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T.M. GENDRON

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APPROXIMATE ROOTS OF PSEUDO-ANOSOV DIFFEOMORPHISMS

by T.M. GENDRON

ABSTRACT. — The Root Conjecture predicts that every pseudo-Anosov diffeomorphism of a closed surface has Teichmüller approximate nth roots for all $n \geqslant 2$. In this paper, we replace the Teichmüller topology by the heights-widths topology – that is induced by convergence of tangent quadratic differentials with respect to both the heights and widths functionals – and show that every pseudo-Anosov diffeomorphism of a closed surface has heights-widths approximate nth roots for all $n \geqslant 2$.

RÉSUMÉ. — La Conjecture de la Racine prévoit que chaque difféomorphisme pseudo-Anosov d'une surface fermée a une racine $n$ième approximative de Teichmüller pour tout $n \geqslant 2$. Dans cet article, on remplace la topologie de Teichmüller par la topologie hauteur-longueur – celle qui est induite par la convergence des différentielles quadratiques tangentes relativement aux fonctionnelles hauteurs et longueurs simultanément – et on prouve que chaque difféomorphisme pseudo-Anosov d'une surface fermée a une racine $n$ième approximative hauteur-longueur pour tout $n \geqslant 2$.

1. Introduction

Let Z be a closed surface of genus $\geqslant 2$ and let $\Phi : Z \to Z$ be a pseudo Anosov diffeomorphism. The induced map Φ_* on the Teichmüller space $\mathscr{T}(Z)$ is an isometry which fixes a Teichmüller geodesic A (the axis of Φ), along which Φ_* acts by a translation of $\log \lambda$ for some $\lambda > 1$ (the entropy of Φ).

Fix a surface Σ of genus 2 and denote by $\mathscr{T}$ the (isometric) direct limit of the Teichmüller spaces $\mathscr{T}(Z)$ of all closed surfaces Z of genus $\geqslant 2$, where the system maps are isometric inclusions induced by unramified coverings, and where each isomorphism class of cover $Z \to \Sigma$ appears exactly once. Thus $\mathscr{T}$ is the universal Teichmüller space parametrizing all marked Riemann surfaces of closed and hyperbolic type.

Keywords: Teichmuller space, pseudo-Anosov diffeomorphism, root conjecture.
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Consider a sequence of pseudo Anosov diffeomorphisms $\{\Psi_i\}$ of closed surfaces Z_i of genus $\geqslant 2$. We call such a sequence an *approximate* n*th root* of Φ if the entropies λ_i of Ψ_i converge to $\sqrt[n]{\lambda}$ and if the axes A_i of Ψ_i converge uniformly on compacta to A in $\mathscr{T}$.

ROOT CONJECTURE. — *Every pseudo Anosov diffeomorphism of a closed surface of genus $\geqslant 2$ has an approximate* n*th root for every natural number $n \geqslant 2$.*

The Root Conjecture affirmed would imply that the universal commensurator mapping class group acts minimally on $\mathscr{T}$: the latter is known as the Ehrenpreis Conjecture [4], [9]. It may also be useful in the search for genuine roots of pseudo Anosov diffeomorphisms, which have been the subject of several recent studies *e.g.* [2], [6]. In this article, we shall prove an analogue of the Root Conjecture in which the Teichmüller topology on $\mathscr{T}$ is replaced by the *heights-widths* topology.

The heights-widths topology may be described as follows. Let $\mathscr{Q}$ be the associated direct limit of the cotangent bundles of quadratic differentials and denote by C the collection of all simple closed curves in surfaces appearing in the system defining $\mathscr{T}$. Given $c \in \mathsf{C}$, we define its q-height and q-width by pulling back q and c to a common surface Z and measuring c with respect to the horizontal and vertical line fields of q, scaled by $(g-1)^{-1}$ where g is the genus of Z. The result is independent of the choice of Z, and the heights-widths topology on $\mathscr{Q}$ is defined by declaring $q_i \to q$ if both of the corresponding functionals converge. See §5 for more details.

The notion of a heights-widths approximate nth root is obtained by replacing the requirement of Teichmüller convergence of axes by heights-widths convergence of tangent quadratic differentials along the axes. The goal of this paper is to prove the following:

MAIN THEOREM. — *Every pseudo Anosov diffeomorphism of a closed surface of genus $\geqslant 2$ has an approximate heights-widths nth root for every natural number $n \geqslant 2$.*

The Main Theorem is proved by construction, making use of an unramified version of the covers considered in [17] to produce small entropy pseudo Anosov diffeomorphisms. One first considers pseudo Anosov diffeomorphisms of the form

$$\Phi = G^{-4N} \circ F^{4N} : Z \longrightarrow Z \tag{1}$$

where F and G are right Dehn twists about a pair of filling simple closed curves c, d which *interlace*, see §§6,8. The latter means that there exist disjoint non-separating curves α, β so that if one takes $2k$ copies of $Z-(\alpha \cup \beta)$ and glues them together in a circular pattern to form a covering surface Z_k, then there are k lifts of each of c and d, each mapping back onto their ancestor with degree 2, and interlacing to form a necklace as indicated in Figure 6. For $k = mn$, Φ as in (1) lifts to the necklace as

$$\tilde{\Phi} \;=\; G_{mn}^{-2N} \circ \cdots \circ G_1^{-2N} \circ F_{mn}^{2N} \circ \cdots \circ F_1^{2N},$$

where F_i, G_i, $i = 1, \ldots, mn$, are the right Dehn twists about the mn lifts of c, d. If one denotes by χ the clockwise rotation of the necklace by an angle of $2\pi/n$, then the necklace root is defined as the sequence

$$\sqrt[n]{\Phi}_m \;=\; \chi \circ G_m^{-2N} \circ \cdots \circ G_1^{-2N} \circ F_m^{2N} \circ \cdots \circ F_1^{2N}$$

where $m = 2, 3, \ldots$. See §9 for more detail.

From the necklace root we consider the related sequence

$$\Psi_m \;=\; \left(\sqrt[nm]{\Phi}_m \right)^m$$

which, despite appearances, is not the same as the necklace root, and which enjoys better convergence properties than the latter. Examination of the carrying matrices of Φ and Ψ_m (see §§7,9) reveals convergence of intersection pairings

$$\mathbb{I}(\mathfrak{f}_m^u, \gamma) \longrightarrow \mathbb{I}(\mathfrak{f}^u, \gamma) \qquad \text{and} \qquad \mathbb{I}(\mathfrak{f}_m^s, \gamma) \longrightarrow \mathbb{I}(\mathfrak{f}^s, \gamma).$$

where $\mathfrak{f}_m^u, \mathfrak{f}^u$ (where $\mathfrak{f}_m^s, \mathfrak{f}^s$) are the unstable laminations (the stable laminations) of Ψ_m, Φ, and where γ is any test simple closed curve in Z. The proof of the latter is given in §10. Using the fact that the axes of pseudo Anosov diffeomorphisms of the form (1) are directionally dense (see §8), we approximate the axis A of an arbitrary pseudo Anosov Φ by a sequence of axes A_i of such pseudo Anosovs $\Phi_i : Z_i \to Z_i$, and extracting a suitable diagonal subsequence of necklace roots of the latter provides the desired lengths-widths root of Φ.

We note that an analogous result for the Weil-Petersson topology has recently been announced by J. Kahn and V. Markovic [12].

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2. The Fundamental System

All surfaces in this paper are assumed closed and of genus at least 2. Let Σ be such surface, to be fixed throughout, and which for convenience we take to be genus 2. For each finite index subgroup H of $\pi_1\Sigma$, choose a cover $\rho_H : Z_H \to \Sigma$ for which

$$(\rho_H)_*\pi_1 Z_H = H.$$

By covering space theory, whenever $H < H'$ there is a unique cover

$$\sigma_{H,H'} : Z_H \to Z_{H'}$$

for which $\rho_H = \rho_{H'} \circ \sigma_{H,H'}$. In this way we obtain an inverse system of finite covering maps indexed by the partially ordered latice of finite index subgroups of $\pi_1\Sigma$. We call this the *fundamental system* and denote it $\mathscr{S} = \mathscr{S}_\Sigma$. We will apply various contravariant functors to $\mathscr{S}$ (Teichmüller space, space of measured laminations, space of quadratic differentials) to obtain by direct limit universal spaces which organize in a genus independent fashion spaces which normally depend on a particular topological surface. It will be clear that these constructions do not depend on the choice of covers ρ_H in their isomorphism classes nor on the choice of base surface Σ. Making use of these genus-independent objects has the advantage of simplifying many statements, and although it is true that everything that we will discuss could be formulated without recourse to them, it can also be said that the Main Theorem is really about them. See [9] for more on this perspective.

3. Measured Laminations

Much of the material here is standard [20], [3], [5], and is reviewed to fix ideas and notation. Let Z be a surface of genus g. A *measured lamination* $\mathfrak{f}$ on Z is a closed 1-dimensional lamination smoothly embedded in Z and possessing a transverse invariant measure $m_\mathfrak{f}$. Two measured laminations are equivalent if they are isotopic through an isotopy taking one measure to the other or if their measures are identically 0. We shall also denote by $\mathfrak{f}$ the equivalence class defined by $\mathfrak{f}$. The set of equivalence classes of measured laminations is denoted $\mathsf{ML}(Z)$. We topologize this set by declaring that a sequence $\{\mathfrak{f}_i\}$ converges to $\mathfrak{f}$ if there exist representative measured laminations for which the measures converge with respect to test transversals.

Let $\mathsf{C}(Z)$ denote the set of isotopy classes of simple closed curves in Z. Note that given $c \in \mathsf{C}(Z)$ and any positive real number r, we may define an element $rc \in \mathsf{ML}(Z)$ which asigns the measure r to any segment intersecting c transversally in a point. In particular we may view $\mathsf{C}(Z) \subset \mathsf{ML}(Z)$ by taking $r = 1$.

Given $\mathfrak{f} \in \mathsf{ML}(Z)$ and $c \in \mathsf{C}(Z)$, the *intersection pairing* is defined

$$\mathbb{I}_Z(\mathfrak{f}, c) \;=\; \inf \int_{\mathfrak{f}\cap c} dm_\mathfrak{f},$$

where the infimum is taken over representatives of the classes of $\mathfrak{f}$ and c. The intersection topology on $\mathsf{ML}(Z)$ is the weak topology defined by the intersection pairing, and it coincides with the topology defined above.

