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CLIFFORD'S THEOREM FOR REAL ALGEBRAIC CURVES

by Jean-Philippe MONNIER (*)

Abstract. — We establish, for smooth projective real curves, an analogue of the classical Clifford inequality known for complex curves. We also study the cases when equality holds.

Résumé. — On démontre, pour les courbes projectives lisses réelles, une version analogue de l'inégalité de Clifford connue pour les courbes complexes. On étudie aussi très précisément les cas où cette inégalité devient une égalité.

1. Introduction

In this note, a real algebraic curve X is a smooth proper geometrically integral scheme over $\mathbb{R}$ of dimension 1. A closed point P of X will be called a real point if the residue field at P is $\mathbb{R}$, and a non-real point if the residue field at P is $\mathbb{C}$. The set of real points $X(\mathbb{R})$ of X decomposes into finitely many connected components, whose number will be denoted by s. By Harnack's Theorem ([4, Th. 11.6.2 p. 245]) we know that $s \leqslant g+1$, where g is the genus of X. A curve with $g+1-k$ real connected components is called an $(M-k)$-curve.

The group $\mathrm{Div}(X)$ of divisors on X is the free abelian group generated by the closed points of X. If D is a divisor on X, we will denote by $\mathcal{O}(D)$ its associated invertible sheaf. The dimension of the space of global sections of this sheaf will be denoted by $\ell(D)$. Let $D \in \mathrm{Div}(X)$, since a principal divisor has an even degree on each connected component of $X(\mathbb{R})$ (*e.g.* [8, Lem. 4.1]), the number $\delta(D)$ (resp. $\beta(D)$) of connected components C

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of $X(\mathbb{R})$ such that the degree of the restriction of D to C is odd (resp. even), is an invariant of the linear system $|D|$ associated to D. If $\ell(D) > 0$, the dimension of the linear system $|D|$ is $\dim |D| = \ell(D) - 1$. Let K be the canonical divisor. If $\ell(K - D) = \dim H^1(X, \mathcal{O}(D)) > 0$, D is said to be special. If not, D is said to be non-special. By Riemann-Roch, if $\deg(D) > 2g - 2$ then D is non-special. Assume D is effective and let d be its degree. If D is non-special then the dimension of the linear system $|D|$ is given by Riemann-Roch. If D is special, then the dimension of the linear system $|D|$ satisfies

$$\dim |D| \leqslant \frac{1}{2}d.$$

This is the well known Clifford inequality for complex curves ([9, Th. 5.4 p. 343]) that stands obviously for real curves.

If X is an M-curve or an $(M-1)$-curve, then Huisman ([10, Th. 3.2]) has shown that

$$\dim |D| \leqslant \frac{1}{2}(d - \delta(D)).$$

We have already proved that Huisman's inequality is also valid for almost all real hyperelliptic curves, for example when $s \neq 2$. But there is a family of real hyperelliptic curves with 2 real connected components for which there exist some special divisors D satisfying $\dim |D| = \frac{1}{2}d > \frac{1}{2}(d - \delta(D))$ ([11, Th. 4.4, Th. 4.5]).

In this note we establish a new Clifford inequality for real curves, completing the inequality given by Huisman.

Theorem A. — *Assume D is effective and special. Then, either*

$$\dim |D| \leqslant \frac{1}{2}(d - \delta(D)) \tag{Clif1}$$

or

$$\dim |D| \leqslant \frac{1}{2}(d - \beta(D)). \tag{Clif2}$$

Moreover, D satisfies the inequality (Clif 1) if either $s \leqslant 1$ or $s \geqslant g$.

The previous theorem implies Huisman's theorem [10, Th. 3.2]. We prove that the inequality (Clif 2) is the best possible complement of the inequality (Clif 1) since we give examples of special divisors that do not satisfy the latter and for which equality holds in (Clif 2). The cases for which equality holds in either (Clif 1) or in (Clif 2) are also studied.

Moreover, looking at divisors that do not satisfy the inequality (Clif 1), we obtain the following theorem.

