbb{C}(\sqrt{q},\sqrt{y})[X,Y]$ is a well-defined invariant that we call the *universal HOMFLY-type invariant* of $\mathcal{L}$.

For $d\geqslant 0$, we denote by $\mathcal{S}_d$ the set of invariants $I:\mathcal{L}_d\to\mathbb{C}(\sqrt{q},\sqrt{y})$ which satisfies the skein relation for $t=\sqrt{y}\sqrt{q}$ and $x=\sqrt{q}-\frac{1}{\sqrt{q}}$. Now, the above terminology "universal HOMFLY-type invariant" is justified by the following.

THEOREM 5.3. — *Let $d\geqslant 0$, and let $L,L'\in\mathcal{L}_d$. We have $\hat{I}(L)=\hat{I}(L')$ if and only if $I(L)=I(L')$ for all $I\in\mathcal{S}_d$.*

Proof. — Let TR'_d be the space of traces on $\{\mathcal{H}(S_dB_n)\}_{n=1}^{+\infty}$ with coefficients in $\mathbb{C}(\sqrt{q},\sqrt{y})$. Clearly, TR'_d is a $\mathbb{C}(\sqrt{q},\sqrt{y})$-vector space, and

$$\mathrm{TR}'_d=\mathbb{C}(\sqrt{q},\sqrt{y})\otimes\mathrm{TR}_d\,.$$

On the other hand, we have

$$\hat{I}(\beta) = \sum_{k=0}^{d} \frac{\sqrt{q}^k}{(d-k)!\,k!} X^k Y^{d-k} \cdot I_{T_k^d}(\beta)\,, \tag{5.3}$$

for all $\beta \in S_d B_n$.

Let $L, L' \in \mathcal{L}_d$ such that $\hat{I}(L) = \hat{I}(L')$. By (5.3), we have $I_{T_k^d}(L) = I_{T_k^d}(L')$ for all $0 \leqslant k \leqslant d$. Let $I \in \mathcal{S}_d$. By Proposition 3.5, there exists $T \in \mathrm{TR}'_d$ such that $I = I_T$. By Theorem 4.7, there exist $\lambda_0, \lambda_1, \ldots, \lambda_d \in \mathbb{C}(\sqrt{q}, \sqrt{y})$ such that

$$T = \lambda_0 T_0^d + \lambda_1 T_1^d + \cdots + \lambda_d T_d^d\,.$$

Then

$$I(L) = \sum_{k=0}^{d} \lambda_k I_{T_k^d}(L) = \sum_{k=0}^{d} \lambda_k I_{T_k^d}(L') = I(L')\,.$$

Now, let $L, L' \in \mathcal{L}_d$ such that $I(L) = I(L')$ for all $I \in \mathcal{S}_d$. We have in particular $I_{T_k^d}(L) = I_{T_k^d}(L')$ for all $0 \leqslant k \leqslant d$, thus, by (5.3), $\hat{I}(L) = \hat{I}(L')$. □

Let A be an abelian group, let $I : \mathcal{L} \to A$ be an invariant, and let $X, Y \in A$. We say that I satisfies the (X, Y) *desingularization relation* if

$$I(L_X) = X \cdot I(L_+) + Y \cdot I(L_0)\,,$$

for all singular links $L_X, L_+, L_0 \in \mathcal{L}$ that have the same link diagram except in a neighborhood of a crossing where they are as in Figure 5.1.

Figure 5.1. The singular links L_X, L_+, and L_0.

We set

$$t = \sqrt{y}\sqrt{q}\,, \quad x = \sqrt{q} - \frac{1}{\sqrt{q}}\,,$$

and we define $\widetilde{\mathrm{tr}}_n : SB_n \to \mathbb{C}(\sqrt{q}, \sqrt{y})[X, Y]$ by

$$\widetilde{\mathrm{tr}}_n(\beta) = (\sqrt{y})^{-\varepsilon(\beta)} \cdot \widehat{\mathrm{tr}}_n(\pi(\beta))\,.$$

With these new notations, $\hat{I}(\beta)$ can be written

$$\hat{I}(\beta) = \left(\frac{1-t^2}{tx}\right)^{n-1} \cdot t^{\varepsilon(\beta)} \cdot \widetilde{\mathrm{tr}}_n(\beta)\,.$$

PROPOSITION 5.4. — *The invariant $\hat{I}$ satisfies the (t,x) skein relation and the (X,Y) desingularization relation.*

Proof. — The fact that $\hat{I}$ satisfies the (t,x) skein relation is proved in the same way as Proposition 3.4. So, we only need to show that $\hat{I}$ satisfies the (X,Y) desingularization relation.

