ANNALES

DE

L'INSTITUT FOURIER

Rama RAWAT & R.K. SRIVASTAVA

Twisted spherical means in annular regions in $\mathbb{C}^n$ and support theorems

cedram

TWISTED SPHERICAL MEANS IN ANNULAR REGIONS IN $\mathbb{C}^n$ AND SUPPORT THEOREMS

by Rama RAWAT & R.K. SRIVASTAVA (*)

————

ABSTRACT. — Let $Z(\text{Ann}(r,R))$ be the class of all continuous functions f on the annulus $\text{Ann}(r,R)$ in $\mathbb{C}^n$ with twisted spherical mean $f \times \mu_s(z) = 0$, whenever $z \in \mathbb{C}^n$ and $s > 0$ satisfy the condition that the sphere $S_s(z) \subseteq \text{Ann}(r,R)$ and ball $B_r(0) \subseteq B_s(z)$. In this paper, we give a characterization for functions in $Z(\text{Ann}(r,R))$ in terms of their spherical harmonic coefficients. We also prove support theorems for the twisted spherical means in $\mathbb{C}^n$ which improve some of the earlier results.

RÉSUMÉ. — Soit $Z(\text{Ann}(r,R))$ la classe de toutes les fonctions continues sur l'anneau $\text{Ann}(r,R)$ de $\mathbb{C}^n$ de moyenne sphérique tordue $f \times \mu_s(z) = 0$, pour tout $z \in \mathbb{C}^n$ et $s > 0$ tels que la sphère $S_s(z) \subseteq \text{Ann}(r,R)$ et la boule $B_r(0) \subseteq B_s(z)$. Dans cet article, nous donnons une caractérisation des fonctions dans $Z(\text{Ann}(r,R))$ en termes de leur coefficients dans le développement en harmoniques sphériques. Nous prouvons également des théorèmes de support pour les moyennes sphériques tordues dans $\mathbb{C}^n$ qui améliorent certains résultats antérieurs.

1. Introduction and the main results

For $s > 0$, let μ_s stand for the normalized surface measure on $\{z \in \mathbb{C}^n : |z| = s\}$. The twisted spherical means of a function f in $L^1_{\text{loc}}(\mathbb{C}^n)$ are defined by

$$(1.1) \qquad f \times \mu_s(z) = \int_{|w|=s} f(z-w) e^{\frac{i}{2} \text{Im}(z.\bar{w})} d\mu_s(w), \ z \in \mathbb{C}^n.$$

These twisted spherical means arise in a natural way from the spherical means on the Heisenberg group $\mathbb{H}^n$. The group $\mathbb{H}^n$, as a manifold is $\mathbb{C}^n \times \mathbb{R}$,

with the group law

$$(z,t)(w,s) = (z + w, t + s + \frac{1}{2}\operatorname{Im} z.\bar{w}).$$

If μ_s is now considered as a measure on $\{(z,0): |z| = s\} \subset \mathbb{H}^n$, then the spherical means of a function f in $L^1_{\mathrm{loc}}(\mathbb{H}^n)$ are defined by

$$(1.2) \qquad f * \mu_s(z,t) = \int_{|w|=s} f((z,t)(-w,0))d\mu_s(w).$$

Let

$$f^\lambda(z) = \int_\mathbb{R} f(z,t)e^{i\lambda t}dt,$$

be the inverse Fourier transform of f in the $\mathbb{R}$ variable.

Then a simple calculation shows that

(1.3)
$$(f * \mu_s)^\lambda = \int_{-\infty}^\infty f * \mu_s(z,t)e^{i\lambda t}dt = \int_{|w|=s} f^\lambda(z - w)e^{\frac{i\lambda}{2}\operatorname{Im}(z.\bar{w})}d\mu_s(w).$$

We can also define the λ-twisted convolution of functions F and G in $L^1(\mathbb{C}^n)$ by

$$F \times_\lambda G(z) = \int_{\mathbb{C}^n} F(z - w)G(w)\ e^{\frac{i\lambda}{2}\operatorname{Im}(z.\bar{w})}dw.$$

Then, (1.3) can be rewritten as

$$(f * \mu_s)^\lambda(z) = f^\lambda \times_\lambda \mu_s(z).$$

Thus, the spherical means $f * \mu_s$ on the Heisenberg group can be studied using the λ-twisted spherical means $f^\lambda \times_\lambda \mu_s$ on $\mathbb{C}^n$. A further scaling argument shows that it is enough to study these means for the case of $\lambda = 1$. From now onwards, we shall write $F \times G$ instead of $F \times_1 G$ and call it the twisted convolution of F and G.

Let $\operatorname{Ann}(r,R) = \{z \in \mathbb{C}^n : r < |z| < R\}$, $0 \leqslant r < R \leqslant \infty$, be an open annulus in $\mathbb{C}^n$. Let $Z(\operatorname{Ann}(r,R))$ be the class of all continuous functions on $\operatorname{Ann}(r,R)$ with the twisted spherical means

$$\int_{|w|=s} f(z - w)e^{\frac{i}{2}\operatorname{Im}(z.\bar{w})}d\mu_s(w) = 0,$$

for all $z \in \mathbb{C}^n$ and $s > 0$ satisfying the condition that the sphere $S_s(z)$ is contained in the annulus $\operatorname{Ann}(r,R)$ and the ball $B_s(z)$ contains the ball $B_r(0)$.

Equivalently, $f \in Z(\operatorname{Ann}(r,R))$ if $f \times \mu_s(z) = 0$, for all $z \in \mathbb{C}^n$ and $s > 0$ for which the sphere $S_s(z)$ is contained in the annulus $\operatorname{Ann}(r,R)$ and the ball $B_s(z)$ contains the ball $B_r(0)$.