Theorem B. — *Let X be a real curve. Let $D \in \mathrm{Div}(X)$ be an effective and special divisor of degree d.*

(i) *If X is hyperelliptic then either*

$$\dim |D| \leqslant \frac{1}{2}(d - \delta(D)) \quad \text{(Clif 1)}$$

or

$$\dim |D| \leqslant \frac{1}{2}\Big(d - \frac{1}{2}(s-2)\Big).$$

(ii) *If X is not hyperelliptic then either*

$$\dim |D| \leqslant \frac{1}{2}(d - \delta(D)) \quad \text{(Clif 1)}$$

or

$$\dim |D| \leqslant \frac{1}{2}\Big(d - \frac{1}{2}(s-1)\Big).$$

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2. Preliminaries

We recall here some classical concepts and more notation that we will be using throughout the paper.

Let X be a real curve. We will denote by $X_{\mathbb{C}}$ the base extension of X to $\mathbb{C}$. The group $\mathrm{Div}(X_{\mathbb{C}})$ of divisors on $X_{\mathbb{C}}$ is the free abelian group on the closed points of $X_{\mathbb{C}}$. The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on the complex variety $X_{\mathbb{C}}$ and also on $\mathrm{Div}(X_{\mathbb{C}})$. We will always indicate this action by a bar. If P is a non-real point of X, identifying $\mathrm{Div}(X)$ and $\mathrm{Div}(X_{\mathbb{C}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$, then $P = Q + \bar{Q}$ with Q a closed point of $X_{\mathbb{C}}$. If D is a divisor on $X_{\mathbb{C}}$, we will denote by $\mathcal{O}(D)$ its associated invertible sheaf and by $\ell_{\mathbb{C}}(D)$ the dimension of the space of global sections of this sheaf. If $D \in \mathrm{Div}(X)$, then $\ell(D) = \ell_{\mathbb{C}}(D)$.

Let $D \in \mathrm{Div}(X)$ be a divisor with the property that $\mathcal{O}(D)$ has at least one nonzero global section. The linear system $|D|$ is called base point free if $\ell(D - P) \neq \ell(D)$ for all closed points P of X. If not, we may write $|D| = E + |D'|$ with E a non zero effective divisor called the base divisor of $|D|$, and with $|D'|$ base point free. A closed point P of X is called a base point of $|D|$ if P belongs to the support of the base divisor of $|D|$. We notice that

$$\dim |D| = \dim |D'|.$$

As usual, a g^r_d is an r-dimensional complete linear system of degree d on X (or $X_{\mathbb{C}}$). Let $|D|$ be a g^r_d on X and assume $|D|$ is base point free. The linear system $|D|$ defines a morphism $\varphi : X \to \mathbb{P}^r_{\mathbb{R}}$ onto a non-degenerate (but maybe singular) curve in $\mathbb{P}^r_{\mathbb{R}}$. If φ is birational (resp. an isomorphism) onto $\varphi(X)$, the g^r_d (or D) is called simple (resp. very ample). Let X' be the normalization of $f(X)$, and assume D is not simple *i.e.*, $|D - P|$ has a base point for any closed point P of $X_{\mathbb{C}}$. Thus, the induced morphism $\varphi : X \to X'$ is a non-trivial covering map of degree $k \geqslant 2$. In particular, there is $D' \in \mathrm{Div}(X')$ such that $|D'|$ is a $g^r_{\frac{d}{k}}$ and such that $D = \varphi^*(D')$, *i.e.*, D is induced by X'. If g' denote the genus of X', $|D|$ is classically called compounded of an involution of order k and genus g'. In the case $g' > 0$, we speak of an irrational involution on X.

The reader is referred to [3] and [9] for more details on special divisors. Concerning real curves, the reader may consult [4] and [8]. For $a \in \mathbb{R}$ we denote by $[a]$ the integral part of a, *i.e.*, the biggest integer $\leqslant a$.

3. Clifford inequalities for real curves

We state the following result of Huisman [10, Th. 3.2].