There is a natural action of $\mathbb{R}_+$ on $\mathsf{ML}(Z)$ defined by scaling the measure. The space of projective classes of measured laminations is denoted $\mathsf{PL}(Z)$, and by a theorem of Thurston [21], [5], is homeomorphic to a sphere of dimension $6g-7$. We have $\mathsf{C}(Z) \subset \mathsf{PL}(Z)$ with dense image. The intersection pairing extends to a pairing $\mathsf{ML}(Z) \times \mathsf{ML}(Z) \to [0,\infty)$ via the formula [13]

$$\mathbb{I}_Z(\mathfrak{f},\mathfrak{g}) \;=\; \inf \int_{\mathfrak{f}\cap\mathfrak{g}} dm_{\mathfrak{f}} \otimes dm_{\mathfrak{g}}.$$

A word is in order here regarding the allied concept of a *measured foliation*, a singular foliation f of Z equipped with a transverse invariant measure: these typically arise as trajectories of quadratic differentials [19]. Two measured foliations are equivalent if after a finite number of Whitehead moves are applied to their singular leaves, they are isotopic through an isotopy taking one measure to the other. There is a bijective correpondence between classes of measured foliations and classes of measured laminations [10]. For example, to obtain a measured lamination starting with a measured foliation f, one chooses a nonsingular leaf from each minimal component of f, replacing each such leaf with a geodesic representative (with respect to, say, a hyperbolic metric) and completes the resulting space. Our default will be to work with measured laminations, and whenever a measured foliation happens to arise, we will assume it has been converted into its associated measured lamination.

The association $Z \mapsto \mathsf{ML}(Z)$ is a contravariant functor in the category of surfaces and finite covers, associating to a cover an inclusion of spaces of measured laminations. Applying this functor to $\mathscr{S}$ and taking direct limits, we obtain a space ML. We denote by $\mathsf{C} \subset \mathsf{ML}$ the union of the images of the $\mathsf{C}(Z)$ for all Z occurring in the system $\mathscr{S}$.

We may define an intersection pairing on ML by scaling the intersection pairings occurring on the surfaces Z in $\mathscr{S}$. Namely, if Z is of genus g and $\mathfrak{f}$, $\mathfrak{g} \in \mathsf{ML}(Z)$, write

$$\mathbb{I}(\mathfrak{f},\mathfrak{g}) = \frac{1}{g-1}\mathbb{I}_Z(\mathfrak{f},\mathfrak{g}).$$

Then if $\rho : W \to Z$ is a finite cover and $\rho^*\mathfrak{f},\rho^*\mathfrak{g} \in \mathsf{ML}(W)$ are the pull-back laminations we have $\mathbb{I}(\mathfrak{f},\mathfrak{g}) = \mathbb{I}(\rho^*\mathfrak{f},\rho^*\mathfrak{g})$. In this way we obtain a well-defined functional

$$\mathbb{I} : \mathsf{ML} \times \mathsf{ML} \to [0,\infty).$$

4. Tracks

We recall here facts about train tracks and bigon tracks, see [20], [18]. Let $\tau \subset Z$ be a smooth 1-dimensional branched manifold: thus τ is a 1-dimensional CW-complex in which the interiors of edges are smooth curves, and the field of tangent lines $T_x\tau$, $x \in \tau \setminus \{\text{vertices}\}$, extends to a continuous line field on τ. We say that τ is a *train track* if it satisfies the following additional properties:

1. The valency of any vertex is at least 3, except for simple closed curve components, which have a single vertex of valence 2.
2. If $D(S)$ is the double of a component $S \subset Z \setminus \tau$, then the Euler characteristic of $D(S)$ is negative.

The vertices of a train track are called *switches*.

A *bigon track* is a smooth 1-dimensional branched manifold $\tau \subset Z$ satisfying item 1. and which satisfies 2. after collapsing bigon complementary regions to curves. In this paper, bigon tracks will typically arise as follows. Let $\mathscr{C} = \{c_1, \ldots, c_k\}$, $\mathscr{D} = \{d_1, \ldots, d_l\}$ be two collections of pair-wise disjoint and homotopically distinct simple closed curves, which have the property that every element of $\mathscr{C}$ has non-trivial intersection with at least one element of $\mathscr{D}$, and *vice verca*, and that the complement of $\mathscr{C} \cup \mathscr{D}$ in Z is a union of topological discs. We call $\mathscr{C}$, $\mathscr{D}$ a *filling pair* of multi curves. By turning each intersection of a $\mathscr{C}$-curve with a $\mathscr{D}$-curve into a 4-valent vertex as in Figure 1, we obtain a bigon track. Since such bigon tracks will be the only ones appearing in this article, we will assume from now on that all switches in bigon tracks have valency no more than four.

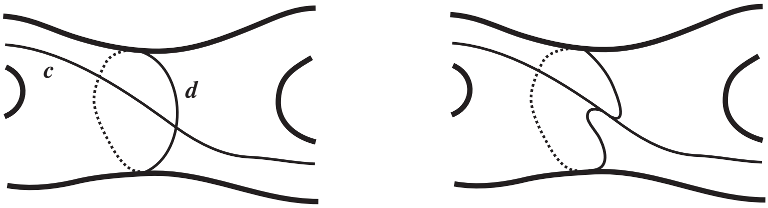

Figure 1. Creating a bigon track from a filling pair of curves.

Denote by $E = E_\tau$ the set of edges of the bigon track τ. In a small disk neighborhood of a switch v, the ends of edges incident to v may be divided into two classes, which for convenience we refer to as "incoming" and "outgoing": each class consists of ends that are asymptotic to one another, and the decision of naming one class incoming, the other outgoing, is arbitrary. We write $e \in \mathsf{in}(v)$ or $e \in \mathsf{out}(v)$ if the edge e has an end belonging to the appropriate class. (Note: it can happen that e belongs to both $\mathsf{in}(v)$ and $\mathsf{out}(v)$.) A *switch-additive measure* on τ is a function $\upsilon : E \to \mathbb{R}_+$ for which

$$\sum_{e \in \mathsf{in}(v)} \upsilon(e) \;=\; \sum_{e \in \mathsf{out}(v)} \upsilon(e)$$

for all switches v. The set of all switch-additive measures forms a linear cone $\mathfrak{C}_\tau$ in $\mathbb{R}_+^E$.

Let $N(\tau)$ be a tubular neighborhood of τ equipped with a (singular) foliation by line segments transverse to τ. A measured lamination $\mathfrak{f} \subset Z$ is said to be *carried* by τ if it may by isotoped into $N(\tau)$ transverse to its foliation. We write in this case $\mathfrak{f} < \tau$, and note that the measure of $\mathfrak{f}$ induces a switch-addititive measure on τ. The subspace of isotopy classes of measured laminations $\mathsf{ML}(Z)$ carried by τ is denoted $\mathsf{ML}_\tau(Z)$. There is an open surjection

$$\mathfrak{C}_\tau \longrightarrow \mathsf{ML}_\tau(Z) \tag{2}$$

which is a homeomorphism if τ is a train track. We refer to the switch additive measure υ as a *track coordinate* for $\mathfrak{f}$ with respect to τ if it maps onto $\mathfrak{f}$ via (2).

Let τ, κ be bigon tracks in Z that intersect transversally and minimally with edge sets E_τ and E_κ; let $\mathfrak{f}, \mathfrak{g}$ be measured laminations carried by them, parametrized by switch-additive measures υ, ω. Then the intersection pairing may be calculated by the following formula:

$$\mathbb{I}_Z(\mathfrak{f}, \mathfrak{g}) \;=\; \sum_{e \in E_\tau} \sum_{e' \in E_\kappa} \upsilon(e)\omega(e')|e \cap e'|. \tag{3}$$

It is useful to re-express (3) as a sum over edges in E_τ only. Thus if we write

$$\omega(e) \;=\; \sum_{e' \in E_\kappa} \omega(e')|e \cap e'|$$

then

$$\mathbb{I}_Z(\mathfrak{f}, \mathfrak{g}) \;=\; \sum_{e \in E_\tau} \upsilon(e)\omega(e). \tag{4}$$

5. Teichmüller Space

References for material in this section are [1], [8], [9]. For any surface Z let $\mathscr{T}(Z)$ denote its Teichmüller space. A finite covering map $\rho : W \to Z$ is amenable, hence induces an isometric inclusion $\rho^* : \mathscr{T}(Z) \hookrightarrow \mathscr{T}(W)$ [15]. The system $\mathscr{S}$ induces thus a direct limit of Teichmüller spaces by *isometric* inclusions,

$$\mathscr{T} = \varinjlim \mathscr{T}(Z),$$

which we equip with the direct limit metric.

Given Z a surface occurring in the system $\mathscr{S}$, let $\mu \in \mathscr{T}(Z)$ be represented by a marked surface $Z \to Z_\mu$. Denote by $Q_\mu(Z)$ the space of holomorphic quadratic differentials on Z_μ, which may be identified with the cotangent space to $\mathscr{T}(Z)$ at μ; write the associated contangent bundle $\mathscr{Q}(Z) = \bigcup Q_\mu(Z)$. There is then an induced direct limit of bundles of holomorphic quadratic differentials

$$\mathscr{Q} := \varinjlim \mathscr{Q}(Z).$$

To any quadratic diferential $q \in \mathscr{Q}$ one associates two transverse, measured laminations $\mathfrak{f}^h$ and $\mathfrak{f}^v \in \mathsf{ML}$: those that correspond to the horizontal and vertical trajectories of q. We have the following version of the theorem of Hubbard and Masur [11]:

THEOREM 5.1. — *Any transverse pair of measured laminations* $\mathfrak{f}, \mathfrak{g} \in \mathsf{ML}$ *determines a unique quadratic differential* $q \in \mathscr{Q}$.

Proof. — This follows immediately from the classical result, since $\mathfrak{f}, \mathfrak{g}$ may be viewed as lifts of measured laminations coming from a single surface Z in $\mathscr{S}$. $\square$

Let $q, q_i \in \mathscr{Q}$, $i = 1, 2, \ldots$, be quadratic differentials. The *heights-widths topology* on $\mathscr{Q}$ is defined by declaring that $q_i \rightarrow q$ if

$$\mathfrak{f}_i^h \rightarrow \mathfrak{f}^h \qquad \text{and} \qquad \mathfrak{f}_i^v \rightarrow \mathfrak{f}^v$$

in the intersection topology. The heights-widths topology in turn induces a topology on $\mathscr{T}$, where $\mu_i \rightarrow \mu$ if there exist contangent vectors q_i, q based at μ_i, μ for which $q_i \rightarrow q$ in the heights-widths topology.

6. Pseudo Anosov Diffeomorphisms

A (homotopy class of) diffeomorphism $\Phi : Z \rightarrow Z$ induces a homeomorphism of $\mathsf{ML}(Z)$ via pullback of measures, in particular inducing a homeomorphism of $\mathsf{PL}(Z)$. According to Thurston's classification of surface diffeomorphisms, [21], [5], [3], Φ is called *pseudo Anosov* if its induced action on $\mathsf{PL}(Z)$ fixes precisely two classes $[\mathfrak{f}^u]$ and $[\mathfrak{f}^s]$. If λ is the entropy of Φ, then $\lambda > 1$; and if $\mathfrak{f}^u \in [\mathfrak{f}^u]$ and $\mathfrak{f}^s \in [\mathfrak{f}^s]$ are representative laminations in $\mathsf{ML}(Z)$, then there is a representative diffeomorphism in the homotopy class of Φ (also denoted Φ) such that $\Phi(\mathfrak{f}^u) = \lambda \mathfrak{f}^u$ and $\Phi(\mathfrak{f}^s) = \lambda^{-1}\mathfrak{f}^s$. The (projective classes of the) laminations $\mathfrak{f}^u$, $\mathfrak{f}^s$ are called respectively the *unstable lamination* and the *stable lamination* of Φ.