Our main result, Theorem 1.1, gives a necessary and sufficient condition for a function f to be in $Z(\mathrm{Ann}(r, R))$ in terms of its spherical harmonic coefficients. As a corollary, we shall also prove some support theorems, for the twisted spherical means, which improve results in [5].

This work is motivated, in spirit, by the work of Epstein and Kliener [2] on the spherical means in annular regions in Euclidean spaces. For some other closely related work on spherical means see [1], [4].

To state our results, we shall require the following basic facts from the theory of bigraded spherical harmonics, (see [7], p. 12). We shall use the notation $K = U(n)$ and $M = U(n-1)$. Then $S^{2n-1} \cong K/M$ under the map $kM \to k.e_n$, $k \in U(n)$ where $e_n = (0, 0, \ldots, 1) \in \mathbb{C}^n$. Let $\hat{K}_M$ denote the set of all the equivalence classes of irreducible unitary representations of K which have a nonzero M-fixed vector. For our set up of K and M, it is known that for each representation in $\hat{K}_M$ has a unique nonzero M-fixed vector, up to a scalar multiple.

For a $\delta \in \hat{K}_M$, which is realized on V_δ, let $\{e_1, \ldots, e_{d(\delta)}\}$ be an orthonormal basis of V_δ, with e_1 as the $M-$ fixed vector. Let $t_{ij}^\delta(k) = \langle e_i, \delta(k)e_j \rangle$, $k \in K$ and $\langle , \rangle$ stand for the innerproduct on V_δ. By Peter-Weyl theorem, it follows that $\{\sqrt{d(\delta)}t_{j1}^\delta : 1 \leqslant j \leqslant d(\delta), \delta \in \hat{K}_M\}$ is an orthonormal basis of $L^2(K/M)$, (see [7], p. 14 for details). Define $Y_j^\delta(\omega) = \sqrt{d(\delta)}t_{j1}^\delta(k)$, where $\omega = k.e_n \in S^{2n-1}$, $k \in K$. It then follows that $\{Y_j^\delta : 1 \leqslant j \leqslant d(\delta), \delta \in \hat{K}_M,\}$ forms an orthonormal basis for $L^2(S^{2n-1})$.

For our purposes, we need a concrete realization of the representations in $\hat{K}_M$, which can be done in the following way. See [6], p. 253, for details.

Let $\mathbb{Z}^+$ denote the set of all non negative integers. For $p, q \in \mathbb{Z}^+$, let $P_{p,q}$ denote the space of all polynomials P in z and $\bar{z}$ of the form

$$P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

Let $H_{p,q} = \{P \in P_{p,q} : \Delta P = 0\}$ where Δ is the standard Laplacian on $\mathbb{C}^n$. The elements of $H_{p,q}$ are called the bigraded solid harmonics on $\mathbb{C}^n$. The group K acts on $H_{p,q}$ in a natural way. It is easy to see that the space $H_{p,q}$ is K-invariant. Let $\pi_{p,q}$ denote the corresponding representation of K on $H_{p,q}$. Then, representations in $\hat{K}_M$ can be identified, up to unitary equivalence, with the collection $\{\pi_{p,q} : p, q \in \mathbb{Z}^+.\}$

Define the bigraded spherical harmonics on the sphere S^{2n-1} by $Y_j^{p,q}(\omega) = \sqrt{d(p,q)}t_{j1}^{p;q}(\sigma)$, where $\omega = k.e_n \in S^{2n-1}$, $k \in K$ and $d(p,q)$ is the dimension of $H_{p,q}$. Then $\{Y_j^{p,q} : 1 \leqslant j \leqslant d(p,q), p, q \in \mathbb{Z}^+\}$ forms an orthonormal basis for $L^2(S^{2n-1})$.

Therefore, for a continuous function f on $\mathbb{C}^n$, writing $z = \rho\omega$, where $\rho > 0$ and $\omega \in S^{2n-1}$, we can expand the function f in terms of spherical harmonics as

$$(1.4) \qquad f(\rho\omega) = \sum_p \sum_q \sum_{j=1}^{d(p,q)} a_j^{p,q}(\rho) Y_j^{p,q}(\omega).$$

The functions $a_j^{p,q}$ are called the spherical harmonic coefficients of the function f.

The $(p,q)^{th}$ spherical harmonic projection, $\Pi_{p,q}(f)$, of the function f is then defined as

$$(1.5) \qquad \Pi_{p,q}(f)(\rho,\omega) = \sum_{j=1}^{d(p,q)} a_j^{p,q}(\rho) Y_j^{p,q}(\omega).$$

We will replace the spherical harmonic $Y_j^{p,q}(\omega)$ on the sphere by the solid harmonic $P_j^{p,q}(z) = |z|^{p+q} Y_j^{p,q}(\frac{z}{|z|})$ on $\mathbb{C}^n$ and accordingly for a function f, define $\tilde{a}_j^{p,q}(\rho) = \rho^{-(p+q)} a_j^{p,q}(\rho)$, where $a_j^{p,q}$ are defined by equation 1.4. We shall continue to call the functions $\tilde{a}_j^{p,q}$ the spherical harmonic coefficients of f.

Our main result is the following characterization theorem.

