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GROUP SCHEMES OVER ARTINIAN RINGS AND APPLICATIONS

by Ioan BERBEC (*)

ABSTRACT. — Let n be a positive integer and A' a complete characteristic zero discrete valuation ring with maximal ideal $\mathfrak{m}$, absolute ramification index $e < p-1$ and perfect residue field k of characteristic $p > 2$. In this paper we classify smooth finite dimensional formal *p-faithful* groups over $A'_n = A'/\mathfrak{m}^n A'$, *i.e.* groups on which the "multiplication by p" morphism is faithfully flat, in particular p-divisible groups. As applications, we prove that p-divisible groups over k, and the morphisms between them, lift canonically to A'/pA', and we study liftings to characteristic zero of certain connected p-divisible groups of dimension d and height h over $k = \overline{k}$, with d and h coprime. When $e = 1$, we classify finite flat group schemes over $A'/p^2 A'$ of p-power order and prove that a finite flat group scheme over $A'/p^n A'$ of p-power order, having flat p^i-torsion for every $i \geqslant 1$, lifts to A'.

RÉSUMÉ. — Soit n un entier positif et A' un anneau de valuation discrète complet de caractéristique zéro avec idéal maximal $\mathfrak{m}$, indice de ramification absolu $e < p-1$ et corps résiduel parfait k de caractéristique $p > 2$. Dans cet article nous classifions les groupes formels lisses *p-fidèles* de dimension finie sur $A'_n = A'/\mathfrak{m}^n A'$, *i.e.* les groupes sur lesquels le morphisme "multiplication par p " est fidèlement plat, en particulier les groupes p-divisibles. Comme application, nous prouvons que les groupes p-divisibles sur k, et les morphismes entre eux, se relèvent canoniquement à A'/pA', et nous étudions les relèvements en caractéristique zéro de certains groupes p-divisibles connexes de dimension d et hauteur h sur $k = \overline{k}$, ou d et h sont étrangers. Quand $e = 1$, nous classifions les schémas en groupes finis et plats sur $A'/p^2 A'$ d'ordre une puissance de p et nous prouvons que tous les schémas en groupes finis et plats sur $A'/p^n A'$ d'ordre une puissance de p, avec p^i-torsion plate pour chaque $i \geqslant 1$, se relèvent à A'.

Introduction

Let $p > 2$ be a prime. Let A' be a complete characteristic 0 discrete valuation ring with absolute ramification index $e = e(A') < p-1$ and perfect

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residue field of characteristic p. Let $\mathfrak{m}$ be its maximal ideal, $k = A'/\mathfrak{m}$ and n a positive integer. In this paper we classify smooth finite dimensional (commutative) formal *p-faithful* groups, *i.e.* groups on which the "multiplication by p" morphism is faithfully flat, in particular p-divisible groups, over $A'_n = A'/\mathfrak{m}^n A'$, and use it to derive other classification results and some applications.

Fontaine classified smooth p-groups over A', *cf.* Theorem 1.11 below. We use his work to achieve the classification of smooth p-groups over A'_n, *cf.* Theorem 2.8. We associate to any such group a so-called *smooth Honda system over* A'_n, *i.e.* linear algebra data constructed from the Dieudonné module of the special fiber of the group. In general, we can prove that this correspondence is essentially surjective and full. In order to prove that this correspondence is also faithful, and thus achieve our classification, we have to restrict to p-faithful groups. While essential surjectivity follows more or less easily from Fontaine's work, fully faithfulness is nontrivial, reflecting phenomena specific to groups over A'_n, *cf.* Lemma 2.10. In the end, we prove that our classification is compatible with Fontaine's. More precisely, we prove that if a p-faithful group Γ over A' is classified, via Fontaine, by the pair $(\mathcal{L}, M)$ then its base change Γ_n to A'_n is classified by $(\mathcal{L}/\mathfrak{m}^{n-1}\mathcal{L}, M)$, *cf.* Proposition 2.12.

In the case $n = e$, from the algebraic properties of the Honda system associated to a p-divisible group over A'_e, we deduce the following proposition. It is implied by Proposition 2.17 and Corollary 2.18.

Proposition. — *For every p-divisible group Γ over k there exists a canonical p-divisible group Γ^{can} over A'_e such that $\Gamma^{\mathrm{can}} \times \operatorname{Spec} k \simeq \Gamma$. Moreover, any morphism $f\colon \Gamma \to \Gamma'$ between p-divisible groups over k lifts canonically to a morphism $f^{\mathrm{can}}\colon \Gamma^{\mathrm{can}} \to (\Gamma')^{\mathrm{can}}$ of p-divisible groups over A'_e. In particular, any abelian variety (resp. finite group scheme) over k lifts canonically to an abelian scheme (resp. finite flat group scheme) over A'_e.*

We apply our classification to the study of liftings to characteristic zero of Manin's groups $G_{d,h-d}$, *cf.* Remark 4.1, where $d < h$ are coprime and k is algebraically closed. The group $G_{d,h-d}$ is a connected p-divisible group of dimension d and absolute height h over k. All connected p-divisible groups of height h and dimension 1 are isomorphic to $G_{1,h}$. In general, all p-divisible groups of dimension d and height h are isogeneous to $G_{d,h-d}$, *cf.* [13], p. 3. We prove, *cf.* Theorem 4.4, the following result.

Theorem 1. — *Let $d < h$ be two coprime positive integers, let k be an algebraically closed field of characteristic p, let $\mathcal{O}$ be the ring of integers in* *a degree h extension $\mathbb{K}$ of $\mathbb{Q}_p$, with absolute ramification index $e < p-1$, and let A' be the ring of integers in a degree e, totally ramified extension of the fraction field of the Witt ring $W(k)$, which contains the maximal unramified extension of $\mathcal{O}$.*

There exists a p-divisible group Γ over A' such that $\Gamma \times \operatorname{Spec} k \simeq G_{d,h-d}$ and $\operatorname{End}_{A'-\mathrm{gr}}(\Gamma) = \mathcal{O}$ if and only if $h \geqslant ed$. In this case:

(i) *There are exactly e^d/g isomorphism classes of such Γ's, where g is the number of automorphisms of $\mathbb{K}$ which fix its maximal unramified subextension.*

(ii) *For every such Γ and every $n \geqslant 1$*

$$\operatorname{End}_{A'_n-\mathrm{gr}}(\Gamma_{A'_n}) = \mathcal{O} + \pi^{n-1} \operatorname{End}_{k-\mathrm{gr}}(\Gamma_k)$$

where $\Gamma_{A'_n}$ (resp. Γ_k) is the base change to A'_n (resp. k) of Γ and π is a uniformizer of $\mathcal{O}$.

We refer the reader to Section 4 for details concerning this result. This Theorem, via Honda systems, becomes a beautiful, yet nontrivial, exercise in semilinear algebra. Over bases with low ramification, Part (i) of the Theorem generalizes to arbitrary dimension results of Lubin, [10], Theorem 4.3.2, and Part (ii) generalizes results of Gross, [6], §3, and Yu, [15], Section 14.

Another application of our classification of p-divisible groups over A'_n is the study of finite flat group schemes over A'_n of p-power order, *finite groups* in the sequel, in the case $e = 1$. Our main tool is Oort's result, *cf.* Theorem 3.1, which states that any finite group embeds into a p-divisible one.

Fontaine associated to a finite group over A' a so-called *finite Honda system (L, M) over A'* that classifies the group, with M being the Dieudonné module of the special fiber of the group, *cf.* [3], Theorem 1.4. We associate to a finite group over A'_n a *finite Honda system over A'_n*, consisting of a triple (L^n, L_n, M), with $(L^n, M[p^{n-1}])$ and $(L_n, M/p^{n-1}M)$ being finite Honda systems over A', *cf.* Definition 3.6 and Proposition 3.7. In the case $n = 2$ we are able to prove that this correspondence classifies finite groups over A'/p^2A', *cf.* Corollary 3.10. For general n we can prove that this correspondence classifies a certain class of finite groups over A'_n among which are the truncated Barsotti-Tate groups of level $s \geqslant 1$, *cf.* Remark 3.11 (2).

One of the main differences between the situation over A' and the situation over A'_n is that the "special fiber" functor from finite groups over A'_n to finite groups over k is not faithful. This implies, in particular, that the category of finite groups over A'_n is not abelian, *cf.* Remark 3.11 (1). Nevertheless, we are able to prove, *cf.* Theorem 3.16, that the "special fiber" functor is faithful on the morphisms that lift to A'_{n+m} for m large enough.

We also prove, *cf.* Theorem 3.13, the following result.

Theorem 2. — *Suppose A' is unramified. Let G be a finite flat group scheme over A'/p^nA' of p-power order.*

(i) *If the p^i-torsion subgroup $G[p^i]$ is flat for every $i \geqslant 1$ then G lifts to A'.*

(ii) *The torsion subgroup $G[p^i]$ is flat for i between* 1 *and some positive integer r if and only if G lifts to $A'/p^{n+r}A'$.*

A future generalization of this paper would be to include higher ($e \geqslant p-1$) ramification on the base. We think that Breuil's techniques, *cf.* [2], can be used to achieve this.

Here is the structure of this paper: in Section 1 we introduce notations and we review concepts and results of Fontaine and Conrad that we will use in our paper. In Section 2 we classify smooth formal p-groups. In Section 3 we study finite groups. In Subsection 3.1 we show how most of the results can be carried out mutatis mutandis in the case of finite groups over A'_n with $e \geqslant 2$ and n of the form $qe+1$. In Section 4 we study liftings to characteristic zero of Manin's groups and their endomorphisms $\bmod \mathfrak{m}^n$.

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1. Notations and Preliminaries

The main references for this paper are Fontaine's book [5] and Conrad's article [3]. For the convenience of the reader we review here all the definitions and results we use from the above papers.

Throughout this paper:

$n \geqslant 2$ is a positive integer.

$p \geqslant 3$ is a fixed prime number.

k is a perfect field of characteristic p.

$A = W(k)$ is the ring of Witt vectors of k. We let $(A', \mathfrak{m})$ be the valuation ring of a finite *totally ramified* extension K' of the fraction field

K of A, with $e = e(A')$ the absolute ramification index of A', and $A'_n = A'/\mathfrak{m}^n$. We fix a uniformizer π of A'.

If $\mathcal{R}$ (resp. R) is an A' (resp A'_n) algebra, we denote by $\mathcal{R}_k$ (resp. R_k) the special fiber $\mathcal{R} \otimes_{A'} k$ (resp. $R \otimes_{A'_n} k$) of $\mathcal{R}$ (resp. R) and by $\mathcal{R}_{K'} = \mathcal{R}_K$ the generic fiber $\mathcal{R} \otimes_{A'} K$ of $\mathcal{R}$.

$D_k = A[F, V]$ is the Dieudonné ring, *i.e.* the variables satisfy $FV = VF = p$, $F\alpha = \sigma(\alpha)$, $V\alpha = \sigma^{-1}(\alpha)$, for all $\alpha \in A$, where $\sigma \colon A \to A$ is the Frobenius morphism.

For us, a *group* is a *group scheme*, formal or finite.

A *pseudo-compact* ring S is a separated and complete linearly topologized ring such that the ring S/I is artinian for all open ideals I of S. Obviously k with the discrete topology, A' and A'_n with the p-adic topology are pseudo-compact.

Definition 1.1. — *Let $(S, \mathfrak{m})$ be a local pseudo-compact ring with residue characteristic p.*

1. *A* formal S-group functor F *is a functor defined on finite S-algebras with values in abelian groups. Thus all our groups are* ***commutative****.*

2. *A* formal S-group *is a pro-representable formal S-group functor. A* formal p-group G over S *is a formal S-group G such that $G \simeq \varinjlim G[p^i]$. We say that a formal S-group G is* smooth *if for all finite S-algebras R and all square zero ideals I of R the canonical map from $G(R)$ to $G(R/I)$ is surjective.*

3. *We say that a smooth formal p-group G over S is* p-faithful *if the "multiplication by p" morphism $[p] \colon G \to G$ is faithfully flat.*

4. *We say that a p-faithful group G over S is* p-divisible *of height h if $G[p^i]$ has order p^{ih} for all $i \geqslant 1$.*

5. *A* finite flat group scheme of p-power order over S *is a formal p-group which is a finite flat scheme over S.*

A *profinite S-module M* is a linearly topologized S-module such that for any open submodule M' the quotient M/M' is an S-module of finite length. A *profinite S-algebra B* is an S-algebra such that B is a profinite S-module.

We briefly review the theory of Witt covectors from [5], Chapter II, §§1-4.

For any commutative ring S (the reader should have in mind k, A or A_n finite algebras as a typical example) we define the *S-valued Witt covectors $CW(S)$* to be the set of sequences $\mathbf{a} = (\dots, a_{-i}, \dots, a_0)$ of elements $a_{-i} \in S$ verifying the condition: *there is an integer $r \geqslant 0$ such that the ideal of S generated by the a_{-i}'s for $i \geqslant r$ is nilpotent.* Letting

$S_m \in \mathbb{Z}[X_0, \dots, X_m, Y_0, \dots, Y_m]$ denote the m^{th} addition polynomial for Witt vectors, *cf.* [5], pp. 71–72, and choosing $\mathbf{a}$ and $\mathbf{b}$ in $CW(S)$, the nilpotence condition ensures that the sequence

$$\left\{S_m(a_{-i-m}, \dots, a_{-i}, b_{-i-m}, \dots, b_{-i})\right\}_{m \geqslant 0}$$

is stationary. Denoting the limit by c_{-i} it is true that $\mathbf{c} = (c_{-i})$ is in $CW(S)$, *cf.* [5], Chapter II, Proposition 1.1. Defining

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$

makes $CW(S)$ into a commutative group with identity $(\dots, 0, \dots, 0)$, *cf.* [5], Chapter II, Proposition 1.4.

We refer the reader to [5], Chapter II, §1.6 for the natural topology of $CW(S)$. We note that $CW(S)$ is complete and separable with respect to this topology and that $CW^u(S) = \{\mathbf{a};\ a_{-i} = 0 \text{ for large } i\}$ is a dense subgroup. Moreover, for every morphism of commutative rings $\varphi\colon S \to S'$ the map

$$CW(\varphi)\colon CW(S) \to CW(S')$$

defined by

$$CW(\varphi)\big((\dots, a_{-i}, \dots, a_0)\big) = \big(\dots, \varphi(a_{-i}), \dots, \varphi(a_0)\big)$$

is continuous. Thus, CW is a functor from the category of commutative rings to the category of topological groups. It can be extended in an obvious way to the category of separable, complete linearly topologized commutative rings, *cf.* [5], Chapter II, §1.7.

Now we specialize to k-algebras S. In this case, $CW_k(S) = CW(S)$ admits a unique structure of topological module over A, such that for all x in k, with Teichmüller lift $[x] = (x, 0, \dots, 0, \dots) \in A$, we have

$$[x] \cdot \mathbf{a} = \big(\dots, x^{p^{-i}} a_{-i}, \dots, x^{p^{-1}} a_{-1}, a_0\big).$$

The operations $F, V\colon CW_k(S) \to CW_k(S)$ given by

$$F(\mathbf{a}) = (\dots, a_{-i}^p, \dots, a_0^p), \quad V(\mathbf{a}) = (\dots, a_{-i-1}, \dots, a_{-1})$$

are additive, continuous and satisfy the relations $FV = VF = p$, $F\alpha = \sigma(\alpha)$, $V\alpha = \sigma^{-1}(\alpha)$, with $\alpha \in A$. In other words, $CW_k(S)$ is a topological D_k-module. This is all functorial in S.

The group functor CW_k on finite k-algebras is pro-representable.We denote by $\widehat{CW_k}$ the group scheme that represents it, *cf.* [5], Chapter II, §4.2.

For any formal p-group G over k we define its *Dieudonné module*

$$\mathscr{M}(G) = \operatorname{Hom}_{k-\mathrm{gr}}(G, \widehat{CW_k})$$

as the group of formal k-group morphisms from G to $\widehat{CW_k}$.

Remark 1.2. — By viewing morphisms of formal group schemes between G and $\widehat{CW_k}$ as morphisms of schemes, we obtain an embedding

$$\mathscr{M}(G) \hookrightarrow \mathrm{Hom}_{k-\mathrm{sch}}(G, \widehat{CW_k}) \simeq CW_k(R)$$

where R is the affine algebra of G. Moreover, if Δ is the comultiplication of R then

$$\mathscr{M}(G) = \big\{\mathbf{a} \in CW_k(R);\ CW(\Delta)(\mathbf{a}) = \mathbf{a}\widehat{\otimes}1 + 1\widehat{\otimes}\mathbf{a}\big\}.$$

This allows us to view $\mathscr{M}(G)$ as a closed topological D_k-submodule of $CW_k(R)$.

All of the standard properties of the classical Dieudonné module theory are proven in [5], Chapter III based on this definition. The main result of this theory, [5], Theorem 1, p. 127, is comprised in the following theorem.

Theorem 1.3. — *The functor $\mathscr{M}$ sets up a duality of abelian categories between formal p-groups over k and certain topological D_k-modules.*

Now, for any separable, complete linearly topologized A-algebra S, in particular for A_n-algebras, $CW(S)$ has a natural structure of topological A-module, which is uniquely determined by

$$[x] \cdot \mathbf{a} = \Big(\dots, \sigma^{-i}([x])a_{-i}, \dots, \sigma^{-1}([x])a_{-1}, [x]a_0\Big)$$

for every $x \in k$ and every $\mathbf{a} = (\dots, a_{-i}, \dots, a_0) \in CW^u(S)$, *cf.* [5], Chapter II, §2.4. We denote this A-module by $CW_A(S)$.

Recall from [5], Chapter II, §5 and Chapter IV, §3 the following definitions and notations:

Definition 1.4.

(i) *A* p-adic A'-algebra $\mathcal{R}$ *is a separable, complete linearly topologized A'-algebra, with the topology being the p-adic one, such that p is not a zero divisor in $\mathcal{R}$.*

(ii) *A* special *A'-algebra $\mathcal{R}$ is a profinite formally smooth A'-algebra locally of finite dimension, i.e. a profinite A'-algebra whose every local component is isomorphic to a power series ring in a finite number of indeterminates with coefficients in the ring of integers of a finite unramified extension of K'.*

For a p-adic A-algebra $\mathcal{R}$ one can construct a continuous A-linear map

$$\widehat{w}_{\mathcal{R}} \colon CW_A(\mathcal{R}) \to \mathcal{R}_K \tag{1.1}$$

the topology on $\mathcal{R}_K$ being the p-adic one, defined by

$$\widehat{w}_{\mathcal{R}}\big((\ldots,\widehat{a}_{-i},\ldots,\widehat{a}_0)\big)=\sum_{i=0}^{\infty}p^{-i}(\widehat{a}_{-i})^{p^i}$$

which induces an A-linear continuous map

$$w_{\mathcal{R}}\colon CW_k(\mathcal{R}_k)\to\frac{\mathcal{R}_K}{p\mathcal{R}} \tag{1.2}$$

defined by

$$w_{\mathcal{R}}\big((\ldots,a_{-i},\ldots,a_0)\big)=\overline{\sum_{i=0}^{\infty}p^{-i}(\widehat{a}_{-i})^{p^i}}$$

where $\widehat{a}_{-i}\in\mathcal{R}$ is an arbitrary lift of a_{-i}, *cf.* [5], Chapter II, §§5.1-2.

If $(\mathcal{R},\mathfrak{m})$ is a local special A'-algebra then we let $\widehat{\mathcal{R}}_K^{an}$ be the separable completion of $\mathcal{R}_K$ with respect to the ideals $J_s=\sum_{i=1}^{\infty}p^{-i+1}\mathfrak{m}^{is}$, for $s\geqslant 1$, *i.e.* $\widehat{\mathcal{R}}_K^{an}=\varprojlim\mathcal{R}_K/J_s$. If $\mathcal{R}$ is an arbitrary special A'-algebra and if $\mathcal{R}=\prod\mathcal{R}_{\mathfrak{m}}$ is the decomposition of $\mathcal{R}$ into local components, we let $\widehat{\mathcal{R}}_K^{an}$ be $\prod(\mathcal{R}_{\mathfrak{m}})_K^{an}$, where $\mathcal{R}_{\mathfrak{m}}$ is $\varprojlim(\mathcal{R}/I)_{\mathfrak{m}/I}$, with I running through all open ideals of $\mathcal{R}$ contained in the open maximal ideal $\mathfrak{m}$. Let us denote by $\Omega_{A'}(\mathcal{R})$ (resp. $\Omega_{A'}(\widehat{\mathcal{R}}_K^{an})$) the module of continuous A'-differentials of $\mathcal{R}$ (resp. $\widehat{\mathcal{R}}_K^{an}$). We let

$$P(\mathcal{R})=\Big\{\alpha\in\widehat{\mathcal{R}}_K^{an};d(\alpha)\in\Omega_{A'}(\mathcal{R})\Big\} \tag{1.3}$$

where $d\colon\widehat{\mathcal{R}}_K^{an}\to\Omega_{A'}(\widehat{\mathcal{R}}_K^{an})$ is the canonical morphism.

For a special A-algebra $\mathcal{R}$ one can construct an A-linear continuous map

$$\widehat{w}_{\mathcal{R}}\colon CW_A(\mathcal{R})\to\widehat{\mathcal{R}}_K^{an} \tag{1.4}$$

defined by the same formula as (1.1) above, whose image is $P(\mathcal{R})$, *cf.* [5], Chapter II, Proposition 5.5, and which induces an A-linear continuous isomorphism

$$w_{\mathcal{R}}\colon CW_k(\mathcal{R}_k)\to\frac{P(\mathcal{R})}{p\mathcal{R}} \tag{1.5}$$

defined as (1.2) above.

1.1. Group schemes over discrete valuation rings

In this Subsection we review Fontaine's classification of smooth p-groups and Conrad's classification of finite flat groups over discrete valuation rings with ramification index $e<p-1$.

Definition 1.5. — *Let M be a D_k-module.*

(i) *Let $M^{(1)}$ be the D_k-module, whose underlying space is M, with A-action given by $a \cdot x := \sigma^{-1}(a)x$, for every $a \in A$ and $x \in M$, and with F and V acting as before. Thus F and V can be seen as A-linear maps*

$$M \xrightarrow{V} M^{(1)}\ ,\ M \xleftarrow{F} M^{(1)}\ .$$

(ii) *Define $M_{A'}$ to be the direct limit of the following diagram of A'-modules*

$$\begin{tikzcd} \mathfrak{m} \otimes_A M \arrow[r, "V_1"] \arrow[d, "\varphi_0"'] & p^{-1}\mathfrak{m} \otimes_A M^{(1)} \\ A' \otimes_A M & A' \otimes_A M^{(1)} \arrow[l, "F_1"] \arrow[u, "\varphi_1"'] \end{tikzcd}$$

where the vertical maps are the obvious "inclusions", $V_1(\lambda \otimes x) = p^{-1}\lambda \otimes V(x)$ and $F_1(\lambda \otimes x) = \lambda \otimes F(x)$, with F, V the usual operators.

(iii) *It is obvious how to associate to a D_k-morphism $\varphi\colon M \to M'$ an A'-morphism $\varphi_{A'}\colon M_{A'} \to M'_{A'}$.*

Remark 1.6.

(i) More explicitly, $M_{A'}$ is the quotient of $A' \otimes_A M \oplus p^{-1}\mathfrak{m} \otimes_A M^{(1)}$ by the submodule

$$\{(\varphi_0(u) - F_1(w), \varphi_1(w) - V_1(u)); u \in \mathfrak{m} \otimes_A M, w \in A' \otimes_A M^{(1)}\}.$$

In particular, it is easy to see that any element in $M_{A'}$ can be written as $(1 \otimes m_0, \sum_{i=1}^{e-1} p^{-1}\pi^i \otimes m_i)$, *cf.* also [3], Lemma 2.2.

(ii) We denote the image of the natural morphism $p^{-1}\mathfrak{m} \otimes_A M^{(1)} \to M_{A'}$ by $M_{A'}[1]$.

(iii) In the case $A' = A$ there is a canonical isomorphism between M and M'_A, via which $M_{A'}[1]$ corresponds to FM. The reader should read M instead of M_A and FM instead of $M_A[1]$ in this case, in all the statements we make.

We have the following basic result, *cf.* [5], Chapter IV, Proposition 2.3.

Proposition 1.7. — *The natural map $M/FM \to M_{A'}/M_{A'}[1]$, given by $[x] \mapsto [(1 \otimes x, 0)]$, is an isomorphism of k-vector spaces.*

Let $\mathcal{R}$ be an A'-special algebra. We defined, *cf.* (1.4), an A-linear morphism $\widehat{w}_{\mathcal{R}}\colon CW_A(\mathcal{R}) \to P(\mathcal{R})$. It is clear that $\widehat{w}_{\mathcal{R}}(CW_A(\mathfrak{m}\mathcal{R})) \subset \mathfrak{m}\mathcal{R}$ $(e<p-1)$, hence $\widehat{w}_{\mathcal{R}}$ induces a morphism $w'_{\mathcal{R}}\colon CW_k(\mathcal{R}_k) \to P(\mathcal{R})/\mathfrak{m}\mathcal{R}$, which in

turn, via extension of scalars, induces a morphism $w''_{\mathcal{R}} \colon A' \otimes_A CW_k(\mathcal{R}_k) \to P(\mathcal{R})/\mathfrak{m}\mathcal{R}$. Now, $A' \otimes_A CW_k(\mathcal{R}_k)$ surjects onto

$$\bigl(CW_k(\mathcal{R}_k)\bigr)_{A'} =: CW_{k,A'}(\mathcal{R}_k),$$

cf. [5], Chapter IV, Proposition 2.5. Fontaine proved that $w''_{\mathcal{R}}$ induces an A'-linear map

$$w_{\mathcal{R}} \colon CW_{k,A'}(\mathcal{R}_k) \to \frac{P(\mathcal{R})}{\mathfrak{m}\mathcal{R}} \tag{1.6}$$

which is an isomorphism, *cf.* [5], Chapter IV, Proposition 3.2.