Theorem 3.1. — *Assume X is an M-curve or an $(M-1)$-curve. Let $D \in \mathrm{Div}(X)$ be an effective and special divisor of degree d. Then*

$$\dim |D| \leqslant \frac{1}{2}(d - \delta(D)).$$

We have already mentioned that Huisman's inequality (Clif 1) is not valid for all real curves [11, Th. 4.5]: Let X be the real hyperelliptic curve of odd genus g which is the smooth completion of the affine plane curve given by the real polynomial equation $y^2 = f(x)$, where f is a monic polynomial of degree $2g+2$ with no real roots. The number of connected components s of $X(\mathbb{R})$ is 2. Let D be an element of the hyperelliptic g^1_2. Let r be an odd integer such that $0 < r < g-1$, then rD is a special divisor of degree $2r$ such that $\dim |rD| = r > \frac{1}{2}(2r - \delta(rD)) = r - 1$.

The aim of the the paper is to provide another inequality for the special divisors that do not satisfy Huisman's inequality.

Before proving the theorems stated in the introduction, we need to establish some preliminary results.

Lemma 3.2. — *Let D be a divisor of degree $d > 0$ such that $\ell(D) > 0$ and such that $d = \delta(D)$.*

1) *If* $d < s$ *then* $\dim |D| = 0$.
2) *If* $d = s$ *then* $\dim |D| \leqslant 1$. *In addition, if* $\dim |D| = 1$ *then* D *is base point free.*

Proof. — Since $\ell(D) > 0$, we may assume that D is effective. Since $d = \delta(D)$, $D = P_1 + \ldots + P_d$ with $P_1, \ldots, P_d$ some real points of X such that no two of them belong to the same connected component of $X(\mathbb{R})$.

Assume $d < s$ and $\dim |D| > 0$. Choose a real point P in one of the $s - d$ real connected components that do not contain any of the points $P_1, \ldots, P_d$. Then $\mathcal{O}(D - P)$ has a nonzero global section and $D - P$ should be linearly equivalent to an effective divisor D' of degree $d - 1$ satisfying $\delta(D') = d + 1$. This is impossible, proving 1).

Assume $d = s$, then $\dim |D| \leqslant 1 + \dim |P_2 + \ldots + P_s| \leqslant 1$ by 1). Suppose $\dim |D| = 1$ and $|D|$ is not base point free. If $|D|$ has a real base point P, then $\dim |D - P| = 1$ and $\deg(D - P) = \delta(D - P) = s - 1$, contradicting 1). If $|D|$ has a non-real base point Q, then $\ell(D - Q) > 0$ and $D - Q$ should be linearly equivalent to an effective divisor D' of degree $s - 2$ satisfying $\delta(D') = s$, which is again impossible. □

The following lemma is due to Huisman ([10] Th. 3.1).

Lemma 3.3. — *Let* $D \in \mathrm{Div}(X)$ *be an effective divisor of degree* d *and assume* $d + \delta(D) < 2s$. *Then*

$$\dim |D| \leqslant \frac{1}{2}(d - \delta(D)).$$

Proof. — Set $k = \frac{1}{2}(d - \delta(D))$. Then $k \geqslant 0$, since D is effective. We have $\delta(D) + k < s$ by the hypotheses. Choose $P_1, \ldots, P_k$ real points among the $\beta(D)$ real connected components on which the degree of the restriction of D is even, such that no two of these points belong to the same real connected component. Let $D' = D - P_1 - \ldots - P_k$. Then $\deg(D') = d - k = \delta(D') = \delta(D) + k < s$. By Lemma 3.2, if $\ell(D') > 0$, then $\dim |D'| = 0$. Finally, $\dim |D| \leqslant \dim |D'| + k \leqslant \frac{1}{2}(d - \delta(D))$. □

Generalizing the previous lemma, we get:

Lemma 3.4. — *Let* $D \in \mathrm{Div}(X)$ *be a divisor of degree* d *such that* $\ell(D) > 0$. *Assume that* $d + \delta(D) < 2s + 2k$ *with* $k \in \mathbb{N}$. *Then*

$$\dim |D| \leqslant \frac{1}{2}(d - \delta(D)) + k.$$

Proof. — We proceed by induction on k. The case $k = 0$ is given by Lemma 3.3.

So, assume that $k > 0$ and that $d + \delta(D) < 2s + 2k$. Since $\ell(D) > 0$, we may assume that D is effective.

If $d+\delta(D)<2s+2k-2$, the proof is done by induction hypothesis. Since $d=\delta(D) \mod 2$, we assume that $d+\delta(D)=2s+2k-2$.