Using Theorem 5.1, it follows that $\mathfrak{f}^u$, $\mathfrak{f}^s$ determine a Teichmüller geodesic $A \subset \mathscr{T}(Z)$ whose tangent quadratic differentials are determined by horizontal and vertical laminations that are positive multiples of $\mathfrak{f}^u$ and $\mathfrak{f}^s$. Along A, the induced map on Teichmüller space Φ_* is a translation by $\log \lambda$.

In this paper, we will be interested in the following class of pseudo Anosov diffeomorphisms. Let $\mathscr{C} = \{c_1, \ldots, c_k\}$, $\mathscr{D} = \{d_1, \ldots, d_l\}$ be a filling pair of multicurves (see §4 for the definition). For $c \in \mathscr{C}$, $d \in \mathscr{D}$, let F_c resp. G_d denote the right Dehn twist about c resp. d. If $\mathsf{c} = \{c_{\alpha_1}, \ldots, c_{\alpha_i}\} \subset \mathscr{C}$, $\mathsf{d} = \{d_{\beta_1}, \ldots, d_{\beta_j}\} \subset \mathscr{D}$ are submulticurves, and if $\mathsf{M} = (M_1, \ldots, M_i) > 0$, $\mathsf{N} = (N_1, \ldots, N_j) > 0$ are vectors of positive integers, we denote

$$F_{\mathsf{c}}^{\mathsf{M}} = F_{c_{\alpha_i}}^{M_i} \circ \cdots \circ F_{c_{\alpha_1}}^{M_1} \qquad \text{and} \qquad G_{\mathsf{d}}^{-\mathsf{N}} = G_{d_{\beta_j}}^{-N_j} \circ \cdots \circ G_{d_{\beta_1}}^{-N_1}$$

Then if $\mathsf{c}_1 \cup \cdots \cup \mathsf{c}_s = \mathscr{C}$ and $\mathsf{d}_1 \cup \cdots \cup \mathsf{d}_s = \mathscr{D}$, a diffeomorphism of the form

$$\Phi \;=\; \Phi_s \circ \cdots \circ \Phi_1, \tag{5}$$

where

$$\Phi_i = G_{\mathsf{d}_i}^{-\mathsf{N}_i} \circ F_{\mathsf{c}_i}^{\mathsf{M}_i}$$

and where the exponents are positive integer vectors, is pseudo Anosov, [20], [16]. (Here we are allowing the possibility that $\mathsf{c}_1 = \emptyset$, $\mathsf{d}_s = \emptyset$.) We call these pseudo Anosov diffeomorphisms of *Thurston-Penner type*.

7. Carrying Matrices

We now describe linear models of pseudo Anosov diffeomorphisms. Let $\Phi : Z \to Z$ be pseudo Anosov and let τ be a bigon track in Z equipped with a foliated tubular neighborhood $N(\tau)$ as in §4. We say that Φ acts on τ if $\Phi(\tau)$ may be isotoped into $N(\tau)$ transverse to its foliation. We write then $\Phi(\tau) < \tau$. Fix a leaf $t_i \subset N(\tau)$ through each edge e_i of τ. The *carrying matrix* of Φ is by definition $M_\Phi = (a_{ij})$ where

$$a_{ij} \;=\; |t_i \cap \Phi(e_j)|.$$

M_Φ induces an inclusion $\mathfrak{C}_{\Phi(\tau)} \hookrightarrow \mathfrak{C}_\tau$ which when precomposed with the pushforward map $\mathfrak{C}_\tau \to \mathfrak{C}_{\Phi(\tau)}$ defines a linear map

$$M_\Phi : \mathfrak{C}_\tau \longrightarrow \mathfrak{C}_\tau.$$

Note that the carrying matrix M_Φ is non-negative and irreducible. Such a matrix has a unique eigenvalue of greatest modulus, which is positive-real and simple [7]. This eigenvalue is called the *Perron root*. A corresponding eigenvector may be taken positive in its projective class, and is called a *Perron vector*. For M_Φ, the Perron root coincides with the entropy λ of Φ, and the Perron vector parametrizes in track coordinates an unstable measured lamination $\mathfrak{f}^u$ of Φ. Note also that the carrying matrix of the inverse Φ^{-1} is M_Φ^T. See [18] for more discussion of this.

When Φ is a pseudo Anosov of Thurston-Penner type, we may calculate the carrying matrix using the bigon track defined in §4. When performing a Dehn twist about a curve c, we choose a cylindrical neighborhood C whose core is c. Denoting $\partial C = \partial^0 \sqcup \partial^1$ the two boundary components, the twist will act as the identity on ∂^0, by a rotation of 2π about ∂^1, and by a rotation of an intermediate angle about c. If c has an even number r of edges, we may assume that they have been ordered cyclically, so that the first $r/2$ map to the last $r/2$ and *vice verca*. Thus if e is an edge belonging to a $\mathscr{D}$-curve which has an end on c, this end will be dragged along a string of $r/2$ consecutive edges. The column corresponding to e will receive a 1 in the row indexed by a c-edge e' whenever one of its two ends has been dragged along e'. In particular, each twist of c will contribute a value of $0,1$ or 2 in the e'th row of the eth column. See Figure 2 where both ends of the edge e meet at the same vertex of c: each end is dragged along one of the two edges of c, so that the subvector of the e-column indexed by the rows labeled by c-edges is $[1,1]^T$. In Figure 3, it is not hard to see that there exists a d-edge e both of whose ends get dragged along some of the same edges of c when a single Dehn twist along c is performed. Such an edge will index a column vector having c-row entries containing all three of the values $0,1$ and 2.

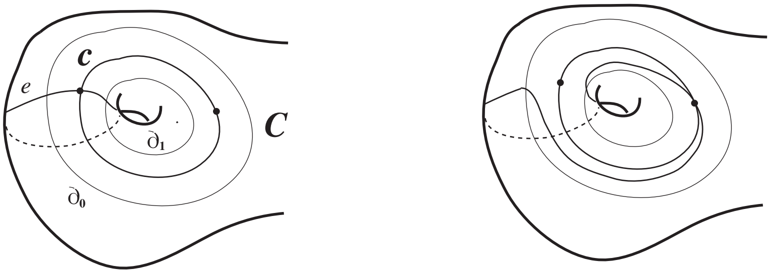

Figure 2. Action of a single Dehn twist

These calculations simplify greatly when one works with even powers of Dehn twists about c, for then an end e meeting c is dragged along the entirety of c. For example, if we twist about c twice, the subvector of the column indexed by e that corresponds to edges contained in c will be of the form

$$\begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}$$

where $a = 0,1,2$ depending on whether e has 0, 1 or 2 ends meeting c.

Example 7.1. — Let $\mathscr{C} = \{c\}$, $\mathscr{D} = \{d\}$ (*e.g.* see Figure 3) with $|c \cap d| = r$, and let $\Phi = G_d^{-2N} \circ F_c^{2N}$. Let $\mathbf{1}_r$ be the $r \times r$ matrix all of whose entries are 1, and let $\boldsymbol{I}_r$ be the $r \times r$ identity matrix. Then we claim that

$$M_\Phi = \begin{pmatrix} \boldsymbol{I}_r & 2N \cdot \mathbf{1}_r \\ 2N \cdot \mathbf{1}_r & r(2N)^2 \cdot \mathbf{1}_r + \boldsymbol{I}_r \end{pmatrix}. \tag{6}$$

To see this, we calculate the carrying matrices M_c, M_d of F_c^{2N}, G_d^{-2N} and use the fact that

$$M_\Phi = M_d \cdot M_c. \tag{7}$$

By the discussion of the previous paragraphs and the fact that both ends of any edge must enter the same curve, we have

$$M_c = \begin{pmatrix} \boldsymbol{I}_r & 2N\cdot\mathbf{1}_r \\ \mathbf{0}_r & \boldsymbol{I}_r \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} \boldsymbol{I}_r & \mathbf{0}_r \\ 2N\cdot\mathbf{1}_r & \boldsymbol{I}_r \end{pmatrix}$$

where $\mathbf{0}_r$ is the $r\times r$ zero matrix. Performing the product in (7) and noting that $\mathbf{1}_r\cdot\mathbf{1}_r = r\mathbf{1}_r$ yields (6).

Notice that in this case the Perron data of M_Φ is completely determined by that of the "curve matrix"

$$C_\Phi = \begin{pmatrix} 1 & r2N \\ r2N & (r2N)^2+1 \end{pmatrix},$$

so-called because it records only the action of Φ on the curves c, d. Indeed, if λ is the Perron root of C_Φ and $(a,b)^T$ is a Perron vector, than λ is also the Perron root of M_Φ with the $2r\times 1$ Perron vector $(a,\dots,a,b,\dots,b)^T$.

8. Directional Density

A family $\mathscr{P}(Z)$ of pseudo Anosov diffeomorphisms of Z is said to be *directionally dense* (in $\mathscr{Q}(Z)$) if the set of quadratic differentials tangent to axes of elements of $\mathscr{P}(Z)$ is Teichmüller dense in $\mathscr{Q}(Z)$. By [14], the family of all pseudo Anosov maps of Z is directionally dense.

Let $\mathscr{P}(Z)$ be the family of pseudo Anosovs Φ of the shape

$$\Phi = G^{-4N}\circ F^{4N} : Z\to Z,$$

where

- F, G are right Dehn twists about simple closed curves c,d in Z,
- (c,d) is a filling pair,
- c is non-separating.

LEMMA 8.1. — *$\mathscr{P}(Z)$ is directionally dense.*

Proof. — Given (c,d) a filling pair of simple closed curves in Z, denote by $\ell_{c,d}$ the Teichmüller geodesic determined by the pair. In Masur's original argument (see Proposition 3.2 in [14]), it is shown that the family of pseudo Anosov diffeomorphisms of the type $\Phi_N = G_d^{-N}\circ F_c^N$ is directionally dense in $\mathscr{Q}(Z)$. This is done by showing that

1. the axis A_N of Φ_N converges to the Teichmüller geodesic $\ell_{c,d}$ defined by the pair (c,d) as $N\to\infty$.
2. The collection of $\ell_{c,d}$ for (c,d) filling is directionally dense.