Let $L_X, L_+, L_0 \in \mathcal{L}$ be three singular links that have the same link diagram except in a neighborhood of a crossing where they are as in Figure 5.1. A careful reading of Birman's proof of Theorem 2.2 shows that there exist a singular braid (β, n) and an index $1 \leqslant i \leqslant n-1$ such that $L_X = \widehat{\tau_i\beta}$, $L_+ = \widehat{\sigma_i\beta}$, and $L_0 = \hat{\beta}$. On the other hand, Proposition 5.1(4) implies that

$$\widetilde{\mathrm{tr}}_n(\tau_i\beta) = Xt \cdot \widetilde{\mathrm{tr}}_n(\sigma_i\beta) + Y \cdot \widetilde{\mathrm{tr}}_n(\beta)\,.$$

Hence

$$\begin{aligned} X \cdot \hat{I}(L_+) + Y \cdot \hat{I}(L_0) &= \left(\frac{1-t^2}{tx}\right)^{n-1} \cdot t^{\varepsilon(\tau_i\beta)} \cdot (Xt \cdot \widetilde{\mathrm{tr}}_n(\sigma_i\beta) + Y \cdot \widetilde{\mathrm{tr}}_n(\beta)) \\ &= \left(\frac{1-t^2}{tx}\right)^{n-1} \cdot t^{\varepsilon(\tau_i\beta)} \cdot \widetilde{\mathrm{tr}}_n(\tau_i\beta) \\ &= \hat{I}(L_X)\,. \end{aligned}$$

□

Now, the following shows that our invariant $\hat{I}$ is a reasonable extension of the HOMFLY polynomial to the singular links.

THEOREM 5.5. — *There exists a unique invariant $\hat{I} : \mathcal{L} \to \mathbb{C}(\sqrt{q}, \sqrt{y})[X,Y]$ which satisfies the (t,x) skein relation and the (X,Y) desingularization relation, and which takes the value 1 on the trivial knot. Moreover, $\hat{I}(L) \in \mathbb{C}[t^{\pm 1}, x^{\pm 1}, X, Y]$ for all $L \in \mathcal{L}$.*

Proof. — The existence of the invariant is given by Proposition 5.4.

Suppose that $\hat{I}' : \mathcal{L} \to \mathbb{C}(\sqrt{q}, \sqrt{y})[X,Y]$ is an invariant which satisfies the (t,x) skein relation and the (X,Y) desingularization relation, and which takes the value 1 on the trivial knot. Let $L \in \mathcal{L}_d$ be a singular link with d singular points. We prove by induction on $d \geqslant 0$ that $\hat{I}'(L) = \hat{I}(L)$, and that this element belongs to $\mathbb{C}[t^{\pm 1}, x^{\pm 1}, X, Y]$.

The case $d = 0$ is well-known (see [7], [3]). We assume $d \geqslant 1$. Let P be a singular point of L. Set $L_X = L$, and let L_+ and L_0 be the singular

links having the same link diagram as L except in a neighborhood of P where they are as in Figure 5.1. Then, by induction and by the (X,Y) desingularization relation, we have

$$\hat{I}'(L) = X \cdot \hat{I}'(L_+) + Y \cdot \hat{I}'(L_0) = X \cdot \hat{I}(L_+) + Y \cdot \hat{I}(L_0) = \hat{I}(L)\,.$$

Moreover, again by induction,

$$\hat{I}(L) = X \cdot \hat{I}(L_+) + Y \cdot \hat{I}(L_0) \in \mathbb{C}[t^{\pm 1}, x^{\pm 1}, X, Y]\,.$$

□

In [9] Kauffman and Vogel proved that there exists an invariant $I_{KV} : \mathcal{L} \to \mathbb{C}(a,A,B)$ defined by the equalities

$$\begin{aligned} a \cdot I_{KV}(L_+) &= A \cdot I_{KV}(L_0) + I_{KV}(L_X)\,, \\ a^{-1} \cdot I_{KV}(L_-) &= B \cdot I_{KV}(L_0) + I_{KV}(L_X)\,, \end{aligned} \tag{5.4}$$

for all links $L_X, L_+, L_-, L_0 \in \mathcal{L}$ that have the same link diagram except in a neighborhood of a crossing where they are as in Figure 3.1 or as in Figure 5.1. It is easily checked that (5.4) is equivalent to

$$\begin{aligned} a \cdot I_{KV}(L_+) - a^{-1} \cdot I_{KV}(L_-) &= (A-B) \cdot I_{KV}(L_0)\,, \\ I_{KV}(L_X) &= a \cdot I_{KV}(L_+) - A \cdot I_{KV}(L_0)\,, \end{aligned} \tag{5.5}$$

thus I_{KV} is defined by the $(a^{-1}, A-B)$ skein relation and the (a, A^{-1}) desingularization relation. In other words, we have:

LEMMA 5.6. — *The invariant I_{KV} can be obtained from the universal HOMFLY-type invariant $\hat{I}$ by setting $t = a^{-1}$, $x = A - B$, $X = a$, and $Y = -A$.*

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