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MORALES-RAMIS THEOREMS VIA MALGRANGE PSEUDOGROUP

by Guy CASALE (*)

ABSTRACT. — In this article we give an obstruction to integrability by quadratures of an ordinary differential equation on the differential Galois group of variational equations of any order along a particular solution. In Hamiltonian situation the condition on the Galois group gives Morales-Ramis-Simó theorem. The main tools used are Malgrange pseudogroup of a vector field and Artin approximation theorem.

RÉSUMÉ. — Dans cet article, nous montrons que les équations variationnelles le long d'une solution d'une équation différentielle intégrable par quadratures ont un groupe de Galois différentielle virtuellement résoluble. Dans le cas particulier des systèmes hamiltoniens intégrables au sens de Liouville la preuve redonne le théorème de Morales-Ramis-Simó. La preuve consiste à montrer que le groupe de Galois de l'équation variationnelle est un quotient d'un sous groupe d'un groupe d'isotropie du pseudogroupe de Malgrange de l'équation non linéaire. On relie ensuite les propriétés de ce groupe d'isotropie en un point spécial à celles du groupe d'isotropie au point générique en utilisant le théorème d'approximation d'Artin.

Introduction

Morales-Ramis theorems give conditions for integrability in sense of Liouville of a Hamiltonian system in terms of the differential Galois group of the linearized system along a particular solution. First theorem of this kind was obtained by Ziglin [28] in terms of monodromy of the variational equation. Later this condition was translated in terms of differential Galois group. Following previous work of Churchill, Rod and Singer [9, 10], Morales and Simó [19] and Morales and Ramis [21], Morales, Ramis and Simó prove in [22] that the variational equations of any order of an integrable Hamiltonian system have virtually abelian (= almost commutative)

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differential Galois groups. Several extensions of this theorem to other kind of integrability (in Bogoyavlensky's sense [4], in Jacobi's sense [25], in non-commutative sense [14] or for discrete dynamical systems [8]) give the same abelianity condition.

In this article we prove the following generalization of these results in algebraic setting for integrability by quadratures.

DEFINITION 0.1. — *Let $E(t, y, y', \dots, y^{(n)}) \in \mathbb{C}[t, y', \dots, y^{(n)}]$ be an order n differential equation given by an irreducible polynomial.*

A Liouvillian solution is a solution f in a differential extension K_N of $K_0 = \mathbb{C}(t)$ built by successive elementary extensions $K_{i-1} \subset K_i$, $1 \leq i \leq N$, of the form $K_i = K_{i-1}(u_i)$ with u_i algebraic over K_{i-1} or $u'_i \in K_{i-1}$ or $\frac{u'_i}{u_i} \in K_{i-1}$.

The equation E is said to be integrable by quadrature if there is a Liouvillian solution f with $\text{transc.deg. } \mathbb{C}(t, f, f', \dots, f^{(n-1)})/\mathbb{C}(t) = n$.

THEOREM 0.2. — *If a rational differential equation is integrable by quadratures then the Galois group of its variational equation of order q along an algebraic solution is virtually solvable.*

If N elementary extensions are needed to build the general solution then the N th derived Lie algebra of the Galois group is null.

The key arguments are the use of Malgrange pseudogroup (Galois D -groupoid of [16]) of a vector field and Artin approximation theorem [2] (see 1.7 p 2599) to replace Ziglin Lemma as it is done in [22]. They are organized as follow. First we prove that the Galois group of the variational equation is a quotient of a subgroup of the isotropy group of Malgrange pseudogroup at a generic point of the particular solution. The fact that the isotropy group of Malgrange pseudogroup of an equation integrable by quadratures at a generic point is virtually solvable is not very difficult to prove but a generic point on a curve is not generic. To prove virtual solvability of the isotropy group at a non generic point we use Artin approximation theorem.

In a first part, definitions and basic theorems about algebraic "Lie pseudogroups" following [16, 24] are given. In a second part we recall the definitions of Malgrange pseudogroup of a rational vector field and Galois group of a linear differential equation. We give some relations between them by means of variational equations. The main theorem is theorem 2.4. From this theorem we get usual Galoisian obstructions to integrability and exhibit new ones in the third part. Two examples of applications are given in the fourth part.

It would be interesting to insert in the setting described here results of Zung [29] and Ito [13].

Results of this article originate from discussions with B. Malgrange. I would like to thank him for his enthusiasm for sharing mathematical ideas and good mood.

1. Definitions

Definitions and missing proofs of this section can be found in [12, 16, 15, 24, 26].

1.1. Frame bundles

Let V be the affine space over \mathbb{C} with coordinates r_1, \dots, r_d and $(\mathbb{C}^d, 0)$ be the germ of analytic space at 0 with coordinates x_1, \dots, x_d . An order q frame on V is a q -jet

$$j_q r = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq q}} r_i^\alpha \frac{x^\alpha}{\alpha!}$$

of germ of biholomorphism $r : (\mathbb{C}^d, 0) \rightarrow V$. The space of q -frames is denoted by $R_q V$ and its coordinate ring is

$$\mathbb{C}[R_q V] = \mathbb{C} \left[r_i^\alpha, \frac{1}{\det(r_i^{\epsilon(j)})} \mid 1 \leq i \leq d, \alpha \in \mathbb{N}^d, |\alpha| \leq q \right]$$

where $\epsilon(j)$ is the multiindex $(0, \dots, \underset{j^{th}}{1}, \dots, 0)$. One gets projections $\pi_q^{q+1} : R_{q+1} V \rightarrow R_q V$ from inclusions $\mathbb{C}[R_q V] \subset \mathbb{C}[R_{q+1} V]$ and projections $\pi^q : R_q V \rightarrow V$ from identifications $r_i = r_i^{(0, \dots, 0)}$. Elementary properties of this space can be found in [12] and [24, p. 285 with different notations]

The q -frames space is a principal bundle over V with structural group

$$\Gamma_q = \{ j_q g \mid g : (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}^d, 0) \text{ biholomorphism} \}$$

acting by ‘source composition’: $j_q g \cdot j_q r = j_q (r \circ g)$.