It is clear then that we may replace N by $4N$ without losing directionally density. By Corollary 2.6 in [14], the subfamily obtained by demanding that c is nonseparating is also directionally dense. □

Now given a nonseparating simple closed curve $\gamma \subset Z$, let $\rho_\gamma : Z_\gamma \to Z$ be the degree 2 cover obtained by cutting two copies of Z along γ and gluing ends. We say that a pair (c,d) of filling, simple closed curves is *interlacing* if there exists a pair of disjoint, nonseparating simple closed curves α, β such that $\rho_\alpha^{-1}(c)$ is connected whereas $\rho_\alpha^{-1}(d)$ is not and $\rho_\beta^{-1}(d)$ is connected whereas $\rho_\beta^{-1}(c)$ is not. See Figure 3.

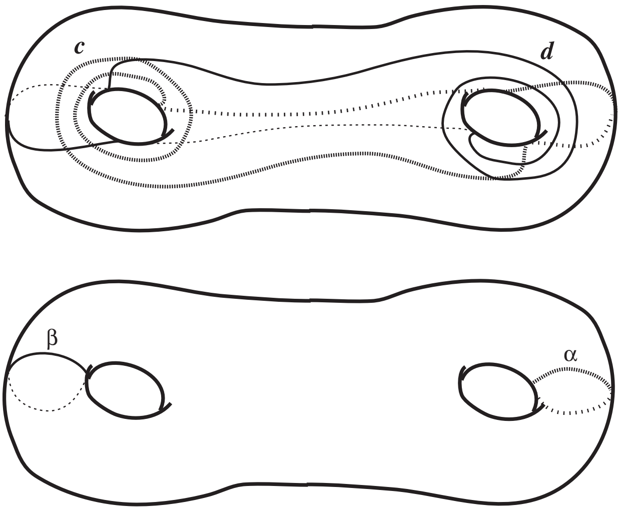

Figure 3. A filling pair c,d and a pair α,β interlacing them.

Let $\mathscr{P}_{\rm int}(Z)$ be the subfamily of the family $\mathscr{P}(Z)$ of Lemma 8.1 consisting of pseudo Anosov diffeomorphisms for which (c,d) is an interlacing pair.

LEMMA 8.2. — *$\mathscr{P}_{\rm int}(Z)$ is directionally dense.*

Proof. — We recall that Lemma 8.1 states that the set of Teichmüller geodesics $\ell_{c,d}$ defined by pairs (c,d) of filling, simple closed curves in Z for which c nonseparating, is directionally dense in $\mathscr{Q}(Z)$. Pick one such pair (c,d) and let F, G be the right Dehn twists about c, d. If (c,d) is interlacing, we are done. If (c,d) is not an interlacing pair, there exists a simple closed curve δ for which the pair (c,δ) is interlacing, though not necessarily filling. Indeed, one may assume after a homeomorphism that c is the curve appearing in Figure 4; then taking δ, α, β as indicated there, (c,δ) is interlacing with respect to the pair $(\alpha,\ \beta)$. Note that since $|c \cap \delta| = 0$ we must have that $|d \cap \delta| > 0$, since (c,d) is a filling pair. Now for j large, $\delta_j = G^j(\delta)$ is close to d (in the topology of $\mathsf{PL}(Z)$), hence (c,δ_j) is eventually filling. If j is in addition even, G^j lifts to the total space of any degree 2 cover of Z, thus the pair (c,δ_j) is interlacing with respect to (α,β), the same curves interlacing (c,δ). Indeed, if for example the cover ρ_α satisfies $\rho_\alpha^{-1}(\delta)$ is (dis)connected, the same is true of $\rho_\alpha^{-1}(\delta_j)$ since

$$\rho_\alpha^{-1}(\delta_j) = \rho_\alpha^{-1}(G^j(\delta)) = \widetilde{G^j}(\rho_\alpha^{-1}(\delta))$$

where $\widetilde{G^j}$ is the lift of G^j to the total space of ρ_α. For N large, the pseudo Anosov $\Phi_j = G_{\delta_j}^{-4N} \circ F^{4N}$ has axis close to the Teichmüller geodesic ℓ_{c,δ_j}, but this is close to $\ell_{c,d}$ by choice of j. This proves the lemma. $\square$

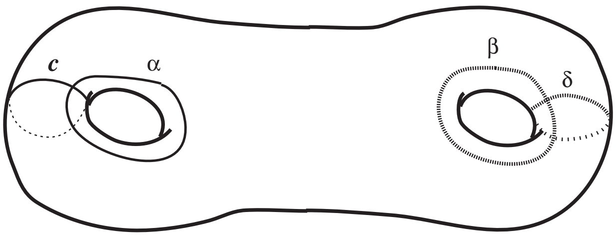

Figure 4. Every nonseparating c is a member of a (not necessarily filling) interlacing pair.

If $\mathscr{P}$ is any family of pseudo Anosov diffeomorphisms of closed surfaces, then we say that $\mathscr{P}$ is directionally dense in $\mathscr{Q}$ if the set of quadratic differentials tangent to axes of elements of $\mathscr{P}$ is Teichmüller dense in $\mathscr{Q}$. We have immediately

COROLLARY 8.3. — *The family $\mathscr{P}_{\mathrm{int}} = \bigcup \mathscr{P}_{\mathrm{int}}(Z)$, where the union is taken over all surfaces Z occurring in the system $\mathscr{S}$, is directionally dense in $\mathscr{Q}$.*

9. Necklace Roots

Fix $n \in \mathbb{N}$ and let $\Phi \in \mathscr{P}_{\mathrm{int}}(Z)$, the family appearing in Lemma 8.2, so that in particular Φ is a pseudo Anosov of the form $\Phi = G^{-4N} \circ F^{4N} : Z \to Z$. Let

$$\rho_{mn} : Z_{mn} \to Z$$

be the cover obtained by cutting $2mn$ copies of Z along a pair α,β interlacing c,d and gluing in a circular fashion. We call ρ_{mn} the *necklace cover* associated to (c,d). In Figure 5, we illustrate the construction of the necklace Z_{mn} and the formation of the lifts of the curve c. In Figure 6 we display the finished necklace.

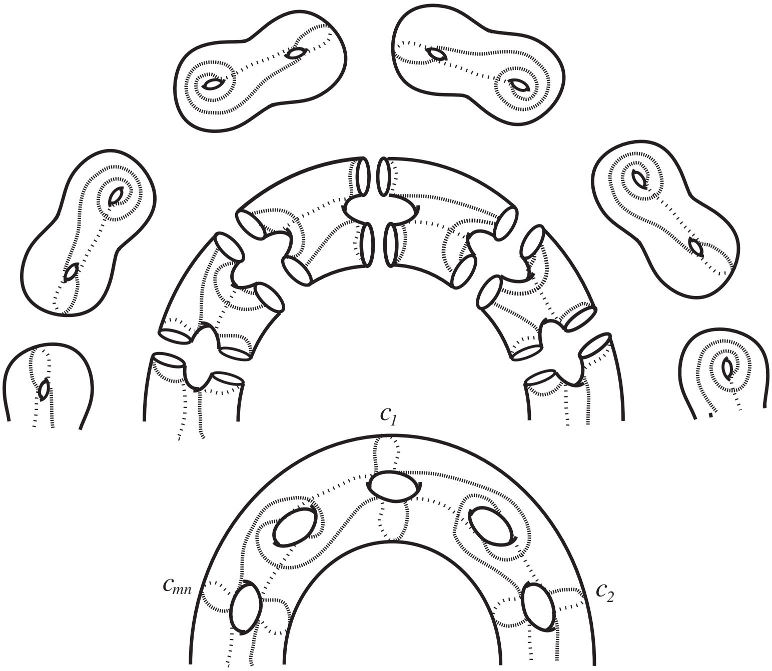

Figure 5. Making the necklace.

There are mn lifts $c_1,\ldots,c_{mn}$ and $d_1,\ldots,d_{mn}$ of c and d, each mapping with degree two onto their ancestor. On Z_{mn}, Φ lifts to

$$\tilde{\Phi} \;=\; G_{mn}^{-2N}\circ\cdots\circ G_1^{-2N}\circ F_{mn}^{2N}\circ\cdots\circ F_1^{2N}$$

where F_i, G_i is the right Dehn twist about c_i, d_i. Let χ denote the clockwise rotation of Z_{mn} by an angle of $2\pi/n$, so that the pair c_i, d_i is taken to c_{i+m}, d_{i+m} (indices taken mod mn). We define the *necklace* n*th root* to be the sequence of diffeomorphisms $\{\sqrt[n]{\Phi}_m\}$ where

$$\sqrt[n]{\Phi}_m \;=\; \chi\circ G_m^{-2N}\circ\cdots\circ G_1^{-2N}\circ F_m^{2N}\circ\cdots\circ F_1^{2N},$$

$m=2,3,\ldots$. The necklace nth root is the basic contruction used in the formation of heights-widths approximate roots. The construction of $\sqrt[n]{\Phi}_m$ is a variation of one that first appeared in [17], where branched rather than unbranched covers were used. Although the construction of the cover ρ_{mn} (and hence $\sqrt[n]{\Phi}_m$) formally depends on the choice of the pair (α,β), it will be clear in the arguments that follow that the latter plays no role, and so we shall not mention the pair (α,β) again.

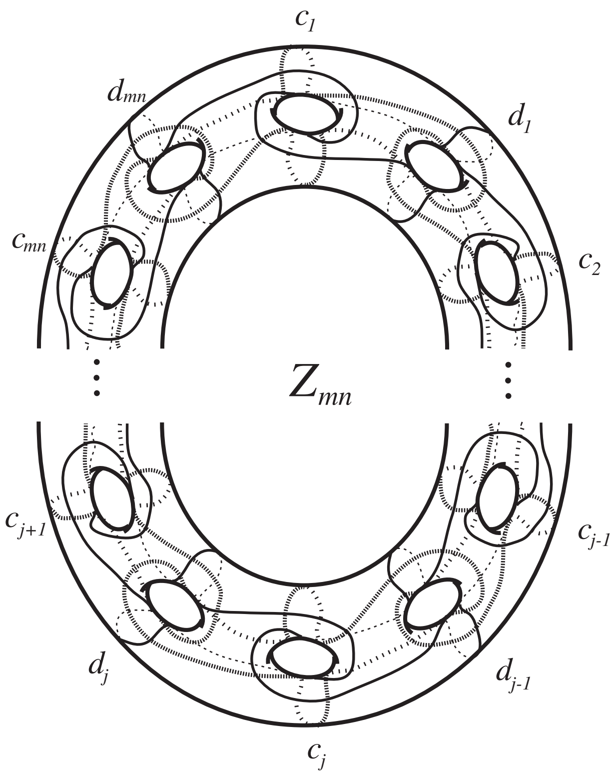

Figure 6. The finished necklace.