For future reference, we note that for a p-adic A'-algebra $\mathcal{S}$ we can construct, as in the case of A'-special algebras, an A'-linear map

$$w_{\mathcal{S}} \colon CW_{k,A'}(\mathcal{S}_k) \to \frac{\mathcal{S}_K}{\mathfrak{m}\mathcal{S}} \tag{1.7}$$

starting with $\widehat{w}_{\mathcal{S}} \colon CW_A(\mathcal{S}) \to \mathcal{S}_K$, *cf.* (1.1) above.

Now let G be a smooth formal p-group over A' and let $\mathcal{R}$ be its affine algebra. Then $G_k = G \times \operatorname{Spec} k$, its special fiber, has affine algebra $\mathcal{R}_k$. Let $M = \mathscr{M}(G_k)$ be the Dieudonné module of G_k. We denote by Δ the comultiplication of $\mathcal{R}$ and by $\widehat{\Delta}$ the extension of Δ to $\widehat{\mathcal{R}}_K^{an}$. Let $\delta(\alpha) := \alpha\widehat{\otimes}1 + 1\widehat{\otimes}\alpha - \widehat{\Delta}(\alpha)$ for $\alpha \in \widehat{\mathcal{R}}_K^{an}$. Let:

$$\mathcal{L}_1 = \bigl\{\alpha \in P(\mathcal{R});\ \delta(\alpha) \in \mathfrak{m}\mathcal{R}\widehat{\otimes}_{A'}\mathcal{R}\bigr\} \text{ and } \mathcal{L} = \bigl\{\alpha \in P(\mathcal{R});\ \delta(\alpha) = 0\bigr\}. \tag{1.8}$$

Fontaine proved the following result.

Proposition 1.8.

(i) *The natural morphism $M_{A'} \to CW_{k,A'}(\mathcal{R}_k)$, induced by the inclusion $M \subset CW_k(\mathcal{R}_k)$, is an injection.*

(ii) *The map $w_{\mathcal{R}}$, cf. (1.6) above, induces an isomorphism w between $M_{A'}$ and $\mathcal{L}_1/\mathfrak{m}\mathcal{R}$.*

Remark 1.9. — Conrad proved, *cf.* [3] last part of Lemma 2.7, that $M_{A'}$ is an A'-submodule of $CW_{k,A'}(\mathcal{R}_k)$ also in the case when M is the Dieudonné module of a finite group.

We now take some time to describe the map $w_{\mathcal{R}}$ on the less obvious part of $M_{A'}$, namely on $M_{A'}[1] = \operatorname{Im}(p^{-1}\mathfrak{m} \otimes_A M^{(1)} \to M_{A'})$. Let $x = (0, p^{-1}\pi^i \otimes \mathbf{a})$ be in $M_{A'}[1]$, where $\mathbf{a} = (\ldots, a_{-j}, \ldots, a_0) \in M^{(1)} \subset CW_k(\mathcal{R}_k)^{(1)}$ and i is between 1 and $e-1$. Then, as an element of $CW_{k,A'}(\mathcal{R}_k)$ via the natural inclusion $M_{A'} \hookrightarrow CW_{k,A'}(\mathcal{R}_k)$, x is equal to $(\pi^i \otimes \mathbf{b}, 0)$, where $\mathbf{b} = (\ldots, a_{-j}, \ldots, a_0, a_1)$ with $a_1 \in \mathcal{R}_k$ an arbitrary element. This is so because $p^{-1}\pi^i\otimes\mathbf{a} = V_1(\pi^i\otimes\mathbf{b})$ in $p^{-1}\mathfrak{m}\otimes_A CW_k(\mathcal{R}_k)^{(1)}$, *cf.* Definition 1.5 (ii) and Remark 1.6 (i). Therefore

$$w_{\mathcal{R}}(x) = \overline{\pi^i \sum_{j=0}^{\infty} p^{-j-1}(\widehat{a}_{-j})^{p^{j+1}}} = \overline{\pi^i\beta} \in \frac{\mathcal{L}_1}{\mathfrak{m}\mathcal{R}} \tag{1.9}$$

where $\widehat{a}_{-j} \in \mathcal{R}$ is any lift of a_{-j}.

Before we introduce the category of classifying objects, we note that a *profinite D_k-module M* is a D_k-module M which is A-profinite and is such that its open D_k-submodules form a fundamental system of neighborhoods of 0. Now, the category of classifying objects $\Lambda^l_{A'}$, *cf.* [5], Chapter IV, §4.3, is defined as follows:

Definition 1.10.

1)

a. *The objects are triples $(\mathcal{L}, M, \rho)$, where*

i) *M is a profinite D_k-module on which the action of F is injective such that the quotient M/FM is a finite dimensional k-vector space,*

ii) *$\mathcal{L}$ is a free A'-module of finite rank,*

iii) *$\rho\colon \mathcal{L} \to M_{A'}$ is A'-linear such that the induced morphism $\overline{\rho}\colon \mathcal{L}/\mathfrak{m}\mathcal{L} \to M_{A'}/M_{A'}[1] \xrightarrow{\sim} M/FM$ is an isomorphism of k-vector spaces.*

b. *A morphism $u\colon (\mathcal{L}, M, \rho) \to (\mathcal{L}', M', \rho')$ is a couple $(u_{\mathcal{L}}, u_M)$ with $u_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{L}'$ (resp. $u_M\colon M \to M'$) an A' (resp. D_k) linear morphism such that $u_{M,A'} \circ \rho = \rho' \circ u_{\mathcal{L}}$.*

2) *The category of* smooth Honda systems $\mathrm{H}^d_{A'}$ over A' *has as objects pairs $(\mathcal{L}, M)$ with M and $\mathcal{L}$ as in 1(a) and with $\mathcal{L}$ included in $M_{A'}$. The morphisms are the obvious ones.*

3) *We denote by $\mathcal{SF}_{A'}$ the category of smooth finite dimensional formal p-groups over A'.*

Fontaine defined a functor $\mathcal{L}M_{A'}$ (resp. $\mathcal{L}M^d_{A'}$) from the category $\mathcal{SF}_{A'}$ (resp. of p-divisible groups over A') to the category $\Lambda^l_{A'}$ (resp. $\mathrm{H}^d_{A'}$) by $\mathcal{L}M_{A'}(G) = (\mathcal{L}, M, \rho)$ (resp. $\mathcal{L}M^d_{A'}(G) = (\rho(\mathcal{L}), M)$), with ρ the composition

$$\mathcal{L} \hookrightarrow \mathcal{L}_1 \to \frac{\mathcal{L}_1}{\mathfrak{m}\mathcal{R}} \xrightarrow{\sim} M_{A'} \tag{1.10}$$

where $\mathcal{L}$, $\mathcal{L}_1$ and M are as in (1.8) above.

Then he proved, *cf.* [5], Chapter IV, Theorem 2 (resp. Proposition 5.1), the following theorem.

THEOREM 1.11. — *The functor $\mathcal{L}M_{A'}$ (resp. $\mathcal{L}M^d_{A'}$) induces a duality of categories between the category $\mathcal{SF}_{A'}$ (resp. of p-divisible groups over A') and the category $\Lambda^l_{A'}$ (resp. $\mathrm{H}^d_{A'}$).*

We now review Conrad's classification of finite flat group schemes over A'.

DEFINITION 1.12.

1) *Let $\mathrm{SH}^f_{A'}$ be the category of* finite Honda systems over A' *whose objects are triples (L, M, j) where:*

i) *M is a D_k-module with finite A'-length,*

ii) *$L \xrightarrow{j} M_{A'}$ is a morphism of A'-modules such that the natural k-linear map $L/\mathfrak{m}L \to M_{A'}/M_{A'}[1]$ is an isomorphism of k-vector spaces and such that $\mathcal{V} \circ j$ is injective, where*

$$\mathcal{V} \colon M_{A'} \to A' \otimes_A M^{(1)}$$

is induced by the maps

$$\mathrm{id} \otimes V \colon A' \otimes_A M \to A' \otimes_A M^{(1)},$$
$$p \otimes \mathrm{id} \colon p^{-1}\mathfrak{m} \otimes_A M^{(1)} \to A' \otimes_A M^{(1)}.$$

A morphism $u \colon (L, M, j) \to (L', M', j')$ is a pair $u = (u_L, u_M)$, with $u_M \colon M \to M'$ a continuous D_k-linear morphism and $u_L \colon L \to L'$ an A'-linear morphism such that $u_{M,A'} \circ j = j' \circ u_L$.

2) *We denote by $\mathcal{FF}_{A'}$ the category of finite flat group schemes over A' of p-power order.*

Let G be a finite flat group scheme over A' of p-power order, with affine A'-algebra R. Let $M = \mathscr{M}(G_k)$ be the Dieudonné module of G_k and let $L \subseteq M_{A'}$ denote the kernel of the A'-linear composite map

$$M_{A'} \hookrightarrow CW_{k,A'}(R_k) \xrightarrow{w_R} \frac{R_K}{\mathfrak{m}R} \tag{1.11}$$

where w_R is the map (1.7) above.

Conrad defined, *cf.* [3], §3, a functor

$$LM_{A'} \colon \mathcal{FF}_{A'} \to \mathrm{SH}^f_{A'}$$

by $LM_{A'}(G) := (L, M)$. Moreover, he proved, *cf.* [3], Theorem 3.6, the following theorem.

THEOREM 1.13. — *The functor $LM_{A'}$ is fully faithful and essentially surjective.*

2. Smooth p-groups

In this Section we classify smooth p-faithful groups over A'_n. We start by defining the category of classifying objects.

Definition 2.1.

1) *Let $\Lambda_{A'_n}$ be the category whose objects are triples (L_n, M, ρ) where:*

i) *M is a profinite D_k-module on which the action of F is injective such that the quotient M/FM is a finite dimensional k-vector space;*

ii) *L_n is a free A'_{n-1}-module;*

iii) *$\rho\colon L_n \to M_{A'}/\mathfrak{m}^{n-1}M_{A'}$ is A'_{n-1}-linear such that the induced morphism*

$$\overline{\rho}\colon \frac{L_n}{\mathfrak{m}L_n} \to \frac{M_{A'}}{\mathfrak{m}M_{A'}} \to \frac{M_{A'}}{M_{A'}[1]} \to \frac{M}{FM}$$

is an isomorphism of k-vector spaces.

A morphism $u\colon (L_n, M, \rho) \to (L'_n, M', \rho')$ is a pair (u_L, u_M), with $u_L\colon L_n \to L'_n$ (resp. $u_M\colon M \to M'$) an A'_{n-1} (resp. D_k) linear morphism, for which the following diagram is commutative

$$\begin{array}{ccc} L_n & \xrightarrow{u_L} & L'_n \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho'} \\ \frac{M_{A'}}{\mathfrak{m}^{n-1}M_{A'}} & \xrightarrow{\overline{u}_{M,A'}} & \frac{M'_{A'}}{\mathfrak{m}^{n-1}M'_{A'}} \end{array}$$

where $\overline{u}_{M,A'}$ is induced by $u_{M,A'}$.

2) *Let $\Lambda^f_{A'_n}$ be the full subcategory of $\Lambda_{A'_n}$ of objects (L_n, M, ρ) such that the “multiplication by p” map $[p]\colon M \to M$ is injective.*

3) *We denote by $\mathcal{SF}_{A'_n}$ (resp. $\mathcal{SFF}_{A'_n}$) the category of smooth finite dimensional formal p-groups (resp. p-faithful groups) over A'_n. Recall, cf. Definition 1.1, that a smooth formal p-group is p-faithful if the “multiplication by p” morphism is faithfully flat.*

Now we want to construct a functor

$$\mathcal{L}M_n\colon \mathcal{SF}_{A'_n} \to \Lambda_{A'_n}.$$

For this, let G be a smooth formal p-group over A'_n and let R be its affine algebra. Then $G_k = G \times \operatorname{Spec} k$, its special fiber, has affine algebra R_k. Let $M = \mathscr{M}(G_k)$ be the Dieudonné module of G_k. Let $\mathcal{R}$ be a smooth A'-lift of R. We know it is unique up to non-unique isomorphism. We denote by Δ (resp. Δ_k) the comultiplication of R (resp. R_k). Let $\widehat{\Delta}\colon \mathcal{R} \to \mathcal{R}\widehat{\otimes}_{A'}\mathcal{R}$ be an A'-algebra morphism that lifts Δ. We also denote by $\widehat{\Delta}$ the extension of $\widehat{\Delta}$ to $\widehat{\mathcal{R}}_K^{an}$. For $n \geqslant r \geqslant 1$ define:

$$\mathcal{L}_r = \left\{\alpha \in P(\mathcal{R});\ \delta(\alpha) \in \mathfrak{m}^r \mathcal{R}\widehat{\otimes}_{A'}\mathcal{R}\right\} \tag{2.1}$$

with $P(\mathcal{R})$ and δ as in (1.3) and (1.8) above, respectively.

Remark 2.2. — Our $\mathcal{L}_1$ is Fontaine's $\mathfrak{M}\mathfrak{H}_{A'}(G)$, *cf.* [5], pp. 166–167 and p. 202.

LEMMA 2.3. — *The sets $\mathcal{L}_r$ are independent of the lift $\widehat{\Delta}$, i.e. they only depend on G and the lift $\mathcal{R}$.*

Proof. — Suppose $\widehat{\Delta}$ and $\widehat{\Delta}_1 \colon \mathcal{R} \to \mathcal{R}\widehat{\otimes}_{A'}\mathcal{R}$ are two A'-lifts of Δ which are uniquely extended to $\widehat{\mathcal{R}}_K^{an}$. Let δ and δ_1 be the corresponding morphisms. For every $\alpha \in \mathcal{R}$ we have $\widehat{\Delta}(\alpha) \equiv \widehat{\Delta}_1(\alpha) \pmod{\mathfrak{m}^n \mathcal{R}\widehat{\otimes}\mathcal{R}}$, that is $\widehat{\Delta}\ (\alpha) = \widehat{\Delta}_1\ (\alpha) + \pi^n x$ with $x \in \mathcal{R}\widehat{\otimes}_{A'}\mathcal{R}$. Therefore, since $\widehat{\Delta}$ and $\widehat{\Delta}_1$ are morphisms of algebras,

$$\widehat{\Delta}\ (\alpha^{p^i}) = (\widehat{\Delta}\ (\alpha))^{p^i} = (\widehat{\Delta}_1(\alpha))^{p^i} + p^i\pi^n y = \widehat{\Delta}_1(\alpha^{p^i}) + p^i\pi^n y \tag{2.2}$$

with $y \in \mathcal{R}\widehat{\otimes}_{A'}\mathcal{R}$. So $p^{-i}\widehat{\Delta}\ (\alpha^{p^i}) \equiv p^{-i}\widehat{\Delta}_1(\alpha^{p^i}) \pmod{\mathfrak{m}^n \mathcal{R}\widehat{\otimes}\mathcal{R}}$ for all integers $i \geqslant 0$. In the unramified setting every element of $P(\mathcal{R})$ is an infinite sum of elements of the form $p^{-i}\beta^{\,p^i}$ with $\beta \in \mathcal{R}$, *cf.* (1.4). In the ramified setting the situation is similar, because $A' \otimes_A CW_A(\mathcal{R})$ surjects on $P(\mathcal{R})$, *cf.* (1.6). Hence every element of $P(\mathcal{R})$ is an A'-combination of infinite sums of elements of the form $p^{-i}\beta^{p^i}$ with $\beta \in \mathcal{R}$. Therefore we get that $\delta(\alpha) - \delta_1(\alpha) \in \mathfrak{m}^n \mathcal{R}\widehat{\otimes}\mathcal{R}$. □

Let $\widetilde{\rho}$ be the composition

$$P(\mathcal{R}) \xrightarrow{\text{proj.}} P(\mathcal{R})/\mathfrak{m}\mathcal{R} \xrightarrow{w_{\mathcal{R}}^{-1}} CW_{k,A'}(\mathcal{R}_k)$$

where $w_{\mathcal{R}}$ is the map (1.6) above.

The following Lemma follows directly from [5], Chapter IV, Lemmas 1.2 and 4.3.

LEMMA 2.4. — *Let r be an integer between 1 and $n-1$ and let $\alpha \in \mathcal{L}_r$. There exists an element $\gamma \in \mathcal{L}_1$ such that $\widetilde{\rho}(\gamma) \in M_{A'}[1]$ and $(\alpha - \pi^{r-1}\gamma) \in \mathcal{L}_{r+1}$.*

Let $\widetilde{\rho}_0$ be the composition

$$\mathcal{L}_n \hookrightarrow \mathcal{L}_1 \xrightarrow{\text{proj.}} \mathcal{L}_1/\mathfrak{m}\mathcal{R} \xrightarrow{w^{-1}} M_{A'} \tag{2.3}$$

where, w is the map in Proposition 1.8 (ii) above, and let ρ_0 be the induced map

$$\rho_0\colon \frac{\mathcal{L}_n}{\mathfrak{m}^{n-1}\mathcal{L}_1} \to \frac{M_{A'}}{\mathfrak{m}^{n-1}M_{A'}}. \tag{2.4}$$

Note that these maps depend only on G and the lift $\mathcal{R}$.

We have the following key result.

Lemma 2.5.

(i) *The morphism ρ_0 induces a map*

$$\overline{\rho}_0\colon \frac{\mathcal{L}_n}{\mathfrak{m}\mathcal{L}_n+\mathfrak{m}^{n-1}\mathcal{L}_1} \simeq \frac{\frac{\mathcal{L}_n}{\mathfrak{m}^{n-1}\mathcal{L}_1}}{\mathfrak{m}\cdot\frac{\mathcal{L}_n}{\mathfrak{m}^{n-1}\mathcal{L}_1}} \to \frac{\frac{M_{A'}}{\mathfrak{m}^{n-1}M_{A'}}}{\mathfrak{m}\cdot\frac{M_{A'}}{\mathfrak{m}^{n-1}M_{A'}}} \xrightarrow{\text{proj.}} \frac{M_{A'}}{M_{A'}[1]} \xrightarrow{\sim} \frac{M}{FM}$$

which is an isomorphism of k-vector spaces.

(ii) $\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ *is a free A'_{n-1}-module.*

Proof.

(i) Since the surjectivity of $\overline{\rho}_0\colon \mathcal{L}_n/(\mathfrak{m}\mathcal{L}_n+\mathfrak{m}^{n-1}\mathcal{L}_1)\to M_{A'}/M_{A'}[1]$ follows from Lemma 2.4, *cf.* also [5], Chapter IV, Proposition 1.1, all we have to prove is that it is injective. For this, we need to prove that $\widetilde{\rho}^{-1}(M_{A'}[1])\cap\mathcal{L}_n=\mathfrak{m}\mathcal{L}_n+\mathfrak{m}^{n-1}\mathcal{L}_1$ (we can view $\overline{\rho}_0$ as being induced by $\widetilde{\rho}\colon P(\mathcal{R})\to CW_{k,A'}(\mathcal{R}_k)$).

"$\subseteq$" Let $\alpha\in\mathcal{L}_n$ such that $\widetilde{\rho}(\alpha)\in M_{A'}[1]$. This means, *cf.* (1.9), that $\alpha=\pi\beta$ for some $\beta\in P(\mathcal{R})$. Since α is in $\mathcal{L}_n$ it follows that β is in $\mathcal{L}_{n-1}$. On the other hand, from Lemma 2.4 we know there is a $\gamma\in\mathcal{L}_1$ such that $\widetilde{\rho}(\gamma)\in M_{A'}[1]$ and $(\beta-\pi^{n-2}\gamma)\in\mathcal{L}_n$. Hence

$$\alpha=\pi\beta=\pi(\beta-\pi^{n-2}\gamma+\pi^{n-2}\gamma)=\pi(\beta-\pi^{n-2}\gamma)+\pi^{n-1}\gamma\in\mathfrak{m}\mathcal{L}_n+\mathfrak{m}^{n-1}\mathcal{L}_1.$$

This proves the first inclusion. Since the other inclusion "$\supseteq$" is obvious, it follows that $\overline{\rho}_0$ is injective.

For future reference, we observe that $\widetilde{\rho}$ induces a surjective morphism, also denoted $\widetilde{\rho}$

$$\widetilde{\rho}\colon \mathcal{L}_n \twoheadrightarrow \frac{M_{A'}}{M_{A'}[1]} \xrightarrow{\sim} \frac{M}{FM}. \tag{2.5}$$

(ii) First of all, from (i) and some linear algebra follows that $\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ is a finite A' (hence A'_{n-1})-module. From (2.5) above, it follows that $\widetilde{\rho}$ induces a surjective morphism $\rho_1\colon \mathcal{L}_n/\mathfrak{m}\mathcal{L}_{n-1}\to M/FM$. Since $\mathfrak{m}\mathcal{L}_n+$

$\mathfrak{m}^{n-1}\mathcal{L}_1 \subset \mathfrak{m}\mathcal{L}_{n-1}$ we have a commutative diagram

$$\begin{tikzcd} \frac{\mathcal{L}_n}{\mathfrak{m}\mathcal{L}_{n-1}} \arrow[r, "\rho_1"] & \frac{M}{FM} \\ \frac{\mathcal{L}_n}{\mathfrak{m}\mathcal{L}_n+\mathfrak{m}^{n-1}\mathcal{L}_1} \arrow[u, two heads] \arrow[ur, two heads, "\bar{\rho}_0"'] & \end{tikzcd} .$$

Since $\bar{\rho}_0$ is an isomorphism, in particular injective, it follows that ρ_1 is injective, hence an isomorphism since it was already surjective. It also follows that the vertical arrow in the above diagram is an isomorphism. Therefore $\mathcal{L}_n/\mathfrak{m}\mathcal{L}_{n-1} = \mathcal{L}_n/(\mathfrak{m}\mathcal{L}_n + \mathfrak{m}^{n-1}\mathcal{L}_1)$.

Now, let $\{\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_d\} \subset \mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ be a lift of a basis of $\mathcal{L}_n/(\mathfrak{m}\mathcal{L}_n + \mathfrak{m}^{n-1}\mathcal{L}_1)$. We claim that $\pi^{n-2} \cdot \overline{\alpha}_j \neq \overline{0}$ in $\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$, for all j between 1 and d. Indeed, suppose $\pi^{n-2} \cdot \overline{\alpha}_j = \overline{0}$ in $\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ for some j. Then it follows that $\pi^{n-2}\alpha_j \in \mathfrak{m}^{n-1}\mathcal{L}_1$. Hence $\pi^{n-2}\alpha_j = \pi^{n-1}\beta$ for some β in $\mathcal{L}_1$. So $\alpha_j = \pi\beta$ and it follows that $\beta \in \mathcal{L}_{n-1}$, because $\alpha_j \in \mathcal{L}_n$. Therefore $\alpha_j \in \mathfrak{m}\mathcal{L}_{n-1}$ and $[\overline{\alpha}_j] = [\overline{0}]$ in $\mathcal{L}_n/\mathfrak{m}\mathcal{L}_{n-1} = \mathcal{L}_n/(\mathfrak{m}\mathcal{L}_n + \mathfrak{m}^{n-1}\mathcal{L}_1)$ which is a contradiction since $[\overline{\alpha}_j]$ is part of a basis of $\mathcal{L}_n/(\mathfrak{m}\mathcal{L}_n + \mathfrak{m}^{n-1}\mathcal{L}_1)$.

From the above claim, the fact that $\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ is a finite A'-module and the structure of modules over a principal ideal domain, it follows that $\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ is A'_{n-1}-free. □

With the previous Lemma we got really close to the definition of our classifying functor $\mathcal{L}M_n$. We are not there yet because all the objects we constructed depend on the lift $\mathcal{R}$. So our goal in the sequel is to embed $\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ in an object depending only on G. This is rather technical and will follow from the following lemma.

Lemma 2.6.

(i) *There is a natural surjective map*

$$A'\widehat{\otimes}_A CW_A(R) \to P(\mathcal{R})/\mathfrak{m}^{n-1}P(\mathcal{R})$$

of A'-modules, whose kernel $\mathcal{K}$ does not depend on the lift $\mathcal{R}$.

(ii) *There is a natural injective map*

$$\frac{\mathcal{L}_n}{\mathfrak{m}^{n-1}\mathcal{L}_1} \to \frac{A'\widehat{\otimes}_A CW_A(R)}{\mathcal{K}}$$

whose image L_n does not depend on the lift $\mathcal{R}$ and is "functorial", i.e. there is a functor $\mathbf{L}_n \colon \mathcal{SF}_{A'_n} \to A'_{n-1}$-Mod such that $\mathbf{L}_n(G) = L_n$.