THEOREM 1.1. — *Let $f(z)$ be a continuous function on* $\mathrm{Ann}(r,R)$. *Then a necessary and sufficient condition for f to be in $Z(\mathrm{Ann}(r,R))$ is that for all $p,q \in \mathbb{Z}^+$, $1 \leqslant j \leqslant d(p,q)$, the spherical harmonic coefficients $\tilde{a}_j^{p,q}$ of f satisfy the following conditions:*

(1) *For $p = 0, q = 0$, and $r < \rho < R$,*

$$\tilde{a}_j^{0,0}(\rho) = 0.$$

(2) *For $p,q \geqslant 1$, and $r < \rho < R$, there exists $c_i, d_k \in \mathbb{C}$, such that*

$$\tilde{a}_j^{p,q}(\rho) = \sum_{i=1}^{p} c_i e^{\frac{1}{4}\rho^2} \rho^{-2(p+q+n-i)} + \sum_{k=1}^{q} d_k e^{-\frac{1}{4}\rho^2} \rho^{-2(p+q+n-k)}.$$

(3) *For $q = 0$ and $p \geqslant 1$ or $p = 0$ and $q \geqslant 1$, and $r < \rho < R$, there exists $c_i, d_k \in \mathbb{C}$, such that*

$$\tilde{a}_j^{p,0}(\rho) = \sum_{i=1}^{p} c_i e^{\frac{1}{4}\rho^2} \rho^{-2(p+n-i)}, \tilde{a}_j^{0,q}(\rho) = \sum_{k=1}^{q} d_k e^{-\frac{1}{4}\rho^2} \rho^{-2(q+n-k)}.$$

Using the above characterization for the case when $R = \infty$, we also prove the following support theorems for the twisted spherical means.

THEOREM 1.2. — *Let f be a continuous function on $\mathbb{C}^n$ such that for each $k = 0, 1, 2, \cdots, |z|^k e^{\frac{1}{4}|z|^2}|f(z)| \leqslant C_k$. Then f is supported in $|z| \leqslant r$ if and only if $f \times \mu_s(z) = 0$ for $s > r + |z|$ and for every $z \in \mathbb{C}^n$.*

THEOREM 1.3. — *Let f be a continuous function on $\mathbb{C}$. Then f is supported in $|z| \leqslant r$ if and only if $f \times \mu_s(z) = \mu_s \times f(z) = 0$ for $s > r + |z|$ and for every $z \in \mathbb{C}$.*

2. Preliminaries

We begin with the observation that the $U(n)$-invariance of the annulus and the measure μ_s implies that for any f in $Z(\mathrm{Ann}(r, R))$ and $p, q \in \mathbb{Z}^+$, $\Pi_{p,q}(f)$, as defined in equation 1.5, also belongs to $Z(\mathrm{Ann}(r, R))$. In fact the following stronger result is true.

LEMMA 2.1. — *Suppose $f \in Z(\mathrm{Ann}(r, R))$. Then for $p, q \in \mathbb{Z}^+$,*

$$a_j^{p,q}(|z|)Y_i^{p,q}(\omega) \in Z(\mathrm{Ann}(r, R)), 1 \leqslant i, j \leqslant d_{p,q}.$$

In particular, if $f \in Z(\mathrm{Ann}(r, R))$, then $\Pi_{p,q}(f) \in Z(\mathrm{Ann}(r, R))$ for all $p, q \in \mathbb{Z}^+$.

Proof. — For $k \in U(n), \omega \in S^{2n-1}$, we have

$$Y_i^{p,q}(k^{-1}\omega) = \sum_{j=0}^{d(p,q)} \overline{t_{ji}^{p,q}(k)} Y_j^{p,q}(\omega).$$

Using the orthogonality of the matrix entries, we have

$$(2.1) \qquad a_j^{p,q}(|z|)Y_i^{p,q}(\omega) = d(p, q) \int_{U(n)} f(k^{-1}z)t_{ij}^{p,q}(k)dk$$

for $1 \leqslant i, j \leqslant d(p, q)$.

The proof now follows from the $U(n)$-invariance of the annulus and the measure μ_s. $\qquad \square$

We shall also frequently need the following lemma to decompose a homogeneous polynomial into sum of homogeneous harmonic polynomials uniquely.

LEMMA 2.2. — *Let $P \in P_{p,q}$. Then we can write $P(z) = P_0(z) + |z|^2 P_1(z) + \cdots + |z|^{2l} P_l(z)$ where $P_k \in H_{p-k,q-k}$, and $l \leqslant \min(p, q)$.*

For a proof of this lemma see [7], p. 66.

Let $p, q, l, m \in \mathbb{Z}^+$. Define the space $H_{p,q} \cdot H_{l,m}$ to be the vector space of finite sums of the form $\sum P_i Q_i$ where $P_i \in H_{p,q}$ and $Q_i \in H_{l,m}$. Let

$$\nu = \nu(p, q, l, m) = \min(p, m) + \min(l, q).$$

Then the following lemma has been proved in [6], p. 253.

LEMMA 2.3. — $H_{p,q}.H_{l,m} \subset \sum_{j=0}^{\nu} H_{p+l-j,q+m-j}$ where $\nu = \nu(p, q, l, m)$.

As in the proof of the Euclidean case [2], to characterize functions in $Z(\text{Ann}(r, R))$ it would be enough to characterize the spherical harmonic coefficients of smooth functions in $Z(\text{Ann}(r, R))$. This can be done using the following approximation argument. Let ϕ be a nonnegative, radial, smooth, compactly supported function supported in the unit ball in $\mathbb{C}^n$ with $\int_{\mathbb{C}^n} \phi = 1$.