Let Q be a non-real point. We have $\deg(D-Q)+\delta(D-Q)<2s+2k-2$ and $\delta(D-Q)=\delta(D)$. If $\ell(D-Q)>0$, by the induction hypothesis, we get $\dim|D-Q|\leqslant\frac{1}{2}(\deg(D-Q)-\delta(D-Q))+k-1=\frac{1}{2}(d-2-\delta(D))+k-1$. Hence $\dim|D|\leqslant\dim|D-Q|+2\leqslant\frac{1}{2}(d-\delta(D))+k$. Now, if $\ell(D-Q)=0$ then $\dim|D|\leqslant 1\leqslant\frac{1}{2}(d-\delta(D))+k$, since $k>0$ and $d\geqslant\delta(D)$ D being effective. □

The following lemma will allow us to restrict the study to base point free linear systems.

Lemma 3.5. — *Let $D\in\mathrm{Div}(X)$ be an effective divisor of degree d. Let E be the base divisor of $|D|$. Let $|D'|=|D-E|$ be the degree d' base point free part of $|D|$.*

(i) *If* $\dim|D'|\leqslant\frac{1}{2}(d'-\delta(D'))+k$ *for a positive integer k, then* $\dim|D|\leqslant\frac{1}{2}(d-\delta(D))+k$.
(ii) *If* $\dim|D'|\leqslant\frac{1}{2}(d'-\beta(D'))-k$ *for a positive integer k, then* $\dim|D|\leqslant\frac{1}{2}(d-\beta(D))-k$.

Proof. — Write $D=D'+E$ where E is the base divisor of $|D|$. Assume $D'\in\mathrm{Div}(X)$ is an effective divisor of degree d' satisfying

$$\dim|D'|\leqslant\frac{1}{2}(d'-\delta(D'))+k$$

for a positive integer k. Since $\dim|D|=\dim|D'|$ and E is effective, we have $\delta(D'+E)\leqslant\delta(D')+\deg(E)$. Then $\dim|D|=\dim|D'+E|\leqslant\frac{1}{2}(d'-\delta(D'))+k\leqslant\frac{1}{2}(\deg(D')+\deg(E)-\delta(D')-\deg(E))+k\leqslant\frac{1}{2}(\deg(D'+E)-\delta(D'+E))+k$ proving statement (i).

For statement (ii), the proof is similar using that $\beta(D'+E)\leqslant\beta(D')+\deg(E)$. □

Let D be a special divisor. Recall that $\delta(D)=\delta(K-D)$ and that $\beta(D)=\beta(K-D)$ since $\delta(K)=0$ (see [8, Cor. 4.3]). The next lemma will allow us to study special divisors of degree $\leqslant g-1$.

Lemma 3.6. — *Let $D\in\mathrm{Div}(X)$ be an effective and special divisor of degree d.*

(i) *If* $\dim|D|\leqslant\frac{1}{2}(d-\delta(D))+k$ *for a positive integer k, then* $\dim|K-D|\leqslant\frac{1}{2}(\deg(K-D)-\delta(K-D))+k$.
(ii) *If* $\dim|D|\leqslant\frac{1}{2}(d-\beta(D))-k$ *for a positive integer k, then* $\dim|K-D|\leqslant\frac{1}{2}(\deg(K-D)-\beta(K-D))-k$.

Proof. — It is a straightforward calculation using Riemann-Roch. □

The following lemma concerns covering maps of degree 2 between real curves.

Lemma 3.7. — *Let $\varphi : X \to X'$ be a covering map of degree 2 between two real curves X and X'. If there exists a real point $P \in X'(\mathbb{R})$ such that $\varphi^{-1}(P) = \{P_1, P_2\}$, with P_1 and P_2 real points not contained in the same connected component of $X(\mathbb{R})$, then $\varphi(C_1) = \varphi(C_2) = C$ and $\varphi^{-1}(C) = C_1 \cup C_2$, with C, C_1, C_2 the real connected components containing the points P, P_1, P_2 respectively.*

Proof. — Since C_1 and C_2 are connected, we have $\varphi(C_1) \subseteq C$ and $\varphi(C_2) \subseteq C$. Moreover $\varphi(C_1)$ and $\varphi(C_2)$ are closed connected subsets of C since X is complete. The morphism φ is étale at P, hence there is an open neighbourhood U of P such that for any $Q \in U$ we have $\varphi^*(Q) = Q_1 + Q_2$ with $Q_i \in C_i$ for $i = 1, 2$. In fact, this situation does not change when we run along C since $C_1 \cap C_2 = \emptyset$ and thus C cannot have a branch point. $\square$

We state the main result of the paper.