(iii) *Let $\rho \colon L_n \to M_{A'}/\mathfrak{m}^{n-1}M_{A'}$ be induced by ρ_0 ($\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ and L_n are isomorphic). Then for every morphism $f \colon G \to G'$ in $\mathcal{SF}_{A'_n}$ the* *diagram*

$$\begin{array}{ccc} L_n & \xrightarrow{\mathbf{L}_n(f)} & L'_n \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho'} \\ \dfrac{M_{A'}}{\mathfrak{m}^{n-1}M_{A'}} & \xrightarrow{\overline{\mathscr{M}}(f)} & \dfrac{M'_{A'}}{\mathfrak{m}^{n-1}M'_{A'}} \end{array} \tag{2.6}$$

is commutative, where $\mathscr{M}$ is the Dieudonné module functor and $\overline{\mathscr{M}}(f)$ is induced by $\mathscr{M}(f)_{A'}\colon M_{A'} \to M'_{A'}$.

Proof. — This Lemma is quite straightforward. All we will do is draw some diagrams which will make everything clear.

(i) We have the following diagram of A-modules with exact columns

$$\begin{tikzcd} CW_A(\mathfrak{m}^n\mathcal{R}) \arrow[r] \arrow[d, hook] & \mathfrak{m}^{n-1}P(\mathcal{R}) \arrow[d, hook] \\ CW_A(\mathcal{R}) \arrow[r, "\widehat{w}_{\mathcal{R}}"] \arrow[d, two heads] & P(\mathcal{R}) \arrow[d, two heads] \\ CW_A(R) & P(\mathcal{R})/\mathfrak{m}^{n-1}P(\mathcal{R}) \end{tikzcd}$$

where $\widehat{w}_{\mathcal{R}}$ is the map (1.4) above and the top horizontal map is the restriction of $\widehat{w}_{\mathcal{R}}$. Actually $\widehat{w}_{\mathcal{R}}(CW_A(\mathfrak{m}^n\mathcal{R})) \subset \mathfrak{m}^n\mathcal{R} \subset \mathfrak{m}^{n-1}P(\mathcal{R})$. From the above diagram we get an induced morphism

$$w'_n(\mathcal{R})\colon CW_A(R) \to P(\mathcal{R})/\mathfrak{m}^{n-1}P(\mathcal{R})$$

of A-modules, which yields, via extension of scalars, an A'-morphism

$$w_n(\mathcal{R})\colon A'\widehat{\otimes}_A CW_A(R) \to P(\mathcal{R})/\mathfrak{m}^{n-1}P(\mathcal{R}).$$

Now for any other A'-lift $\mathcal{R}_1$ of R there exists a non-unique A'-isomorphism $\mathcal{R} \xrightarrow{\varphi} \mathcal{R}_1$. It is easy to see that we have the following commutative diagram

$$\begin{tikzcd} & P(\mathcal{R})/\mathfrak{m}^{n-1}P(\mathcal{R}) \arrow[dd, "\varphi^*"] \\ A'\widehat{\otimes}_A CW_A(R) \arrow[ur, "w_n(\mathcal{R})"] \arrow[dr, "w_n(\mathcal{R}_1)"'] & \\ & P(\mathcal{R}_1)/\mathfrak{m}^{n-1}P(\mathcal{R}_1) \end{tikzcd} \tag{2.7}$$

where φ^* is induced by φ. This shows that the A'-module

$$\mathcal{K} = \ker(w_n(\mathcal{R})) = \ker(w_n(\mathcal{R}_1))$$

does not depend on the lift $\mathcal{R}$. This proves (i).

(ii) Since $\mathcal{L}_n \cap \mathfrak{m}^{n-1}P(\mathcal{R}) = \mathfrak{m}^{n-1}\mathcal{L}_1$ we get a canonical map

$$\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1 \hookrightarrow P(\mathcal{R})/\mathfrak{m}^{n-1}P(\mathcal{R}),$$

which induces an injective map

$$\frac{\mathcal{L}_n}{\mathfrak{m}^{n-1}\mathcal{L}_1} \hookrightarrow \frac{P(\mathcal{R})}{\mathfrak{m}^{n-1}P(\mathcal{R})} \xrightarrow{\sim} \frac{A'\widehat{\otimes}_A CW_A(R)}{\mathcal{K}}. \tag{2.8}$$

Now, a diagram similar to (2.7) above, with $(A'\widehat{\otimes}_A CW_A(R))/\mathcal{K}$ instead of $A'\widehat{\otimes}_A CW_A(R)$ and with $\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ (resp. $\mathcal{L}'_n/\mathfrak{m}^{n-1}\mathcal{L}'_1$) instead of $P(\mathcal{R})/\mathfrak{m}^{n-1}P(\mathcal{R})$ (resp. $P(\mathcal{R}')/\mathfrak{m}^{n-1}P(\mathcal{R}')$) will convince the reader that the image of (2.8) does not depend on the lift $\mathcal{R}$, where the objects $\mathcal{L}'_1$ and $\mathcal{L}'_n$ correspond to $\mathcal{R}'$.

We are left with defining the functor $\mathbf{L}_n\colon \mathcal{SF}_{A'_n} \to A'_{n-1}$-Mod. Since we said how it acts on objects, $\mathbf{L}_n(G) = L_n =$ image of (2.8), we only have to define it on morphisms. Let $f\colon G' \to G$ be a morphism in $\mathcal{SF}_{A'_n}$, let $\varphi\colon R \to R'$ be the induced morphism of A'_n-algebras, and let $\phi\colon \mathcal{R} \to \mathcal{R}'$ be a lift of φ where $\mathcal{R}$ (resp. $\mathcal{R}'$) is an A'-special lift of R (resp. R'). We also denote by ϕ the unique extension of ϕ to $\widehat{\mathcal{R}}^{an}_K$ and its restriction to $P(\mathcal{R})$. It is obvious that $\phi(\mathcal{L}_r) \subset \mathcal{L}'_r$ for all $1 \leqslant r \leqslant n$. If $\overline{\phi}\colon \mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1 \to \mathcal{L}'_n/\mathfrak{m}^{n-1}\mathcal{L}'_1$ is induced by ϕ we define $\mathbf{L}_n(f)\colon L_n \to L'_n$ to be the induced morphism. Recall that $L_n \simeq \mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ and $L'_n \simeq \mathcal{L}'_n/\mathfrak{m}^{n-1}\mathcal{L}'_1$. It is clear that this construction is functorial. This proves (ii).

(iii) We want to prove that the diagram (2.6) is commutative. We keep the same notations as in the proof of (ii) above. The commutativity of (2.6) follows from the following enlarged diagram

$$\begin{tikzcd}
L_n \arrow[r, "\sim"] \arrow[d, "\mathbf{L}_n(f)"'] & \frac{\mathcal{L}_n}{\mathfrak{m}^{n-1}\mathcal{L}_1} \arrow[r, hook] \arrow[d, "\overline{\phi}"] & \frac{\mathcal{L}_1}{\mathfrak{m}^{n-1}\mathcal{L}_1} \arrow[r, two heads] \arrow[d, "\overline{\phi}"] & \frac{\mathcal{L}_1}{\mathfrak{m}\mathcal{R}+\mathfrak{m}^{n-1}\mathcal{L}_1} \arrow[r, "\sim"] \arrow[d, "\overline{\phi}"] & \frac{M_{A'}}{\mathfrak{m}^{n-1}M_{A'}} \arrow[d, "\overline{\mathscr{M}}(f)"] \\
L'_n \arrow[r, "\sim"] & \frac{\mathcal{L}'_n}{\mathfrak{m}^{n-1}\mathcal{L}'_1} \arrow[r, hook] & \frac{\mathcal{L}'_1}{\mathfrak{m}^{n-1}\mathcal{L}'_1} \arrow[r, two heads] & \frac{\mathcal{L}'_1}{\mathfrak{m}\mathcal{R}'+\mathfrak{m}^{n-1}\mathcal{L}'_1} \arrow[r, "\sim"] & \frac{M'_{A'}}{\mathfrak{m}^{n-1}M'_{A'}}
\end{tikzcd}$$

in which all the squares are commutative and in which the composition of the maps on the top (resp. bottom) row is ρ (resp. ρ'). Here we used the fact that $\mathcal{L}_1/\mathfrak{m}\mathcal{R} \xrightarrow{\sim} M_{A'}$, *cf.* Proposition 1.8 (ii), to get the isomorphisms between the fourth and the fifth column in the above diagram. ☐

Now we are able to make the following definition.

DEFINITION 2.7. — *Define the functor $\mathcal{L}M_n\colon \mathcal{SF}_{A'_n} \to \Lambda_{A'_n}$ as follows:*
i) *for an object G of $\mathcal{SF}_{A'_n}$ we let*

$$\mathcal{L}M_n(G) := (L_n, M, \rho)$$

where M is the Dieudonné module of G_k and L_n and ρ are as in Lemma 2.6;

ii) *for a morphism $f\colon G \to G'$ in $\mathcal{SF}_{A'_n}$ we let*

$$\mathcal{L}M_n(f) = (\mathbf{L}_n(f), \mathscr{M}(f))$$

where $\mathbf{L}_n$ is as in Lemma 2.6.

Remark. — Lemmas 2.5 and 2.6 guarantee that the functor $\mathcal{L}M_n$ is well defined. Also, it is clear that by restricting $\mathcal{L}M_n$ to $\mathcal{SFF}_{A'_n}$ we get a functor

$$\mathcal{L}M_n^f\colon \mathcal{SFF}_{A'_n} \to \Lambda^f_{A'_n}.$$

The main result of this Section is the following theorem.

Theorem 2.8.

A. *The functor $\mathcal{L}M_n$ is essentially surjective and full.*

B. *The functor $\mathcal{L}M_n^f$ induces a duality of categories between the category $\mathcal{SFF}_{A'_n}$ of smooth finite dimensional p-faithful groups over A'_n and the category $\Lambda^f_{A'_n}$.*

Proof.

A. We first prove that $\mathcal{L}M_n$ is essentially surjective. The reader who is familiar with the proof of the main result of Fontaine, *cf.* Theorem 1.11 above, will realize that our proof of essential surjectivity is built on his, *cf.* [5], §1 and §4.8. Let (L_n, M, ρ) be an object in $\mathcal{SF}_{A'_n}$. From Theorem 1.3 we know there exists a smooth formal p-group G_k of finite dimension over k such that its Dieudonné module $M_0 = \mathscr{M}(G_k)$ is isomorphic to M. We fix such an isomorphism $i_M\colon M \simeq M_0$. Let R_k be the affine k-algebra of G_k, let $\mathcal{R}$ be an A'-special lift of R_k and let $R = \mathcal{R}/\mathfrak{m}^n\mathcal{R}$. Note that R is unique up to non-unique isomorphism. As in Lemma 2.3, the set $\mathcal{L}_1 = \{\alpha \in P(\mathcal{R}); \delta(\alpha) \in \mathfrak{m}\mathcal{R}\widehat{\otimes}\mathcal{R}\} \subset P(\mathcal{R})$ is well defined and depends only on R_k. Since L_n is A'_{n-1}-free, we can embed it into $\mathcal{L}_1/\mathfrak{m}^{n-1}\mathcal{L}_1 \subset P(\mathcal{R})/\mathfrak{m}^{n-1}\mathcal{L}_1$. We choose an isomorphism $i_L\colon L_n \to L_0$ of L_n onto an A'_{n-1}-submodule of $\mathcal{L}_1/\mathfrak{m}^{n-1}\mathcal{L}_1$ such that the following diagram

$$\begin{tikzcd} L_n \arrow[r, "i_L"] \arrow[d, "\rho"'] & L_0 \arrow[r, hook] & \frac{\mathcal{L}_1}{\mathfrak{m}^{n-1}\mathcal{L}_1} \arrow[d] \\ \frac{M_{A'}}{\mathfrak{m}^{n-1}M_{A'}} \arrow[rr, "\overline{i_M}"] & & \frac{(M_0)_{A'}}{\mathfrak{m}^{n-1}(M_0)_{A'}} \end{tikzcd}$$

is commutative. Here $\overline{i_M}$ is induced by $(i_M)_{A'}$ and the right vertical arrow is induced by the isomorphism

$$\frac{P(\mathcal{R})}{\mathfrak{m}\mathcal{R}} \supset \frac{\mathcal{L}_1}{\mathfrak{m}\mathcal{R}} \xrightarrow{w^{-1}} (M_0)_{A'} \subset CW_{k,A'}(R_k)$$

cf. Proposition 1.8 (ii).

Let $\mathcal{L}_0 \subset \mathcal{L}_1$ be an A'-free module of rank $d = \mathrm{rk}_{A'_{n-1}}(L_n) = \dim(G_k)$ that lifts L_0 and let $\mathcal{L}_n = \mathcal{L}_0 + \mathfrak{m}^{n-1}\mathcal{L}_1 \subset \mathcal{L}_1 \subset P(\mathcal{R})$. It is easy to see that $\mathcal{L}_n$ does not depend on the particular lift $\mathcal{L}_0$. Furthermore it is not hard to see that we have

$$\frac{\mathcal{L}_0}{\mathfrak{m}\mathcal{L}_0} \simeq \frac{L_0}{\mathfrak{m}L_0} \simeq \frac{L_n}{\mathfrak{m}L_n} \tag{2.9}$$

and

$$\frac{\mathcal{L}_0}{\mathfrak{m}^{n-1}\mathcal{L}_0} \simeq \frac{\mathcal{L}_n}{\mathfrak{m}^{n-1}\mathcal{L}_1} = L_0 \simeq L_n. \tag{2.10}$$

We denote by p_0 the projection from $\mathcal{L}_0$ onto L_n via the above isomorphism. Due to the particular choice of i_L we know that the following diagram

$$\begin{tikzcd}
\mathcal{L}_0 \arrow[rr, "\widehat{\rho_0}"] \arrow[d, "p_0"'] & & (M_0)_{A'} \arrow[d] \\
L_n \arrow[r, "\rho"] & \frac{M_{A'}}{\mathfrak{m}^{n-1}M_{A'}} \arrow[r, "\overline{i_M}"] & \frac{(M_0)_{A'}}{\mathfrak{m}^{n-1}(M_0)_{A'}}
\end{tikzcd}$$

commutes, where $\widehat{\rho_0}$ is the composition

$$\mathcal{L}_0 \hookrightarrow \mathcal{L}_1 \to \frac{\mathcal{L}_1}{\mathfrak{m}\mathcal{R}} \simeq (M_0)_{A'}.$$

From now on, we identify M with M_0 and L_n with L_0 via i_M and i_L. Let $\mathcal{S}$ be a p-adic A'-algebra:

- we denote by $N_{\mathcal{L}_0}(\mathcal{S})$ (resp. $N^0_{\mathcal{L}_0}(\mathcal{S})$) the abelian group $\mathrm{Hom}_{A'}(\mathcal{L}_0, \mathcal{S}_K/\mathfrak{m}^n\mathcal{S})$ (resp. $\mathrm{Hom}_{A'}(\mathcal{L}_0, \mathcal{S}_K/\mathfrak{m}\mathcal{S})$) of A'-linear morphisms from $\mathcal{L}_0$ to $\mathcal{S}_K/\mathfrak{m}^n\mathcal{S}$ (resp. $\mathcal{S}_K/\mathfrak{m}\mathcal{S}$);
- we denote by $G_M(\mathcal{S})$ the abelian group $\mathrm{Hom}^{\mathrm{cont}}_{D_k}(M, CW_k(\mathcal{S}_k))$ of continuous D_k-linear morphisms from M to $CW_k(\mathcal{S}_k)$;
- we denote by φ_{ρ_0} the map from $G_M(\mathcal{S})$ to $N^0_{\mathcal{L}_0}(\mathcal{S})$ which associates to $u \in G_M(\mathcal{S})$ the composition
$$\mathcal{L}_0 \xrightarrow{\widehat{\rho_0}} M_{A'} \xrightarrow{u_{A'}} CW_{k,A'}(\mathcal{S}_k) \xrightarrow{w_{\mathcal{S}}} \frac{\mathcal{S}_K}{\mathfrak{m}\mathcal{S}}$$
cf. (1.7) above for the definition of $w_{\mathcal{S}}$;
- finally, we denote by $G_{(\mathcal{L}_0, M, \widehat{\rho_0})}(\mathcal{S})$ the fiber product $N_{\mathcal{L}_0}(\mathcal{S}) \times_{N^0_{\mathcal{L}_0}(\mathcal{S})} G_M(\mathcal{S})$, where the morphism from $N_{\mathcal{L}_0}(\mathcal{S})$ to $N^0_{\mathcal{L}_0}(\mathcal{S})$ is the one induced by the projection from $\mathcal{S}_K/\mathfrak{m}^n\mathcal{S}$ to $\mathcal{S}_K/\mathfrak{m}\mathcal{S}$ and the morphism from $G_M(\mathcal{S})$ to $N^0_{\mathcal{L}_0}(\mathcal{S})$ is φ_{ρ_0}.

It is clear that everything is functorial in $\mathcal{S}$. Thus we have defined a functor $G_{(\mathcal{L}_0,M,\widehat{\rho}_0)}$ from p-adic A'-algebras to abelian groups. We can also define another group functor $G_{(\mathcal{L}_n,M,\widehat{\rho})}$ from p-adic A'-algebras to abelian groups by replacing in the above construction $\mathcal{L}_0$ with $\mathcal{L}_n/\mathfrak{m}^n\mathcal{R}$ and $\widehat{\rho}_0$ with $\widehat{\rho}$, where $\widehat{\rho}$ is the composition

$$\frac{\mathcal{L}_n}{\mathfrak{m}^n\mathcal{R}} \to \frac{\mathcal{L}_1}{\mathfrak{m}\mathcal{R}} \simeq M_{A'}.$$

For any p-adic A'-ring $\mathcal{S}$ we denote by $X_R(\mathcal{S}_n)$ the set of continuous A'_n-algebra morphisms from R to $\mathcal{S}_n = \mathcal{S}/\mathfrak{m}^n\mathcal{S}$. We want to define a morphism $\eta(\mathcal{S})\colon X_R(\mathcal{S}_n) \to G_{(\mathcal{L}_n,M,\widehat{\rho})}(\mathcal{S})$. Let x be in $X_R(\mathcal{S}_n)$ and let $\widehat{x}\colon \mathcal{R} \to \mathcal{S}$ be an A'-lift of x. We also denote by $\widehat{x}$ the unique extension of $\widehat{x}$ to a morphism from $\widehat{\mathcal{R}}_K^{an}$ to $\mathcal{S}_K$ and its restriction to $P(\mathcal{R})$. We define $x_{\mathcal{L}_n}$ to be the composition

$$\frac{\mathcal{L}_n}{\mathfrak{m}^n\mathcal{R}} \hookrightarrow \frac{P(\mathcal{R})}{\mathfrak{m}^n\mathcal{R}} \xrightarrow{\overline{\widehat{x}}} \frac{\mathcal{S}_K}{\mathfrak{m}^n\mathcal{S}}. \tag{2.11}$$

It is easy to see that $x_{\mathcal{L}_n}$ does not depend on the lift $\widehat{x}$ of x. Let x_M be the composition

$$M \hookrightarrow CW_k(R_k) \overset{CW(x_k)}{\longrightarrow} CW_k(\mathcal{S}_k) \tag{2.12}$$

where $x_k\colon R_k \to \mathcal{S}_k$ is induced by x. We define $\eta(\mathcal{S})(x) = (x_{\mathcal{L}_n}, x_M)$.

It is clear from the following commutative diagram

(2.13)

$$\begin{tikzcd}
\mathcal{L}_n/\mathfrak{m}^n\mathcal{R} \arrow[r, hook] \arrow[d, "\widehat{\rho}"'] & P(\mathcal{R})/\mathfrak{m}^n\mathcal{R} \arrow[rr, "\overline{\widehat{x}}"] & & \mathcal{S}_K/\mathfrak{m}^n\mathcal{S} \arrow[d] \\
M_{A'} \arrow[r, hook] & CW_{k,A'}(R_k) \arrow[r, "(CW_k(x_k))_{A'}"] & CW_{k,A'}(\mathcal{S}_k) \arrow[r, "w_{\mathcal{S}}"] & \mathcal{S}_K/\mathfrak{m}\mathcal{S}
\end{tikzcd}$$

that $\eta(\mathcal{S})$ is well defined. It is obvious that η is functorial in $\mathcal{S}$.

Moreover, the inclusion $\mathcal{L}_0 \hookrightarrow \mathcal{L}_n$ defines a morphism of group functors $\mathrm{res}_{\mathcal{L}_0}\colon G_{(\mathcal{L}_n,M,\widehat{\rho})} \to G_{(\mathcal{L}_0,M,\widehat{\rho}_0)}$, which in turn defines a morphism $\eta_{\mathcal{L}_0}(\mathcal{S})\colon X_R(\mathcal{S}_n) \to G_{(\mathcal{L}_0,M,\widehat{\rho}_0)}(\mathcal{S})$, for every p-adic A'-ring $\mathcal{S}$, by letting

$$\eta_{\mathcal{L}_0}(\mathcal{S}) = \mathrm{res}_{\mathcal{L}_0}(\mathcal{S}) \circ \eta(\mathcal{S}). \tag{2.14}$$

It is clear that $\eta_{\mathcal{L}_0}$ is functorial in $\mathcal{S}$.

Now, the fact that $\mathcal{L}M_n$ is essentially surjective follows from the following lemma.

Lemma 2.9. — *For all p-adic A'-algebras $\mathcal{S}$ the map $\eta_{\mathcal{L}_0}(\mathcal{S})$ is bijective. There is a unique structure of smooth formal p-group on $G = \mathrm{Spf}_{A'_n} R$*

induced by $\eta_{\mathcal{L}_0}$ which is independent of the particular lift $\mathcal{L}_0$, i.e. it depends only on the triple (L_n, M, ρ). Moreover $\mathcal{L}M_n(G) \simeq (L_n, M, \rho)$.

Proof of Lemma 2.9. — This Lemma follows basically from [5], Chapter IV, §1.6 and §4.8. We limit ourselves to pointing out the main steps in the proof. We use Fontaine's notations.

We prove that $\eta_{\mathcal{L}_0}(\mathcal{S})$ is bijective. Let (ξ, γ) be an element in $G_{(\mathcal{L}_0, M, \widehat{\rho_0})}(\mathcal{S})$. We want to prove that there exists a unique element x in $X_R(\mathcal{S}_n)$ such that $\eta_{\mathcal{L}_0}(\mathcal{S})(x) = (\xi, \gamma)$. First of all, by Theorem 1.3 above, there exists a unique continuous k-algebra morphism $x_k\colon R_k \to \mathcal{S}_k$ such that $\gamma = CW(x_k)$. So we want to prove that there exists a unique $x\colon R \to \mathcal{S}_n$, which reduces to x_k, such that $x_{\mathcal{L}_0} = \xi$, where $x_{\mathcal{L}_0}$ is the restriction of $x_{\mathcal{L}_n}$ to $\mathcal{L}_0$, *cf.* (2.11) for the definition of $x_{\mathcal{L}_n}$.

We have $R_k = R_k^{\mathrm{et}} \widehat{\otimes}_k R_k^{\mathrm{c}}$. Let $\mathcal{R}^{\mathrm{et}}$ denote the lifting of R_k^{et} in $\mathcal{R}$ and let us choose a local subring $\mathcal{R}^{\mathrm{c}}$ of $\mathcal{R}$ which lifts R_k^{c}. There is a natural isomorphism of $\mathcal{R}^{\mathrm{et}} \widehat{\otimes}_{A'} \mathcal{R}^{\mathrm{c}}$ into $\mathcal{R}$. We chose coordinates $\underline{X} = (X_1, X_2, \dots, X_d)$ of $\mathcal{R}^{\mathrm{c}}$. Thus $\mathcal{R}^{\mathrm{c}}$ is identified with the ring $A'[[\underline{X}]]$ and R_k^{c} with $k[[\overline{\underline{X}}]]$, where $\overline{X}_j$ is the image of X_j in R_k^{c} for $1 \leqslant j \leqslant d$.

It is obvious that a lift $x\colon \mathcal{R} \to \mathcal{S}$ of x_k is uniquely determined by a d-tuple $\sigma = (s_1, s_2, \dots, s_d) \in \mathcal{S}^d$, where $s_j \in \mathcal{S}$ is a lift of $\overline{s}_j = x_k^{\mathrm{c}}(\overline{X}_j) \in \mathcal{S}_k$. Thus there is a bijection between morphisms $x\colon R \to \mathcal{S}_n$ that lift x_k and d-tuples $\sigma \in \mathcal{S}^d$, which are well defined modulo $\mathfrak{m}^n\mathcal{S}$. We will denote by x^σ the morphism from $\mathcal{R}$ to $\mathcal{S}$ corresponding to σ.

Since $\eta_{\mathcal{L}_0}(\mathcal{S})(x) = (x_{\mathcal{L}_0}, x_M) = (x_{\mathcal{L}_0}, \gamma) \in G_{(\mathcal{L}_0, M, \widehat{\rho_0})}(\mathcal{S})$, it follows that $x_{\mathcal{L}_0}(\alpha) - \xi(\alpha) \in \mathfrak{m}\mathcal{S}$ for all $\alpha \in \mathcal{L}_0$. We see that the bijectivity of $\eta_{\mathcal{L}_0}(\mathcal{S})$ will follow from the following result.