Thanks to Faà di Bruno formulas, these groups and actions are algebraic. Because of projections π_q^{q+1} one can define the formal frame bundle $RV = \varprojlim R_q V$ with structural group $\Gamma = \varprojlim \Gamma_q$.

1.2. Groupoids

The algebraic variety

$$\text{Aut}_q V = \{j_q\varphi \mid \varphi : (V, a) \rightarrow (V, b) \text{ biholomorphism}\}$$

with coordinate ring

$$\mathbb{C}[\text{Aut}_q V] = \mathbb{C}\left[r_i, \varphi_j^\alpha, \frac{1}{\det(\varphi_j^{\epsilon(k)})} \mid 1 \leq i, j \leq d, \alpha \in \mathbb{N}^d, |\alpha| \leq q\right]$$

is a groupoid. The groupoid structure is given by the following morphisms

- sources and targets $(s, t) : \text{Aut}_q V \rightarrow V \times V$,
- composition $c : \text{Aut}_q V \times_{sVt} \text{Aut}_q V \rightarrow \text{Aut}_q V$;
 $c(j_q\varphi_1, j_q\varphi_2) = j_q(\varphi_1 \circ \varphi_2)$,
- inverse $i : \text{Aut}_q V \rightarrow \text{Aut}_q V$; $i(j_q\varphi) = j_q\varphi^{-1}$,
- identity $\text{id} : V \rightarrow \text{Aut}_q V$; $\text{id}(r) = j_q \text{id}_r$,

satisfying natural commutative diagrams [15], [24, p 270].

This groupoid acts on R_qV by “target composition”:

$$\begin{aligned} \circ : R_qV \times_{V^s} \text{Aut}_q V &\rightarrow R_qV \\ j_q r \circ j_q \varphi &= j_q(\varphi \circ r). \end{aligned}$$

A subgroupoid \mathcal{G}_q of $\text{Aut}_q V$ is a closed algebraic subvariety such that the induced morphisms give a groupoid structure on \mathcal{G}_q [24, definition 2.2].

A singular subgroupoid \mathcal{G}_q with singularities on a closed subvariety S of V is a closed subvariety of $\text{Aut}_q V$ whose localisation with source and target out of S gives a subgroupoid of $\text{Aut}_q(V - S)$ [16, definition 4.1.1].

A variant of Chevalley theorem [11, theorem 8.1.], [24, proposition 2.3.6.] for this type of groupoids is the following.

THEOREM 1.1. — *Let \mathcal{G}_q be a singular subgroupoid of $\text{Aut}_q V$. There are H_1, \dots, H_n in $\mathbb{C}(R_qV)$ such that, out of its singular locus S ,*

$$\mathcal{G}_q = \{j_q\varphi \mid H_i(\cdot \circ j_q\varphi) = H_i(\cdot)\}.$$

Singular subgroupoids are essentially characterized by their field of rational invariants $F_q \subset \mathbb{C}(R_qV)$.

Groupoids $\text{Aut}_q V$ have “Lie algebras” (usually called Lie algebroids)

$$\mathfrak{aut}_q V = \{j_q Y \mid Y \text{ holomorphic vector field on } (V, a)\}.$$

We will not directly use the Lie algebroid structure of $\mathfrak{aut}_q V$ but the fiber-wise bracket

$$\begin{aligned} \mathfrak{aut}_q V \times_V \mathfrak{aut}_q V &\rightarrow \mathfrak{aut}_{q-1} V \\ (j_q Y_1, j_q Y_2) &\mapsto j_{q-1}[Y_1, Y_2] \end{aligned}$$

The Lie algebra of a groupoid \mathcal{G}_q will be described in next section.

1.3. Prolongations and invariants

Let $\varphi : U_1 \rightarrow U_2$ be a biholomorphism between two open sets of V . It induces a biholomorphism

$$\begin{aligned} R_q\varphi & : R_qU_1 \rightarrow R_qU_2 \\ j_q r & \mapsto j_q(\varphi \circ r) \end{aligned}$$

called the order q prolongation of φ .

Let X be a holomorphic vector field on an open set U of V . Prolongations of its flows define a local 1-parameter action on R_qU . The infinitesimal generator of this action is R_qX the prolongation of X .

These prolongations are defined by polynomial formulas and can be extend to formal biholomorphism $\widehat{\varphi} : \widehat{V}, a \rightarrow \widehat{V}, b$ (and to formal vector fields on \widehat{V}, a). The prolongation is $R_q\widehat{\varphi} : (R_q\widehat{V}, R_qV_a) \rightarrow (R_q\widehat{V}, R_qV_b)$ a formal biholomorphism from a formal neighborhood of frames at $a \in V$ to formal neighborhood of frames at b .