LEMMA 9.1. — *$\sqrt[n]{\Phi}_m$ is pseudo Anosov for all m.*

Proof. — For $i = 1, \ldots, n$, let

$$T_i \;=\; G_{im}^{-2N} \circ \cdots \circ G_{(i-1)m+1}^{-2N} \circ F_{im}^{2N} \circ \cdots \circ F_{(i-1)m+1}^{2N}.$$

Then

$$T_i = \chi^{i-2} \circ \sqrt[n]{\Phi}_m \circ \chi^{-(i-1)}$$

and therefore

$$(\sqrt[n]{\Phi}_m)^n \;=\; T_2 \circ \cdots \circ T_n \circ T_1.$$

The latter is of Thurston-Penner type, hence [16] $(\sqrt[n]{\Phi}_m)^n$ is pseudo Anosov, implying $\sqrt[n]{\Phi}_m$ is pseudo Anosov as well. □

For each $m = 2, 3, \ldots$ let $\sqrt[mn]{\Phi}_m$ denote the mth element in the mnth necklace root of Φ. Thus $\sqrt[mn]{\Phi}_m$ is a diffeomorphism defined on $Z_{m^2 n}$. Define the sequence of pseudo Anosov diffeomorphisms $\{\Psi_m\}$ by

$$\Psi_m \;=\; \left(\sqrt[mn]{\Phi_m}\right)^m.$$

Observe that the stable and unstable laminations of Ψ_m and $\sqrt[nm]{\Phi_m}$ are equal.

Note 9.2. — Ψ_m is not the same as $\sqrt[n]{\Phi_m}$. In fact, if we lift $\sqrt[n]{\Phi_m}$ to Z_{m^2n} – where Ψ_m is defined – we see that this lift twists along m disjoint "blocks" of curves, each block consisting of a succession of m lifts of c and d. On the other hand, Ψ_m consists of twists along one block consisting of a succession of m^2 lifts of c and d. As it happens, the stable and unstable laminations of the family $\{\Psi_m\}$ have better convergence properties than those of $\{\sqrt[n]{\Phi_m}\}$.

We now describe the carrying matrices of Φ, its lift $\tilde{\Phi}$ to Z_{m^2n} and that of Ψ_m. Let $r = |c \cap d|$. Recall (see *Example* 7.1 of §7) that the carrying matrix of Φ is

$$M_\Phi \;=\; \begin{pmatrix} \boldsymbol{I}_r & 4N\cdot\mathbf{1}_r \\ 4N\cdot\mathbf{1}_r & r(4N)^2\cdot\mathbf{1}_r+\boldsymbol{I}_r \end{pmatrix}.$$

Let Z_{m^2n} be the surface where Ψ_m is defined. Since the exponents in the definition of Φ are even, it follows that Φ lifts to a pseudo Anosov diffeomorphism $\tilde{\Phi}$ of Z_{m^2n}. The curve families $\mathscr{C} = \{c_1,\dots,c_{m^2n}\}$, $\mathscr{D} = \{d_1,\dots d_{m^2n}\}$ are filling, and each curve intersects r times each of its neighbors *e.g.* $|c_i \cap d_i| = |c_i \cap d_{i-1}| = r$ (indices taken modulo m^2n). Thus the corresponding bigon track will contain $2r$ edges along each curve.

For each i, let us refer to d_{i-1} as the preceding neighbor of c_i and to d_i as the successive neighbor of c_i; similarly, the preceding neighbor of d_i is c_i and its successive neighbor is c_{i+1}. Notice that in a given curve, some edges will join only the preceding neighbor, some only the successive neighbor and some both. We order the c edges so that the first x edges are effected only by twists along the preceding neighbor, the next y are effected by twists along both neighbours, and the last z edges are only twisted by the successive neighbor. Similarly, we order the d edges so that the first s edges are effected only by twists along the preceding neighbor, the next t are effected by twists along both neighbours, and the last u edges are only twisted by the successive neighbor. By construction of the cover, the triples (x,y,z), (s,t,u) depend only on c, d (and not any particular lift c_i, d_i), and we have $x+y+z = 2r = s+t+u$.

The carrying matrix $M_{\tilde{\Phi}}$ of $\tilde{\Phi}$ is obtained in the same way as that of Φ: that is, one can calculate the carrying matrices of each of the compositions $F_{m^2n}^{2N}\circ\cdots\circ F_1^{2N}$, $G_{m^2n}^{-2N}\circ\cdots\circ G_1^{-2N}$ and multiply them as in *Example* 7.1. We have indicated the result in Figure 7, where we have grouped edges according to which curve they belong to: thus, for example, the column label c_1 indexes the $2r$ columns corresponding to the edges belonging to c_1: the first x of which will be twisted by

d_{m^2n}, the last z of which are twisted by d_1 and the middle y of which are twisted by both d_{m^2n} and d_1. The boxed matrix labeled I is the $2rm^2n \times 2rm^2n$ identity matrix. The values of the other boxed matrices are indicated in Figures 8 and 9, where a rectangle containing a value a indicates a matrix all of whose entries are a. For example, the rectangle containing 2 in the definition of the boxed matrix B_1 signifies a $2r \times s$ matrix all of whose entries are 2. The matrix I occurring in the description of the boxed matrix D_2 is the $2r \times 2r$ identity matrix.

(To verify this result, one notes that the carrying matrix of $F_{m^2n}^{2N} \circ \cdots \circ F_1^{2N}$ is of the form

$$M_c = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}$$

where $\boldsymbol{B}$ is the content of the upper right hand quadrant in Figure 7, containing the pattern of "B"-matrices. Similarly, the carrying matrix of $G_{m^2n}^{-2N} \circ \cdots \circ G_1^{-2N}$ is of the form

$$M_d = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{I} \end{pmatrix}$$

where $\boldsymbol{C}$ is the content of the lower left hand quadrant in Figure 7, containing the pattern of "C"-matrices. The product $M_d \cdot M_c$ is the matrix shown in Figure 7.)

Note that the Perron data of $M_{\tilde{\Phi}}$ is completely determined by that of Φ. Namely, if λ is the Perron root and $\upsilon = (a, ..., a, b, ...b)^T$ the $2r \times 1$ Perron vector for M_Φ, then λ is also the Perron root of $M_{\tilde{\Phi}}$ with a $4rm^2n \times 1$ Perron vector of the shape $\tilde{\upsilon} = (a, \ldots, a, b, \ldots, b)^T$. Indeed, the result of multiplying $\tilde{\upsilon}$ by a c-row is

$$a + N(2s + t + t + 2u)b = a + r4Nb = \lambda a$$

and the result of multiplying $\tilde{\upsilon}$ by a d-row is

$$\begin{aligned}(r4N)a \quad + \quad & \big\{N^2\left((2x+y)(2s+t) + (y+2z)(t+2u)\right) + \\ & 2N^2\left((y+2z)s + (x+y+z)t + (2x+y)u\right) + 1\big\}\, b\end{aligned}$$

Collecting terms and using the fact that $s + t + u = x + y + z = 2r$ gives

$$(r4N)a + \big\{N^2(4(x+y+z)(s+t+u)) + 1\big\}\, b = (r4N)a + ((4rN)^2 + 1)b = \lambda b$$

as claimed.

To calculate the carrying matrix of $(\Psi_m)^n = (\sqrt[mn]{\Phi_m})^{mn}$, we use the formula

$$(\sqrt[mn]{\Phi_m})^{mn} = (T_2 \circ \cdots \circ T_{mn}) \circ T_1$$

where T_i is as in Lemma 9.1: that is, we calculate the carrying matrix of $(\Psi_m)^n$ as the product

$$M_{(\Psi_m)^n} = M_{T_2 \circ \cdots \circ T_{mn}} \cdot M_{T_1}.$$

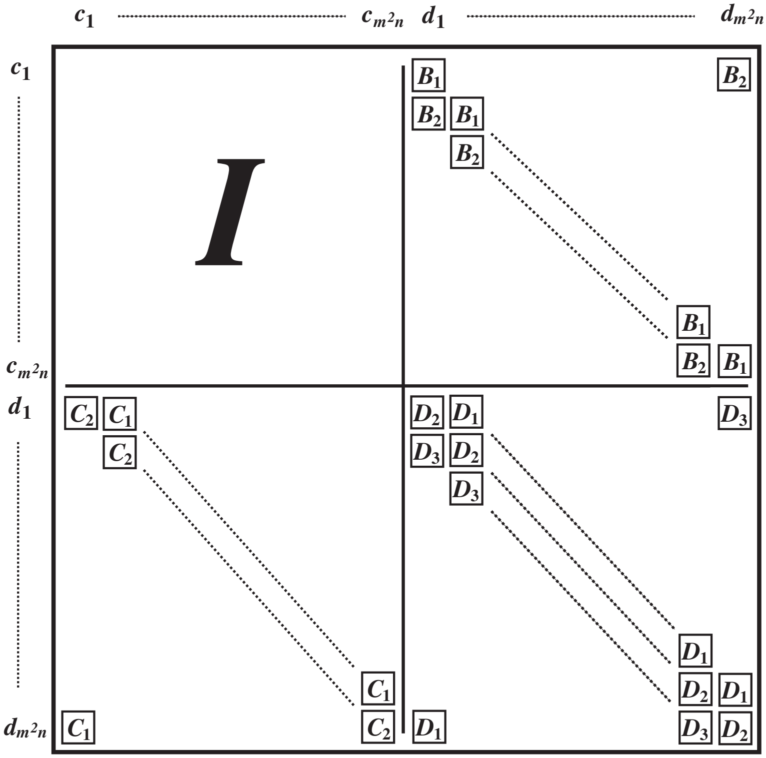

Figure 7. The Carrying Matrix for $\tilde{\Phi}$.

$$\boxed{B_1} = N\cdot\left(\overbrace{\boxed{2}}^{s}\overbrace{\boxed{1}}^{t}\overbrace{\boxed{0}}^{u}\right) \qquad \boxed{C_1} = N\cdot\left(\overbrace{\boxed{2}}^{x}\overbrace{\boxed{1}}^{y}\overbrace{\boxed{0}}^{z}\right)$$

$$\boxed{B_2} = N\cdot\left(\overbrace{\boxed{0}}^{s}\overbrace{\boxed{1}}^{t}\overbrace{\boxed{2}}^{u}\right) \qquad \boxed{C_2} = N\cdot\left(\overbrace{\boxed{0}}^{x}\overbrace{\boxed{1}}^{y}\overbrace{\boxed{2}}^{z}\right)$$

$$s+t+u = 2r \qquad x+y+z = 2r$$

Figure 8. Values of the B and C Boxed Matrices

$$\boxed{D_1} = \boxed{C_1}\cdot\boxed{B_1} = N^2(2x+y)\cdot\left(\overbrace{\boxed{2}}^{s}\,\overbrace{\boxed{1}}^{t}\,\overbrace{\boxed{0}}^{u}\right)$$

$$\boxed{D_3} = \boxed{C_2}\cdot\boxed{B_2} = N^2(y+2z)\cdot\left(\overbrace{\boxed{0}}^{s}\,\overbrace{\boxed{1}}^{t}\,\overbrace{\boxed{2}}^{u}\right)$$

$$\boxed{D_2} = \boxed{C_2}\cdot\boxed{B_1} + \boxed{C_1}\cdot\boxed{B_2} + \boldsymbol{I}$$

$$= 2N^2\cdot\left(\overbrace{\boxed{y+2z}}^{s}\,\overbrace{\boxed{\begin{array}{c}x+y+z\\ \| \\ 2r\end{array}}}^{t}\,\overbrace{\boxed{2x+y}}^{u}\right) + \boldsymbol{I}$$

Figure 9. Values of the D Boxed Matrices

The carrying matrix of T_1 is indicated in Figure 10 and that of $T_2 \circ \cdots \circ T_{mn}$ in Figure 11, where the black boxed matrix D_2 is shown in Figure 12 and a boxed matrix labeled "I" indicates an identity matrix of the appropriate dimension. Their product, the carrying matrix of $(\Psi_m)^n$, is displayed in Figure 13 where the content of the black boxed matrices are displayed successively in Figures 14, 15, 16 and 17. Note that the black boxed matrices are non-negative and *independent* of m, n. Figure 18 contains the carrying matrix of $\sqrt[mn]{\Phi_m}$: we note that the latter may be obtained as the product of the permutation matrix corresponding to χ with the carrying matrix M_{T_1}.