Sublemma 2.9.1. — *Let r be an integer between 1 and $n-1$. Suppose there exists a d-tuple $\sigma_0 = (s_1^0, s_2^0, \dots, s_d^0) \in \mathcal{S}^d$ lifting the $\overline{s}_j$'s, which is uniquely determined modulo $\mathfrak{m}^r\mathcal{S}$ and which is such that $x_{\mathcal{L}_0}^{\sigma_0}(\alpha) - \xi(\alpha) \in \mathfrak{m}^r\mathcal{S}$ for all $\alpha \in \mathcal{L}_0$. Then there exists a d-tuple $\sigma = (s_1, s_2, \dots, s_d) \in \mathcal{S}^d$ lifting the $\overline{s}_j$'s, which is uniquely determined modulo $\mathfrak{m}^{r+1}\mathcal{S}$ and which is such that $x_{\mathcal{L}_0}^{\sigma}(\alpha) - \xi(\alpha) \in \mathfrak{m}^{r+1}\mathcal{S}$ for all $\alpha \in \mathcal{L}_0$.*

This Sublemma, in the unramified case, is actually the result on the bottom of p. 178 in [5]. A careful reading of the proof of Fontaine's result shows that the only condition on $\mathcal{L}_0$ for the result to hold is that $\mathcal{L}_0/p\mathcal{L}_0$ be isomorphic to M/FM. This condition is satisfied in our case since $\mathcal{L}_0/p\mathcal{L}_0 \simeq L_n/pL_n \simeq M/FM$, *cf.* (2.9) above.

The proof in the ramified case goes through similarly. To convince the reader of this, we show that the main steps in the above proof have a correspondent in the ramified case, too. For space reasons, we use the same notations as in [5], Chapter IV, §1.6. Here are the main steps:

(i) Let $\alpha \in \mathcal{L}_1$. Then $\alpha = w_{\mathcal{R}}(y)$, with $y = (1 \otimes \mathbf{a}_0, \sum_{i=1}^{e-1} p^{-1}\pi^i \otimes \mathbf{a}_i)$ in $M_{A'}$, with $\mathbf{a}_0 = (\dots, a_{-m,0}, \dots, a_{-1,0}, a_{0,0}) \in M$ and $\mathbf{a}_i = (\dots, a_{-m,i}, \dots, a_{-2,i}, a_{-1,i}) \in M$ for all $1 \leqslant i \leqslant e-1$, *cf.* Remark 1.6 (i), Proposition 1.8 and (1.9). It follows that

$$\alpha = \widehat{a}_{0,0} + \sum_{i=0}^{e-1} \pi^i \left(\sum_{m=1}^{\infty} p^{-m} \widehat{a}_{-m,i}^{p^m} \right) + \pi\beta \tag{2.15}$$

for some lifts $\widehat{a}_{-m,i} \in \mathcal{R}$ of the $a_{-m,i}$'s. Moreover, since $M = M^{\mathrm{c}} \oplus M^{\mathrm{et}}$, we can actually write α like

$$\alpha = \alpha^{\mathrm{c}} + \alpha^{\mathrm{et}} + \pi\beta$$

with α^{c} (resp. α^{et}) as in (2.15) above, but with the $\widehat{a}_{-m,j}$'s in $\mathcal{R}^{\mathrm{c}}$ (resp. $\mathcal{R}^{\mathrm{et}}$) and $\beta \in \mathcal{R}$.

(ii) Let $\{\alpha_1, \dots, \alpha_d\}$ be an A'-basis of $\mathcal{L}_0$. Then, with notations as in (i) above, it is trivial that $\frac{\partial \alpha_i^{\mathrm{c}}}{\partial X_u}$ is in $\mathcal{R}^{\mathrm{c}}$, for all i and u between 1 and d. Moreover, the fact that $\mathcal{L}_0/\mathfrak{m}\mathcal{L}_0 \xrightarrow{\sim} M_{A'}/M_{A'}[1] \xrightarrow{\sim} M/FM$, *cf.* (2.9) above, guarantees that the matrix $(\frac{\partial \alpha_i^{\mathrm{c}}}{\partial X_u})_{1 \leqslant i,u \leqslant d}$ is invertible.

(iii) One can repeat the argument in [5] bottom of p. 179 - top of p. 180 to show that, if $x_i = x_i^0 + \pi^r y_i$ in $\mathcal{S}$, for $1 \leqslant i \leqslant d$, then

$$\alpha_i^{\mathrm{c}}(x_1, \dots, x_d) - \alpha_i^{\mathrm{c}}(x_1^0, \dots, x_d^0) \equiv \pi^r \sum_{u=1}^{d} \frac{\partial \alpha_i^{\mathrm{c}}}{\partial X_u}(x_1^0, \dots, x_d^0) \cdot y_u \pmod{\mathfrak{m}^{r+1}\mathcal{S}}$$

for all $1 \leqslant i \leqslant d$. The argument still works because $e < p-1$. This ends our sketch of the proof of the Sublemma.

The rest of Lemma 2.9 is standard. □

We now prove that $\mathcal{L}M_n$ is full. Let G be an A'_n-group, R its affine algebra, $\mathcal{R}$ an A'-special lift of R, and let $(L_n, M, \rho) = \mathcal{L}M_n(G)$. Let $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_n$ be as before.

From Lemma 2.9 and from (2.14) it follows that, for all p-adic A'-rings $\mathcal{S}$, the map

$$\eta(\mathcal{S})\colon G(\mathcal{S}_n) \to G_{(\mathcal{L}_n, M, \widehat{\rho})}(\mathcal{S})$$

is an injective group homomorphism.

Let G' be another A'_n-group with affine algebra R', let $\mathcal{R}'$ be an A'-special lift of R', let $\mathcal{L}'_1$ and $\mathcal{L}'_n$ be as in (2.1) above, and let $(L'_n, M', \rho') = \mathcal{L}M_n(G')$. Let $(\zeta, \gamma)\colon (L_n, M, \rho) \to (L'_n, M', \rho')$ be a morphism in $\Lambda_{A'_n}$. We want to construct a morphism $f\colon G' \to G$ such that $\mathcal{L}M_n(f) = (\zeta, \gamma)$. In order to do that, we need the following important technical result.

Lemma 2.10. — *There exists a morphism $\widehat{\zeta}_0\colon \mathcal{L}_0 \to \mathcal{L}'_n$ which makes the following diagram*

$$\begin{tikzcd}
\mathcal{L}_0 \arrow[r, hook] \arrow[dddr, "\widehat{\zeta}_0"'] & \mathcal{L}_n \arrow[rrr, "\widehat{\rho}"] \arrow[dr, "\mathrm{pr}"'] & & & M_{A'} \arrow[dl, "p_M"] \arrow[ddd, "\gamma_{A'}"] \\
& & L_n \arrow[r, "\rho"] \arrow[d, "\zeta"'] & \frac{M_{A'}}{\mathfrak{m}^{n-1}M_{A'}} \arrow[d, "\overline{\gamma}"] & \\
& & L'_n \arrow[r, "\rho'"] & \frac{M'_{A'}}{\mathfrak{m}^{n-1}M'_{A'}} & \\
& \mathcal{L}'_n \arrow[ur, "\mathrm{pr}'"] \arrow[rrr, "\widehat{\rho}'"'] & & & M'_{A'} \arrow[ul, "p_{M'}"']
\end{tikzcd}$$

commute. Moreover, there exists a unique A'-linear morphism $\widetilde{\zeta}\colon \mathcal{L}_n \to \mathcal{L}'_n/(\mathfrak{m}^{n-1}\mathcal{L}'_1 \cap \mathfrak{m}\mathcal{R}')$ lifting ζ, which makes the diagram

$$\begin{tikzcd}
\mathcal{L}_n \arrow[rr, "\widehat{\rho}"] \arrow[d, "\widetilde{\zeta}"'] & & M_{A'} \arrow[d, "\widetilde{\gamma}"] \\
\frac{\mathcal{L}'_n}{\mathfrak{m}^{n-1}\mathcal{L}'_1 \cap \mathfrak{m}\mathcal{R}'} \arrow[r, two heads] & L'_n \arrow[r, "\rho'"] & \frac{M'_{A'}}{\mathfrak{m}^{n-1}M'_{A'}}
\end{tikzcd}$$

commute, where $\widetilde{\gamma} = p_{M'} \circ \gamma_{A'}$.

Proof of Lemma 2.10. — Let $\{e_1, \ldots, e_d\} \subset \mathcal{L}_0$ be an A'-basis of $\mathcal{L}_0$. There are two conditions that $\widehat{\zeta}_0$ has to satisfy: $\mathrm{pr}' \circ \widehat{\zeta}_0 = \zeta \circ \mathrm{pr}$ and $\widehat{\rho}' \circ \widehat{\zeta}_0 = \gamma_{A'} \circ \widehat{\rho}$. From these two conditions we see that in order to construct $\widehat{\zeta}_0$ we need to come up with a set of elements $\{h_1, \ldots, h_d\} \subset \mathcal{L}'_n$ such that h_i is a lift of $\zeta(\overline{e}_i)$ and $\widehat{\rho}'(h_i) = \gamma_{A'}(\widehat{\rho}(e_i))$, for all $1 \leqslant i \leqslant d$, where $\overline{e}_i = \mathrm{pr}(e_i)$.

Let $g_i \in \mathcal{L}'_n$ be an arbitrary lift of $\zeta(\overline{e}_i)$, for every i. We have

$$(p_{M'} \circ \widehat{\rho}')(g_i) = (\rho' \circ \mathrm{pr}')(g_i) = \rho'(\zeta(\overline{e}_i)) = \overline{\gamma}(\rho(\overline{e}_i)) = (p_{M'} \circ \gamma_{A'} \circ \widehat{\rho})(e_i).$$

So $\widehat{\rho}'(g_i) - (\gamma_{A'} \circ \widehat{\rho})(e_i) \in \mathfrak{m}^{n-1}M'_{A'}$. We know $w_{\mathcal{R}'}(\mathcal{L}'_1) = M'_{A'}$, *cf.* Proposition 1.8 (ii). Hence, there exists $g'_i \in \mathcal{L}'_1$ such that

$$(\gamma_{A'} \circ \widehat{\rho})(e_i) = \widehat{\rho}'(g_i) + \widehat{\rho}'(\pi^{n-1}g'_i).$$

Now it is clear that the elements $(h_i)_{1 \leqslant i \leqslant d}$, defined by $h_i = g_i + \pi^{n-1}g'_i$, satisfy the above requirements. Define $\widehat{\zeta}_0\colon \mathcal{L}_0 \to \mathcal{L}'_n$ by sending e_i to h_i for all i's.

Let $\alpha \in \mathcal{L}_n$. Suppose there exist u and v in $\mathcal{L}'_n$ such that

$$\mathrm{pr}'(u) = \mathrm{pr}'(v) = \zeta(\overline{\alpha}) \quad \text{and} \quad \widehat{\rho}'(u) = \widehat{\rho}'(v) = \gamma_{A'}(\widehat{\rho}(\alpha)).$$

It follows that $u-v\in\pi^{n-1}\mathcal{L}'_1$ and $u-v\in\operatorname{Ker}(\widehat{\rho}') = \mathcal{L}'_n\cap\mathfrak{m}\mathcal{R}'$, *cf.* Proposition 1.8 (ii). Thus $u-v$ has to be in $\mathfrak{m}^{n-1}\mathcal{L}'_1\cap\mathcal{L}'_n\cap\mathfrak{m}\mathcal{R}' = \mathfrak{m}^{n-1}\mathcal{L}'_1\cap\mathfrak{m}\mathcal{R}'$.

On the other hand, from the first part of the Proof we see that we can always find an element u_α in $\mathcal{L}'_n$ such that

$$\pi'(u_\alpha) = \zeta(\overline{\alpha}) \ \text{ and } \ \widehat{\rho}'(u_\alpha) = \gamma_{A'}(\widehat{\rho}(\alpha)).$$

Thus, the correspondence $\alpha\mapsto[u_\alpha]$ defines a well defined A'-linear morphism $\widetilde{\zeta}\colon\mathcal{L}_n \to \mathcal{L}'_n/(\pi^{n-1}\mathcal{L}'_1\cap\pi\mathcal{R}')$, which is uniquely determined by (ζ,γ). □

Now, $\widehat{\zeta}_0$ and γ induce a morphism of group functors

$$\phi\colon G'_{(\mathcal{L}'_n,M',\widehat{\rho}')}\to G_{(\mathcal{L}_0,M,\widehat{\rho}_0)}.$$

Since $\eta_{\mathcal{L}_0}$ is an isomorphism between G and $G_{(\mathcal{L}_0,M,\widehat{\rho}_0)}$, the morphism ϕ induces a morphism of group functors

$$f\colon G'\to G$$

that makes the diagram

$$\begin{tikzcd} G' \arrow[r, "f"] \arrow[d, "\eta'"'] & G \arrow[d, "\eta_{\mathcal{L}_0}"] \\ G'_{(\mathcal{L}'_n,M',\widehat{\rho}')} \arrow[r, "\phi"] & G_{(\mathcal{L}_0,M,\widehat{\rho}_0)} \end{tikzcd} \tag{2.16}$$

commute.

It is clear from the first diagram in Lemma 2.10 that $\mathcal{L}M_n(f) = (\zeta,\gamma)$. Thus $\mathcal{L}M_n$ is full.

B. We will prove that $\mathcal{L}M_n^f$ is essentially surjective and fully faithful. By Part **A**, all we need to prove is faithfulness. Let $f\colon G'\to G$ be a morphism in $\mathcal{SFF}_{A'_n}$ such that $\mathcal{L}M_n^f(f) = (0,0)$ in $\operatorname{Hom}_{\Lambda^f_{A'_n}}\big((L_n,M,\rho),(L'_n,M',\rho')\big)$, where $(L_n,M,\rho) = \mathcal{L}M_n^f(G)$ and $(L'_n,M',\rho') = \mathcal{L}M_n^f(G')$. We need the following result.

Lemma 2.11. — *If G is p-faithful then*

$$\mathfrak{m}^{n-1}\mathcal{L}_1\cap\mathfrak{m}\mathcal{R} = \mathfrak{m}^n\mathcal{R}.$$

Proof of Lemma 2.11. — Since $[p]\colon G_k\to G_k$ is faithful, it follows that $[p]\colon M\to M$ is injective. This, after some computations using the explicit description of $M_{A'}$ given in Remark 1.6 (i), implies that $[p]_{A'}\colon M_{A'}\to M_{A'}$ is also injective. Since $[p]_{A'}$ is the "multiplication by p" map on $M_{A'}$ and

$(\pi^e) = (p)$, it follows that the "multiplication by π" map on $M_{A'}$ is injective. Hence, we get an isomorphism

$$[\pi]\colon \frac{\mathcal{L}_1}{\mathfrak{m}\mathcal{R}} \to \pi \cdot \frac{\mathcal{L}_1}{\mathfrak{m}\mathcal{R}} = \frac{\mathfrak{m}\mathcal{L}_1 + \mathfrak{m}\mathcal{R}}{\mathfrak{m}\mathcal{R}} = \frac{\mathfrak{m}\mathcal{L}_1}{\mathfrak{m}\mathcal{L}_1 \cap \mathfrak{m}\mathcal{R}}.$$

Since $u\colon \mathcal{L}_1/\mathfrak{m}\mathcal{R} \to \mathfrak{m}\mathcal{L}_1/\mathfrak{m}^2\mathcal{R}$ is also an isomorphism, where $u(\widetilde{\alpha}) = \overline{\pi\alpha}$, we get that $\mathfrak{m}\mathcal{L}_1 \cap \mathfrak{m}\mathcal{R} = \mathfrak{m}^2\mathcal{R}$. The Lemma follows by finite induction. $\square$

Let $f^*\colon R \to R'$ be induced by f and let $\varphi\colon \mathcal{R} \to \mathcal{R}'$ be a lift of f^*. We also denote by φ the unique extension of φ to $P(\mathcal{R})$. It is clear that $\varphi(\mathcal{L}_1) \subset \mathcal{L}'_1$ and $\varphi(\mathcal{L}_n) \subset \mathcal{L}'_n$. In particular f induces a morphism $\overline{\varphi}\colon \mathcal{L}_n \to \mathcal{L}'_n/\mathfrak{m}^n\mathcal{R}'$ which reduces to the zero morphism from L_n to L'_n and which makes the second diagram in Lemma 2.10 commute. Since, by Lemma 2.11, $\mathfrak{m}^{n-1}\mathcal{L}'_1 \cap \mathfrak{m}\mathcal{R}' = \mathfrak{m}^n\mathcal{R}'$ we can apply Lemma 2.10 to deduce that the morphism $\overline{\varphi}\colon \mathcal{L}_n \to \mathcal{L}'_n/\mathfrak{m}^n\mathcal{R}'$ is the zero morphism. Therefore

$$\varphi(\mathcal{L}_n) \subset \mathfrak{m}^n\mathcal{R}'. \tag{2.17}$$

Let $\mathcal{S}$ be a p-adic A'-ring. Using (2.17), we get that the morphism $\widetilde{\varphi}(\mathcal{S})\colon G_{(\mathcal{L}'_n,M',\widehat{\rho'})}(\mathcal{S}) \to G_{(\mathcal{L}_n,M,\widehat{\rho})}(\mathcal{S})$, induced by $\varphi\colon \mathcal{L}_n \to \mathcal{L}'_n$ and $0 = \mathscr{M}(f)\colon M \to M'$, is the zero morphism. Now, since in the commutative diagram

$$\begin{tikzcd}
G'(\mathcal{S}) \arrow[r, "f(\mathcal{S})"] \arrow[d, "\eta'(\mathcal{S})"'] & G(\mathcal{S}) \arrow[d, "\eta(\mathcal{S})"] \\
G'_{(\mathcal{L}'_n,M',\widehat{\rho'})}(\mathcal{S}) \arrow[r, "\widetilde{\varphi}(\mathcal{S})=0"] & G_{(\mathcal{L}_n,M,\widehat{\rho})}(\mathcal{S})
\end{tikzcd}$$

the vertical arrows are injective, $f(\mathcal{S})$ has to be zero. Hence f is the zero morphism between G' and G. Thus $\mathcal{L}M_n^f$ is faithful.

With this we achieved the proof of our Theorem. $\square$

Proposition 2.12. — *Consider the following diagram of functors:*

$$\begin{tikzcd}
\mathcal{SF}_{A'} \arrow[r, "\mathcal{L}M_{A'}"] \arrow[d, "B_n"'] & \Lambda^l_{A'} \arrow[d, "Q_n"] \\
\mathcal{SF}_{A'_n} \arrow[r, "\mathcal{L}M_n"] & \Lambda_{A'_n}
\end{tikzcd}$$

where B_n is the base change from A' to A'_n and Q_n sends an object $(\mathcal{L}, M, \rho)$ of $\Lambda^l_{A'}$ to the object $(\mathcal{L}/\mathfrak{m}^{n-1}\mathcal{L}, M, \overline{\rho})$ of $\Lambda_{A'_n}$, with

$$\overline{\rho}\colon \mathcal{L}/\mathfrak{m}^{n-1}\mathcal{L} \to M_{A'}/\mathfrak{m}^{n-1}M_{A'}$$

induced by ρ.

The functors $F = Q_n \circ \mathcal{L}M_{A'}$ and $E = \mathcal{L}M_n \circ B_n$ are isomorphic.

Proof. — Let G be an object in $\Lambda^l_{A'}$. With usual notations, we have $F(G) = (\mathcal{L}/\mathfrak{m}^{n-1}\mathcal{L}, M, \overline{\rho})$ (resp. $E(G) = (L_n, M, \rho_n)$), where $(\mathcal{L}, M, \rho)$ (resp. (L_n, M, ρ_n)) is as in (1.8) (resp. Definition 2.7).

We have the following commutative diagram of A'-modules

$$\begin{tikzcd}
\frac{\mathcal{L}}{\mathfrak{m}^{n-1}\mathcal{L}} \arrow[r, hook] \arrow[d, "\sim"'] \arrow[rrd, dotted, "u"] \arrow[rdd, dotted] & \frac{P(\mathcal{R})}{\mathfrak{m}^{n-1}P(\mathcal{R})} \arrow[r, "\sim"] & \frac{A'\widehat{\otimes}_A CW_A(R)}{\mathcal{K}} \\
\frac{\mathcal{L}_n}{\mathfrak{m}^{n-1}\mathcal{L}_1} \arrow[ru, hook] \arrow[rr, "\sim"] \arrow[rd] & & L_n \arrow[u, hook] \arrow[ld, "\rho_n"] \\
& \frac{M_{A'}}{\mathfrak{m}^{n-1}M_{A'}} &
\end{tikzcd} \tag{2.18}$$

cf. Lemma 2.6 for the objects in the right column. Note that the morphism $\mathcal{L}/\mathfrak{m}^{n-1}\mathcal{L} \to \mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1$ is an isomorphism, because it is so mod p and the modules are A'_{n-1}-flat.

Define a morphism $\mu(G)\colon F(G) \to E(G)$, by $\mu(G) = (u, \mathrm{id}_M)$. It is clear that $\mu(G)$ is well defined and bijective and that μ is a morphism of functors. □

We have the following consequence of the Proof of Theorem 2.8 and of Proposition 2.12.

Corollary 2.13. — *Any smooth finite dimensional formal p-group over A'_n lifts to a smooth finite dimensional formal p-group over A'.*

Now, we restrict Theorem 2.8 to the case of p-divisible groups, *cf.* Definition 1.1 (4). For this, we first introduce the classifying category, the category of *smooth Honda systems over A'_n*.

Definition 2.14. — *Let $H^{\mathrm{d}}_{A'_n}$ be the full subcategory of $\Lambda^f_{A'_n}$ whose objects are pairs $(\mathcal{L}_n, M)$, with M an A'-free module and*

$$\mathcal{L}_n \subseteq M_{A'}/(\mathfrak{m}^{n-1}M_{A'}).$$

Define a functor $\mathcal{L}M^d_n$ from the category of p-divisible groups over A'_n to the category of smooth Honda systems over A'_n by

$$\mathcal{L}M^d_n(G) = (\rho_0(\mathcal{L}_n/\mathfrak{m}^{n-1}\mathcal{L}_1), M),$$

cf. (2.1) and (2.4).

From Theorem 2.8 we get the following result.

Corollary 2.15. — *The functor $\mathcal{L}M^d_n$ induces a duality of categories between the category of p-divisible groups over A'_n and the category $H^{\mathrm{d}}_{A'_n}$ of smooth Honda systems over A'_n.*

Remark 2.16. — Let G be a p-divisible group over A' and let $\mathcal{L}M^d_{A'}(G) = (\mathcal{L}, M)$, *cf.* (1.10). From Proposition 2.12 it follows that

$$\mathcal{L}M^d_n(G_n) = (\mathcal{L}/\mathfrak{m}^{n-1}\mathcal{L}, M).$$

The following result is a special feature of p-divisible groups over $A'_e = A'/pA'$.

Proposition 2.17. — *For every p-divisible group Γ over k there exists a canonical lift Γ^{can} over A'_e. Moreover, for every morphism $f\colon \Gamma \to \Gamma'$ between p-divisible groups over k there exists a canonical morphism $f^{\mathrm{can}}\colon \Gamma^{\mathrm{can}} \to (\Gamma')^{\mathrm{can}}$ over A'_e that lifts it.*

Proof. — Let M be the Dieudonné module of Γ. Then, we claim that the natural morphism

$$\frac{A'}{\pi^{e-1}A'} \otimes_A \frac{M}{FM} \to \frac{M_{A'}}{\pi^{e-1}M_{A'}}$$

given by $\sum_{j=0}^{e-2} \pi^j \otimes m_j \mapsto [(\sum_{j=0}^{e-2} \pi^j \otimes m_j, 0)]$, is injective and its image $L^{\mathrm{can}} \subseteq M_{A'}/\pi^{e-1}M_{A'}$ is A'_{e-1}-free.

It is easy to see that it is well defined, *i.e.* that $\sum_{j=0}^{e-2} \pi^j \otimes Fm_j$ is mapped to zero. Indeed, this follows from the following relation

$$\Big(\sum_{j=0}^{e-2} \pi^j \otimes Fm_j, 0\Big) = \sum_{j=0}^{e-2}\big(\pi^j \otimes Fm_j, -\pi^j \otimes m_j\big) + \pi^{e-1}\sum_{j=0}^{e-2}\big(0, p^{-1}\pi^{j+1} \otimes m_j\big)$$

in which the right hand side is clearly zero in $M_{A'}/\pi^{e-1}M_{A'}$, *cf.* Remark 1.6 (i).

We now prove that it is injective. Assume $[(\sum_{j=0}^{e-2} \pi^j \otimes m_j, 0)]$ is zero in $M_{A'}/\pi^{e-1}M_{A'}$. This means that we have the following relation

$$\begin{aligned}\Big(\sum_{j=0}^{e-2} \pi^j \otimes m_j, 0\Big) &= \pi^{e-1}\Big(\sum_{s=0}^{e-1} \pi^s \otimes m'_s, \sum_{t=1}^{e} p^{-1}\pi^t \otimes n_t\Big)\\ &\quad + \Big(\sum_{t=1}^{e} \pi^t \otimes x_t - \sum_{s=0}^{e-1} \pi^s \otimes Fy_s, \sum_{s=0}^{e-1} \pi^s \otimes y_s - \sum_{t=1}^{e} p^{-1}\pi^t \otimes Vx_t\Big)\end{aligned}$$

in $A' \otimes_A M \oplus p^{-1}\mathfrak{m} \otimes_A M^{(1)}$, for some m''s, n's, x's and y's in M. In the case $e = 3$, the reader would have to trust us that similar things happen in the general case, this gives the set of equations

$$\begin{aligned} m_0 &= pm'_1 + px_3 - Fy_0, & 0 &= pn_2 + py_1 - Vx_1,\\ m_1 &= pm'_2 + x_1 - Fy_1, & 0 &= pn_3 + py_2 - Vx_2,\\ 0 &= m'_0 + x_2 - Fy_2, & 0 &= n_1 + y_0 - Vx_3.\end{aligned}$$

Clearly $m_0 \in FM$ and, because $Vx_1 \in pM = VFM$ and V_M is injective, it follows that $x_1 \in FM$, hence $m_1 \in FM$.