Cartan derivations are given by the action of $\frac{\partial}{\partial x_i}$ on $\mathbb{C}[R_qV]$, the ring of PDE in d functions, r_1, \dots, r_d of d variables x_1, \dots, x_d in the neighborhood of 0:

$$\begin{aligned} D_i & : \mathbb{C}[R_qV] \rightarrow \mathbb{C}[R_{q+1}V] \\ r_j^\alpha & \mapsto r_j^{\alpha + \epsilon(i)} \end{aligned} .$$

The proof of the following lemma is left to the reader following [24, pp. 258–270].

LEMMA 1.2.

- Let $\varphi : U_1 \rightarrow U_2$ be a local biholomorphism on V and $(\mathcal{R}_q\varphi)^* : \mathbb{C}[R_qU_2] \rightarrow \mathbb{C}[R_qU_1]$ the induced isomorphism of rings then

$$D_i \circ (\mathcal{R}_q\varphi)^* = (\mathcal{R}_{q+1}\varphi)^* \circ D_i.$$

- Let X be a local holomorphic vector field $U \subset V$ then

$$D_i \circ \mathcal{R}_qX = \mathcal{R}_{q+1}X \circ D_i.$$

- The order q prolongation of a vector field $X = \sum_j a_j \frac{\partial}{\partial r_j}$ is

$$R_qX = \sum_{\substack{0 \leq j \leq d \\ \alpha \in \mathbb{N}^d}} D^\alpha a_j \frac{\partial}{\partial r_j^\alpha}.$$

Example 1.3. — Let V be the affine line over \mathbb{C} with coordinate ring $\mathbb{C}[r]$ the order q frame bundle is $R_q V = V \times \mathbb{C}^* \times \mathbb{C}^{q-1}$ with coordinate ring $\mathbb{C}[r, r', \frac{1}{r}, \dots, r^{(q)}]$.

If $\varphi : U_1 \rightarrow U_2$ is a biholomorphism between open sets of V its third prolongation is $R_3\varphi : U_1 \times \mathbb{C}^* \times \mathbb{C}^2 \rightarrow U_2 \times \mathbb{C}^* \times \mathbb{C}^2$ and $R_3\varphi(r, r', r'', r''')$ is

$$(\varphi(r), \varphi'(r)r', \varphi''(r)r'^2 + \varphi'(r)r'', \varphi'''(r)r'^3 + 3\varphi''(r)r'r'' + \varphi'(r)r''').$$

Example 1.4. — Let V be the affine space of dimension d over \mathbb{C} with coordinate ring $\mathbb{C}[r_1, \dots, r_d]$ the order 1 frame bundle is $R_1V = V \times GL_d(\mathbb{C})$ with coordinate ring $\mathbb{C} \left[r_1, \dots, r_d, r_1^1, \dots, r_d^d, \frac{1}{\det(r_i^j)} \right]$.

If $X = \sum a_j(r) \frac{\partial}{\partial r_j}$ then

$$R_1X = \sum a_j(r) \frac{\partial}{\partial r_j} + \sum \frac{\partial a_j}{\partial r_i}(r) r_i^k \frac{\partial}{\partial r_j^k}.$$

When $r(t)$ is a trajectory of X then the restriction of R_1X above this trajectory is

$$\frac{\partial}{\partial t} + \sum \frac{\partial a_j}{\partial r_i}(r(t)) r_i^k \frac{\partial}{\partial r_j^k}$$

i.e., the first variational equation of X along $r(t)$ in fundamental form.

Let \mathcal{G}_q be a singular subgroupoid with invariants field F_q . Its first prolongation \mathcal{G}_{q+1} is the singular subgroupoid defined by the subfield of $\mathbb{C}(R_{q+1}V)$ generated by F_q and $D_i F_q$ for all i .

The field of rational functions of any order $\mathbb{C}(RV) = \varinjlim \mathbb{C}(R_q V)$ with Cartan derivations is a differential field. The differential field F generated by all the F_q defines a subvariety \mathcal{S} of $\text{Aut } V = \varprojlim \text{Aut}_q V$ by formulas of theorem 1.1 whose projection on $\text{Aut}_q V$, $(\mathcal{S})_q$ can be smaller than \mathcal{G}_q .

By a theorem of B. Malgrange [16, theorem 4.4.1.] the subvariety of $\text{Aut } V$ defined by invariance of a differential subfield F of $\mathbb{C}(RV)$ defines for any q a singular subgroupoid with singularities on S independant of q . Let F be a differential subfield of $\mathbb{C}(RV)$ and $F_q = F \cap \mathbb{C}(R_q V)$. Let us define

$$\begin{aligned} \text{Iso}(F) = \{ \text{formal biholomorphism } \varphi : \widehat{V}, a \rightarrow \widehat{V}, b \mid \forall q, \forall H \in F_q, \\ H \circ R_q \varphi = H \} \end{aligned}$$

whose ‘‘Lie algebra’’ is

$$\mathfrak{iso}(F) = \{ \text{formal vector field } Y \text{ on } \widehat{V}, a \mid \forall q, \forall H \in F_q, R_q Y \cdot H = 0 \}$$

and $\text{Iso}_q, \mathfrak{iso}_q$ the closure of their projections on order q jets spaces.