Theorem 3.8. — *Let D be an effective and special divisor of degree d, and let $k \in \mathbb{N}$. Then either*

$$\dim |D| \leqslant \frac{1}{2}(d - \delta(D)) + k$$

or

$$\dim |D| \leqslant \frac{1}{2}(d - \beta(D)) - k.$$

Moreover D satisfies the first inequality if either $s \leqslant 4k + 1$ or $s \geqslant g - 2k$.

Proof. — We may assume that $|D|$ is base point free and that $0 \leqslant d \leqslant g - 1$ by Lemmas 3.5 and 3.6. Let $r = \dim |D|$. If we can show that

$$r \geqslant \frac{1}{2}(d - \delta(D)) + k + 1 \tag{3.1}$$

and

$$r > \frac{1}{2}(d - \beta(D)) - k \tag{3.2}$$

do not hold simultaneously, we shall have proved the first part of the theorem.

By Lemma 3.4, we have

$$d + \delta(D) \geqslant 2s + 2k. \tag{3.3}$$

Using (3.3) and (3.2) we obtain

$$r > \frac{1}{2}(d - s + \delta(D)) - k \geqslant \frac{1}{2}s. \tag{3.4}$$

By (3.1) and (3.2)

$$\delta(D) \geqslant d - 2r + 2k + 2,$$

and

$$\beta(D) = s - \delta(D) \geqslant d - 2r - 2k + 1.$$

Hence

$$s \geqslant 2d - 4r + 3$$

and using (3.4) we obtain

$$r > \frac{1}{3}d + \frac{1}{2}. \tag{3.5}$$

There are two cases to be looked at:

i) First, D is simple.

In this case, X is mapped birationally by $|D|$ onto a curve of degree d in $\mathbb{P}^r_{\mathbb{R}}$. First of all, $r \geqslant 2$ since a curve of genus 0 does not carry an effective special divisor, so that $g \geqslant 1$ and $r \neq 1$. According to the Castelnuovo bound [3, (2.3) p. 116] for the genus of a curve that admits a birational mapping onto a non-degenerate curve of degree d in $\mathbb{P}^r_{\mathbb{R}}$, we have

$$g \leqslant m\Big(d - 1 - \frac{1}{2}(m+1)(r-1)\Big) \tag{3.6}$$

where $m = \left[\frac{d-1}{r-1}\right]$. By Clifford's theorem, $r \leqslant \frac{d}{2}$. In particular, $d \geqslant 4$. Then from (3.5)

$$m = \left[\frac{d-1}{r-1}\right] \leqslant \left[\frac{d-1}{\frac{1}{3}d - \frac{1}{2}}\right] \leqslant \left[3\frac{d-1}{d-\frac{3}{2}}\right] = 3.$$

If $m = 1$, then $\frac{d-1}{r-1} < 2$, hence $r > \frac{d+1}{2}$, contradicting the Clifford inequality. If $m = 2$ (resp. $m = 3$), replacing in (3.6) and using (3.5), we get $d > g + \frac{1}{2}$ (resp. $d > g$), contradicting the fact that D was supposed of degree $\leqslant g - 1$.

ii) Second, D is not simple.

Consider the map $f : X \to \mathbb{P}^r_{\mathbb{R}}$ associated to $|D|$. Let X' be the normalization of $f(X)$. Then the induced morphism $\varphi : X \to X'$ is a non-trivial covering map of degree $t \geqslant 2$ and there is $D' \in \mathrm{Div}(X')$ such that $|D'|$ is a $g^r_{\frac{d}{t}}$ and such that $D = \varphi^*(D')$.

To finish the first part of the proof, we proceed in three steps:

Step 1. We prove that D' is non-special and that $t = 2$.