So the map is injective and since $A'/\pi^{e-1}A' \otimes M/FM$ is A'_{e-1}-free of rank $d = \dim_k M/FM$, we get that (L^{can}, M) is a smooth Honda system over A'_e. Therefore, it gives rise to a p-divisible group Γ^{can} over A'_e, which lifts Γ.

Now let $f\colon \Gamma \to \Gamma'$ be a morphism between p-divisible groups over k and let $\varphi\colon M_2 \to M_1$ be the corresponding D_k-linear map between their Dieudonné modules. It is clear that $\overline{\varphi}_{A'}(L_2^{\mathrm{can}}) \subseteq L_1^{\mathrm{can}}$, where $\overline{\varphi}_{A'}\colon M_{A'}/(\mathfrak{m}^{n-1}M_{A'}) \to M'_{A'}/(\mathfrak{m}^{n-1}M'_{A'})$. Therefore $\varphi\colon (L_2^{\mathrm{can}}, M_2) \to (L_1^{\mathrm{can}}, M_1)$ is a morphism of smooth Honda systems over A'_e, which gives rise to a group morphism $f^{\mathrm{can}}\colon \Gamma^{\mathrm{can}} \to (\Gamma')^{\mathrm{can}}$ over A'_e that lifts f. It is clear that the correspondences $\Gamma \mapsto \Gamma^{\mathrm{can}}$ and $f \mapsto f^{\mathrm{can}}$ are functorial, *i.e.* canonical. □

Corollary 2.18.

(i) *Any abelian variety over k and any morphism between abelian varieties over k, lifts canonically to A'_e.*

(ii) *Any finite group over k lifts canonically to a finite flat group over A'_e.*

Proof. — Part (i) follows from the above Proposition via Serre-Tate, *cf.* [9], Theorem 1.2.1. Regarding Part (ii), we note that any finite group G over k is the kernel of an isogeny $\phi\colon \Gamma \to \Gamma'$ between p-divisible groups over k, *cf.* Theorem 3.1 below. Then $G^{\mathrm{can}} := \ker(\phi^{\mathrm{can}})$ is the canonical lift of G to A'_e. □

3. Finite flat group schemes

We begin by recalling some fundamental concepts about formal groups over a pseudo-compact noetherian local ring. Thus a morphism between formal groups $u\colon G \to H$ is a *monomorphism* or a *closed immersion* if the morphism induced between the affine algebras is surjective. A morphism $u\colon G \to H$ between two formal groups is an *epimorphism* if the morphism induced between the affine algebras is flat and faithful with respect to the completed tensor product, *cf.* [4], Exposé $\mathrm{VII_B}$, §1.3.1 (note that over noetherian bases "topologically flat" is the same as "flat"). An *isogeny* is an epimorphism with finite kernel, between two p-divisible groups.

In our case, *i.e.* over A'_n or A', we see that a morphism between formal groups that are flat over the base is an epimorphism (resp. isogeny, resp. monomorphism) if and only if it is so $\bmod \mathfrak{m}$. Another way to express this is in terms of Dieudonné modules. Thus, via Theorem 1.3, we see that a morphism is an epimorphism (resp. monomorphism) if and only if the induced morphism between the Dieudonné modules of the special fibers of the groups is injective (resp. surjective).

Also there exists the quotient of a formal group by a flat subgroup, *cf.* loc. cit. §2.4. It is a standard fact that the quotient of a p-divisible group by a finite flat subgroup is a p-divisible group, *cf.* for instance [1], Lemme 3.3.12.

As we announced in the Introduction, we make heavy use of the following result of Oort, *cf.* [12].

Theorem 3.1. — *Every finite flat group scheme over a noetherian complete local ring, with perfect residue field of characteristic p, is the kernel of an isogeny between two p-divisible groups.*

Remark. — See Remark on pp. 112–113 of [1] for details. Also see Theorem 3.1.1 in [1] for a generalization due to Raynaud.

Recall from Section 2 that for a p-divisible group Γ over A'_n we denote by R_Γ its affine A'_n-algebra, by $\mathcal{R}_\Gamma$ an A'-special lift of R_Γ, by $\mathcal{L}_{r,\Gamma}$ the set $\{\alpha \in P(\mathcal{R}_\Gamma); \delta(\alpha) \in \mathfrak{m}^r \mathcal{R}_\Gamma \widehat{\otimes}_{A'} \mathcal{R}_\Gamma\}$, by M_Γ the Dieudonné module of Γ_k and by $(L_{n,\Gamma}, M_\Gamma)$ the smooth Honda system of Γ. Moreover, we introduce the following further notations.

Definition 3.2. — *Let (L_n, M) be a smooth Honda system over A'_n. We call an* $\mathcal{L}$-lift *of L_n a free A'-submodule $\mathcal{L}$ of $M_{A'}$ of rank equal to $\dim_k M/FM$ such that*

$$\mathrm{Im}\Big(\mathcal{L} \hookrightarrow M_{A'} \to \frac{M_{A'}}{\mathfrak{m}^{n-1} M_{A'}}\Big) = L_n.$$

Also, we denote by $\mathcal{L}_{|n|}$ the preimage $\pi_n^{-1}(L_n)$ of L_n inside $M_{A'}$, where $\pi_n \colon M_{A'} \to M_{A'}/(\mathfrak{m}^{n-1} M_{A'})$ is the natural projection.

The following result establishes the basic properties of these notions.

Proposition 3.3.

1. *Let (L_n, M) be a smooth Honda system over A'_n. For every $\mathcal{L}$-lift $\mathcal{L}$ of L_n we have*
   (i) $\mathcal{L}/\mathfrak{m}\mathcal{L} \simeq M_{A'}/M_{A'}[1] \simeq M/FM.$
   (ii) $\mathcal{L}_{|n|} = \mathcal{L} + \mathfrak{m}^{n-1} M_{A'}.$

2. *Let $\phi \colon \Gamma \to \Gamma_1$ be an isogeny between two p-divisible groups over A'_n.*
   (i) *The isogeny ϕ lifts to A' if and only if there exist $\mathcal{L}$-lifts $\mathcal{L}_\Gamma$ (resp. $\mathcal{L}_{\Gamma_1}$) of $L_{n,\Gamma}$ (resp. L_{n,Γ_1}) such that the injection $M_{\Gamma_1} \to M_\Gamma$ sends $\mathcal{L}_{\Gamma_1}$ into $\mathcal{L}_\Gamma$.*

(ii) *Let G be the kernel of ϕ. Then G is killed by p^r, for some positive integer $r \geqslant 1$, if and only if $p^r M_\Gamma \subseteq M_{\Gamma_1}$ and $p^r \mathcal{L}_{|n|,\Gamma} \subseteq \mathcal{L}_{|n|,\Gamma_1}$.*

Proof. — Part 1 is standard and Part 2(i) follows from Proposition 2.12. The kernel of ϕ is killed by p^r if and only if $\Gamma \xrightarrow{p^r} \Gamma$ factors like $\Gamma \xrightarrow{\phi} \Gamma_1 \to \Gamma$. This, in turn, is equivalent to the inclusions $p^r M_\Gamma \subseteq M_{\Gamma_1} \subseteq M_\Gamma$ and $p^r \mathcal{L}_{|n|,\Gamma} \subseteq \mathcal{L}_{|n|,\Gamma_1} \subseteq \mathcal{L}_{|n|,\Gamma}$. □

Remark 3.4. — For a p-divisible group Γ over A'_n, with smooth Honda system $(L_{n,\Gamma}, M)$, the isomorphism classes of $\mathcal{L}$-lifts of $L_{n,\Gamma}$ are in bijection with the isomorphism classes of p-divisible groups over A' that lift Γ. Indeed, an $\mathcal{L}$-lift $\mathcal{L}$ of $L_{n,\Gamma}$ together with M_Γ make up a smooth Honda system over A' which, via Remark 2.16, corresponds to a p-divisible group $\Gamma^{\mathcal{L}}$ over A' that lifts Γ.

Lemma 3.5. — *Let*

$$0 \to M_2 \to M_1 \to M \to 0 \tag{3.1}$$

be an exact sequence of D_k-modules.

(i) *Suppose we are given a smooth (resp. finite) Honda system $(\mathcal{L}_1, M_1)$ (resp. (L, M)) over A such that $\operatorname{Im}(\mathcal{L}_1 \subseteq M_1 \to M) = L$. Then $(\mathcal{L}_1 \cap M_2, M_2)$ is a smooth Honda system over A.*

(ii) *Suppose we are given smooth (resp. finite) Honda systems $(L_{n,i}, M_i)$ (resp. (L, M)) over A_n (resp. A), with $i = 1, 2$, such that the sequence induced by (3.1)*

$$L_{n,2} \to L_{n,1} \to L \to 0$$

is exact. Moreover suppose that $pM_1 \subseteq M_2$ and $p\mathcal{L}_{|n|,1} \subseteq \mathcal{L}_{|n|,2}$.

Let $\beta_1, \ldots, \beta_u$ (resp. $\gamma_{u+1}, \ldots, \gamma_d$) be in $\mathcal{L}_{|n|,2}$ (resp. $\mathcal{L}_{|n|,1}$) such that their images in $L_{n,1}/pL_{n,1}$ (resp. L) form a k-basis of $\ker(L_{n,1}/pL_{n,1} \to L)$ (resp. L), where $d = \dim_k M_1/FM_1 = \dim_k M_2/FM_2$. Then the A-module

$$\mathcal{L}_1 = \langle \beta_1, \ldots, \beta_u, \gamma_{u+1}, \ldots, \gamma_d \rangle_A$$

is an $\mathcal{L}$-lift of $L_{n,1}$ and the A-module

$$\mathcal{L}_2 = \langle \beta_1, \ldots, \beta_u, p\gamma_{u+1}, \ldots, p\gamma_d \rangle_A$$

is an $\mathcal{L}$-lift of $L_{n,2}$, which is equal to $\mathcal{L}_1 \cap M_2$.

(iii) *Let $r \geqslant 1$ be an integer and let $m \geqslant n + r$. Suppose we are given smooth Honda systems $(L_{m,1}, M_1)$ and $(L_{m,2}, M_2)$ over A_m such that $p^r M = 0$ and $\operatorname{Im}(L_{m,2} \subseteq M_2/p^{m-1}M_2 \to M_1/p^{m-1}M_1) \subseteq L_{m,1}$. Then for every $\mathcal{L}$-lift $\mathcal{L}_1$ of $L_{m,1}$ the A-module $\mathcal{L}_2 = \mathcal{L}_1 \cap M_2$ is an $\mathcal{L}$-lift of $L_{n,2} = L_{m,2}/p^{n-1}L_{m,2} \subseteq M_2/p^{n-1}M_2$.*

(iv) *Suppose we are given finite Honda systems* (L_2, M_2) *and* (L, M) *over* A. *Furthermore, suppose that*

$$\dim_k M_1/pM_1 = \dim_k M_1/FM_1 + \dim_k M_1/VM_1.$$

If L_1 *is an* A*-submodule of* M_1 *such that the sequence induced by (3.1)*

$$0 \to L_2 \to L_1 \to L \to 0$$

is exact, then (L_1, M_1) *is a finite Honda system over* A.

Remark. — Before we give the proof of the Lemma we make it more comprehensible by translating it in terms of groups. Thus, (i) gives the explicit smooth Honda system of the quotient Γ/G of a p-divisible group Γ over A by a finite flat subgroup G. Number (ii) states that an isogeny $\phi\colon \Gamma \to \Gamma_1$ over A_n, such that $\ker\phi$ is killed by p, lifts to A. Number (iii) states that given an isogeny $\phi\colon \Gamma \to \Gamma_1$ over $A/p^{n+r}A$ such that $\ker\phi$ is killed by p^r, its base change to A/p^nA lifts to A. Number (iv) will help us construct Honda systems.

Proof.

(i) It is clear that M_2 (resp. $\mathcal{L}_2 = \mathcal{L}_1 \cap M_2$) is A-free and it has the same A-rank as M_1 (resp. $\mathcal{L}_1$). This is so because M (resp. L) is an A-module of finite A-length and M_1 (resp. $\mathcal{L}_1$) is A-free. On the other hand, we have the equalities $\mathrm{rk}_A(\mathcal{L}_1) = \dim_k M_1/FM_1 = \dim_k M_2/FM_2$. Indeed, the first equality comes from the fact that $(\mathcal{L}_1, M_1)$ is Honda and the second comes from the fact that the sequence (3.1) corresponds via Theorem 1.3 to an exact sequence $0 \to G \to \Gamma_1 \to \Gamma_2 \to$ of groups over k, with G finite and Γ_1 and Γ_2 smooth, necessary of the same dimension, dimension which is equal to $\dim_k M_1/FM_1 = \dim_k M_2/FM_2$, *cf.* [5], Chapter III, Proposition 6.1(ii). Therefore, in order to prove that $(\mathcal{L}_2, M_2)$ is Honda it is enough to prove that the natural map $\mathcal{L}_2/p\mathcal{L}_2 \to M_2/FM_2$ is injective. Let x be in $\mathcal{L}_2 \cap FM_2$. Then x is also in $\mathcal{L}_1 \cap FM_1 = p\mathcal{L}_1$. So $x = p\alpha$ for some α in $\mathcal{L}_1$. Since $x = Fy$ for some y in M_2, we get that $V\alpha = y$. Therefore $V_M\overline{\alpha}$ is zero inside M. But $\overline{\alpha}$ is in L and $L \cap \ker V_M$ is zero, hence $\overline{\alpha}$ is zero. It follows that α is in M_2, hence in $\mathcal{L}_1 \cap M_2 = \mathcal{L}_2$. Thus $\mathcal{L}_2 \cap FM_2 = p\mathcal{L}_2$ and we are done.

(ii) Since from the way we constructed it, $\mathcal{L}_1$ is an $\mathcal{L}$-lift of $L_{n,1}$ that surjects onto L, it follows from Proposition 3.3, 1(i), that $(\mathcal{L}_1, M_1)$ is a smooth Honda system over A, hence by (i) we get that $(\mathcal{L}_1 \cap M_2, M_2)$ is a smooth Honda system over A. All we have to do is prove that $\mathcal{L}_2 = \mathcal{L}_1 \cap M_2$. Then, since by hypothesis $\mathcal{L}_2$ is included in $\mathcal{L}_{|n|,2}$, it follows that $\mathcal{L}_2/(p^{n-1}\mathcal{L}_2)$ injects into $\mathcal{L}_{|n|,2}/(p^{n-1}M_2) = L_{n,2}$ and since they become isomorphic mod p they must be equal. Note that the sequence of A-modules induced by (3.1)

$$0 \to \mathcal{L}_1 \cap M_2 \to \mathcal{L}_1 \to L \to 0$$

is exact. Since by hypothesis $\mathcal{L}_2$ is included in $\mathcal{L}_1 \cap M_2$ the fact that they are equal is a standard linear algebra exercise.

(iii) By hypothesis the morphism $M_2 \to M_1$ induces a morphism

$$\Big(L_{m,2}, \frac{M_2}{p^{m-1}M_2}\Big) \to \Big(L_{m,1}, \frac{M_1}{p^{m-1}M_1}\Big) \tag{3.2}$$

of finite Honda systems over A. Since the category of finite Honda systems over A is abelian, *cf.* [3], Lemma 1.3, it follows that the coimage and the image of (3.2) are equal. Since the coimage is

$$\Big(L_{m,1} \cap \frac{M_2 + p^{m-1}M_1}{p^{m-1}M_1}, \frac{M_2 + p^{m-1}M_1}{p^{m-1}M_1}\Big)$$

and the image is

$$\Big(\frac{\mathcal{L}_{|m|,2} + p^{m-1}M_1}{p^{m-1}M_1}, \frac{M_2 + p^{m-1}M_1}{p^{m-1}M_1}\Big) \tag{3.3}$$

and $L_{m,1} = \mathcal{L}_{|m|,1}/p^{m-1}M_1$, we get the relation

$$\mathcal{L}_{|m|,2} + p^{m-1}M_1 = \mathcal{L}_{|m|,1} \cap M_2 + p^{m-1}M_1. \tag{3.4}$$

Since $p^r M_1$ is included in M_2 we get that $\mathcal{L}_{|m|,1} \cap M_2$ is included in $\mathcal{L}_{|m|,2} + p^{n-1}M_2$ which, via Proposition 3.3, 1(ii), is equal to $\mathcal{L}_{|n|,2}$.

Let (L, M) be the cokernel of the morphism (3.2) and let $\mathcal{L}_1$ be an $\mathcal{L}$-lift of $L_{m,1}$. From (i) it follows that $(\mathcal{L}_1 \cap M_2, M_2)$ is a smooth Honda system over A. Since $\mathcal{L}_1 \cap M_2$ is included in $\mathcal{L}_{|m|,1} \cap M_2$, which in turn is included in $\mathcal{L}_{|n|,2}$, it follows that $\mathcal{L}_1 \cap M_2$ is an $\mathcal{L}$-lift of $L_{n,2}$.

(iv) We have to prove two things: that V_{M_1} is injective on L_1 and that L_1/pL_1 is isomorphic to M_1/FM_1. Let α be in L_1 such that $V_{M_1}\alpha$ is zero. Then $V_M\overline{\alpha}$ is zero in M and since $\overline{\alpha}$ is in L we get that $\overline{\alpha}$ is zero. It follows that α is in L_2 and since $V_{M_2}\alpha = V_{M_1}\alpha = 0$ we get that α is zero.

Since V_{M_1} is injective on L_1 it follows that $L_1[p] \oplus \ker V_{M_1}$ injects into $M_1[p]$, where $L_1[p]$ (resp. $M_1[p]$) is the submodule of L_1 (resp. M_1) of elements killed by p. This fact implies the relations

$$\begin{aligned} \dim_k L_1/pL_1 = \dim_k L_1[p] &\leqslant \dim_k M_1/pM_1 - \dim_k M_1/VM_1 \\ &= \dim_k M_1/FM_1. \end{aligned}$$

Thus in order to complete the proof we need to show that the natural map $L_1/pL_1 \to M_1/FM_1$ is surjective. Let x be in M_1. Since L_1 (resp. M_1) surjects onto L (resp. M) and L surjects onto M/FM there exists α

in L_1 and y in M_1 such that $\overline{x} = \overline{\alpha} + F_M\overline{y}$ in M. It follows that there exists z in M_2 such that $x = \alpha + F_{M_1}y + z$ in M_1. Since L_2 surjects onto M_2/FM_2 there exists β in L_2 and w in M_2 such that $z = \beta + F_{M_2}w$. We get $x = (\alpha+\beta) + F_{M_1}(y+w)$. This completes the proof. □

Now we construct two functors on finite groups over A_n. Let G be an object in $\mathcal{FF}_{A_n}$, with affine A_n-algebra R_G, and let M be its Dieudonné module. By Theorem 3.1, G sits in an exact sequence of formal group schemes over A_n

$$0 \to G \to \Gamma \to \Gamma_1 \to 0 \tag{3.5}$$

where Γ and Γ_1 are p-divisible groups. We call such a sequence a *presentation of G*.

The sequence (3.5) induces the following exact sequence of Dieudonné modules

$$0 \to M_{\Gamma_1} \to M_\Gamma \to M \to 0 \tag{3.6}$$

which, in turn, induces the following exact sequence of D_k-modules with finite A-length

$$0 \to \frac{M_{\Gamma_1} \cap p^{n-1}M_\Gamma}{p^{n-1}M_{\Gamma_1}} \to \frac{M_{\Gamma_1}}{p^{n-1}M_{\Gamma_1}} \to \frac{M_\Gamma}{p^{n-1}M_\Gamma} \to \frac{M}{p^{n-1}M} \to 0. \tag{3.7}$$

Note that because $\Gamma \to \Gamma_1$ is a morphism of p-divisible groups over A_n, the morphism $M_{\Gamma_1} \to M_\Gamma$ induces a morphism $(L_{n,\Gamma_1}, M_{\Gamma_1}) \to (L_{n,\Gamma}, M_\Gamma)$ of smooth Honda systems over A_n and, in particular a morphism

$$\Big(L_{n,\Gamma_1}, \frac{M_{\Gamma_1}}{p^{n-1}M_{\Gamma_1}}\Big) \to \Big(L_{n,\Gamma}, \frac{M_\Gamma}{p^{n-1}M_\Gamma}\Big) \tag{3.8}$$

of finite Honda systems over A. Let

$$\Big(L_n, \frac{M}{p^{n-1}M}\Big) \ \text{ and } \ \Big(L^{(n)}, \frac{M_{\Gamma_1} \cap p^{n-1}M_\Gamma}{p^{n-1}M_{\Gamma_1}}\Big) \tag{3.9}$$

be the cokernel and the kernel, respectively, of the morphism (3.8), in the abelian category of finite Honda systems over A. Note that under the isomorphism

$$\frac{M_\Gamma}{M_{\Gamma_1}} \xrightarrow{p^{n-1}} \frac{p^{n-1}M_\Gamma}{p^{n-1}M_{\Gamma_1}} \tag{3.10}$$

the D_k-submodule $M[p^{n-1}]$ of M, of elements killed by p^{n-1}, corresponds to $(M_{\Gamma_1} \cap p^{n-1}M_\Gamma)/(p^{n-1}M_{\Gamma_1})$. Let

$$\big(L^n, M[p^{n-1}]\big) \simeq \Big(L^{(n)}, \frac{M_{\Gamma_1} \cap p^{n-1}M_\Gamma}{p^{n-1}M_{\Gamma_1}}\Big)$$

be the finite Honda system over A induced by the isomorphism (3.10).

Definition 3.6. — *Let* $\mathrm{SH}^f_{A_n}$ *(resp.* $\mathrm{SH}^r_{A_n}$ *) be the category of* finite *(resp.* restricted*)* Honda systems over A_n *whose objects are triples (resp. pairs)* (L^n, L_n, M) *(resp.* (L_n, M)*), where* M *is a* D_k*-module with finite* A*-length and* $(L^n, M[p^{n-1}])$ *(resp.* $(L_n, M/p^{n-1}M)$*) is a finite Honda system over* A*, and whose morphisms are the obvious ones.*

We define two functors LM^f_n (resp. LM^r_n) from $\mathcal{FF}_{A_n}$ to $\mathrm{SH}^f_{A_n}$ (resp. $\mathrm{SH}^r_{A_n}$) which send a group G to

$$LM^f_n(G) = (L^n, L_n, M) \quad \text{and} \quad LM^r_n(G) = (L_n, M)$$

with L^n, L_n and M as in the paragraph before Definition 3.6, and a morphism $f\colon G' \to G$ to $\mathscr{M}(f_k)\colon M_G \to M_{G'}$.

Proposition 3.7.

1. *The functors* LM^f_n *and* LM^r_n *are well defined.*
2. *The functor* LM^r_n *is essentially surjective.*

Proof.

1. It suffices to prove that LM^f_n is well defined. We have to prove two things: that LM^f_n is well defined on objects, *i.e.* its definition does not depend on the particular presentation (3.5) of G, and that it sends a morphism of groups into a morphism of finite Honda systems. We assume for the moment that LM^f_n is well defined on objects and prove that it is a functor.

Let $f\colon G \to H$ be a morphism of finite groups over A_n. Let

$$0 \to G \to \Gamma \to \Gamma_1 \to 0 \;\; ; \;\; 0 \to H \to \Lambda \to \Lambda_1 \to 0 \tag{3.11}$$

be presentations of G and H. Let ι_f be the composition

$$G \xrightarrow{\Delta} G \times G \xrightarrow{1\times f} G \times H \to \Gamma \times \Lambda. \tag{3.12}$$

It gives rise to a morphism v_f between Dieudonné modules

$$M_\Gamma \oplus M_\Lambda \to M_G \oplus M_H \xrightarrow{1\oplus \overline{f}} M_G \oplus M_G \to M_G. \tag{3.13}$$

Note that the morphism $M_G \oplus M_G \to M_G$ is the addition. It follows that v_f is surjective, hence ι_f is a monomorphism. Denote $\Gamma \times \Lambda$ by Υ and let Υ_1 be the p-divisible group $\operatorname{coker} \iota_f$. From (3.11) and (3.12) we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] & \Gamma \arrow[r] & \Gamma_1 \arrow[r] & 0\,. \\
0 \arrow[r] & G \arrow[u, "\mathrm{id}"] \arrow[r, "\iota_f"] \arrow[d, "f"'] & \Gamma \times \Lambda \arrow[u, "\mathrm{pr}_1"'] \arrow[r] \arrow[d, "\mathrm{pr}_2"] & \Upsilon_1 \arrow[u] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] & \Lambda \arrow[r] & \Lambda_1 \arrow[r] & 0
\end{tikzcd} \tag{3.14}$$

The bottom right square of diagram (3.14) yields, in particular, the following commutative diagram

$$\begin{tikzcd}
(L_{n,\Lambda_1}, \frac{M_{\Lambda_1}}{p^{n-1}M_{\Lambda_1}}) \arrow[r] \arrow[d] & (L_{n,\Lambda}, \frac{M_{\Lambda}}{p^{n-1}M_{\Lambda}}) \arrow[d] \\
(L_{n,\Upsilon_1}, \frac{M_{\Upsilon_1}}{p^{n-1}M_{\Upsilon_1}}) \arrow[r] & (L_{n,\Upsilon}, \frac{M_{\Upsilon}}{p^{n-1}M_{\Upsilon}})
\end{tikzcd} \tag{3.15}$$

of finite Honda systems over A, *cf.* also (3.8). We get induced morphisms between the kernels (resp. cokernels) of the horizontal morphisms in (3.15), which express exactly the fact that $LM_n^f(f)$ is a morphism of finite Honda systems over A_n.