Let $\phi_\epsilon(z) = \epsilon^{-2n}\phi(\frac{z}{\epsilon})$. Then the function

$$S_\epsilon(f)(z) = \int_{\mathbb{C}^n} f(z - w)\phi_\epsilon(w)e^{\frac{i}{2}\text{Im}(z.\bar{w})}dw$$

is smooth and it is easy to see that $S_\epsilon(f)$ lies in $Z(\text{Ann}(r+\epsilon, R-\epsilon))$ for each $\epsilon > 0$. Since f is continuous, $S_\epsilon(f)$ converges to f uniformly on compact sets. Therefore, for each p, q,

$$\lim_{\epsilon \to 0} \Pi_{p,q}(S_\epsilon(f)) = \Pi_{p,q}(f).$$

Henceforth, we would assume, without loss of generality, that the functions in $Z(\text{Ann}(r, R))$ are also smooth in the annulus $\text{Ann}(r, R)$. This would allow us to differentiate the functions in $Z(\text{Ann}(r, R))$ arbitrarily.

Let us define the $2n$ vector fields on $\mathbb{C}^n$ by

$$Z_j = \frac{\partial}{\partial z_j} - \frac{1}{4}\bar{z}_j, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} + \frac{1}{4}z_j, \quad j = 1, 2, \cdots \cdots n.$$

These vector fields together with the identity generate an algebra which is isomorphic to the $(2n+1)$ dimensional Heisenberg algebra. For the twisted convolution on $\mathbb{C}^n$, they play a role similar to that of the Lie algebra of left invariant vector fields on a Lie group.

It is easy to verify that if $f \in Z(\text{Ann}(r, R))$, then

$$Z_j(f \times \mu_s) = Z_j f \times \mu_s \text{ and } \bar{Z}_j(f \times \mu_s) = \bar{Z}_j f \times \mu_s.$$

As a consequence, $Z_j f$ and $\bar{Z}_j f$ both belong to $Z(\text{Ann}(r, R))$.

3. The Proofs

We shall first prove the necessary part of Theorem 1.1. For this, by Lemma 2.1, it is enough to prove the following theorem.

THEOREM 3.1. — Let f be a smooth function on $\operatorname{Ann}(r, R)$ of the form $f(z) = \tilde{a}(\rho)P(z)$, where $|z| = \rho$ and $P \in H_{p,q}$. Then, for f to be in $Z(\operatorname{Ann}(r, R))$ it is necessary that $\tilde{a}$ satisfies the following conditions.

(1) If $p = 0, q = 0$ and $r < \rho < R$, then $\tilde{a}(\rho) = 0$.

(2) If $p, q \geqslant 1$ and $r < \rho < R$, then there exists $c_i, d_k \in \mathbb{C}$, such that

$$\tilde{a}(\rho) = \sum_{i=1}^{p} c_i e^{\frac{1}{4}\rho^2} \rho^{-2(p+q+n-i)} + \sum_{k=1}^{q} d_k e^{-\frac{1}{4}\rho^2} \rho^{-2(p+q+n-k)}.$$

(3) If $q = 0$ and $p \geqslant 1$ and $r < \rho < R$, then there exists $c_i \in \mathbb{C}$, such that

$$\tilde{a}(\rho) = \sum_{i=1}^{p} c_i e^{\frac{1}{4}\rho^2} \rho^{-2(p+n-i)}.$$

(4) If $p = 0$ and $q \geqslant 1$, and $r < \rho < R$, then there exists $d_k \in \mathbb{C}$, such that

$$\tilde{a}(\rho) = \sum_{k=1}^{q} d_k e^{-\frac{1}{4}\rho^2} \rho^{-2(q+n-k)}.$$

Proof. — If $p = 0, q = 0$, then

$$\tilde{a}(\rho) = \int_{|w|=\rho} f(w) d\mu_\rho(w) = f \times \mu_\rho(0) = 0 \text{ for } R > \rho > r,$$

and the condition on $\tilde{a}_{0,0}$ follows.

For the other cases, we proceed in the following way. Since $\bar{Z}_j f \in Z(\operatorname{Ann}(r, R))$, computing

$$\bar{Z}_j f = \frac{\partial f}{\partial \bar{z}_j} + \frac{1}{4} z_j f,$$

we have

$$\bar{Z}_j f = \frac{z_j}{2\rho} \frac{\partial \tilde{a}}{\partial \rho} P + \tilde{a} \frac{\partial P}{\partial \bar{z}_j} + \frac{1}{4} \tilde{a} z_j P,$$

i.e.,

(3.1) $$\bar{Z}_j f = \frac{1}{2} \left(\frac{1}{\rho} \frac{\partial \tilde{a}}{\partial \rho} + \frac{1}{2} \tilde{a} \right) z_j P + \tilde{a} \frac{\partial P}{\partial \bar{z}_j}.$$

Also

$$\triangle_z(z_j P) = 4\sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \bar{z}_k}(z_j P)$$

$$= 4\frac{\partial^2}{\partial z_j \partial \bar{z}_j}(z_j P) + 4\sum_{k\neq j}\frac{\partial^2}{\partial z_k \partial \bar{z}_k}(z_j P)$$

$$= 4\frac{\partial P}{\partial \bar{z}_j} + z_j \triangle_z(P).$$

Since P is harmonic, we have

(3.2) $$\triangle_z(z_j P) = 4\frac{\partial P}{\partial \bar{z}_j}.$$

We shall need the identity

(3.3) $$\triangle_z(|z|^2 \frac{\partial P}{\partial \bar{z}_j}) = 4(n + p + \overline{q-1})\frac{\partial P}{\partial \bar{z}_j}.$$