THEOREM 1.5. — *The subspace $\text{Iso}(F)$ of $\text{Aut } V$ is stable by composition and inversion. The subspace $\text{iso}(F)$ of $\text{aut } V$ is stable by Lie bracket.*

This is a set theoretical stability. The proalgebraic variety $\text{Iso}(F)$ is singular subgroupoid and the singularities are unavoidable. But the theorem says that the set of formal solutions is a set theoretical groupoid. Before giving the proof let show an example.

Example 1.6. — V is the affine line \mathbb{A}^1 , we look at formal diffeomorphisms preserving the 1-form $\frac{dr}{r}$ (this gives an order 1 invariant). The differential equation satisfied by such transformations φ is $E(r, \varphi) = r \frac{d\varphi}{dr} - \varphi = 0$. $E(r, \varphi_1 \circ \varphi_2)$ is a consequence of $E(r, \varphi_2)$ and $E(\varphi_2, \varphi_1)$ only after localisation of sources and targets out of 0. This is a singular groupoid with singularity at 0 but invertible solutions are $\varphi(r) = \lambda r$ for $\lambda \in \mathbb{C}^*$ and form a groupoid (even a group!).

In many proofs of the third part and in the following proof of theorem 1.5 the following theorem is used

ARTIN APPROXIMATION THEOREM [2] 1.7. — *Consider an arbitrary system of analytic equations*

$$(E) \quad f(x, y) = 0$$

where $f(x, y) = (f_1(x, y), \dots, f_m(x, y))$ are convergent series in the variables $x = (x_1, \dots, x_n), y = (y_1, \dots, y_N)$. Suppose that $\bar{y}(x) = (\bar{y}_1(x), \dots, \bar{y}_N(x))$ are formal power series without constant term which solve (E) For any integer q there exists a convergent series solution $y(x) = (y_1(x), \dots, y_N(x))$ of (E) such that for all i $j_q(\bar{y}_i(x) - y_i(x)) = 0$.

Proof of Theorem 1.5. — By Malgrange theorem (above mentionned), the theorem is clear if sources and targets of two composable elements of $\text{Iso}(F)$ are not in the singular locus S . By analytic continuation, it is also clear for convergent elements. This implies the theorem because of the following.

Let φ_1 and φ_2 be a composable couple of elements of $\text{Iso}(F)$. For any q and $i = 1, 2$, $j_q(\varphi_i)$ are formal sections of $\text{Iso}_q(F)$ over V . By Artin approximation theorem 1.7 there are holomorphic nonholonomic sections ψ_i on neighborhoods U_1 of a and U_2 of $\varphi_1(a)$ such that $j_q\psi_1 = j_q(j_q(\varphi_1))$ at a and $j_q\psi_2 = j_q(j_q(\varphi_2))$ at $\varphi_1(a)$. These new sections ψ_i are no more jets of sections.

For each $\tilde{a} \in U_1$, $\psi_1(\tilde{a}) \in \text{Aut}_q V$ is an order q jet of biholomorphism with source \tilde{a} and target a point $t(\psi_1(\tilde{a}))$ near $\varphi_1(a)$. Because the q -jet of

$t(\psi_1) : U_1 \rightarrow V$ at a equals the q -jet of φ_1 at a , U_1 can be chosen small enough so that $t(\psi_1)(U_1)$ is an open set included in U_2 .

Then one can compose $\psi_2 \circ \psi_1$ pointwise. Because $\text{Iso}_q(F)$ is a groupoid out of some singular locus S , $\psi_2 \circ \psi_1$ is a section of $\text{Iso}_q(F)$ out of S . $\text{Iso}_q(F)$ is closed so $\psi_2 \circ \psi_1$ is a section of $\text{Iso}_q(F)$ and in particular $\psi_2(a) \circ \psi_1(a) = j_q(\varphi_2 \circ \varphi_1)$ belongs to $\text{Iso}_q(F)$. This is true for all integer q thus $\varphi_2 \circ \varphi_1 \in \text{Iso}(F)$.

Same arguments are used to prove the second part of the theorem. \square

Remark 1.8. — Following B. Malgrange, one can give another proof by using Ritt approximation theorem [26] of formal solutions of PDE by convergent solutions (not defined at the same point) instead of Artin approximation theorem of these solutions by nonholonomic sections (defined at the same point).

2. Galois theories

2.1. “Nonlinear” Galois theory

Let X be a rational vector field on V its field of order q differential invariants is

$$\text{Inv}_q(X) = \{H \in \mathbb{C}(R_q V) \mid R_q X \cdot H = 0\}.$$

Let $\text{Inv}(X)$ be the differential field of all differential invariant of any order then Malgrange pseudogroup of a rational vector field X is

$$\text{Mal } X = \text{Iso}(\text{Inv}(X))$$

whose Lie algebra is

$$\text{mal } X = \text{iso}(\text{Inv}(X)).$$

For $(a, b) \in V \times V$, $\text{Mal } X_{(a,b)}$ is the subspace of formal biholomorphisms with source a and target b . One gets the following corollary of theorem 1.5.