We now prove that LM_n^f is well defined on objects. For this, take H (resp. f) equal to G (resp. id) in the above construction. Let $(L^n(\Gamma), L_n(\Gamma), M)$ and $(L^n(\Lambda), L_n(\Lambda), M)$ be the finite Honda systems induced by the two presentations (3.11) of G. Also let $(L^n(\Gamma\times\Lambda), L_n(\Gamma\times\Lambda), M)$ be the finite Honda system induced by the middle row of diagram (3.14). From (3.13) we see that

$$L_n(\Gamma\times\Lambda) = L_n(\Gamma) + L_n(\Lambda) \subseteq \frac{M}{p^{n-1}M}. \tag{3.16}$$

Since the three L's are isomorphic mod p they must be equal. Thus $L_n(\Gamma) = L_n(\Lambda)$.

From (3.12) we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, "\Delta"'] & \Gamma\times\Lambda \arrow[r] \arrow[d, "\mathrm{id}"'] & \Upsilon_1 \arrow[r] \arrow[d] & 0\,. \\
0 \arrow[r] & G\times G \arrow[r] & \Gamma\times\Lambda \arrow[r] & \Gamma_1\times\Lambda_1 \arrow[r] & 0
\end{tikzcd} \tag{3.17}$$

which gives us two things. The first is that the induced morphism

$$\frac{M_\Gamma}{M_{\Gamma_1}} \oplus \frac{M_\Lambda}{M_{\Lambda_1}} \to \frac{M_\Gamma\oplus M_\Lambda}{M_{\Upsilon_1}} \tag{3.18}$$

is the addition from $M\oplus M$ to M. The second is that we have an induced morphism of finite Honda systems over A between

$$\ker\Bigg(\Big(L_{n,\Gamma_1\times\Lambda_1}, \frac{M_{\Gamma_1}\oplus M_{\Lambda_1}}{p^{n-1}(M_{\Gamma_1}\oplus M_{\Lambda_1})}\Big) \to \Big(L_{n,\Gamma\times\Lambda}, \frac{M_\Gamma\oplus M_\Lambda}{p^{n-1}(M_\Gamma\oplus M_\Lambda)}\Big)\Bigg)$$

which is $(L^{(n)}(\Gamma)\oplus L^{(n)}(\Lambda), \frac{M_{\Gamma_1}\cap p^{n-1}M_\Gamma}{p^{n-1}M_{\Gamma_1}} \oplus \frac{M_{\Lambda_1}\cap p^{n-1}M_\Lambda}{p^{n-1}M_{\Lambda_1}})$, and

$$\ker\Bigg(\Big(L_{n,\Upsilon_1}, \frac{M_{\Upsilon_1}}{p^{n-1}M_{\Upsilon_1}}\Big) \to \Big(L_{n,\Gamma\times\Lambda}, \frac{M_\Gamma\oplus M_\Lambda}{p^{n-1}(M_\Gamma\oplus M_\Lambda)}\Big)\Bigg)$$

which is $\left(L^{(n)}(\Gamma\times\Lambda), \frac{M_{\Upsilon_1}\cap p^{n-1}(M_\Gamma\oplus M_\Lambda)}{p^{n-1}M_{\Upsilon_1}}\right)$, *cf.* also (3.8) and (3.15). In particular, the A-module $L^{(n)}(\Gamma)\oplus L^{(n)}(\Lambda)$ maps inside $L^{(n)}(\Gamma\times\Lambda)$. From the following commutative diagram

$$\begin{tikzcd} \frac{M_{\Gamma_1}\cap p^{n-1}M_\Gamma}{p^{n-1}M_{\Gamma_1}} \oplus \frac{M_{\Lambda_1}\cap p^{n-1}M_\Lambda}{p^{n-1}M_{\Lambda_1}} \arrow[r] \arrow[d, hook] & \frac{M_{\Upsilon_1}\cap p^{n-1}(M_\Gamma\oplus M_\Lambda)}{p^{n-1}M_{\Upsilon_1}} \arrow[d, hook] \\ \frac{p^{n-1}M_\Gamma}{p^{n-1}M_{\Gamma_1}} \oplus \frac{p^{n-1}M_\Lambda}{p^{n-1}M_{\Lambda_1}} \arrow[r] \arrow[d, "\sim"'] & \frac{p^{n-1}(M_\Gamma\oplus M_\Lambda)}{p^{n-1}M_{\Upsilon_1}} \arrow[d, "\sim"'] \\ \frac{M_\Gamma}{M_{\Gamma_1}} \oplus \frac{M_\Lambda}{M_{\Lambda_1}} \arrow[r] & \frac{M_\Gamma\oplus M_\Lambda}{M_{\Upsilon_1}} \end{tikzcd} \tag{3.19}$$

and from (3.18) we get the inclusion

$$L^n(\Gamma)+L^n(\Lambda)\subseteq L^n(\Gamma\times\Lambda)\subseteq M[p^{n-1}].$$

Since the L's are isomorphic $\bmod p$ we get that they are equal. Thus $L^n(\Gamma)=L^n(\Lambda)$.

2. Let (L_n, M) be a restricted Honda system over A_n. By Theorem 3.1 and Theorem 1.3, M sits in an exact sequence $0\to M_2\to M_1\to M\to 0$ of D_k-modules, with M_i being A-free. From [3], Theorem 1.4, Step 4, applied to the exact sequence of D_k-modules $0\to M_2+p^{n-1}M_1\to M_1\to M/p^{n-1}M\to 0$, there exists a smooth Honda system $(\mathcal{L}_1, M_1)$ that surjects onto $(L_n, M/p^{n-1}M)$. Define $L_{n,1}=\mathcal{L}_1/p^{n-1}\mathcal{L}_1$. Similarly, we get a smooth Honda system $(\mathcal{L}_2, M_2)$ that surjects onto the finite Honda system $\ker\big((L_{n,1}, M_1/p^{n-1}M_1)\to(L_n, M/p^{n-1}M)\big)$ over A. Define $L_{n,2}=\mathcal{L}_2/p^{n-1}\mathcal{L}_2$. Then the morphism

$$(L_{n,2}, M_2)\to(L_{n,1}, M_1)$$

of smooth Honda systems over A_n gives rise to an isogeny $\Gamma_1\to\Gamma_2$ over A_n whose kernel has restricted Honda system equal to (L_n, M). □

For future reference, we state the following practical consequence of the fact that LM_n^r is well defined.

Corollary 3.8. — *Let $G\to\Gamma$ be a monomorphism over A_n, with G finite and Γ a p-divisible group. Then the restricted Honda system of G is the image of the smooth Honda system of Γ.*

Proposition 3.9. — *The functor LM_2^f is essentially surjective and full.*

Proof. — Let (L_1, L_2, M) be a finite Honda system over A_2. As in the proof of Proposition 3.7 Part 2, we construct an exact sequence $0\to M_2\to M_1\to M\to 0$ of D_k-modules, with M_i being A-free, and a smooth Honda system $(L_{2,1}, M_1)$ over A_2 that surjects onto (L_2, M). We let $L_{2,2} \subseteq M_2/pM_2$ be any k-vector space that fills in the empty space of the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \frac{M_2\cap pM_1}{pM_2} \arrow[r] & \frac{M_2}{pM_2} \arrow[r] & \frac{M_2+pM_1}{pM_1} \arrow[r] & 0 \\
0 \arrow[r] & L_1^{(1)} \arrow[u, hook] \arrow[r] & {} \arrow[u, hook] \arrow[r] & L' \arrow[u, hook] \arrow[r] & 0
\end{tikzcd}$$

where $(L', (M_2{+}pM_1)/pM_1)$ is the kernel of $(L_{2,1}, M_1/pM_1) \to (L_2, M/pM)$ and $(L_1^{(1)}, (M_2{\cap}pM_1)/pM_2)$ corresponds to $(L_1, M[p])$ via the isomorphism (3.10). By Lemma 3.5 (iv), $(L_{2,2}, M_2)$ is a smooth Honda system over A_2. The morphism

$$(L_{2,2}, M_2) \to (L_{2,1}, M_1)$$

of smooth Honda systems over A_2 gives rise to an isogeny $\Gamma_1 \to \Gamma_2$ over A_2, whose kernel has finite Honda system equal to (L_1, L_2, M). This proves that LM_2^f is essentially surjective.

Let G and H be two finite groups over A_2, let (L^G, L_G, M_G) and (L^H, L_H, M_H) be their finite Honda systems, respectively, and let

$$0 \to G \to \Gamma \to \Gamma_1 \to 0\ ;\quad 0 \to H \to \Lambda \to \Lambda_1 \to 0 \tag{3.20}$$

be some presentations of them.

Let $(L^H, L_H, M_H) \xrightarrow{u} (L^G, L_G, M_G)$ be a morphism in $\mathrm{SH}^f_{A_2}$. We seek to reconstruct diagram (3.14) from u and (3.20), in order to get a morphism $f\colon G \to H$ that maps to u. For this, it is enough to construct the middle and the right columns. The only unknown is Υ_1. We know its Dieudonné module M. It is the kernel of the surjective composition

$$M_\Gamma \oplus M_\Lambda \to M_G \oplus M_H \xrightarrow{1\oplus u} M_G \oplus M_G \to M_G.$$

We need to construct a k-vector subspace L_2 of M/pM such that the morphisms

$$(L_{2,\Gamma_1\times\Lambda_1}, M_{\Gamma_1\times\Lambda_1}) \to (L_2, M) \to (L_{2,\Gamma\times\Lambda}, M_{\Gamma\times\Lambda}) \tag{3.21}$$

are morphisms of smooth Honda systems over A_2.

Consider the commutative diagram of D_k-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M_{\Gamma_1} \oplus M_{\Lambda_1} \arrow[r] \arrow[d] & M_\Gamma \oplus M_\Lambda \arrow[r] \arrow[d] & M_G \oplus M_H \arrow[r] \arrow[d] & 0\ . \\
0 \arrow[r] & M \arrow[r] & M_\Gamma \oplus M_\Lambda \arrow[r] & M_G \arrow[r] & 0
\end{tikzcd} \tag{3.22}$$

To ease notation, we denote $\Gamma\times\Lambda$ (resp. $\Gamma_1\times\Lambda_1$) by Ξ (resp. Ξ_1). Let (L',M') (resp. (L'',M'')) be the kernel of

$$\big(L_{2,\Xi}, M_\Xi/pM_\Xi\big)\to\big(L_G\oplus L_H,(M_G\oplus M_H)/p(M_G\oplus M_H)\big)$$

(resp. $(L_{2,\Xi}, M_\Xi/pM_\Xi)\to(L_G, M_G/pM_G)$). From the snake lemma it follows that the induced morphism $L'\to L''$ is injective.

Diagram (3.22) induces, in particular, the following commutative diagrams with exact rows of k-vector spaces

$$\begin{tikzcd}
\frac{M_{\Xi_1}\cap pM_\Xi}{pM_{\Xi_1}} \arrow[r, hook] \arrow[d] & \frac{M_{\Xi_1}}{pM_{\Xi_1}} \arrow[r, two heads] \arrow[d] & M' \arrow[d, hook] \\
\frac{M\cap pM_\Xi}{pM} \arrow[r, hook] & \frac{M}{pM} \arrow[r, two heads] & M''
\end{tikzcd}
\qquad
\begin{tikzcd}
L^{(2)}_\Xi \arrow[r, hook] \arrow[d] & L_{2,\Xi_1} \arrow[r, two heads] \arrow[d] & L' \arrow[d, hook] \\
L^{(2)} \arrow[r, hook] & {} \arrow[r, two heads] & L''
\end{tikzcd}
\tag{3.23}$$

where the right diagram is included in the left one, $L^{(2)}$ is induced by L^G via the isomorphism $pM_\Xi/pM\simeq M_\Xi/M\simeq M_G$ and $L^{(2)}_\Xi=L^{(2)}_\Gamma\oplus L^{(2)}_\Lambda\simeq L^G\oplus L^H$, *cf.* also (3.9) above. The fact that the image of $L^{(2)}_\Xi\subseteq(M_{\Xi_1}\cap pM_\Xi)/pM_{\Xi_1}\to(M\cap pM_\Xi)/pM$ lies inside $L^{(2)}$ follows from the inclusion $u(L^H)\subseteq L^G$ and the fact that the image is actually $L^{(2)}$ follows from (3.19) and (3.22). It follows that the sequence

$$0\to L^{(2)}\to L_1\to L_1'\to 0$$

is exact, where L_1 (resp. L_1') is the image of $L_{2,\Xi_1}\subseteq M_{\Xi_1}/pM_{\Xi_1}\to M/pM$ (resp. $L'\to L''$). Thus there exists a k-vector subspace L_2 of M/pM that fills in the empty space of the right diagram in (3.23). It follows from Lemma 3.5 (iv) that (L_2,M) is a smooth Honda system over A_2.

Let Υ_1 be the p-divisible group over A_2 that corresponds to (L_2,M). From (3.21) we get a commutative diagram of smooth Honda systems over A_2

$$\begin{tikzcd}
(L_{2,\Gamma_1}, M_{\Gamma_1}) \arrow[r] \arrow[d] & (L_{2,\Gamma}, M_\Gamma) \arrow[d, "i_1"'] \\
(L_2, M) \arrow[r] & (L_{2,\Gamma}\oplus L_{2,\Lambda}, M_\Gamma\oplus M_\Lambda) \\
(L_{2,\Lambda_1}, M_{\Lambda_1}) \arrow[r] \arrow[u] & (L_{2,\Lambda}, M_\Lambda) \arrow[u, "i_2"]
\end{tikzcd}$$

which induces a commutative diagram with exact rows of groups over A_2

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] & \Gamma \arrow[r] & \Gamma_1 \arrow[r] & 0\, . \\
0 \arrow[r] & G' \arrow[r] \arrow[u, dotted] \arrow[d, dotted] & \Gamma\times\Lambda \arrow[r] \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"'] & \Upsilon_1 \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] & \Lambda \arrow[r] & \Lambda_1 \arrow[r] & 0
\end{tikzcd}
\tag{3.24}$$

It follows from the universal property of the kernel that the dotted arrows in the above diagram are well defined morphisms of finite groups over A_2. In fact $G' \to G$ is an isomorphism because it is so $\bmod p$. If we define f to be the composition $G \to G' \to H$ it follows that $LM_n^f(f) = u$. $\square$

Corollary 3.10. — *Two finite groups over A_2 are isomorphic if and only if their finite Honda systems are isomorphic.*

Proof. — The "only if" implication being trivial, we need only prove the "if" implication. Start with an isomorphism between the finite Honda systems. By Proposition 3.9, it gives rise to a morphism between the groups, which is an isomorphism $\bmod p$, hence an isomorphism. $\square$

Remark 3.11.

1. If we consider a finite Honda system (L', L, M) over A_2 with $pM = 0$ and $L' \neq L$ we get a finite group G over A_2 with the property that the morphism $G \xrightarrow{p} G$ is not zero but it is zero $\bmod p$. This is one instance of the non-faithfulness of the special fiber functor. Other more general examples can be given. Note also that for such a group G the p-torsion subgroup $G[p]$ is not flat.

2. For arbitrary n, we see from the proof of Proposition 3.9 that LM_n^f is essentially surjective onto the finite Honda systems (L', L, M) over A_n, with L and L' free A_{n-1}-modules, and that LM_n^f is full, hence classifying by Corollary 3.10, when restricted to the finite groups over A_n that map to such Honda systems. Indeed, the thing that made the proof work was that the L's in the Honda systems were $A/p^{n-1}A$-free.

In particular, Proposition 3.9 and Corollary 3.10 hold also for LM_n^f restricted to the category of truncated Barsotti-Tate groups of level $s \geqslant n$ over A_n (the case of levels less than n is trivially true).

3. Since for truncated Barsotti-Tate groups the kernel and the cokernel of the "multiplication by p^{n-1}" morphism are isomorphic in a functorial way, we see that the restricted Honda system is enough to classify these groups. It follows that a p-divisible group (resp. truncated Barsotti-Tate group) over A_n is determined by its reduction $\bmod p$ and by its p^{n-1} torsion subgroup. This, over A_n, recovers a classical result of Grothendieck, *cf.* [8], Corollary 4.7.

Definition 3.12. — *We say that a finite group G over A'_n is* fully flat *if all the p-torsion subgroups $\{G[p^i]\}_{i \geqslant 1}$ are flat.*

Theorem 3.13. — *Let $\phi\colon \Gamma \to \Gamma_1$ be an isogeny between two p-divisible groups over A_n and let G be its kernel.*

(i) *If G is fully flat then ϕ lifts to A.*

(ii) *The torsion subgroup $G[p^r]$ is fully flat for some integer $r \geqslant 1$ if and only if G lifts to $A/p^{n+r}A$.*

Proof.

(i) We define the following sequence of finite groups

$$G_0 = G, \quad G_{i+1} = \frac{G_i}{G_i[p]}, \; i \geqslant 0. \tag{3.25}$$

We have the isomorphisms

$$G_i \simeq \frac{G}{G[p^i]} \quad ; \quad G_i[p] \simeq \frac{G[p^{i+1}]}{G[p^i]}. \tag{3.26}$$

Let G and $G[p]$ sit in exact sequences

$$\begin{tikzcd} 0 \arrow[r] & G[p] \arrow[r] \arrow[d, hook] & \Gamma_0 \arrow[r] \arrow[d, "\mathrm{id}"'] & \Gamma_1 \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & G \arrow[r] & \Gamma_0 \arrow[r] & \Gamma_0' \arrow[r] & 0 \end{tikzcd}$$

with the Γ's p-divisible. From the properties of the cokernel we get an induced morphism $\Gamma_1 \to \Gamma_0'$ which, by the snake lemma, is an isogeny with kernel G_1.

Therefore, in general, for every $i \geqslant 0$ we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & G_i[p] \arrow[r] \arrow[d, hook] & \Gamma_i \arrow[r] \arrow[d, "\mathrm{id}"'] & \Gamma_{i+1} \arrow[r] \arrow[d] & 0 \; . \\ 0 \arrow[r] & G_i \arrow[r] & \Gamma_i \arrow[r] & \Gamma_0' \arrow[r] & 0 \end{tikzcd} \tag{3.27}$$

From (3.26) and (3.27) we get the following relations between the Dieudonné modules

$$M_{G_i} = p^i M_G = \frac{p^i M_{\Gamma_0} + M_{\Gamma_0'}}{M_{\Gamma_0'}}, \quad M_{\Gamma_i} = p^i M_{\Gamma_0} + M_{\Gamma_0'}.$$

Suppose G_r is zero for some positive integer r. From (3.27) and Proposition 3.3, 2(ii), we get

$$p\mathcal{L}_{|n|,\Gamma_i} \subseteq \mathcal{L}_{|n|,\Gamma_{i+1}} \subseteq \mathcal{L}_{|n|,\Gamma_i}, \quad 0 \leqslant i \leqslant r-1 \tag{3.28}$$

where, by convention, Γ_r is equal to Γ_0'.

Note that since the kernel of $\Gamma_0 \to \Gamma_i$ is $G[p^i]$, which is killed by p^i, the isogeny $\Gamma_0 \xrightarrow{p^i} \Gamma_0$ factors like $\Gamma_0 \to \Gamma_i \to \Gamma_0$. We claim that the composition $G_i \to \Gamma_i \to \Gamma_0$ is a monomorphism. Indeed, the corresponding morphism between the Dieudonné modules is $p^i M_{\Gamma_0} \to M_{\Gamma_i} \to p^i M_G$ which is clearly surjective. Therefore, by Corollary 3.8, the restricted Honda system of G_i is (L_i, M_{G_i}), with

$$L_i = \operatorname{Im}\Big(\mathcal{L}_{|n|,\Gamma_0} \subseteq M_{\Gamma_0} \xrightarrow{p^i} p^i M_G \to \frac{p^i M_G}{p^{i+n-1}M_G}\Big). \tag{3.29}$$

It follows, in particular, that for every $i \geqslant 0$ the A-module L_i surjects onto L_{i+1} via the surjective morphism $p^i M_G/p^{i+n-1}M_G \xrightarrow{p} p^{i+1}M_G/p^{i+n}M_G$ given by $[x] \mapsto \{px\}$. For future reference we denote by d (resp. d_i) the dimension of the k-vector space $L_{n,\Gamma_0}/pL_{n,\Gamma_0}$ (resp. L_i/pL_i). We have the inequalities

$$d \geqslant d_0 \geqslant d_1 \geqslant \cdots \geqslant d_{r-2} \geqslant d_{r-1}.$$

On the other hand, from diagram (3.27) and the fact that the functor LM_n^r is well defined, *cf.* Proposition 3.7, we get the following exact sequences of A-modules

$$L_{n,\Gamma'_0} \to L_{n,\Gamma_i} \to L_i \to 0\ ;\ L_{n,\Gamma_{i+1}} \to L_{n,\Gamma_i} \to \frac{L_i}{pL_i} \to 0. \tag{3.30}$$

Here we implicitly used the fact, which we leave as an exercise to the reader, that $(L_i/pL_i, p^i M_G/p^{i+1}M_G)$ is the restricted Honda system of $G_i[p]$.

The sequences (3.30) yield the following commutative diagram with exact rows of k-vector spaces

$$\begin{tikzcd}
\frac{L_{n,\Gamma'_0}}{pL_{n,\Gamma'_0}} \arrow[rr] \arrow[dr, two heads] \arrow[dd] & & \frac{L_{n,\Gamma_{r-1}}}{pL_{n,\Gamma_{r-1}}} \arrow[r] \arrow[dd] & L_{r-1} \arrow[r] & 0 \\
& L'_{r-1} \arrow[ur, hook] \arrow[dd, dotted, two heads] & & & \\
\frac{L_{n,\Gamma_{r-1}}}{pL_{n,\Gamma_{r-1}}} \arrow[rr] \arrow[dr, two heads] \arrow[ddd, dotted, no head] & & \frac{L_{n,\Gamma_{r-2}}}{pL_{n,\Gamma_{r-2}}} \arrow[r] \arrow[ddd, dotted, no head] & \frac{L_{r-2}}{pL_{r-2}} \arrow[r] \arrow[ddd, dotted, no head] & 0 \\
& L'_{r-2} \arrow[ur, hook] \arrow[dd, dotted, no head] & & & \\
& & & & \\
\frac{L_{n,\Gamma_2}}{pL_{n,\Gamma_2}} \arrow[rr] \arrow[dr, two heads] \arrow[dd] & & \frac{L_{n,\Gamma_1}}{pL_{n,\Gamma_1}} \arrow[r] \arrow[dd] & \frac{L_1}{pL_1} \arrow[r] & 0 \\
& L'_1 \arrow[ur, hook] \arrow[dd, dotted, two heads] & & & \\
\frac{L_{n,\Gamma_1}}{pL_{n,\Gamma_1}} \arrow[rr] \arrow[dr, two heads] & & \frac{L_{n,\Gamma_0}}{pL_{n,\Gamma_0}} \arrow[r] & \frac{L_0}{pL_0} \arrow[r] & 0 \\
& L'_0 \arrow[ur, hook] & & &
\end{tikzcd} \tag{3.31}$$

where L'_i is by definition

$$\ker\big(L_{n,\Gamma_i}/pL_{n,\Gamma_i} \to L_i/pL_i\big) = \mathrm{Im}\big(L_{n,\Gamma_{i+1}}/pL_{n,\Gamma_{i+1}} \to L_{n,\Gamma_i}/pL_{n,\Gamma_i}\big).$$

Note that the kernel of $L_{n,\Gamma_i} \to L_i$, which by (3.30) is the image of $L_{n,\Gamma'_0} \to L_{n,\Gamma_i}$, surjects onto the kernel of $L_{n,\Gamma_i}/pL_{n,\Gamma_i} \to L_i/pL_i$, which is L'_i. Since the morphism $L_{n,\Gamma'_0} \to L_{n,\Gamma_i}$ factors through all the L_{n,Γ_j}'s with $j \geqslant i$, we get that L'_j surjects onto L'_i for all $j \geqslant i$.

Next we prove the following statement by decreasing induction on i between $r-1$ and 0.