For this, note that

$$\triangle_z\left(|z|^2 P\right) = 4\sum_{k=1}^{n}\frac{\partial^2}{\partial z_k \partial \bar{z}_k}\left(|z|^2 P\right)$$

$$= 4\sum_{k=1}^{n}\frac{\partial}{\partial \bar{z}_k}\left(\bar{z}_k P + |z|^2\frac{\partial P}{\partial \bar{z}_k}\right)$$

$$= 4\sum_{k=1}^{n}\left[P + \bar{z}_k\frac{\partial P}{\partial \bar{z}_k} + z_k\frac{\partial P}{\partial z_k}\right]$$

$$= 4(n + q + p)P.$$

Since $\frac{\partial P}{\partial \bar{z}_j}$ is a homogeneous harmonic polynomial of degree $p + (q-1)$, we have (3.3). By Lemma 2.2, $z_j P(z) \in P_{p+1,q}$ has a unique representation

(3.4) $$z_j P(z) = P_0(z) + |z|^2 P_1(z) + \cdots + |z|^{2l} P_l(z)$$

where $P_k \in H_{p+1-k,q-k}, 1 \leqslant k \leqslant l \leqslant \min(p+1, q)$. We shall now show that

(3.5) $$z_j P(z) = P_0(z) + \frac{\rho^2}{(n+p+q-1)}\frac{\partial P}{\partial \bar{z}_j}.$$

From (3.2) and (3.3), we have

$$\Delta_z\left(z_j P - \frac{\rho^2}{(n+p+q-1)}\frac{\partial P}{\partial \bar{z}_j}\right) = 0.$$

We know that representation in (3.4) is unique. Therefore

$$z_j P(z) = \left[z_j P(z) - \frac{|z|^2}{(n+p+q-1)} \frac{\partial P}{\partial \bar{z}_j} \right] + \frac{|z|^2}{(n+p+q-1)} \frac{\partial P}{\partial \bar{z}_j}$$

which is nothing but (3.5). In view of (3.5), (3.7) can be rewritten as

$$\bar{Z}_j f(z) = \frac{1}{2} \left(\frac{1}{\rho} \frac{\partial \tilde{a}}{\partial \rho} + \frac{1}{2} \tilde{a} \right) \left[P_0(z) + \frac{\rho^2}{(n+p+q-1)} \frac{\partial P}{\partial \bar{z}_j} \right] + \tilde{a} \frac{\partial P}{\partial \bar{z}_j}.$$

After rearranging the terms, we have

$$\begin{aligned} \bar{Z}_j f(z) &= \frac{1}{2} \left(\frac{1}{\rho} \frac{\partial \tilde{a}}{\partial \rho} + \frac{1}{2} \tilde{a} \right) P_0 \\ &\quad + \left[\left\{ \frac{1}{2(n+p+q-1)} \left(\rho \frac{\partial}{\partial \rho} + \frac{1}{2} \rho^2 \right) + 1 \right\} \tilde{a} \right] \frac{\partial P}{\partial \bar{z}_j}. \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} Z_j f(z) &= \frac{1}{2} \left(\frac{1}{\rho} \frac{\partial \tilde{a}}{\partial \rho} - \frac{1}{2} \tilde{a} \right) P_0 \\ &\quad + \left[\left\{ \frac{1}{2(n+p+q-1)} \left(\rho \frac{\partial}{\partial \rho} - \frac{1}{2} \rho^2 \right) + 1 \right\} \tilde{a} \right] \frac{\partial P}{\partial z_j}. \end{aligned}$$

Hence the projection $\Pi_{p,q-1}$ of $\bar{Z}_j f$ is

$$(3.6) \quad \Pi_{p,q-1}(\bar{Z}_j f) = \left[\left\{ \frac{1}{2(n+p+q-1)} \left(\rho \frac{\partial}{\partial \rho} + \frac{1}{2} \rho^2 \right) + 1 \right\} \tilde{a} \right] \frac{\partial P}{\partial \bar{z}_j}.$$

Let $p = 0$ and $q = 1$. Then there exists a j such that $\dfrac{\partial P}{\partial \bar{z}_j}$ is a non-zero constant. Therefore, in this case,

$$\Pi_{0,0}(\bar{Z}_j f)(z) = C \left\{ \frac{1}{2n} \left(\rho \frac{\partial}{\partial \rho} + \frac{1}{2} \rho^2 \right) + 1 \right\} \tilde{a}(\rho)$$

is in $Z(\mathrm{Ann}(r, R))$. Evaluating the twisted spherical mean at $z = 0$, we get

$$\left\{ \frac{1}{2n} \left(\rho \frac{\partial}{\partial \rho} + \frac{1}{2} \rho^2 \right) + 1 \right\} \tilde{a} = 0.$$

To solve this equation, we substitute $\tilde{a}(\rho) = e^{-\frac{1}{4}\rho^2} \tilde{b}(\rho)$ and get the differential equation

$$e^{-\frac{1}{4}\rho^2} \left\{ \frac{1}{2n} \rho \frac{\partial}{\partial \rho} + 1 \right\} \tilde{b} = 0.$$

Solving it, we conclude that for the case $p = 0, q = 1$, the coefficient $\tilde{a}(\rho) = c_1 e^{-\frac{1}{4}\rho^2} \rho^{-2n}$.

A simple induction argument gives that for $p = 0$ and $q \geqslant 1$, $\tilde{a}$ satisfies

$$\prod_{i=1}^{q} \left\{ \frac{1}{2(n+q-i)} \left(\rho \frac{\partial}{\partial \rho} + \frac{1}{2} \rho^2 \right) + 1 \right\} \tilde{a} = 0$$

and therefore

$$\tilde{a}(\rho) = \sum_{i=1}^{q} c_i e^{-\frac{1}{4}\rho^2} \rho^{-2(n+q-i)}.$$

Similarly, using equation (3.12), we find that for $p \geqslant 1$, and $q = 0$, we have

$$\tilde{a}(\rho) = \sum_{k=1}^{p} d_k e^{\frac{1}{4}\rho^2} \rho^{-2(n+p-k)}.$$

This completes the description of the coefficients (p, q) when either p or q is zero.