COROLLARY 2.1. — *The formal solution of Malgrange pseudogroup at $a \in V$*

$$\text{Mal } X_{(a,a)} = \{\varphi : \widehat{(V, a)} \rightarrow \widehat{(V, a)} \mid \varphi \in \text{Mal } X\}$$

is a group with Lie algebra

$$\text{mal } X_a^0 = \{Y \text{ on } \widehat{V, a} \mid Y \in \text{mal } X, Y(a) = 0\}.$$

Remark 2.2. — These groups may be different depending on a belongs to the singular locus S of $\text{Mal } X$ or not. However, as we will see in the last section, they share lots of properties.

2.2. Linear Galois theory

2.2.1. Principal version

Let \mathcal{C} be an algebraic curve over \mathbb{C} , $E \xrightarrow{\pi} \mathcal{C}$ a principal G -bundle, i.e., $E \times_{\mathcal{C}} E \sim E \times G$ over E for the first projection and G is an algebraic linear group. For a π -projectable, G -invariant rational vector field Y on E with $\pi_* Y \neq 0$, PY denotes a closed minimal Y -invariant subvariety of E dominating \mathcal{C} and $\text{Gal } Y$ its stabilizer in G .

- Two such PY are isomorphic under action of G and called Picard-Vessiot varieties of Y . The field extension $\mathbb{C}(\mathcal{C}) \subset \mathbb{C}(PY)$ is usually called the Picard-Vessiot extension for Y .
- The group $\text{Gal } Y$ is well defined up to conjugation in G . It is the Galois group of Y .
- Common level sets of all rational first integrals of Y in $\mathbb{C}(E)$ dominating \mathcal{C} are Picard-Vessiot varieties.

Malgrange pseudogroup of such a Y is simple to describe. Let Z_1, \dots, Z_N be infinitesimal generators of the action of G then Y, Z_1, \dots, Z_N is a Y -invariant rational parallelism of E i.e., a basis of the $\mathbb{C}(\mathcal{C})$ vector space of rational vector field on E such that $[Y, Z_i] = 0$. Let $\mathbb{C}(E)^Y$ be the field of rational first integrals of Y . One has

$$\text{Mal } Y = \{ \varphi \mid \varphi^* Y = Y, \forall i \varphi^* Z_i = Z_i, \forall F \in \mathbb{C}(E)^Y F \circ \varphi = F \}.$$

The inclusion ‘ \subset ’ is clear from the definition. To prove the other inclusion one remarks that Y and Z ’s give rise to lots of order 1 invariants. Because they form a basis, $\mathbb{C}(R_1 E)$ is generated over $\mathbb{C}(E)$ by these invariants. This implies that $\mathbb{C}(R_q E)$ is generated over $\mathbb{C}(E)$ by derivatives of these invariants. Each new differential invariant for Y reduces modulo this field of invariants to order 0 invariant i.e., to a rational first integral of Y .

Let $\text{Mal } Y_a$ be the restriction of this pseudogroup to the fiber E_a at generic $a \in \mathcal{C}$. The fiber E_a is isomorphic to G and one can choose this isomorphism to send PY_a on $\text{Gal } Y$.

This isomorphism conjugates the action of G on E_a to the left translation on G . Because the action of $\text{Mal } Y$ commutes to left translation on G each $\varphi \in \text{Mal } Y_a$ is the restriction on some open set of right translation by a $g_\varphi \in G$. But $\text{Mal } Y_a$ must preserve $\text{Gal } Y$ so $g_\varphi \in \text{Gal } Y$. We have proved the following theorem.

THEOREM 2.3. — *Under this isomorphism $\text{Mal } Y_a$ equals $\text{Gal } Y$ as pseudogroup generated by a subgroup of G .*

2.2.2. Non principal version

Galois group can be defined for more general bundles with special kind of connections. They are built from principal bundles by “fiber reduction”.

Let $E \rightarrow \mathcal{C}$ be a G -principal bundle and F be an algebraic variety with a left action of G with G a algebraic linear group. This group G acts on $F \times E$ by $g(p, e) = (pg^{-1}, ge)$. The bundle $P = (F \times E)/G$ has structural group G and fibers isomorphic to F . If Y is a π -projectable, G -invariant rational vector field on E with $\pi_*Y \neq 0$ one defines \tilde{Y} on $F \times E$ such that $pr_{1*}\tilde{Y} = 0$ and $pr_{2*}\tilde{Y} = Y$ and Y_P on P by projection.

Galois group of Y_P is by definition the Galois group of Y and one can prove that it is well defined. From [7] Malgrange pseudogroup of Y_P is isomorphic to a subpseudogroup of Malgrange pseudogroup of \tilde{Y} which is itself isomorphic to the one of Y .

2.3. Variational equations

2.3.1. Principal variational equations

Let X be a rational vector field on V and \mathcal{C} be a algebraic X -invariant curve with $X|_{\mathcal{C}} \neq 0$. Its prolongations are rational vector fields R_qX on frames bundles R_qV . The restriction of the frames bundles over \mathcal{C} are Γ_q -principal bundles over \mathcal{C} with projectable Γ_q -invariant vector fields given by the restrictions of R_qX over \mathcal{C} .