Let λ_m be the Perron root of M_{Ψ_m} and let υ_m be its Perron vector, normalized so that its L^1 norm (sum of its entries) is equal to $2m^2n = \deg(\rho_{m^2n})$. We shall abbreviate (following our convention for carrying matrices)

$$\upsilon_m = (a_1, \ldots, a_{m^2n}, b_1, \ldots, b_{m^2n})^T$$

where a_i (where b_i) is a $2r \times 1$ column vector which contains the entries of υ_m that correspond to edges belonging to the curve c_i (the curve d_i). Let us assume that

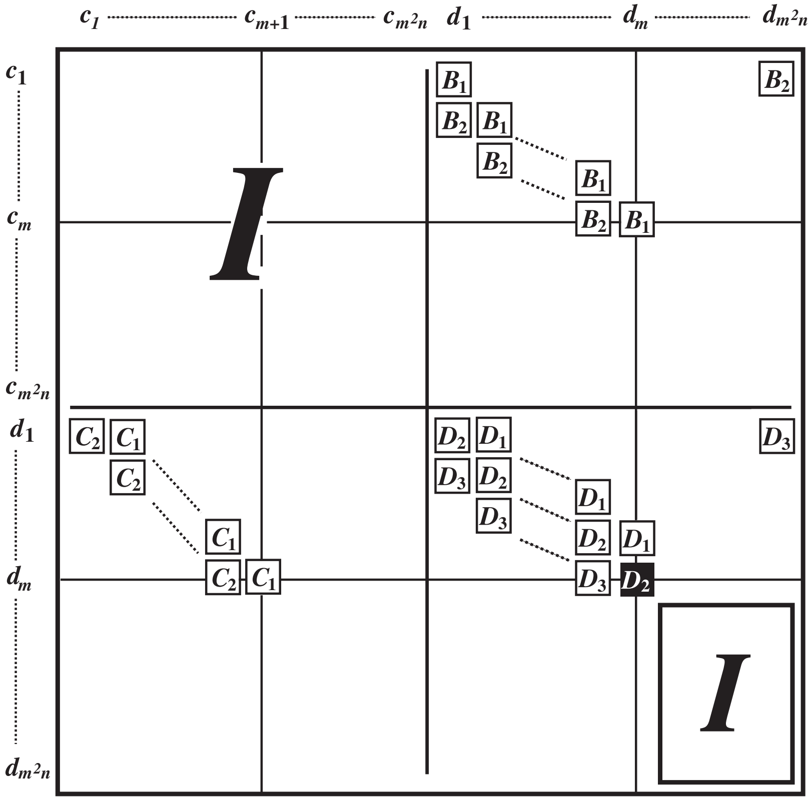

Figure 10. The Carrying Matrix for T_1

the Perron vector $\tilde{\upsilon}$ of the carrying matrix of $\tilde{\Phi}$ has been scaled to have L^1 norm $2m^2n$ as well. We recall that this spectral data has the following interpretation:

- The Perron roots λ_m, λ are equal to the entropies of Ψ_m, Φ.
- Let τ_m be the bigon track formed from the curve families $\mathscr{C}$, $\mathscr{D}$. The measures formed from the Perron vectors υ_m, $\tilde{\upsilon}$ parametrize the unstable laminations $\mathfrak{f}^u_m$, $\tilde{\mathfrak{f}}^u$ in the cone $\mathfrak{C}_{\tau_m}$.

Note that the column sums of $M_{(\Psi_m)^n}$ have uniform upper and lower bounds $L > l > 1$. We thus obtain the bound [7]

$$1 < l < (\lambda_m)^n < L.$$

We may then assume, after passing to a subsequence if necessary, that the λ_m converge to some value $\lambda_* > 1$.

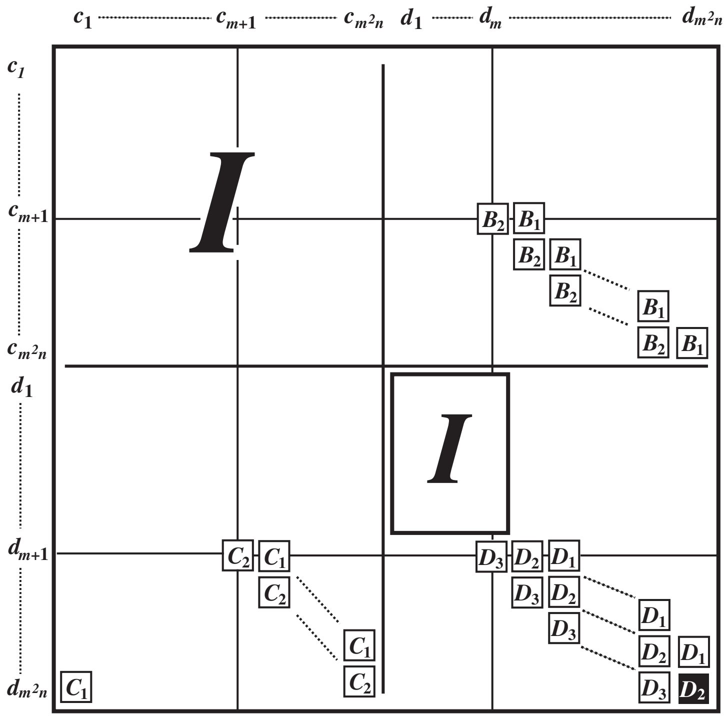

Figure 11. The Carrying Matrix for $T_2 \circ \cdots \circ T_{mn}$.

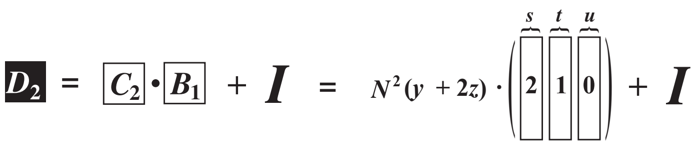

Figure 12. The Black Boxed Matrix D_2

We shall need to control the vector entries $a_m, a_{m+1}, b_{m-1}, b_m, b_{m+1}$ of υ_m, which correspond to the "clasp" of the necklace:

LEMMA 9.3. — *Let x_m be any entry belonging to one of the vectors a_m, a_{m+1}, b_{m-1}, b_m, b_{m+1}. Then $x_m = o(2m^2n)$.*

Proof. — We must show that the corresponding vector entries of the probability vector $\frac{1}{2m^2n}\upsilon_m$ tend to the zero vector. We denote these entries also by $a_m, a_{m+1}, b_{m-1}, b_m, b_{m+1}$. Let ξ_m be the Perron root of $M_{\sqrt[mn]{\Phi_m}}$: thus $(\xi_m)^m = \lambda_m$.

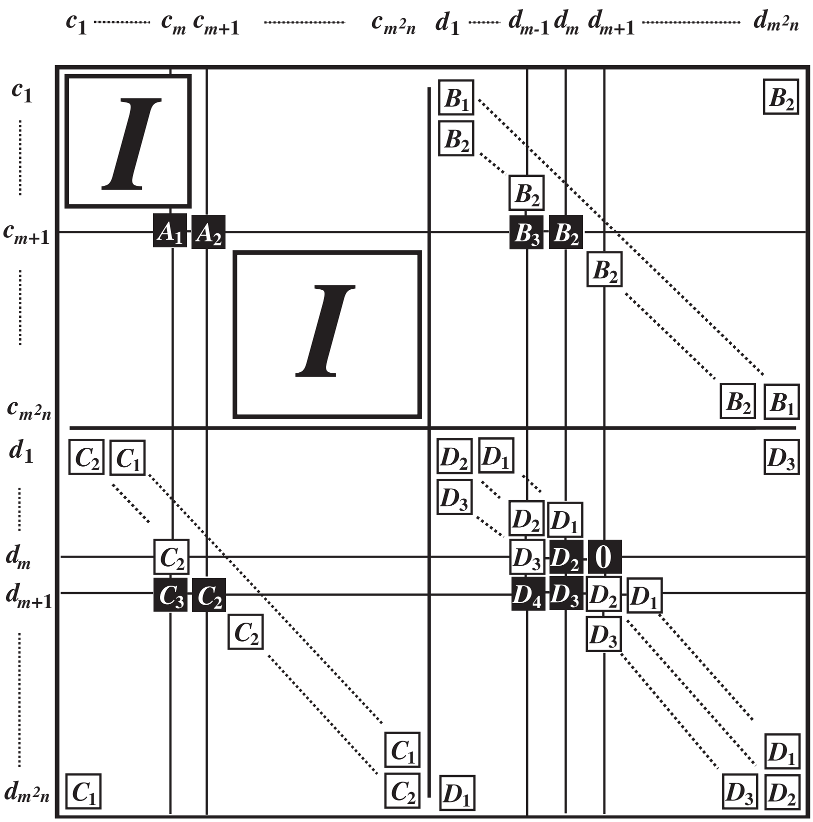

Figure 13. The Carrying Matrix for $(\Psi_m)^n$.

$$\blacksquare A_1 = \boxed{B_2}\cdot\boxed{C_2} = N^2(t+2u)\cdot\left(\overbrace{\boxed{0}}^{x}\ \overbrace{\boxed{1}}^{y}\ \overbrace{\boxed{2}}^{z}\right)$$

$$\blacksquare A_2 = \boxed{B_2}\cdot\boxed{C_1} + I = N^2(t+2u)\cdot\left(\overbrace{\boxed{2}}^{x}\ \overbrace{\boxed{1}}^{y}\ \overbrace{\boxed{0}}^{z}\right) + I$$