Next we take up the case when $p = 1, q = 1$. This can be reduced to case of $p = 0, q = 0$, by means of the operators Z_j and $\bar{Z}_j$.

For this, using Lemma 2.1, without loss of generality, assume that function is of the form, $f(z) = \tilde{a}(\rho) z_1 \bar{z}_2 \in Z(\text{Ann}(r, R))$. Applying the operators $Z_1 \bar{Z}_2$ and taking the $(0, 0)^{th}$ projection, we have

$$\left\{ \frac{1}{2(n+1)} \left(\rho \frac{\partial}{\partial \rho} - \frac{1}{2} \rho^2 \right) + 1 \right\} \left\{ \frac{1}{2(n+1)} \left(\rho \frac{\partial}{\partial \rho} + \frac{1}{2} \rho^2 \right) + 1 \right\} \tilde{a} = 0.$$

Solving this differential equation, we get

$$\tilde{a}(\rho) = c_1 e^{\frac{1}{4}\rho^2} \rho^{-2(n+1)} + d_1 e^{-\frac{1}{4}\rho^2} \rho^{-2(n+1)}.$$

Finally, for the arbitrary p, q, again using Lemma 2.1, we can again assume that the function is of the form, $f(z) = \tilde{a}(\rho) z_1^p \bar{z}_2^q \in Z(\text{Ann}(r, R))$.

Applying the operator $Z_1^p \bar{Z}_2^q$ and taking $(0, 0)^{th}$ projection, we have

$$\prod_{i=1}^{p} \left\{ A_i \left(\rho \frac{\partial}{\partial \rho} - \frac{1}{2} \rho^2 \right) + 1 \right\} \prod_{k=1}^{q} \left\{ B_k \left(\rho \frac{\partial}{\partial \rho} + \frac{1}{2} \rho^2 \right) + 1 \right\} \tilde{a} = 0,$$

where $A_i = (2(n+p+q-i))^{-1}$ and $B_k = (2(n+p+q-k))^{-1}$.

Solving this, we get

$$\tilde{a}(\rho) = \sum_{i=1}^{p} c_i e^{\frac{1}{4}\rho^2} \rho^{-2(n+p+q-i)} + \sum_{k=1}^{q} d_k e^{-\frac{1}{4}\rho^2} \rho^{-2(n+p+q-k))}.$$

This completes the proof of the theorem.   □

Now we shall prove the sufficient part of Theorem 1.1. The proof of this part runs exactly the same way as that worked out for an example in [4]. Nonetheless, for the sake of completeness, we give it here for the general

case. This proof will be using the result of Epstein and Kliener [2] on the spherical means on $\mathbb{R}^d$, which we briefly describe here.

For a function g on $\mathbb{R}^d$ we have the spherical harmonic expansion

$$g(x) = g(\rho\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} a_{kl}(\rho) Y_k^l(\omega)$$

where $\rho = |x|$ and $\{Y_k^l(\omega) : l = 1, 2, \cdots \cdots d_k\}$ is an orthonormal basis for the space V_k of homogeneous harmonic polynomials of degree k restricted to the unit sphere. For each k, the space V_k is invariant under the action of $SO(d)$. When $d = 2m$ for some m, it is invariant under the the action of the unitary group $U(m)$ as well, and under this action of $U(m)$ the space V_k breaks up into an orthogonal direct sum of $H_{p,q}$'s where $p + q = k$. Let σ_s stand for the normalized surface measure on the sphere of radius s centered at the origin contained in $\mathbb{R}^d$. The main result in [2] implies the following theorem for the special case of the annulus $\{x \in \mathbb{R}^d : |x| > B\}$:

THEOREM 3.2. — *A continuous function g on $\mathbb{R}^d$ satisfies*

$$\int_{|y|=s} g(x+y) d\sigma_s(y) = 0 \quad \text{for} \quad s > |x| + B \quad \text{for all} \quad x \in \mathbb{R}^d$$

if and only if

$$a_{kl}(\rho) = \sum_{i=0}^{k-1} \alpha_{kl}^i \rho^{k-d-2i}, \quad \alpha_{kl}^i \in \mathbb{C},$$

for all $k > 0$, $1 \leqslant l \leqslant d_k$, and $a_0(\rho) = 0$ whenever $\rho > B$.

Next we take up the proof of the sufficient part of Theorem 1.1.

THEOREM 3.3. — *Suppose h is a function defined on $\mathrm{Ann}(r, \infty)$ by*
$$h(z) = \frac{e^{\frac{1}{4}|z|^2} P(z)}{|z|^{2(n+p+q-i)}}, \text{ where } P \in H_{p,q} \text{ and } 1 \leqslant i \leqslant p. \text{ Then } h \in Z(\mathrm{Ann}(r, \infty)).$$

Proof. — We have to show that $h \times \mu_s(z) = 0$ for all z, s with $|z| + r < s$. Consider,

$$h \times \mu_s(z) = \int_{|w|=s} \frac{e^{\frac{1}{4}|z+w|^2} P(z+w)}{|z+w|^{2(n+p+q-i)}} e^{-\frac{i}{2} \mathrm{Im}(z.\bar{w})} d\mu_s(w).$$