Because $R_{q+1}X$ is π_q^{q+1} -projectable on R_qX , this is also true for Malgrange pseudogroup [7, lemme 4.6.] and Galois group. One has subjective morphisms

$$\text{Mal}(R_{q+1}X|_{\mathcal{C}}) \rightarrow \text{Mal}(R_qX|_{\mathcal{C}}) \quad \text{and} \quad \text{Gal}(R_{q+1}X|_{\mathcal{C}}) \rightarrow \text{Gal}(R_qX|_{\mathcal{C}}).$$

One sets

$$\text{Mal}(RX|_{\mathcal{C}}) = \varprojlim \text{Mal}(R_qX|_{\mathcal{C}}) \quad \text{and} \quad \text{Gal}(RX|_{\mathcal{C}}) = \varprojlim \text{Gal}(R_qX|_{\mathcal{C}}).$$

THEOREM 2.4. — *Let a be a generic point on \mathcal{C} , one gets*

$$\text{Gal}(RX|_{\mathcal{C}}) \subset \text{Mal } X_{(a,a)}.$$

Proof. — Let $\text{Mal}(RX|_{\mathcal{C}})_a$ be the restriction of $\text{Mal}(RX|_{\mathcal{C}})$ on the fiber RV_a . The Γ -principal bundle $RV|_{\mathcal{C}}$ is isomorphic to the subspace of $\text{Aut } V_{(a,\mathcal{C})}$ with source $a \in \mathcal{C}$ and target in \mathcal{C} where $\Gamma = \varprojlim \Gamma_q$ is the group of formal biholomorphisms from (V, a) to (V, a) . Under this identification

- RV_a is Γ ,
- $\text{Gal}(RX|_{\mathcal{C}})$ is a subgroup acting by left translation,
- $\text{Mal}(RX|_{\mathcal{C}})_a$ is isomorphic to $\text{Gal}(RX|_{\mathcal{C}})$ but acts by right translation.

The closed subvariety $\text{Mal } X_{(a, \mathcal{C})}$ with source a and target in \mathcal{C} of $\text{Aut } V_{(a, \mathcal{C})}$ is

- RX -invariant because $\text{Mal } X$ is RX -invariant,
- dominates \mathcal{C} because it contains flows of X along \mathcal{C} .

These imply that $\text{Gal}(RX|_{\mathcal{C}})$ preserves $\text{Mal } X_{(a, a)}$ by left translation in Γ thus $\text{Gal}(RX|_{\mathcal{C}}) \subset \text{Mal } X_{(a, a)}$. □

2.3.2. “Non principal” variational equations

Let $A_q V$ be the space of order q arcs on $V : j_q \gamma$ with $\gamma : (\mathbb{C}, 0) \rightarrow V$ holomorphic. This bundle is associated to the Γ_q principal bundle $R_q V$ by the procedure of 2.2.2 with Γ_q acting on

$$A_q = \{j_q \gamma \mid \gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^d, 0)\}$$

by target composition. Groupoid $\text{Aut}_q V$ acts on $A_q V$ by composition. A rational vector field X on V acts on $A_q V$ as a rational vector field $A_q X$. This vector field can be obtained from “fiber reduction” given in 2.2.2. The restriction of this vector field over a X -invariant curve \mathcal{C} is the usual variational equation.

3. Corollaries

3.1. Abelianity

Abelianity of $\text{mal } X$ implies abelianity of the identity components of variational equations (see next section for a proof). In Hamiltonian context one gets the following theorem, consequence of the “key” Lemma [3].

THEOREM 3.1 (J.-P.Ramis [17]). — *If X is a completely integrable Hamiltonian vector field on a symplectic algebraic variety over \mathbb{C} by means of rational first integrals then $\text{mal } X$ is Abelian.*

Together with theorem 2.4 it implies Morale-Ramis-Simó theorem in algebraic context.

THEOREM 3.2 ([22]). — *If X is a completely integrable Hamiltonian vector field on a symplectic algebraic variety over \mathbb{C} by means of rational first integrals and \mathcal{C} be an algebraic X -invariant curve with $X|_{\mathcal{C}} \neq 0$ then identity component of the Galois group of the order q variational equation is Abelian.*

3.2. Solvability

LEMMA 3.3. — *If the N th derived algebra of $\mathfrak{mal} X$ is null the same is true for any variational equations.*

Proof. — We have to prove that $\mathfrak{mal} X_a$ satisfies this property for any $a \in V$ as soon as it is satisfied at any a out of an hypersurface S on V . We follow the proof of theorem 1.5.

Let Y_1, \dots, Y_{2^N} be 2^N formal vector fields at a solutions of $\mathfrak{mal} X$. By Artin Approximation theorem there are $\tilde{Y}_1, \dots, \tilde{Y}_{2^N}$ be holomorphic nonholonomic sections of $\mathfrak{mal}_{N+q} X$ whose $N+q$ jets at a are given by Y 's. The iterated fiberwise bracket in the N th derived algebra of $\mathfrak{mal} X$ obtained from the \tilde{Y} 's is zero out of S thus everywhere. It is determined at a by the q -jet of the iterated Lie bracket of the formal vector fields. Because it is true for any q it proves the lemma. \square

Let say that differential equation over $\mathbb{C}(t)$ is integrable by quadratures if the general solution belongs to a Liouvillian extension (with possibly new constants).

DEFINITION 3.4. — *Let $E(t, y, y', \dots, y^{(n)}) \in \mathbb{C}[t, y', \dots, y^{(n)}]$ be an order n differential equation given by an irreducible polynomial.*

A Liouvillian solution is a solution f in a differential extension K_N of $K_0 = \mathbb{C}(t)$ build by successive elementary extensions $K_{i-1} \subset K_i$, $1 \leq i \leq N$ of the form $K_i = K_{i-1}(u_i)$ with u_i algebraic over K_{i-1} or $u_i \in K_{i-1}$ or $\frac{u_i'}{u_i} \in K_{i-1}$.