Figure 14. The Black Boxed Matrices A_1 and A_2

$$B_2 = \boxed{B_2} \cdot D_2 = N \cdot \left(\overbrace{\boxed{2R}}^{s} \overbrace{\boxed{R+1}}^{t} \overbrace{\boxed{2}}^{u} \right)$$

$$B_3 = \boxed{B_2} \cdot \boxed{D_3} = NR \cdot \left(\overbrace{\boxed{0}}^{s} \overbrace{\boxed{1}}^{t} \overbrace{\boxed{2}}^{u} \right)$$

$$R = N^2 (t + 2u)(y + 2z)$$

Figure 15. The Black Boxed Matrices B_2 and B_3

$$C_2 = \boxed{D_3} \cdot \boxed{C_1} + \boxed{C_2} = N \cdot \left(\overbrace{\boxed{2R}}^{x} \overbrace{\boxed{R+1}}^{y} \overbrace{\boxed{2}}^{z} \right)$$

$$C_3 = \boxed{D_3} \cdot \boxed{C_2} = NR \cdot \left(\overbrace{\boxed{0}}^{x} \overbrace{\boxed{1}}^{y} \overbrace{\boxed{2}}^{z} \right)$$

$$R = N^2 (t + 2u)(y + 2z)$$

Figure 16. The Black Boxed Matrices C_2 and C_3

$$D_3 = \boxed{D_3} \cdot D_2 = N^2 (y + 2z) \cdot \left(\overbrace{\boxed{2R}}^{s} \overbrace{\boxed{R+1}}^{t} \overbrace{\boxed{2}}^{u} \right)$$

$$D_4 = \boxed{D_3} \cdot \boxed{D_3} = N^2 R (y + 2z) \cdot \left(\overbrace{\boxed{0}}^{s} \overbrace{\boxed{1}}^{t} \overbrace{\boxed{2}}^{u} \right)$$

$$R = N^2 (t + 2u)(y + 2z)$$

Figure 17. The Black Boxed Matrices D_3 and D_4

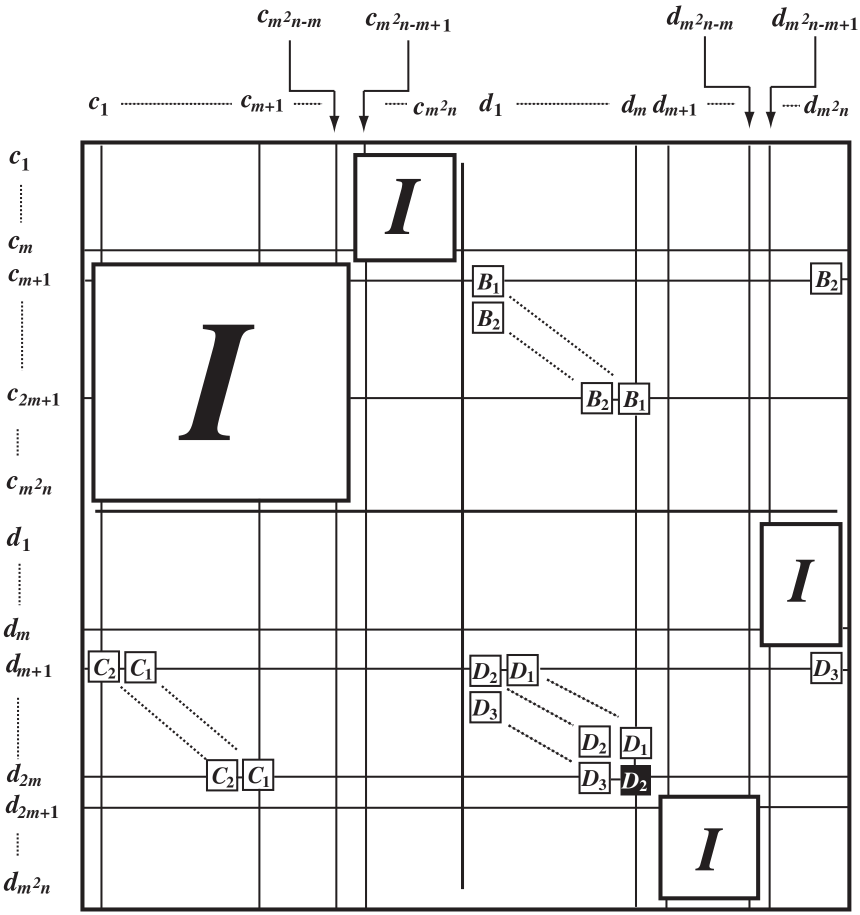

Figure 18. The Carrying Matrix for $\sqrt[mn]{\Phi}_m$.

Claim 9.4. — If one of the vectors $a_m, a_{m+1}, b_{m-1}, b_m, b_{m+1}$ does not converge to the zero vector, then none of them do.

Proof of Claim. — To prove the claim we consider the action of the matrix $M_{(\Psi_m)^n}$ on $\frac{1}{2m^2n}\upsilon_m$. Recall that $\lambda_m \to \lambda_* > 1$ and (after passing to a further subsequence if necessary) we may assume that the vectors $a_m, a_{m+1}, b_{m-1}, b_m$ and b_{m+1} converge (in $\mathbb{R}^{2r}$) to vectors denoted $a_*, a_{*+1}, b_{*-1}, b_*$ and b_{*+1}. It follows that we have the limit eigen-equations displayed in Figure 19, where the vectors labeled $*\pm k$ refer to limits of vectors labeled $m \pm k$. Note that all vector terms in these equations are non-negative. We proceed case by case: suppose that

- a_{*+1} is not the zero vector. The black boxed matrix C_2 is positive (all entries positive) therefore its product with a_{*+1} is a positive vector, hence by the eigen-equation for b_{*+1}, b_{*+1} is positive. Note also that this implies that $B_1 b_{*+1}$ is positive and hence a_{*+1} must also be positive. In addition, $C_1 a_{*+1}$ is positive so that b_* is positive. The latter gives the positivity of a_* and b_{*-1}.
- b_{*+1} is not the zero vector. Then since the matrix D_2 is positive, $D_2 b_{*+1}$ is positive, thus by the eigen-equation for b_{*+1}, b_{*+1} is positive. Since b_{*+1} occurs in the eigen-equation of a_{*+1}, the latter is positive. By the previous paragraph, the remaining vectors are positive as well.
- b_* is not the zero vector. The black-boxed matrix B_2 is positive, implying that a_{*+1} and hence the remaining vectors are positive.
- b_{*-1} is not the zero vector. Since D_2 is positive, the eigen-equation for b_{*-1} shows that the latter is positive. But b_{*-1} appears in the eigen-equation for a_{*+1} so that a_{*+1} and therefore all remaining vectors are positive.
- a_* is not the zero vector. If a_* has a non zero entry amongst its first x-entries, then $C_1 a_*$ is non zero hence b_{*-1} is non zero and all remaining vectors are non zero. If a_* has no non zero entry amongst its first x-entries, then there must be a non zero vector amongst the last $y+z$ entries, implying $C_2 a_*$ is non zero, that b_* is non zero and hence all remaining vectors are non zero.

$\square$

$$
\begin{aligned}
\lambda_* \boldsymbol{a}_* &= \boldsymbol{a}_* + \boxed{B_2}\,\boldsymbol{b}_{*-1} + \boxed{B_1}\,\boldsymbol{b}_* \\
\lambda_* \boldsymbol{a}_{*+1} &= \colorbox{black}{$\color{white}A_1$}\,\boldsymbol{a}_* + \colorbox{black}{$\color{white}A_2$}\,\boldsymbol{a}_{*+1} + \colorbox{black}{$\color{white}B_3$}\,\boldsymbol{b}_{*-1} + \colorbox{black}{$\color{white}B_2$}\,\boldsymbol{b}_* + \boxed{B_1}\,\boldsymbol{b}_{*+1} \\
\lambda_* \boldsymbol{b}_{*-1} &= \boxed{C_2}\,\boldsymbol{a}_{*-1} + \boxed{C_1}\,\boldsymbol{a}_* + \boxed{D_3}\,\boldsymbol{b}_{*-2} + \boxed{D_2}\,\boldsymbol{b}_{*-1} + \boxed{D_1}\,\boldsymbol{b}_* \\
\lambda_* \boldsymbol{b}_* &= \boxed{C_2}\,\boldsymbol{a}_* + \boxed{C_1}\,\boldsymbol{a}_{*+1} + \boxed{D_3}\,\boldsymbol{b}_{*-1} + \colorbox{black}{$\color{white}D_2$}\,\boldsymbol{b}_* \\
\lambda_* \boldsymbol{b}_{*+1} &= \colorbox{black}{$\color{white}C_3$}\,\boldsymbol{a}_* + \colorbox{black}{$\color{white}C_2$}\,\boldsymbol{a}_{*+1} + \boxed{C_1}\,\boldsymbol{a}_{*+2} + \colorbox{black}{$\color{white}D_4$}\,\boldsymbol{b}_{*-1} + \colorbox{black}{$\color{white}D_3$}\,\boldsymbol{b}_* \\
&\quad + \boxed{D_2}\,\boldsymbol{b}_{*+1} + \boxed{D_1}\,\boldsymbol{b}_{*+2}
\end{aligned}
$$

Figure 19. The Limiting Characteristic Equations

Thus, let us suppose that $x_m \in b_m$ does not converge to 0 so that say eventually $x_m \geqslant \delta > 0$. Examination of the action of $M_{\sqrt[mn]{\Phi_m}}$ on $\frac{1}{2m^2n}\upsilon_m$ (see Figure 18) then shows that there is a component x_{m^2n} of b_{m^2n} with $x_{m^2n} = \xi_m x_m$. Iterating this observation gives

$$x_{m^2n-im} = \xi_m^{i+1} x_m$$

for $i = 1, \dots mn-2$, where x_{m^2n-im} is a component of b_{m^2n-im}. However $\xi_m^{i+1} > 1$ for all i, and since $\frac{1}{2m^2n}\upsilon_m$ is a probability vector, this would imply that $(mn-2)\delta \leqslant (mn-2)x_m < 1$, impossible since $m \to \infty$. This proves the lemma. □

10. Existence of Heights-Widths Roots

Let $\Phi : Z \to Z$ be a pseudo Anosov diffeomorphism. Denote by λ its entropy and by A its axis. Recall that we view A as a geodesic in $\mathscr{T}$ via the canonical isometric inclusion $\mathscr{T}(Z) \hookrightarrow \mathscr{T}$.