If D' were a special divisor on X', then $2r \leqslant \frac{d}{t} < \frac{3r}{t}$ (by (3.5) and Clifford's theorem), contradicting $t \geqslant 2$. Hence D' is non-special and $r = \frac{d}{t} - g'$ by Riemann-Roch, where g' denotes the genus of X'. Using (3.5), we get $0 \leqslant g' < \frac{3r-1}{t} - r$ and thus, $t = 2$.

Step 2. We prove that X' is an M-curve, that $\delta(D) = 2\delta(D') = 2g' + 2$ and that $k = 0$.
Since $\varphi : X \to X'$ is a non-trivial covering map of degree 2 and $D = \varphi^*(D')$, we have $\delta(D) \leqslant 2\delta(D') \leqslant 2g' + 2$ by Lemma 3.7 and by Harnack's inequality. Since $r = \frac{1}{2}(d - 2g')$, using the inequality (3.1) we obtain $2g' + 2k + 2 \leqslant \delta(D)$. Thus, $\delta(D) = 2\delta(D') = 2g' + 2$ and $k = 0$.

Step 3. We prove that $s = 2g' + 2$.
Let $P_1, \ldots, P_{g'+1}$ be real points of the support of D' such that $E = \varphi^*(E')$ satisfies $\delta(E) = 2g' + 2$ where $E' = P_1 + \ldots + P_{g'+1}$. By Riemann-Roch, dim $|E'| \geqslant 1$, hence dim $|E| \geqslant 1$. Using Lemma 3.2, $s = 2g' + 2$.

Summing up, $r = \frac{1}{2}(d-2g') \leqslant \frac{1}{2}d = \frac{1}{2}(d-\beta(D))-k$ and this contradicts (3.2).

We give now some conditions on the number s of connected components of $X(\mathbb{R})$ for which the first inequality is always satisfied.

Let D be a special divisor of degree d. By Lemma 3.4, dim $|D| \leqslant \frac{1}{2}(d - \delta(D)) + k$ if $d + \delta(D) \leqslant 2s + 2k - 1$. By Lemma 3.6 and Lemma 3.4, we have dim $|D| \leqslant \frac{1}{2}(d - \delta(D)) + k$ if $2g - 2 - d + \delta(D) \leqslant 2s + 2k - 1$. Consequently, dim $|D| \leqslant \frac{1}{2}(d-\delta(D))+k$ if $2g-2+2\delta(D) \leqslant 4s+4k-2$. Since $g-1+\delta \leqslant g-1+s$, we get dim $|D| \leqslant \frac{1}{2}(d-\delta(D))+k$ if $g-1+s \leqslant 2s+2k-1$ *i.e.*, if $s \geqslant g - 2k$.

Before ending the proof, we have to remark that $\frac{1}{2}(d-\delta(D))+k$ is always an integer and that $\frac{1}{2}(d - \beta(D)) - k$ is an integer if and only if s is even.

Firstly, assume s is even. We know that D satisfies one of the two inequalities of the theorem. Consequently, dim $|D| \leqslant \frac{1}{2}(d - \delta(D)) + k$ if $\frac{1}{2}(d-\delta(D))+k \geqslant \frac{1}{2}(d-\beta(D))-k = \frac{1}{2}(d-s+\delta(D))-k$ *i.e.*, if $2\delta(D)-s \leqslant 4k$. Since $2\delta(D) - s \leqslant s$, it follows that dim $|D| \leqslant \frac{1}{2}(d - \delta(D)) + k$ if $s \leqslant 4k$ and also if $s \leqslant 4k + 1$, since s is even.

Secondly, assume s is odd. Since dim $|D|$, $\frac{1}{2}(d - \delta(D)) + k$ are integers, but $\frac{1}{2}(d - \beta(D)) - k$ is not an integer, the theorem implies that dim $|D| \leqslant \frac{1}{2}(d - \delta(D)) + k$ if $\frac{1}{2}(d - \delta(D)) + k \geqslant \frac{1}{2}(d - \beta(D)) - k - \frac{1}{2}$. Arguing as in the even case, if $s \leqslant 4k + 1$ then dim $|D| \leqslant \frac{1}{2}(d - \delta(D)) + k$, ending the proof. □

In the previous theorem, the case $k = 0$ gives Theorem A stated in the introduction.