FACT 3.14. — *There exist elements $\gamma_1, \dots, \gamma_{d_i}$ in $\mathcal{L}_{|n|,\Gamma_0}$ such that:*

(i) *The images of $p^i\gamma_1, \dots, p^i\gamma_{d_i}$ in L_i/pL_i form a k-basis of this vector space.*

(ii) *The elements $p^{i+1}\gamma_{d_{i+1}+1}, \dots, p^{i+1}\gamma_{d_i}$ are in $\mathcal{L}_{|n|,\Gamma'_0}$ and, if i is strictly less then $r-1$, their images in $L_{n,\Gamma_{i+1}}/pL_{n,\Gamma_{i+1}}$ form a k-basis of $\ker(L'_{i+1} \to L'_i)$.*

Note that $P(r-1)$ is trivial because of (3.28) and (3.29). Assume we have $P(i)$. Let $\xi_{d_i+1}, \dots, \xi_{d_{i-1}}$ be in $\mathcal{L}_{|n|,\Gamma_0}$ such that the images of $p^{i-1}\xi_{d_i+1}, \dots, p^{i-1}\xi_{d_{i-1}}$ in L_{i-1} (resp. L_{i-1}/pL_{i-1}) lie in $\ker(L_{i-1} \xrightarrow{p} L_i)$ (resp. form a k-basis of $\ker(L_{i-1}/pL_{i-1} \xrightarrow{p} L_i/pL_i)$). This is possible because of (3.29) and the fact that the kernel of $L_{i-1} \xrightarrow{p} L_i$ surjects onto the kernel of $L_{i-1}/pL_{i-1} \xrightarrow{p} L_i/pL_i$. It follows that the image of $p^i\xi_j$ is zero inside L_i. Hence

$$\overline{p^i\xi_j} \in \ker(L_{n,\Gamma_i} \to L_i) = \frac{\mathcal{L}_{|n|,\Gamma'_0} + p^{n-1}M_{\Gamma_i}}{p^{n-1}M_{\Gamma_i}}$$

cf. (3.30) and (3.3). Since $M_{\Gamma_i} = p^iM_{\Gamma_0} + M_{\Gamma'_0}$ there exist β_j in $\mathcal{L}_{|n|,\Gamma'_0}$ and x_j (resp. y_j) in M_{Γ_0} (resp. $M_{\Gamma'_0}$) such that

$$p^i\xi_j = \beta_j + p^{n-1}(p^ix_j + y_j).$$

Define $\gamma_j = \xi_j - p^{n-1}x_j$. Then γ_j is in $\mathcal{L}_{|n|,\Gamma_0}$ and

$$p^i\gamma_j = \beta_j + p^{n-1}y_j \in \mathcal{L}_{|n|,\Gamma'_0}.$$

Since the image of $p^{i-1}\gamma_j$ inside $L_{n,\Gamma_{i-1}}$ is the same as that of $p^{i-1}\xi_j$ the only thing that remains to prove is that the elements $\overline{p^i\gamma_{d_i+1}}, \dots, \overline{p^i\gamma_{d_{i-1}}}$ form a k-basis of $\ker(L'_i \to L'_{i-1})$. For this, let $\delta_d, \dots, \delta_{d_{i-1}+1}$ be in $\mathcal{L}_{|n|,\Gamma'_0}$ such that their images $\overline{\delta_d}, \dots, \overline{\delta_{d_{i-1}+1}}$ form a k-basis of L'_{i-1}. Then, by Lemma 3.5 (ii) applied to the isogeny $\Gamma_{i-1} \to \Gamma_i$, the A-module

$$\mathcal{L} = \langle \delta_d, \dots, \delta_{d_{i-1}+1}, p^i\gamma_{d_{i-1}}, \dots, p^i\gamma_1\rangle_A$$

is an $\mathcal{L}$-lift of L_{n,Γ_i}. It follows that the elements

$$\overline{\delta_d}, \dots, \overline{\delta_{d_{i-1}+1}}, \overline{p^i\gamma_{d_{i-1}}}, \dots, \overline{p^i\gamma_{d_i+1}}$$

form a basis of L'_i. Since $p^i\gamma_j$ is in $pM_{\Gamma_{i-1}}$ we get that the elements $\overline{p^i\gamma_{d_{i-1}}}, \dots, \overline{p^i\gamma_{d_i+1}}$ form a basis of $\ker(L'_i \to L'_{i-1})$.

Now, to end the proof of the Theorem, we take elements $\delta_d, \dots, \delta_{d_0+1}$ in $\mathcal{L}_{|n|,\Gamma'_0}$ such that their images in L'_0 form a k-basis of L'_0. We claim that the A-module

$$\mathcal{L}_0 = \langle \delta_d, \dots, \delta_{d_0+1}, \gamma_{d_0}, \dots, \gamma_1 \rangle_A$$

is an $\mathcal{L}$-lift of L_{n,Γ_0} such that $\mathcal{L}_0 \cap M_{\Gamma_i}$ is an $\mathcal{L}$-lift of L_{n,Γ_i}, for all $1 \leqslant i \leqslant r$.

To show how the proof of the last claim goes we assume r is equal to 3 in order to avoid unnecessary technicalities. By Lemma 3.5 (ii) applied to the isogeny $\Gamma_0 \to \Gamma_1$ and by $P(0)$ we get that $\mathcal{L}_0$ is an $\mathcal{L}$-lift of L_{n,Γ_0} and that the A-module

$$\mathcal{L}_1 = \mathcal{L}_0 \cap M_{\Gamma_1} = \langle \delta_d, \dots, \delta_{d_0+1}, p\gamma_{d_0}, \dots, p\gamma_1 \rangle_A$$

is an $\mathcal{L}$-lift of L_{n,Γ_1}.

Then, by $P(1)$ (ii) (resp. (i)), it follows that the elements

$$\overline{\delta_d}, \dots, \overline{\delta_{d_0+1}}, \overline{p\gamma_{d_0}}, \dots, \overline{p\gamma_{d_1+1}}$$

(resp. $\overline{p\gamma_{d_1}}, \dots, \overline{p\gamma_1}$) form a basis of L'_1 (resp. L_1/pL_1). By Lemma 3.5 (ii) applied to the isogeny $\Gamma_1 \to \Gamma_2$ we get that the A-module

$$\mathcal{L}_2 = \mathcal{L}_1 \cap M_{\Gamma_2} = \langle \delta_d, \dots, \delta_{d_0+1}, p\gamma_{d_0}, \dots, p\gamma_{d_1+1}, p^2\gamma_{d_1}, \dots, p^2\gamma_1 \rangle_A$$

is an $\mathcal{L}$-lift of L_{n,Γ_2}.

Similarly, it follows that the elements

$$\overline{\delta_d}, \dots, \overline{\delta_{d_0+1}}, \overline{p\gamma_{d_0}}, \dots, \overline{p\gamma_{d_1+1}}, \overline{p^2\gamma_{d_1}}, \dots, \overline{p^2\gamma_{d_2+1}}$$

form a basis of L'_2 and that $\mathcal{L}'_0 = \mathcal{L}_0 \cap M_{\Gamma'_0}$ is an $\mathcal{L}$-lift of L_{n,Γ_0}, generated by the elements

$$\mathcal{L}'_0 = \langle \delta_d, \dots, \delta_{d_0+1}, p\gamma_{d_0}, \dots, p\gamma_{d_1+1}, p^2\gamma_{d_1}, \dots, p^2\gamma_{d_2+1}, p^3\gamma_{d_2}, \dots, p^3\gamma_1 \rangle_A.$$

Proposition 3.3, 2(i), finishes the proof for us.

(ii) Here we prove the 'only if' part of the statement. The 'if' part follows from Corollary 3.17 below. Suppose $G[p^r]$ is fully flat. As usual, we have the presentations

$$\begin{tikzcd} 0 \arrow[r] & G[p^r] \arrow[r] \arrow[d, hook] & \Gamma_0 \arrow[r] \arrow[d, "\mathrm{id}"'] & \Gamma_r \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G \arrow[r] & \Gamma_0 \arrow[r] & \Gamma'_0 \arrow[r] & 0 \end{tikzcd} \tag{3.32}$$

By (i) there exists an $\mathcal{L}$-lift $\mathcal{L}_{\Gamma_0}$ of L_{n,Γ_0} such that $\mathcal{L}_{\Gamma_0} \cap M_{\Gamma_r}$ is an $\mathcal{L}$-lift of L_{n,Γ_r}. Let $\mathcal{L}' = \langle \alpha_1, \dots, \alpha_d \rangle_A$ be an $\mathcal{L}$-lift of $L_{n,\Gamma_0'}$. Since $\mathcal{L}' \subseteq \mathcal{L}_{|n|,\Gamma_0'} \subseteq \mathcal{L}_{|n|,\Gamma_r} = \mathcal{L}_{\Gamma_0} \cap M_{\Gamma_r} + p^{n-1} M_{\Gamma_r}$ and $M_{\Gamma_r} = p^r M_{\Gamma_0} + M_{\Gamma_0'}$ we can write

$$\alpha_j = \beta_j + p^{n-1} x_j + p^{n+r-1} y_j, \quad 1 \leqslant j \leqslant d$$

with β_j in $\mathcal{L}_{\Gamma_0} \cap M_{\Gamma_r}$ and x_j (resp. y_j) in $M_{\Gamma_0'}$ (resp. M_{Γ_0}). Define $\gamma_j = \alpha_j - p^{n-1} x_j$ for j between 1 and d. Then the A-module $\mathcal{L}_{\Gamma_0'} = \langle \gamma_1, \dots, \gamma_d \rangle_A$ is an $\mathcal{L}$-lift of $L_{n,\Gamma_0'}$ which is included in $\mathcal{L}_{\Gamma_0} + p^{n+r-1} M_{\Gamma_0}$. It follows that the morphism

$$\Big(\frac{\mathcal{L}_{\Gamma_0'}}{p^{n+r-1} \mathcal{L}_{\Gamma_0'}}, M_{\Gamma_0'} \Big) \to \Big(\frac{\mathcal{L}_{\Gamma_0}}{p^{n+r-1} \mathcal{L}_{\Gamma_0}}, M_{\Gamma_0} \Big)$$

of Honda systems over $A/p^{n+r}A$ lifts the morphism $(L_{n,\Gamma_0'}, M_{\Gamma_0}) \to (L_{n,\Gamma}, M_\Gamma)$ of Honda systems over A_n. By Corollary 2.15 we get an isogeny $\Gamma \to \Gamma'$ over A_{n+r} that lifts $\Gamma_0 \to \Gamma_0'$. The kernel of this isogeny lifts G. □

Lemma 3.15. — *Let m and r be two positive integers such that $m \geqslant r$, let Γ be a p-divisible group over A_{n+m} and let $(L_{n+m,\Gamma}, M_\Gamma)$ be its Honda system. For every $\mathcal{L}$-lift $\mathcal{L}$ of $L_{n+m,\Gamma}$ the composition*

$$\mathcal{L}_{n+m,\Gamma} \to \frac{P(\mathcal{R}_\Gamma)}{p^n \mathcal{R}_\Gamma} \to \frac{(\mathcal{R}_r^{\mathcal{L}})_K}{p^n \mathcal{R}_r^{\mathcal{L}}}$$

is zero, where $\mathcal{R}_r^{\mathcal{L}}$ is the affine A-algebra of $\Gamma^{\mathcal{L}}[p^r]$, with $\Gamma^{\mathcal{L}}$ as in Remark 3.4.

Proof. — Note that the affine A-algebra of $\Gamma^{\mathcal{L}}$ is still $\mathcal{R}_\Gamma$, enhanced with a comultiplication given by the group structure on $\Gamma^{\mathcal{L}}$. Also from Theorem 1.11 and (1.8) it follows that the A-module $\mathcal{L}$ is identified with the elements α in $P(\mathcal{R}_\Gamma)$ such that $\delta(\alpha) = 0$. It also follows that $\mathcal{L}_{n+m,\Gamma} = \mathcal{L} + p^{n+m-1} \mathcal{L}_{1,\Gamma}$. Indeed, the right module is included in the left one, they are A_{n+m-1}-free modulo $p^{n+m-1} \mathcal{L}_{1,\Gamma}$ and they are isomorphic mod p, *cf.* Lemma 2.5.

From [5], Remark 2 on page 186, we know that the composition

$$\mathcal{L} \hookrightarrow P(\mathcal{R}_\Gamma) \to (\mathcal{R}_r^{\mathcal{L}})_K$$

is zero. Hence we only have to deal with $p^{n+m-1} \mathcal{L}_{1,\Gamma}$. Let β be in $\mathcal{L}_{1,\Gamma}$. The fact that $\widehat{\Delta}(\beta) - \beta \widehat{\otimes} 1 + 1 \widehat{\otimes} \beta$ is in $p \mathcal{R}_\Gamma \widehat{\otimes}_A \mathcal{R}_\Gamma$ implies that $[p^r](\beta) = p^r \beta + pa$ for some a in $\mathcal{R}_\Gamma$, where $P(\mathcal{R}_\Gamma) \overset{[p^r]}{\to} P(\mathcal{R}_\Gamma)$ is induced by $\Gamma^{\mathcal{L}} \overset{p^r}{\to} \Gamma^{\mathcal{L}}$. Since $[p^r](\beta)$ is mapped to zero inside $(\mathcal{R}_r^{\mathcal{L}})_K$, it follows that $p^r \beta$ is mapped into $p \mathcal{R}_r^{\mathcal{L}}$. Thus $p^{n+m-1} \mathcal{L}_{1,\Gamma}$ is mapped into $p^{n+m-r} \mathcal{R}_r^{\mathcal{L}} \subseteq p^n \mathcal{R}_r^{\mathcal{L}}$. The Lemma follows. □

Theorem 3.16. — *Let m and r be two positive integers.*

1. *Assume $r \leqslant m$ and let G (resp. H) be a fully flat (resp. an arbitrary) finite group over A_{n+m} such that G is killed by p^r. The natural map*

$$\mathrm{Im}\big(\mathrm{Hom}_{A_{n+m}-\mathrm{gr}}(G,H) \to \mathrm{Hom}_{A_n-\mathrm{gr}}(G_n,H_n)\big) \to \mathrm{Hom}_{k-\mathrm{gr}}(G_k,H_k)$$

is injective.

2. *Assume $2r \leqslant m$ and let G and H be arbitrary finite groups over A_{n+m} such that G is killed by p^r. The natural map*

$$\mathrm{Im}(\mathrm{Hom}_{A_{n+m}-\mathrm{gr}}(G,H) \to \mathrm{Hom}_{A_n-\mathrm{gr}}(G_n,H_n)) \to \mathrm{Hom}_{k-\mathrm{gr}}(G_k,H_k)$$

is injective.

Proof.

1. Let $f\colon G \to H$ be a morphism such that $f_k = 0$. Then, by (3.14), f extends to a morphism of presentations

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r, "\iota_f"] \arrow[d, "f"'] & \Upsilon \arrow[r] \arrow[d] & \Upsilon_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H \arrow[r, "\iota_H"] & \Lambda \arrow[r] & \Lambda_1 \arrow[r] & 0 \end{tikzcd} \tag{3.33}$$

By Theorem 3.13 (i), there exists an $\mathcal{L}$-lift $\mathcal{L}$ of $L_{n+m,\Upsilon}$ such that the presentation of G lifts to

$$0 \longrightarrow G^{\mathcal{L}} \xrightarrow{\iota^{\mathcal{L}}} \Upsilon^{\mathcal{L}} \longrightarrow \Upsilon_1^{\mathcal{L}} \longrightarrow 0$$

over A. Since $G^{\mathcal{L}}$ is killed by p^r, it follows that $\iota^{\mathcal{L}}$ factors through $\Upsilon^{\mathcal{L}}[p^r]$. Thus, the morphism $P(\mathcal{R}_\Upsilon) \to (R_{G^{\mathcal{L}}})_K$ factors like

$$P(\mathcal{R}_\Upsilon) \to (\mathcal{R}_r^{\Upsilon^{\mathcal{L}}})_K \to (R_{G^{\mathcal{L}}})_K \tag{3.34}$$

with $\mathcal{R}_r^{\Upsilon^{\mathcal{L}}}$ the affine A-algebra of $\Upsilon^{\mathcal{L}}[p^r]$.

Let $\mathcal{R}_\Lambda \to \mathcal{R}_\Upsilon$ be an A-algebra morphism that lifts $R_\Lambda \to R_\Upsilon$. Then the composition

$$\mathcal{R}_\Lambda \to \mathcal{R}_\Upsilon \to R_{G^{\mathcal{L}}} \tag{3.35}$$

is an A-algebra morphism that lifts the composition $R_\Lambda \xrightarrow{i_H} R_H \xrightarrow{r_f} R_G$.

Let $\mathcal{L}_\Lambda$ be an arbitrary $\mathcal{L}$-lift of $L_{n+m,\Lambda}$. The morphism $P(\mathcal{R}_\Lambda) \to P(\mathcal{R}_\Upsilon)$, induced by $\mathcal{R}_\Lambda \to \mathcal{R}_\Upsilon$, takes $\mathcal{L}_\Lambda$ over into $\mathcal{L}_{n+m,\Upsilon}$. From (3.34), (3.35) and Lemma 3.15 it follows that the composition

$$\mathcal{L}_\Lambda \to \frac{P(\mathcal{R}_\Lambda)}{p^n\mathcal{R}_\Lambda} \to \frac{(R_{G^{\mathcal{L}}})_K}{p^n R_{G^{\mathcal{L}}}}$$

is zero. This together with the fact that f_k is zero imply that $\eta_{\mathcal{L}_\Lambda}(R_{G^{\mathcal{L}}})(\iota_{H,n}\circ f_n)$ is zero, *cf.* (2.13) and (2.14). Since $\eta_{\mathcal{L}_\Lambda}(R_{G^{\mathcal{L}}})$ is bijective by Lemma 2.9, we get that $\iota_{H,n}\circ f_n$ is zero, hence f_n is zero since $\iota_{H,n}$ is a monomorphism.

2. The proof of this part is almost the same as the proof of (1) with the only difference that, since in general ι_f doesn't lift to A, we have to work with $\iota_{f,n+m-r}$ which does lift to A, by Lemma 3.5 (iii). This is why we need $m \geqslant 2r$. □

Corollary 3.17. — *Let m and r be two positive integers such that $m \geqslant r$ and let G be a finite group over A_{n+m}. Then $G[p^r]_n$ is fully flat.*

Proof. — Let $0 \to G \to \Gamma \xrightarrow{\phi} \Gamma_1 \to 0$ be a presentation of G. Then $G[p^r]$ is the kernel of $\phi[p^r]$. Let $\mathcal{L}$ (resp. $\mathcal{L}_1$) be an $\mathcal{L}$-lift of $L_{n+m,\Gamma}$ (resp. L_{n+m,Γ_1}) and let $\Gamma^{\mathcal{L}}$ (resp. $\Gamma_1^{\mathcal{L}_1}$) be the corresponding p-divisible groups over A lifting Γ (resp. Γ_1). It is clear that the Honda system of $\Gamma^{\mathcal{L}}[p^r]$ (resp. $\Gamma_1^{\mathcal{L}_1}[p^r]$) is $(\mathcal{L}/p^r\mathcal{L}, M_\Gamma/p^rM_\Gamma)$ (resp. $(\mathcal{L}_1/p^r\mathcal{L}_1, M_{\Gamma_1}/p^rM_{\Gamma_1})$). Also, because $\mathcal{L}$ is an $\mathcal{L}$-lift, it follows that $\mathcal{L}/p^{n+m-1}\mathcal{L} = L_{n+m,\Gamma}$. And the same holds for $\mathcal{L}_1$.

Since the morphism

$$(L_{n+m,\Gamma_1}, \frac{M_{\Gamma_1}}{p^{n+m-1}M_{\Gamma_1}}) \to (L_{n+m,\Gamma}, \frac{M_\Gamma}{p^{n+m-1}M_\Gamma})$$

is a morphism of finite Honda systems over A, it follows that the morphism

$$(\frac{\mathcal{L}_1}{p^r\mathcal{L}_1}, \frac{M_{\Gamma_1}}{p^rM_{\Gamma_1}}) \to (\frac{\mathcal{L}}{p^r\mathcal{L}}, \frac{M_\Gamma}{p^rM_\Gamma})$$

is also such a morphism. By Theorem 1.13 it gives rise to a morphism

$$\psi\colon \Gamma^{\mathcal{L}}[p^r] \to \Gamma_1^{\mathcal{L}_1}[p^r]$$

of finite groups over A.

Now, we have two morphisms ψ_{n+m} and $\phi[p^r]$ between $\Gamma[p^r]$ and $\Gamma_1[p^r]$ which are equal mod p. By Theorem 3.16 (1), ψ_n equals $\phi[p^r]_n$. Thus $G[p^r]_n$ is the base change to A_n of $\ker\psi$. Since the category $\mathcal{FF}_A$ is abelian, *cf.* [14], Theorem 3.3.3, our Corollary follows. □

3.1. Remarks on the situation in the ramified setting

Most of the results can be carried out mutatis mutandis over A'_n with n of the form $qe+1$.

More precisely, using the exactness properties of the functor $M \mapsto M_{A'}$, *cf.* [5], Chapter IV, §2.3 and Proposition 2.4, and [3], Lemma 2.2, one can define the functors LM_n^f and LM_r in the same manner, *cf.* Definition 3.6, prove that they are well-defined and that LM_n^r is essentially surjective, *cf.* Proposition 3.7.

Then, as in the proof of Proposition 3.9, one can prove that LM_n^f is essentially surjective onto the finite Honda systems (L', L, M) over A'_n, with L and L' free A'/p^qA'-modules, and that LM_n^f is full, hence classifying, when restricted to the finite groups over A'_n that map to such Honda systems. In particular, *cf.* Remark 3.11 (3), LM_n^r classifies the truncated Barsotti-Tate groups over A'_n.

Definition 3.18. — *We say that a finite group G over A'_n is* strongly flat *if it is fully flat and L_i is A'/pA'-free for every $i \geqslant 0$, where $(L_i, M_i) = LM_n^r(G_i[p])$ and G_i is as in (3.25).*

Theorem 3.13 remains true when applied to the strongly flat groups over A'_n. Theorem 3.16 and Corollary 3.17 also remain true with 'fully flat' replaced by 'strongly flat'.

Most probably, with a bit more work one could extend these results to the case of arbitrary n.

4. Almost canonical liftings.

In this Section we assume k is algebraically closed. We let $d < h$ be two relatively prime positive integers. Let Γ_0 be a p-divisible group over k, with Dieudonné module M isomorphic, as a D_k-module, to

$$\left(A^h, \begin{bmatrix} \mathrm{O}_{d,h-d} & p\cdot \mathrm{I}_d \\ \mathrm{I}_{h-d} & \mathrm{O}_{h-d,d} \end{bmatrix} \circ \sigma\right) \tag{4.1}$$

where σ is the absolute Frobenius on $A = W(k)$ and $\mathrm{O}_{i\times j}$ (resp. I_i) is the zero (resp. identity) matrix in the ring of $i \times j$ (resp. $i \times i$) matrices over A. This means that there exists an A-basis of M on which F acts as prescribed by the matrix. We denote by $\{e_1, \ldots, e_h\}$ the elements in this basis. We have the relations

$$Fe_j = e_{d+j}\ ,\quad 1 \leqslant j \leqslant h-d\ ;\quad Fe_{h-d+j} = pe_j\ ,\quad 1 \leqslant j \leqslant d.$$

Remark 4.1. — There is a canonical way of associating to the Dieudonné module (4.1) a connected p-divisible group $G_{d,h-d}$ over $\mathbb{F}_p$ of dimension d and height h, *cf.* [11], §4.2 and [7] (28.5.7). Thus we require Γ_0 to be isomorphic over k to $G_{d,h-d}$.

We denote by W_i the ring $W(\mathbb{F}_{p^i})$ of Witt vectors over $\mathbb{F}_{p^i}$, for $i \geqslant 1$. By requiring that an $h \times h$ matrix over A commute with F, as given in (4.1), we get the following description of the subring $R = \mathrm{End}_{D_k}(M)$ of the ring of $h \times h$ matrices over A

$$\left\{ \begin{bmatrix} a_h & pa_{h-1} & \cdots & pa_2 & pa_1 \\ a_1^{\sigma^u} & a_h^{\sigma^u} & \cdots & pa_3^{\sigma^u} & pa_2^{\sigma^u} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{h-2}^{\sigma^{(h-2)u}} & & \cdots & & pa_{h-1}^{\sigma^{(h-2)u}} \\ a_{h-1}^{\sigma^{(h-1)u}} & a_{h-2}^{\sigma^{(h-1)u}} & \cdots & a_1^{\sigma^{(h-1)u}} & a_h^{\sigma^{(h-1)u}} \end{bmatrix} ; a_i \in W_h \right\} \tag{4.2}$$

where $u \geqslant 1$ is minimal such that $ud = wh + 1$ for some integer w. The $\mathbb{Z}_p$-algebra R is the maximal order in the division algebra $\mathbb{D}_{h,d}$ over $\mathbb{Q}_p$, with invariant d/h. We have

$$R = \left\{ a_h + \sum_{i=1}^{h-1} a_i \gamma^i ;\ a_i \in W_h \right\}$$

where W_h is embedded into R via

(4.3)
$$a \longmapsto \begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a^{\sigma^u} & 0 & \cdots & 0 \\ 0 & 0 & a^{\sigma^{2u}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a^{\sigma^{(h-1)u}} \end{bmatrix} \quad \text{and} \quad \gamma = \begin{bmatrix} 0 & 0 & \cdots & 0 & p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For every $a \in W_h$ we have the relation

$$a\gamma = \gamma a^{\sigma^u}.$$

Let $\mathcal{O}$ be the ring of integers in a degree h extension $\mathbb{K}$ of $\mathbb{Q}_p$, which we embed into $R \subset \mathbb{D}_{h,d}$. Let $e = e(\mathcal{O}) < p-1$ and $f = h/e$. Also let A' be the ring of integers in a degree e totally ramified extension of the fraction field of A. Moreover, we choose A' to contain the maximal unramified extension of $\mathcal{O}$. Let π be a uniformizer of $\mathcal{O}$ and A', and let $\varepsilon \in \mathcal{O}^\times$ be such that $p = \varepsilon\pi^e$. We denote by v the unique valuation on $\mathbb{D}_{h,d}$ normalized such that $v(\pi) = 1$.

Remark 4.2. — For an element $\flat$ in $\mathcal{O}$ we denote by $T_\flat$ its matrix representation inside R as given by (4.2).

Definition 4.3. — *We call an* $\mathcal{O}$-lifting *of* Γ_0 *a* p*-divisible group* Γ *over* A' *such that* $\Gamma_k \simeq \Gamma_0$ *and* $\mathrm{End}_{A'-\mathrm{gr}}(\Gamma) = \mathcal{O}$. *We say that two* $\mathcal{O}$*-liftings are* isomorphic *if they are isomorphic as groups over* A'.