Expanding the term $|z + w|^2$ and simplifying, we see that it is enough to consider the integral

$$\int_{|w|=s} \frac{e^{\bar{z}.w} P(z+w)}{|z+w|^{2(n+p+q-i)}} d\mu_s(w).$$

On expanding the exponential factor, this leads to terms of the form

$$\int_{|w|=s} \frac{w^\alpha P(z+w)}{|z+w|^{2(n+p+q-i)}} d\mu_s(w)$$

where α is a multi-index. Writing $w_1 = z_1 + w_1 - z_1$ etc. and expanding again we see that it is enough to consider terms of the form

$$\int_{|w|=s} \frac{(w+z)^\beta P(z+w)}{|z+w|^{2(n+p+q-i)}} d\mu_s(w).$$

Let $g(z) = \frac{z^\beta P(z)}{|z|^{2(n+p+q-i)}}$. Then, the above expression is

$$\int_{|w|=s} g(z+w) d\sigma_s(w),$$

which is a Euclidean spherical mean of g on the sphere of radius s centered at the origin contained in $\mathbb{R}^{2n}$.

Thus, we need to show that

$$\int_{|w|=s} g(z+w) d\sigma_s(w) = 0,$$

for $s > |z| + r$.

Using the Lemma 2.3, we have the decomposition

$$z^\beta P(z) = P_0(z) + |z|^2 P_1(z) + \cdots + |z|^{2l} P_l(z)$$

where $P_j \in H_{p+|\beta|-j,q-j}$, for $0 \leqslant j \leqslant l$, $l \leqslant \min(|\beta|, q)$.

With this, the function g further decomposes in functions of the form $|z|^{-2(n+p+q-i-j)} P_j(z)$.

Hence, to prove that g satisfies the desired convolution equation, it is enough to show that the function $|z|^{-2(n+p+q-i-j)} P_j(z)$ satisfies it.

Let us rewrite

$$|z|^{-2(n+p+q-i-j)} P_j(z) = \rho^{k-2n-2(p+q-i-j)} Y_k,$$

where $k = p + q + |\beta| - 2j$ and Y_k is a spherical harmonic of degree k on $\mathbb{R}^{2n}$. Using Theorem 3.2, we need to show that $0 \leqslant p + q - i - j \leqslant k - 1$, or equivalently $j - i \leqslant |\beta| - 1$.

If $|\beta| \leqslant q$, then $l = |\beta|$ and $j - i \leqslant j - 1 \leqslant |\beta| - 1$ (since $j \leqslant |\beta|$ and $1 \leqslant i \leqslant p$). For $|\beta| > q$, we get $l = q$. Since $|\beta| > q \geqslant j$, therefore we have $|\beta| - 1 > j - 1 \geqslant j - i$, as $1 \leqslant i \leqslant p$.

This completes the proof. $\square$

Similarly, we can prove the following theorem.

THEOREM 3.4. — *Suppose h is a function defined on $\mathrm{Ann}(r,R)$ by*
$$h(z) = \frac{e^{-\frac{1}{4}|z|^2}P(z)}{|z|^{2(n+p+q-k)}}, \text{ where } P \in H_{p,q} \text{ and } k = 1,\ldots,q. \text{ Then } h \in$$
$Z(\mathrm{Ann}(r,R))$.

Putting together Theorem 3.3 and Theorem 3.4, the sufficient part of Theorem 1.1 follows.

4. Proofs of the support theorems and concluding remarks

We begin by recalling the Helgason's support theorem ([3], p. 16) for Euclidean spherical means.

THEOREM 4.1. — *Let g be a continuous function on $\mathbb{R}^d$ such that for each $k = 0,1,\cdots$, $\sup |x|^k |g(x)| < \infty$. Then g is supported in $\{x \in \mathbb{R}^d : |x| \leqslant B\}$ if and only if*
$$\int_{|y|=s} g(x+y)d\sigma_s(y) = 0 \quad \text{for} \quad s > |x| + B \quad \text{for all} \quad x \in \mathbb{R}^d.$$

Here, as before, σ_s stand for the normalized surface measure on $\{x \in \mathbb{R}^d : |x| = s\}$.

This theorem can now be deduced as a corollary of Theorem 3.2 (also noted in [2]), as the spherical harmonic coefficients of f satisfy the same decay conditions as f.

Next we recall the following support theorems for the twisted spherical means proved in [5] for the twisted spherical means.

THEOREM 4.2. — *Let f be a function on $\mathbb{C}^n$ such that $f(z)e^{\frac{1}{4}|z|^2}$ is in the Schwartz class. Then f is supported in $|z| \leqslant r$ if and only if $f \times \mu_s(z) = 0$ for $s > r + |z|$ for every $z \in \mathbb{C}^n$.*

In the above theorem, the function f is assumed to have exponential decay, which reflects the non-Euclidean nature of the twisted spherical means. Such decay conditions also arise naturally in the integral geometry on the Heisenberg group as can be seen in the results in [1], [4]. However, the differentiability conditions on the function are genuine and cannot be relaxed. This is because the condition that $f(z)e^{\frac{1}{4}|z|^2}$ is in Schwartz class is not translation invariant ([5]). Nonetheless, to do away with the smoothness condition on f, a stronger condition like $|f(z)| \leqslant Ce^{-(\frac{1}{4}+\epsilon)|z|^2}$, for some $\epsilon > 0$ can be imposed. As then we may convolve f on the right with a radial approximate identity to get smooth functions $\{f_\epsilon\}$ which approximate f and also satisfy the vanishing mean conditions.

In contrast, in Theorem 1.2 we do not impose any differentiability conditions on the function nor do we impose a stronger decay condition. Our conditions can be thought of as an exact analogue of the conditions in the Euclidean set up.