The equation E is said to be integrable by quadratures if there is a Liouvillian solution f with $\text{transc.deg.} \mathbb{C}(t, f, f', \dots, f^{(n-1)}) / \mathbb{C}(t) = n$.

Remark 3.5. — It is important to allow new constant in order to get $y'' = 0$ integrable by quadratures.

THEOREM 3.6. — *If a rational ordinary differential equation is integrable by quadratures then its variational equations along algebraic solutions have solvable identity component of their Galois groups.*

Proof. — We have to prove that if X on V is a vector field given by the equation on a phase space, $\text{mal } X_a$ is solvable for a generic $a \in V$. Then apply lemma 3.3 and theorem 2.4 and the proof is done. Let $\mathbb{C}(t) \subset K_1 \cdots \subset K_N$ be a Liouvillian tower such that E has a transcendence degree $d - 1$ solution in K_N over $\mathbb{C}(t)$. For simplicity let us assume that all extensions are transcendental. Each K_i is the field of rational functions on some affine space $\mathbb{A}^1 \times \mathbb{A}^i$ with a vector field $X_i = \frac{\partial}{\partial t} + \sum_{j=1}^i u'_j \frac{\partial}{\partial u_j}$ projectable on X_{i-1} . This means that, as derivation of K_i , X_i preserves the subfield K_{i-1} and $X_i|_{K_{i-1}} = X_{i-1}$. Hypothesis of the theorem are:

- there is a dominant map $V \rightarrow \mathbb{A}^1$ and X is projectable on $\frac{\partial}{\partial t}$,
- there is a dominant map from $\mathbb{A}^1 \times \mathbb{A}^N$ to V over \mathbb{A}^1 and X_N is projectable on X .

From [7, lemme 4.6.] if $\text{mal } X_N$ is solvable so is $\text{mal } X$.

Because of the structure of the tower of extension one can find $N - 1$ forms, $\theta_i = du_i - X_N u_i dt$, $1 \leq i \leq N$, constant on X_N satisfying $d\theta_i = 0 \pmod{(\theta_1, \dots, \theta_{i-1})}$. Because $L_{X_N} \theta_i = 0 \pmod{(\theta_1, \dots, \theta_{i-1})}$ same equations are satisfied by vector field of $\text{mal } X$. Let x_1, \dots, x_N be local (analytic) coordinates such that $dx_i = \theta_i \pmod{(\theta_1, \dots, \theta_{i-1})}$ then $Y \in \text{mal } X$ can be written

$$Y = c_1 \frac{\partial}{\partial x_1} + c_2(x_1) \frac{\partial}{\partial x_2} + \cdots + c_N(x_1, \dots, x_{N-1}) \frac{\partial}{\partial x_N}.$$

The N th derived algebra of this type of Lie algebra of formal vector field is zero. □

For instance, X on V is Jacobi integrable if it has $d - 2$ rational first integrals and an invariant rational d -form. Morales-Ramis type theorem for this kind of integrability was obtained by M. Przytylska in [25] in a particular case. Theorem 2.4 gives the general situation. Computation of vector fields in the Lie algebra of the Malgrange pseudogroup of such a vector field is left to the reader. In suitable local (analytic) coordinates one gets vector field of the form $c_1(x_2, \dots, x_n) \frac{\partial}{\partial x_1} + c_2(x_3, \dots, x_n) \frac{\partial}{\partial x_2}$. The first derived algebra is Abelian.

COROLLARY 3.7. — *If X is a Jacobi integrable rational vector field on an algebraic variety then identity components of Galois groups of variational equations are solvable and their first derived Lie algebras are Abelian.*

3.3. Finiteness

LEMMA 3.8. — *Finiteness of the dimension of $\mathfrak{mal} X$ implies that the dimensions of Galois groups of variational equations are uniformly bounded.*

Proof. — We have to prove that if $\mathfrak{mal} X_a$ is finite dimensional at generic a in V , it is finite dimensional at any a in V with smaller dimension. Then theorem 2.4 can be used to conclude.

Let N be the generic dimension of $\mathfrak{mal} X$. If X_1, \dots, X_{N+1} are $N+1$ elements of $\mathfrak{mal} X_a$, by Artin approximation theorem, there are $\tilde{X}_1, \dots, \tilde{X}_{N+1}$ holomorphic nonholonomic sections of $\mathfrak{mal}_q X$ whose q -jet at a are given by X 's. For each $a \in V - S$ the vectors $\tilde{X}_1(a), \dots, \tilde{X}_{N+1}(a)$ are linearly dependent. By analytic continuation it is also true for any $a \in V$ and any order q . This proves the lemma. \square

From

THEOREM 3.9 (B. Malgrange [18]). — *If X is a completely integrable Hamiltonian vector field on a symplectic algebraic variety over \mathbb{C} by means of rational first integrals then $\mathfrak{mal} X$ is finite dimensional.*

One gets

THEOREM 3.10. — *In the situation of theorem 3.2 the sequence of dimensions*

$$(\dim \text{Gal}(R_q X|_{\mathcal{C}}))_{q \in \mathbb{N}}$$

is bounded uniformly in \mathcal{C} .