The main result of this Section is the following theorem.

Theorem 4.4. — *There exists an $\mathcal{O}$-lifting of Γ_0 if and only if $f \geqslant d$. In this case, we have*

(i) *For every $\mathcal{O}$-lifting Γ and every $n \geqslant 1$*

$$R_{n-1} = \mathrm{End}_{A'_n-\mathrm{gr}}(\Gamma_n) = \mathcal{O} + \pi^{n-1}R$$

where Γ_n is the base change to A'_n of Γ.

(ii) *There are exactly e^d/g isomorphism classes of $\mathcal{O}$-liftings Γ of Γ_0, where g is the number of automorphisms of $\mathbb{K}$ which fix its maximal unramified subextension.*

Remark 4.5. — It is clear from Theorem 2.8 **B** that we have

$$\cdots \subseteq R_n \subseteq R_{n-1} \subseteq \cdots \subseteq R_1 \subseteq R_0 = R.$$

Proof. — The way we prove this Theorem is by translating it in terms of Honda systems. Thus, a p-divisible group Γ over A', which lifts Γ_0, corresponds to a Honda system $(\mathcal{L}, M)$, *cf.* Definition 1.10 (2) and Theorem 1.11. The fact that Γ has endomorphisms by $\mathcal{O} \subseteq R = \mathrm{End}_{D_k}(M) \subset \mathrm{End}_{A'}(M_{A'})$ translates into the fact that for every endomorphism $M \xrightarrow{T} M$ of M we have $T_{A'}(\mathcal{L}) \subseteq \mathcal{L}$. In general, we say that an A'-submodule U of $M_{A'}$ is an *$\mathcal{O}$-module* if $T_{A'}(U) \subseteq U$ for every T in $\mathcal{O} \subset R$. So, all we have to do is to find all the $\mathcal{O}$-modules of $M_{A'}$ which are A'-free of rank d. Furthermore, via Corollary 2.15, we know that

$$(4.4) \quad R_{n-1} = \mathrm{End}_{\mathrm{H}^{\mathrm{d}}_{A'_n}}(\mathcal{L}/\mathfrak{m}^{n-1}\mathcal{L}, M) = \big\{T \in R;\ T_{A'}(\mathcal{L}) \subset \mathcal{L} + \mathfrak{m}^{n-1}M_{A'}\big\}.$$

This will help us prove the statement about the endomorphisms $\mathrm{mod}\, \mathfrak{m}^n$.

We first prove the statement about the existence of $\mathcal{O}$-liftings, then the one concerning endomorphisms and, finally, the one concerning isomorphism classes. In the sequel we will distinguish between the case in which $\mathcal{O}$ is unramified, which is straightforward, and the case in which $\mathcal{O}$ is ramified, which is a bit more involved.

Case $e = 1$. $\mathcal{O} = W_h$. In this case, the $\mathcal{O}$-modules of M of A-rank 1 are exactly

$$Ae_1,\ Ae_2, \ldots,\ Ae_h.$$

It is clear that $\mathcal{L} = Ae_1 + Ae_2 + \cdots + Ae_d$ is the only A-free $\mathcal{O}$-module of M satisfying $\mathcal{L}/p\mathcal{L} \simeq M/FM$. The pair $(\mathcal{L}, M)$ is a Honda system over A which gives rise to a p-divisible group Γ over A that lifts Γ_0. It is also clear in this case that $R_{n-1} = \mathcal{O} + p^{n-1}R$ and that Γ is unique up to A-isomorphism.

Case $e>1$. In this case we have to find an A'-free $\mathcal{O}$-module $\mathcal{L}$ of $M_{A'}$, such that $\mathcal{L}/\pi\mathcal{L}\simeq M/FM$.

We start by describing $\mathrm{Im}(\mathrm{End}_{D_k}(M)\xrightarrow{(-)_{A'}}\mathrm{End}_{A'}(M_{A'}))$. We choose and fix the following A'-basis of $M_{A'}$

$$E_1=(1\otimes e_1,0),\ E_2=(1\otimes e_2,0),\dots,\ E_d=(1\otimes e_d,0)$$

and

$$E_j=(0,p^{-1}\pi\otimes e_{j-d}),\ d+1\leqslant j\leqslant h.$$

We use this basis to identify $M_{A'}$ with $(A')^h$ and $\mathrm{End}_{A'}(M_{A'})$ with the h by h matrices over A'. Using the description of $M_{A'}$ as in Remark 1.6 (i), for every T in R we get the following relation between T and $T_{A'}$:

$$T=\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\ \text{ and }\ T_{A'}=\begin{bmatrix}A_1 & p^{-1}\pi A_2\\ p\pi^{-1}A_3 & A_4\end{bmatrix}\tag{4.5}$$

where $A_1\in M_d(A)$, $A_2\in M_{d,h-d}(A)$, $A_3\in M_{h-d,d}(A)$ and $A_4\in M_{h-d}(A)$.

We choose an embedding $\mathcal{O}\hookrightarrow R$ such that the embedding $W_f\subseteq\mathcal{O}\hookrightarrow R$ corresponds to $W_h\hookrightarrow R$ as given by $a\mapsto T_a$ in (4.3) above. Then, for every element $\theta\in\mathcal{O}$ we have

$$T_\theta=T_{a_h}+\sum_{i=1}^{e-1}T_{a_{if}}\gamma^{if}\tag{4.6}$$

for some a_{if}'s in W_h.

We claim that there exists an element θ in $\mathcal{O}$ which is primitive, *i.e.* $\mathbb{K}=\mathbb{Q}_p(\theta)$, and verifies $v(\theta)=1$. Indeed, using a Primitive Element Theorem type of argument, we can choose θ from among the elements $\{\omega+p^i\zeta\}_{i\geqslant 0}$, where ζ (resp. ω) in $\mathcal{O}$ is a primitive p^f-1 root of unity (resp. root of an Eisenstein polynomial over W_f of degree e), in particular $K_f=\mathbb{Q}_p(\zeta)$ (resp. $\mathbb{K}=K_f(\omega)$), where K_f is the fraction field of W_f. The characteristic polynomial of T_θ (resp. $T_{\theta,A'}$) is the minimal polynomial of θ over $\mathbb{Z}_p$. Hence T_θ (resp. $T_{\theta,A'}$) has h distinct eigenvalues, which are conjugate by $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $\{w_1,w_2,\dots,w_h\}\subseteq(A')^h$ be an A'-basis of $M_{A'}$ formed by eigenvectors of $T_{\theta,A'}$. It follows from (4.6) that the eigenvectors can be split into f subsets $C_l=\{w_l,w_{f+l},\dots,w_{(e-1)f+l}\}$, for $1\leqslant l\leqslant f$, such that the elements of C_l have zeros on all positions 1 through h, which are not congruent to $l \mod f$. In fact, the elements in C_l, when viewed as elements of $(A')^e$ by considering only the entries congruent to $l \mod f$, are the eigenvectors of the e by e matrix T_l obtained from $T_{\theta,A'}$ by considering only the entries (i,j) with $i\equiv j\equiv l \pmod f$. Moreover, the characteristic polynomial of $T_{\theta,A'}$ is the product of the characteristic polynomials $\mu_l(X)$

of the T_l's and $\mu_l(X)$ is the minimal polynomial of $\theta^{\sigma^{(l-1)u}}$ over W_f, for every l between 1 and f.

If $f \geqslant d$ and $l \leqslant d$ then T_l is

$$T_l = \begin{bmatrix} a_h^{\sigma^{(l-1)u}} & \pi a_{h-f}^{\sigma^{(l-1)u}} & \cdots & \pi a_{2f}^{\sigma^{(l-1)u}} & \pi a_f^{\sigma^{(l-1)u}} \\ \varepsilon\pi^{e-1} a_f^{\sigma^{(l-1+f)u}} & a_h^{\sigma^{(l-1+f)u}} & \cdots & p a_{3f}^{\sigma^{(l-1+f)u}} & p a_{2f}^{\sigma^{(l-1+f)u}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon\pi^{e-1} a_{(e-2)f}^{\sigma^{[l-1+(e-2)f]u}} & & \cdots & & p a_{h-f}^{\sigma^{[l-1+(e-2)f]u}} \\ \varepsilon\pi^{e-1} a_{h-f}^{\sigma^{[l-1+(e-1)f]u}} & a_{(e-2)f}^{\sigma^{[l-1+(e-1)f]u}} & \cdots & a_f^{\sigma^{[l-1+(e-1)f]u}} & a_h^{\sigma^{[l-1+(e-1)f]u}} \end{bmatrix}. \tag{4.7}$$

We choose arbitrary elements

$$\begin{array}{c} B_1 \\ 0 \\ \vdots \\ 0 \\ B_{f+1} \\ 0 \\ \vdots \\ 0 \\ B_{h-f+1} \\ 0 \\ \vdots \\ 0 \end{array}, \qquad \begin{array}{c} 0 \\ B_2 \\ 0 \\ \vdots \\ 0 \\ B_{f+2} \\ 0 \\ \vdots \\ 0 \\ B_{h-f+2} \\ \vdots \\ 0 \end{array}, \qquad \cdots \qquad \begin{array}{c} 0 \\ \vdots \\ 0 \\ B_f \\ 0 \\ \vdots \\ 0 \\ B_{2f} \\ 0 \\ \vdots \\ 0 \\ B_h \end{array} \tag{4.8}$$

in each of the subsets C_1, C_2, …, C_f, respectively.

Writing out what it means for the vectors in (4.8), when viewed inside $(A')^e$, to be eigenvectors of the T_l's we get the following values for the valuations of the entries

$$v(B_{if+l}) = \begin{cases} e-i-1 & \text{if } \ 1 \leqslant i \leqslant e-1 \\ 0 & \text{if } \ i = 0 \end{cases}, \quad \text{if} \quad f \geqslant d \tag{4.9}$$

and

$$v(B_l) > 0, \ 1 \leqslant l \leqslant \min(f, d-f), \quad \text{if } f < d. \tag{4.10}$$

From (4.10) it follows that, in the case $f < d$ there are no A'-free $\mathcal{O}$-modules $\mathcal{L}$ of $M_{A'}$ such that $\mathcal{L}/\pi\mathcal{L} \simeq M/FM$. Hence, in this case, there is no $\mathcal{O}$-lifting of Γ_0.

From now on we assume that $f \geqslant d$. Choose one element z_l in C_l for every $1 \leqslant l \leqslant d(\leqslant f)$ and define $\mathcal{L} \subseteq M_{A'}$ to be

$$\mathcal{L} = A'z_1 + A'z_2 + \cdots + A'z_d. \tag{4.11}$$

It is clear that $\mathcal{L}$ is an A'-free $\mathcal{O}$-module of $M_{A'}$ such that $\mathcal{L}/\pi\mathcal{L} \simeq M/FM$. Hence the pair $(\mathcal{L}, M)$ is a Honda system over A', which gives rise to a p-divisible group Γ over A' that lifts Γ_0. Moreover, it is clear that $\mathcal{O} = \{\phi \in R;\, \phi_{A'}(\mathcal{L}) \subseteq \mathcal{L}\} = \mathrm{End}_{\mathrm{H}^{\mathrm{d}}_{A'}}(\mathcal{L}, M) = \mathrm{End}_{A'-\mathrm{gr}}(\Gamma)$. This proves the existence of the $\mathcal{O}$-lifting.

We now prove (i), *i.e.* the statement about the endomorphisms $\mathrm{mod}\, \mathfrak{m}^n$. We start by proving a result that holds for all p-divisible groups over A'. We have the following result.

Proposition 4.6. — *With notations as in the statement of Theorem 4.4, for all integers $n, m \geqslant 0$ we have*

$$\pi^n R_m \subseteq R_{n+m} \ \text{ and } \ \pi^n R \cap R_{n+m} = \pi^n R_m.$$

Proof. — We know, *cf.* (4.4) above, that $R_n = \{T \in R;\, T_{A'}(\mathcal{L}) \subset \mathcal{L} + \mathfrak{m}^n M_{A'}\}$. Let T be in R_m. Then $T_{A'}(\mathcal{L}) \subset \mathcal{L} + \pi^m M_{A'}$, so

$$\begin{aligned}(\pi^n T)_{A'}(\mathcal{L}) = (T^n_{\pi,A'} \circ T_{A'})(\mathcal{L}) &\subset T^n_{\pi,A'}(\mathcal{L} + \pi^m M_{A'}) \\ &= \pi^n \mathcal{L} + \pi^{n+m} M_{A'} \subset \mathcal{L} + \pi^{n+m} M_{A'}\end{aligned}$$

where T_π is the image of $\pi \in \mathcal{O}$ via $\mathcal{O} \hookrightarrow R$. Thus $\pi^n T$ is in R_{n+m} and the first part of the Proposition follows.

For the second part, the only thing to prove is the inclusion "$\subset$". The other inclusion follows from the first part and from Remark 4.5. Let T be in $\pi^n R \cap R_{n+m}$. Then there is a U in R such that $T = \pi^n U$ and such that $\pi^n U_{A'}(\mathcal{L}) \subset \mathcal{L} + \pi^{n+m} M_{A'}$. It is clear that the last relation implies that $\pi^n U_{A'}(\mathcal{L})$ is actually included in $\pi^n \mathcal{L} + \pi^{n+m} M_{A'}$, hence that $U_{A'}(\mathcal{L})$ is included in $\mathcal{L} + \pi^m M_{A'}$. Therefore U is in R_m and T is in $\pi^n R_m$. □

This Proposition, in the case $m = 1$, implies that we always have an injection

$$0 \to \frac{R}{R_1} \xrightarrow{\pi^n} \frac{R_n}{R_{n+1}}. \tag{4.12}$$

Thus $\dim_{k_f} R_n/R_{n+1} \geqslant \dim_{k_f} R/R_1$, where $k_f = \mathbb{F}_{p^f}$.

Coming back to our $\mathcal{O}$-lifting Γ and its Honda system $(\mathcal{L}, M)$, it is clear that $\mathcal{O} + \pi^n R \subseteq R_n$ for all $n \geqslant 0$. So, taking (4.12) into account, all we have to prove is that

$$R_1 \subseteq \mathcal{O} + \pi R. \tag{4.13}$$

We actually prove that $R_1 \subseteq W_f + \pi R$. Let $\phi = b_h + \sum_{i=1}^{h-1} b_i\gamma^i$ be an element in R_1. This means that $\phi_{A'}(\mathcal{L}) \subseteq \mathcal{L} + \pi M_{A'}$. We may assume that ϕ is $b_h + \sum_{i=1}^{f-1} b_i\gamma^i$, because $\sum_{i=f}^{h-1} b_i\gamma^i \in \pi R$. Now, another nontrivial computation, *cf.* also Lemma 4.7 (a) below, shows that the conditions

$$\phi_{A'}(z_j) \equiv c_1^j z_1 + \cdots + c_d^j z_d \pmod{\pi}$$

with $1 \leqslant j \leqslant d$, imply

$$b_h^{\sigma^{fu}} \equiv b_h \pmod{\pi} \quad \text{and} \quad b_i \equiv 0 \pmod{\pi} \quad \text{for} \quad 1 \leqslant i \leqslant f-1.$$

At this point, we used the fact that for $d > 1$ we have $d < f$ and $h+1 > 2f$. Hence $\overline{b}_h$, which is an element of k_h, is actually in k_f. Therefore b_h is in W_f and $\phi - b_h$ is in πR. This proves (4.13) and, with it, the statement about the endomorphisms $\operatorname{mod} \pi^n$.

Now, we prove (ii). We first remark that there are exactly e^d submodules $\mathcal{L}$ of $M_{A'}$ that are A'-free $\mathcal{O}$-modules such that $\mathcal{L}/\pi\mathcal{L} \simeq M/FM$. Indeed, $\mathcal{L}$ would admit an A'-basis of eigenvectors for the action of $\mathcal{O}$, *i.e.* a decomposition into a direct sum of A'-free $\mathcal{O}$-modules of $M_{A'}$ of rank 1. But these submodules are elements of the set $\{A'w_i;\ 1 \leqslant i \leqslant h\}$. Finally the condition $\mathcal{L}/\pi\mathcal{L} \simeq M/FM$ implies that $\mathcal{L}$ is of the form

$$\mathcal{L} = A'w_{i_1} + \cdots + A'w_{i_d} \tag{4.14}$$

with w_{i_l} in C_l for $1 \leqslant l \leqslant d$. Let $I \in \Pi_{l=1}^d \{l + if;\ 0 \leqslant i \leqslant e-1\}$ be a multi-index. Define $\mathcal{L}_I$ as in (4.14) above, where $I = (i_1, \ldots, i_d)$. We use the notation I_l for i_l. The statement about the isomorphism classes of $\mathcal{O}$-liftings follows from the following lemma.

Lemma 4.7. — *Fix a multi-index I.*

(a) *Let $T \in R^\times$ be a D_k-automorphism of M such that $T_{A'}(\mathcal{L}_I) = \mathcal{L}_J$, for some multi-index J. Then*

$$T_{A'}(A'w_{I_l}) = A'w_{J_l} \quad \textit{and} \quad TU = UT$$

for every $1 \leqslant l \leqslant d$ and $U \in W_f \subset \mathcal{O} \subset R$.

(b) *There is a bijection of sets*

$$\big\{J;\ \textit{there exists}\, T \in R^\times \textit{ with } T_{A'}(\mathcal{L}_I) = \mathcal{L}_J\big\} \longleftrightarrow \operatorname{Gal}(\mathbb{K}/K_f)$$

where $\operatorname{Gal}(\mathbb{K}/K_f)$ is the group of automorphisms of $\mathbb{K}$ which fix $K_f = \operatorname{Frac}(W_f)$.

We start by proving (a). Let $T = b_h + \sum_{i=1}^{h-1} b_i\gamma^i$, *cf.* (4.2). Recall, *cf.* (4.8) above, that for all $1 \leqslant l \leqslant d$ we can write

$$w_{\flat_l} = \sum_{i=0}^{e-1} B_{if+l}^{\flat} E_{if+l}$$

where $\flat \in \{I, J\}$ and the $B^{\flat}_{if+l}$'s are elements in A' satisfying (4.9). The fact that $T_{A'}(\mathcal{L}_I)$ is included in $\mathcal{L}_J$ means that, for all l, we can write

$$T_{A'} w_{I_l} = \sum_{r=1}^{d} c_{rl} w_{J_r} \tag{4.15}$$

for some constants c_{rl}'s in A'. Now, since the w_{J_r}'s have zeroes on all positions not congruent to $r \mod f$ and since r in (4.15) runs from 1 to d, it follows that $T_{A'} w_{I_l}$ has zeroes on all positions congruent to $d+1, \dots, f \mod f$.

If $T_{A'} = (T_{ij})_{1 \leqslant i,j \leqslant h}$, for $1 \leqslant r, l \leqslant f$ we let $T_{rl} = (T_{if+r, jf+l})_{0 \leqslant i,j \leqslant e-1}$ be the e by e matrix obtained from $T_{A'}$ by keeping only the elements on the rows congruent to $r \mod f$ and on the columns congruent to $l \mod f$. The fact that $T_{A'} w_{I_l}$ has zeroes on all positions congruent to $d+1, \dots, f \mod f$ translates into the following relations

$$T_{rl} \begin{pmatrix} B^I_l \\ B^I_{f+l} \\ \vdots \\ B^I_{h-f+l} \end{pmatrix} = 0 \quad \text{for} \quad d+1 \leqslant r \leqslant f \quad \text{and} \quad 1 \leqslant l \leqslant d.$$

This implies that $T_{rl} = 0$ and, with it, that $b_{if+r-l} = 0$ for all l between 1 and d, for all r between $d+1$ and f and for all i between 0 and $e-1$. It is easy to see that we have the following equality of sets

$$\{1, 2, \dots, f-1\} = \{r - l;\, d+1 \leqslant r \leqslant f,\, 1 \leqslant l \leqslant d\}.$$

Therefore $b_{if+r} = 0$ for all $1 \leqslant r \leqslant f-1$ and $0 \leqslant i \leqslant e-1$. Hence, $T = b_h + \sum_{i=1}^{e-1} b_{if} \gamma^{if}$. It is clear that this proves (a).

(b) Let $\eta\colon \{J;\ \text{there exists } T \in R^\times \text{ with } \ T_{A'}(\mathcal{L}_I) = \mathcal{L}_J\} \to \mathrm{Gal}(\mathbb{K}/K_f)$ be defined as

$$\eta(J) = \iota_T|_{\mathbb{K}}$$

where T is in $R^\times$ such that $T_{A'}(\mathcal{L}_I) = \mathcal{L}_J$ and $\iota_T\colon \mathbb{D}_{h,d} \to \mathbb{D}_{h,d}$ is the inner automorphism $x \mapsto T^{-1} x T$ of $\mathbb{D}_{h,d}$. Also, let $\delta\colon \mathrm{Gal}(\mathbb{K}/K_f) \to \{J;\ \text{there exists}\, T \in R^\times \text{ with } \ T_{A'}(\mathcal{L}_I) = \mathcal{L}_J\}$ be defined as

$$\delta(\tau) = J$$

where J is defined by the relation $T_{A'}(\mathcal{L}_I) = \mathcal{L}_J$, with $T \in R^\times$ such that ι_T extends $\tau\colon \mathbb{K} \to \mathbb{K}$ (such a T exists by Skolem-Noether).

All we need to do is to prove that η and δ are well defined, because it is clear that they are inverse to each other.

We start with η. The first thing we need to prove is that if $T \in R^\times$ satisfies $T_{A'}(\mathcal{L}_I) = \mathcal{L}_J$ then $\iota_T(\mathbb{K}) \subseteq \mathbb{K}$. Since $T_{\theta, A'}(\mathcal{L}_J) \subseteq \mathcal{L}_J$, it follows that $(T^{-1}T_\theta T)_{A'}(\mathcal{L}_I) \subseteq \mathcal{L}_I$. From (a) it follows, in particular, that

$$(T^{-1}T_\theta T)_{A'}(A'w_{I_1}) \subseteq A'w_{I_1},$$

which, in turn, implies that

$$(T^{-1}T_\theta T)_{A'}T_{\theta,A'}w_{I_1} = T_{\theta,A'}(T^{-1}T_\theta T)_{A'}w_{I_1}.$$

This last relation implies that $T^{-1}T_\theta T$ commutes with T_θ, hence $\iota_T(T_\theta) = T^{-1}T_\theta T$ must be in $\mathcal{O} \hookrightarrow R$. Since $\mathbb{K} = \mathbb{Q}_p(T_\theta) \subset \mathbb{D}_{h,d}$, it follows that $\iota_T(\mathbb{K}) \subseteq \mathbb{K}$. The fact that ι_T fixes K_f also follows from (a).

The second thing we need to prove is that if T_1 and T_2 are elements of $R^\times$ such that $T_{1,A'}(\mathcal{L}_I) = T_{2,A'}(\mathcal{L}_I)$ then $\eta(T_1) = \eta(T_2)$. As above, since $(T_1T_2^{-1})_{A'}(\mathcal{L}_I) = \mathcal{L}_I$, we get that $T_1T_2^{-1}$ commutes with T_θ, hence $T_1T_2^{-1}$ must be in $\mathcal{O} \hookrightarrow R$. It is clear that this implies that $\iota_{T_1}|_\mathbb{K} = \iota_{T_2}|_\mathbb{K}$. Thus, η is well defined.

We prove now that δ is well defined. First we have to prove that if $T \in R^\times$ satisfies $\iota_T(\mathbb{K}) \subseteq \mathbb{K}$ then $T_{A'}(\mathcal{L}_I) = \mathcal{L}_J$ for some J. The fact that $\iota_T|_\mathbb{K}$ is an automorphism of $\mathbb{K}$ implies, in particular, that $\iota_T(T_\theta) = T^{-1}T_\theta T$ is in $\mathcal{O} = \mathrm{End}_{\mathrm{H}^{\mathrm{d}}_{A'}}(\mathcal{L}_I, M)$. Hence $(T^{-1}T_\theta T)_{A'}(\mathcal{L}_I \subseteq \mathcal{L}_I$, which, together with the fact that ι_T fixes K_f, implies that $T_{A'}(\mathcal{L}_I)$ is an A'-free $\mathcal{O}$-module of $M_{A'}$ such that $T_{A'}(\mathcal{L}_I)/\pi T_{A'}(\mathcal{L}_I) \simeq M/FM$. We saw in the paragraph before our Lemma, that this implies that $T_{A'}(\mathcal{L}_I) = \mathcal{L}_J$ for some J.

Finally, we have to prove that if T_1 and T_2 are elements of $R^\times$ such that $\iota_{T_1}|_\mathbb{K} = \iota_{T_2}|_\mathbb{K}$ then $T_{1,A'}(\mathcal{L}_I) = T_{2,A'}(\mathcal{L}_I)$. The equality $\iota_{T_1}|_\mathbb{K} = \iota_{T_2}|_\mathbb{K}$ implies that $T_1^{-1}T_2$ is in $\mathcal{O} = \mathrm{End}_{\mathrm{H}^{\mathrm{d}}_{A'}}(\mathcal{L}_I, M)$. Hence $(T_1^{-1}T_2)_{A'}(\mathcal{L}_I) = \mathcal{L}_I$, *i.e.* $T_{1,A'}(\mathcal{L}_I) = T_{2,A'}(\mathcal{L}_I)$. The proof of Lemma 4.7 and, with it, the proof of our Theorem is complete now. $\square$

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