The proof of Theorem 1.2 follows immediately from Theorem 1.1, as the spherical harmonic coefficients $a_j^{p,q}$ satisfy the same decay conditions as the function f.

When $n = 1$, the authors in [5] have shown that under very weak conditions on f and with a suitable condition involving both sided twisted spherical means the following result holds.

THEOREM 4.3. — *Let f be a locally integrable function on $\mathbb{C}$ satisfying the condition that $|f(z)| \leqslant Ce^{\frac{1}{4}(1-\epsilon)|z|^2}$ for some $\epsilon > 0$. Then f is supported in $|z| \leqslant B$ if and only if $f \times \mu_r(z) = \mu_r \times f(z) = 0$ for $r > B + |z|$ for every $z \in \mathbb{C}$.*

In the version Theorem 1.3 of this support theorem, we do not need any growth conditions on the function.

For a proof of Theorem 1.3, let us consider the space $Z^*(\text{Ann}(r, R))$ of continuous functions f on $\mathbb{C}^n$ with both the twisted spherical means $f \times \mu_s(z) = \mu_s \times f(z) = 0$ for all spheres $S_s(z)$ contained in the $\text{Ann}(r, R)$ and with $B_r(0) \subseteq B_s(z)$. Then Theorem 1.1 can be strengthened to the following result:

THEOREM 4.4. — *A necessary and sufficient condition for a function f to belong to $Z^*(\text{Ann}(r, R))$ is that for $p, q \in \mathbb{Z}^+$ and $1 \leqslant j \leqslant d(p, q)$, the spherical harmonic coefficients $\tilde{a}_j^{p,q}(\rho)$ of f satisfy, for $r < \rho < R$,*

$$\tilde{a}_j^{p,q}(\rho) = \sum_{i=1}^{\min(p,q)} c_i e^{\frac{1}{4}\rho^2} \rho^{-2(p+q+n-i)} + \sum_{k=1}^{\min(p,q)} d_k e^{-\frac{1}{4}\rho^2} \rho^{-2(p+q+n-k)},$$

$$p \neq 0, \ q \neq 0$$

and $\tilde{a}_j^{0,q} = \tilde{a}_j^{p,0} = 0$. Here c_i, d_k are arbitrary constants in $\mathbb{C}$.

Proof. — As $\mu_\rho \times f = \overline{\bar{f} \times \mu_\rho}$, it follows that $f \in Z^*(\text{Ann}(r, R))$ if and only if $\bar{f} \in Z^*(\text{Ann}(r, R))$. Also a (p,q)th spherical harmonic coefficient of f, $\tilde{a}_j^{p,q}(f)$ is related to the corresponding spherical harmonic coefficient of $\bar{f}$ by $\tilde{a}_j^{p,q}(\bar{f}) = \overline{\tilde{a}_j^{p,q}(f)}$. Hence the conclusion follows from Theorem 1.1.   □

The proof of Theorem 1.3 now follows as a corollary of the above theorem and the observation that for $n = 1$, the nonzero spaces $H_{p,q}$ will have either $p = 0$ or $q = 0$.

It is therefore no surprise that the decay condition on f could be completely relaxed for the support theorem on functions on $\mathbb{C}$.

Finally, coming back to the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, let f be a continuous function on $\mathbb{H}^n$ which has the spherical means (as defined in 1.2) $f * \mu_s(z, t) = 0$ for all $t \in \mathbb{R}$ and $z \in \mathbb{C}^n$ satisfying $B_r(0) \subseteq B_s(z)$ and $S_s(z) \subseteq \mathrm{Ann}(r, R)$. The problem of characterizing such functions in general is open. However, if f is of the form $f(z, t) = e^{i\lambda t}\varphi(z), \lambda \in \mathbb{R} \setminus \{0\}$, then an easy modification of the proof of Theorem 1.1 for λ-twisted spherical means, λ in $\mathbb{R} \setminus \{0\}$, gives a characterization for f in terms of the spherical harmonic coefficients of the function φ. For $\lambda = 0$, the problem reduces to the problem on Euclidean spherical means.

BIBLIOGRAPHY

[1] M. L. AGRANOVSKY & R. RAWAT, "Injectivity sets for spherical means on the Heisenberg group", *J. Fourier Anal. Appl.* **5** (1999), no. 4, p. 363-372.

[2] C. L. EPSTEIN & B. KLEINER, "Spherical means in annular regions", *Comm. Pure Appl. Math.* **46** (1993), no. 3, p. 441-451.

[3] S. HELGASON, *The Radon Transform*, Birkhauser, 1983.

[4] E. K. NARAYANAN & S. THANGAVELU, "Injectivity sets for spherical means on the Heisenberg group", *J. Math. Anal. Appl.* **263** (2001), no. 2, p. 565-579.

[5] ———, "A spectral Paley-Wiener theorem for the Heisenberg group and a support theorem for the twisted spherical means on $\mathbb{C}^n$", *Ann. Inst. Fourier, Grenoble* **56** (2006), no. 2, p. 459-473.

[6] W. RUDIN, *Function theory in the unit ball of $\mathbb{C}^n$*, Springer-Verlag, New York-Berlin, 1980.

[7] S. THANGAVELU, "An introduction to the uncertainty principle", in *Prog. Math.*, vol. 217, Birkhauser, Boston, 2004.

Rama RAWAT & R.K. SRIVASTAVA
Indian Institute of Technology
Department of Mathematics and Statistics,
Kanpur 208 016 (India)
rrawat@iitk.ac.in
rksri@iitk.ac.in