There is no uniform bound for all algebraic Hamiltonians but it depends on the geometry of the moment map. For instance if the moment map is the restriction on some open set of an algebraically isotrivial fibration in Abelian varieties then the bound should be the degree of freedom. If the fibration is not isotrivial, e.g., for algebraic complete integrability [1], the bound depends on the Gauss-Manin connexion of such a family.

4. Applications

4.1. Painlevé II equation

Irreducibility of these equations is proved in [23, 27] and implies that these equations are not integrable by quadratures. Here is another proof

of this weaker assertion in a particular case. These computations have been done in [20] to apply usual Morales-Ramis theorem.

Second Painlevé equation depends on a parameter

$$y'' = 2y^3 + xy + \alpha \text{ for } \alpha \in \mathbb{C}.$$

For $\alpha = 0$, it is the vector field

$$\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + (2y^3 + xy) \frac{\partial}{\partial y'}$$

with first order (non principal) variational equation along $\mathcal{C} = \{y = y' = 0\}$

$$\frac{\partial}{\partial x} + r_3 \frac{\partial}{\partial r_2} + xr_2 \frac{\partial}{\partial r_3}$$

on $T\mathbb{C}^3|_{\mathcal{C}}$ with induced coordinates (x, r_1, r_2, r_3) . The rank 2 subsystem on r_2, r_3 is Airy equation with Galois group $SL_2(\mathbb{C})$. This group is not solvable so Painlevé II equation is not integrable by quadratures when $\alpha = 0$. From Okamoto (see [23, 27]), we know that two Painlevé II equations with parameter α and $\alpha + n, n \in \mathbb{Z}$ are isomorphic by a birational change of coordinates on the phase space. Non integrability for $\alpha = 0$ implies non integrability for any interger values of α .

4.2. Lorenz system

Computations presented here originates from [6] where the non integrability in sense of Liouville of a Hamiltonian form of this system is proved by Morales-Ramis theorem. Lorenz system depends on 3 constants σ, ρ and β :

$$\begin{cases} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z. \end{cases}$$

First assume $\beta \neq 0$. An invariant curve is the z -axis \mathcal{C} and $X|_{\mathcal{C}} = -\beta z \frac{\partial}{\partial z}$. Let us consider the following time dependent form of this equation on $\mathbb{C}^3 \times \mathbb{C}$ with coordinates (x, y, z, t)

$$\sigma(x - y) \frac{\partial}{\partial x} + (\rho x - xz - y) \frac{\partial}{\partial y} + (xy - \beta z) \frac{\partial}{\partial z} - \beta t \frac{\partial}{\partial t}$$

an invariant curve is $\mathcal{C} = \{x = y = z - t = 0\}$ and the first variational equation is the linear system

$$-\beta t \frac{d}{dt} A = \begin{bmatrix} -\sigma & \sigma & 0 & 0 \\ \rho - z & -1 & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \end{bmatrix} A.$$

The 2×2 subsystem given by the first block is equivalent to the second order equation

$$\beta^2 t^2 \frac{d^2 a}{dt^2} + (\beta^2 - \beta\sigma - \beta)t \frac{da}{dt} + \sigma(t - \rho + 1)a = 0.$$

Applying [5] (or Maple 11), we get solution

$$t^{\frac{\sigma+1}{2\beta}} J \left(\frac{\sigma^2 - 2\sigma + 1 + a\sigma\rho}{\beta^2}, \frac{2\sigma^{1/2}}{\beta} t^{1/2} \right)$$

where $J(\alpha, x)$ is any solution to Bessel equation $x^2 \frac{d^2 J}{dx^2} + x \frac{dJ}{dx} + (x^2 - \alpha)J = 0$. The Galois group of this equation is $SL_2(\mathbb{C})$.

If $\sigma \neq 0$ then using the vector field

$$\sigma(x - y) \frac{\partial}{\partial x} + (\rho x - xz - y) \frac{\partial}{\partial y} + (xy - \beta z) \frac{\partial}{\partial z} - \sigma t \frac{\partial}{\partial t}$$

an the invariant curve is $\mathcal{C} = \{y = z - \rho = x - t = 0\}$ we get the variational equation

$$-\sigma t \frac{d}{dt} A = \begin{bmatrix} -\sigma & \sigma & 0 & 0 \\ 0 & -1 & t & 0 \\ 0 & t & -\beta & 0 \\ 0 & 0 & & -\sigma \end{bmatrix} A.$$

From the middle 2×2 subsystem we get the second order equation

$$\sigma^2 t^2 \frac{d^2 a}{dt^2} - \sigma(t + 1)t \frac{da}{dt} + ((1 - \beta)t + \sigma)a = 0.$$

Solutions are

$$t^{\frac{1}{2\sigma}} \exp \left(\frac{t}{2\sigma} \right) W \left(\frac{1 - 2\beta}{2\sigma}, \frac{\sigma - 1}{2\sigma}, \frac{t}{\sigma} \right)$$

where $W(k, m, x)$ is any solution of Whittaker equation $x^2 \frac{d^2 W}{dx^2} + x^2 \frac{dW}{dx} + (kx - m^2 + 1/4)W = 0$. Galois group of this equation is $SL_2(\mathbb{C})$.

If $\sigma = \beta = 0$ the Lorenz system is 2×2 linear system with constant coefficients and a parameter x . Such systems can be explicitly solved by exponentials. For any other values of parameters, Lorenz system is not integrable by quadratures.

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