DEFINITION 10.1. — *Let $n > 1$ be a natural number. A* heights-widths approximate nth root *of Φ is a sequence $\{\Psi_m : Z_m \to Z_m\}$ of pseudo Anosov diffeomorphisms for which*

- α) *If λ_m is the entropy of Ψ_m then $\lim \lambda_m = \sqrt[n]{\lambda}$.*
- β) *If A_m is the axis of Ψ_m then there exist tangent quadratic differentials q_m along A_m and q along A such that q_m converges to q in the heights-widths topology.*

We note that in "genus-dependent" language, β) is equivalent to:

- β') There exist tangent quadratic differentials $q = (\mathfrak{f}^u, \mathfrak{f}^s) \in \mathscr{Q}(Z)$, $q_m = (\mathfrak{f}^u_m, \mathfrak{f}^s_m) \in \mathscr{Q}(Z_m)$ along A, A_m such that for each W occurring in $\mathscr{S}$ and $\gamma \in \mathsf{C}(W)$, there exist a sequence of covers $\rho_m : W_m \to Z$, $\sigma_m : W_m \to Z_m$, $\zeta_m : W_m \to W$ such that

$$\begin{aligned}\left|\mathbb{I}_{W_m}(\rho_m^*(\mathfrak{f}^u), \gamma_m) - \mathbb{I}_{W_m}(\sigma_m^*(\mathfrak{f}^u_m), \gamma_m)\right| &= o(g_m - 1) \\ &= \left|\mathbb{I}_{W_m}(\rho_m^*(\mathfrak{f}^s), \gamma_m) - \mathbb{I}_{W_m}(\sigma_m^*(\mathfrak{f}^s_m), \gamma_m)\right|\end{aligned}$$

where $\gamma_m = \zeta_m^{-1}(\gamma)$ and g_m is the genus of W_m.

Note 10.2. — The Teichmüller flow preserves the heights-widths convergence required in item β): that is, $q_m \to q$ implies heights-widths convergence $q_m(t) \to q(t)$ of the time t Teichmüller flowed quadratic differentials (obtained by a uniform stretch). One can express this by saying that $A_m \to A$ "uniformly on compacta" in the heights-widths topology.

In this section we show that Φ has a heights-widths approximate nth root for any $n > 1$, thus proving the Main Theorem.

Assume first that $\Phi \in \mathscr{P}_{\mathrm{int}}(Z)$ and let $\{\Psi_m\}$ be the family of pseudo Anosov diffeomorphisms defined in the previous section. Denote by $\mathfrak{f}^u_m, \mathfrak{f}^s_m$ the unstable and stable laminations of Ψ_m, which we assume are parametrized by the L^1 norm $2m^2n = \deg(\rho_{m^2n})$ Perron vectors of $M_{\Psi_m}, M_{\Psi_m^{-1}}$. Denote by $\mathfrak{f}^u, \mathfrak{f}^s$ the unstable and stable laminations of Φ, corresponding to the probability Perron vectors of M_Ψ, $M_{\Psi^{-1}}$. Recall that $\mathsf{C}(Z)$ denotes the isotopy classes of simple closed curves of Z, which we view as belonging to C.

THEOREM 10.3. — *For all* $\gamma \in \mathsf{C}(Z)$,

$$\mathbb{I}(\mathfrak{f}^u_m, \gamma) \longrightarrow \mathbb{I}(\mathfrak{f}^u, \gamma) \qquad \textit{and} \qquad \mathbb{I}(\mathfrak{f}^s_m, \gamma) \longrightarrow \mathbb{I}(\mathfrak{f}^s, \gamma).$$

Proof. — The proof will be through examination of carrying matrices. Without loss of generality, we may assume that the intersection functional has been rescaled so that $\mathbb{I} = \mathbb{I}_Z$ on $\mathsf{ML}(Z)$. Let

$$\upsilon^{\mathrm{avg}}_m = \frac{1}{2m^2n}(a^{\mathrm{avg}}_m, b^{\mathrm{avg}}_m)$$

where a^{avg}_m is the $1 \times r$ row vector indexed by edges $e \in c$, in which the entry corresponding to e is the sum of the entries of υ_m indexed by edges lying over e. The vector b^{avg}_m is indexed by edges of d and is defined similarly. Since υ_m defines a switch additive measure on the bigon track τ_m, the vector $\upsilon^{\mathrm{avg}}_m$ determines a switch additive probability measure on the bigon track τ formed from c, d. Let $\mathfrak{f}^{\mathrm{avg}}_m$ denote the corresponding measured lamination.

In the case of Φ and its lift $\tilde{\Phi}$, the Perron roots are identical and will be denoted λ. We choose the associated Perron vectors υ and $\tilde{\upsilon}$ so that the former is a probability vector and the latter has L^1 norm $2m^2n$. Thus $\tilde{\upsilon}^{\mathrm{avg}} = \upsilon$, where $\tilde{\upsilon}^{\mathrm{avg}}$ is defined as in the preceding paragraph.

Let us shorten notation, writing $\mathfrak{f} = \mathfrak{f}^u$ and $\mathfrak{f}_m = \mathfrak{f}^u_m$ for the unstable foliations of Φ and Ψ_m. Let $\gamma \in \mathsf{C}(Z)$. Then by (4) and the definition of $\mathbb{I}$ we have

$$\mathbb{I}\big(\mathfrak{f}_m, \gamma\big) \; = \; \frac{1}{\deg(\rho_{m^2n})} \cdot \sum_{e \in \tau_m} \upsilon_m(e)|e \cap \tilde{\gamma}|,$$

where $\tilde{\gamma}$ is the lift of γ to the necklace total space Z_{m^2n}. Thus

$$\mathbb{I}\big(\mathfrak{f}_m, \gamma\big) = \mathbb{I}\big(\mathfrak{f}^{\mathrm{avg}}_m, \gamma\big).$$

However examination of the matrices $M_{\tilde{\Phi}}$ and $M_{(\Psi_m)^n}$ yields

$$\begin{aligned}(\lambda_m)^n \upsilon_m^{\mathrm{avg}} &= ((M_{(\Psi_m)^n}\upsilon_m)^{\mathrm{avg}} \\ &= M_{\Phi}\upsilon_m^{\mathrm{avg}} + ((M_{(\Psi_m)^n} - M_{\tilde{\Phi}})\upsilon_m)^{\mathrm{avg}} \\ &:= M_{\Phi}\upsilon_m^{\mathrm{avg}} + \varepsilon_m.\end{aligned}$$

The entries of the vector ε_m are linear combinations of entries of υ_m that belong to a_m, a_{m+1}, b_{m-1}, b_m, b_{m+1}, scaled by $2m^2 n$. However the coefficients of these linear combinations are independent of m (in view of the fact that the non zero entries of $M_{(\Psi_m)^n} - M_{\tilde{\Phi}}$ are independent of m), hence by Lemma 9.3, ε_m converges to the zero vector. We deduce that $\upsilon_m^{\mathrm{avg}}$ converges to an eigenvector υ_* of M_Φ of eigenvalue λ_*^n (after having passed to a subsequence, if necessary).

Now the entries of υ_* must be non-negative, being a limit of positive probability vectors. In fact it is easy to see, examining the form of M_Φ, that υ_* is positive. However such a matrix has only one positive probability eigenvector, the Perron vector, so that $\upsilon_m^{\mathrm{avg}} \to \upsilon$ and $\lambda_* = \sqrt[n]{\lambda}$. In particular,

$$\lim_{m\to\infty} \mathbb{I}(\mathfrak{f}_m, \gamma) = \lim_{m\to\infty} \mathbb{I}(\mathfrak{f}_m^{\mathrm{avg}}, \gamma) = \mathbb{I}(\mathfrak{f}, \gamma). \tag{8}$$

This takes care of the unstable part of the theorem; the stable part is handled by repeating the above argument for Φ^{-1} and Ψ_m^{-1}. □

Note 10.4. — The intersection convergence proved in Theorem 10.3 extends to test laminations in $\mathsf{ML}(Z) - (\mathbb{R}_+ \cdot \mathfrak{f}) \supset \mathsf{C}(Z)$, uniformly on compacta.

Proof of Main Theorem. — Choose a sequence of covers $Z(\alpha) \to Z$ which are cofinal in the fundamental system $\mathscr{S}$. Since $\mathscr{P}_{\mathrm{int}}$ is directionally dense, there exists a sequence

$$\{\Phi(\alpha) : Z(\alpha) \to Z(\alpha)\} \subset \mathscr{P}_{\mathrm{int}},$$

whose axes $A(\alpha) \to A$ = axis of Φ in the Teichmüller topology. Note that we may assume that this convergence is also with respect to the heights-widths topology. Indeed, let $U_i(\alpha) \subset \mathsf{ML}(Z(\alpha)) - (\mathbb{R}_+ \cdot \mathfrak{f})$, $i = 1, 2, \ldots$ be an exhaustion by compacta, and let $\{V_j = U_{i_j}(\alpha_j)\}$ be a diagonal subsequence which gives an exhaustion of $\mathsf{ML} - (\mathbb{R}_+ \cdot \mathfrak{f}) \supset \mathsf{C}$. Let $\varepsilon_j \to 0$. Then we may choose $\Phi(\alpha_j)$ so that the heights and widths functionals associated to a quadratic differential q_j tangent to the axis A_j of $\Phi(\alpha_j)$ are uniformly ε_j close to those associated to a quadratic differential q tangent to the axis A of Φ, when restricted to curves in V_j.

For each α, fix a natural number $n(\alpha)$ and let $\{\Psi_m(\alpha)\}$ be the sequence of pseudo Anosovs constructed in §9 from the necklace $n(\alpha)$th root. If we denote by $\lambda(\alpha)$ the Perron root of $\Phi(\alpha)$, we may choose the $n(\alpha)$ so that there exists a sequence of natural numbers $N(\alpha)$ for which

$$(\lambda(\alpha))^{N(\alpha)/n(\alpha)} \to \sqrt[n]{\lambda}.$$

Then by Theorem 10.3 and Note 10.4 we may obtain a lengths-widths nth root $\{\Psi_m\}$ of Φ by extracting a suitable diagonal subsequence of $\{(\Psi_m(\alpha))^{N(\alpha)}\}$.

Indeed, let $\{V_j\}$ be the exhaustion of $\mathsf{ML} - (\mathbb{R}_+ \cdot \mathfrak{f})$ defined in the first paragraph. Then for α_j fixed, pick m_j large enough so that intersection pairings of any $\gamma \in V_j$ with the unstable and stable laminations of $\Psi_{m_j}(\alpha_j)$ are uniformly ε_j-close to those with the unstable and stable laminations of Φ. Then the sequence

$$\left\{ \left(\Psi_{m_j}(\alpha_j)\right)^{N(\alpha_j)} \right\}$$

is the desired heights-width root. $\square$

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T.M. GENDRON
Universidad Nacional Autonoma de México
Instituto de Matemáticas
Av. Universidad S/N
Unidad Cuernavaca
C.P. 62210 Cuernavaca Morelos (MÉXICO)
tim@matcuer.unam.mx