Remark 3.9. — The previous theorem implies Huisman's theorem [10, Th. 3.2]. The result of Theorem A is natural since, for real curves without real points, the inequalities (Clif 1) and (Clif 2) both become the classical Clifford inequality. Moreover, Theorem 3.8 shows that in case $r = \dim |D|$ exceeds the right hand term of (Clif 1) by $k > 0$, then r is exceeded by the right hand term of (Clif 2) by at least $k-1$. It suggests that the inequalities (Clif 1) and (Clif 2) are not completely independent.

Classically, in the theory of special divisors, if the Clifford inequality becomes an equality for a divisor different from 0 and from the canonical divisor, then the curve is hyperelliptic. A real hyperelliptic curve is a real curve X such that $X_{\mathbb{C}}$ is hyperelliptic, *i.e.*, $X_{\mathbb{C}}$ has a g_2^1 (a linear system of dimension 1 and degree 2). If $X(\mathbb{R}) \neq \emptyset$, since this g_2^1 is unique, it is a real linear system *i.e.*, X has a g_2^1 (see [11, Lem. 4.2]). As always, we assume that $g \geqslant 2$.

We will prove below that the inequality (Clif 2) is the best possible complement of Huisman's inequality (Clif 1), since in the following proposition we will obtain examples of special divisors of some real hyperelliptic curves that do not satisfy (Clif 1) and for which equality holds in (Clif 2). This proposition is a reformulation of [11, Th. 4.1, Th. 4.5]. We will restrict to the case $X(\mathbb{R}) \neq \emptyset$ since in the case $X(\mathbb{R}) = \emptyset$, the special divisors clearly satisfy (Clif 1) which is the classical Clifford inequality.

Proposition 3.10. — *Let X be a real hyperelliptic curve such that $X(\mathbb{R}) \neq \emptyset$ and let D be an effective special divisor of degree d on X.*

(i) *If $\delta(g_2^1) = 0$ then*

$$\dim |D| \leqslant \frac{1}{2}(d - \delta(D)). \tag{Clif1}$$

Moreover $\delta(D) \leqslant g - 1$.

(ii) *If $\delta(g_2^1) = 2$ then $s = 2$. Moreover either*

$$\dim |D| \leqslant \frac{1}{2}(d - \delta(D)) \tag{Clif1}$$

or

$$\dim |D| = \frac{1}{2}(d - \beta(D)). \tag{Clif2}$$

If (Clif 1) *is not satisfied and equality holds in* (Clif 2) *then $|D| = rg_2^1$, with $0 < r < g - 1$ and r odd.*

Proof. — By Lemma 3.6, we may assume that $d \leqslant g-1$. Set $r = \dim |D|$.

Firstly, we consider the case that $\delta(g_2^1) = 0$. A consequence of the geometric version of the Riemann-Roch Theorem is that any complete and special g_d^r on $X_{\mathbb{C}}$ is of the form

$$rg_2^1 + D',$$

where D' is an effective divisor of degree $d - 2r$ which has no fixed part under the hyperelliptic involution $\imath$ induced by the g_2^1. Since $\delta(g_2^1) = 0$, we get $\delta(D) \leqslant \deg(D') = d - 2r$. Moreover, (Clif 1) becomes an equality if and only if $D' = \sum_{i=1}^{\delta(D)} P_i$ with one P_i in each component of $X(\mathbb{R})$ where the degree of the restriction of D is odd. Since D is effective, we get $g-1 \geqslant d \geqslant \delta(D)$. Hence (i) of the proposition.

Secondly, we assume that $\delta(g_2^1) = 2$. By Lemma 3.7 or the proof of *Step 3* of the previous theorem, $s = 2$ and the hyperelliptic involution exchanges the two connected components of $X(\mathbb{R})$. If r is even, the proof runs as in the case $\delta(g_2^1) = 0$ and we get the inequality (Clif 1). If r is odd, we again write

$$|D| = rg_2^1 + D',$$

where D' is an effective divisor of degree $d - 2r$ which has no fixed part under the hyperelliptic involution. If $\deg(D') \geqslant 2$, then $\dim |D| = r \leqslant \frac{1}{2}(2r+2-\delta(D)) \leqslant \frac{1}{2}(d-\delta(D))$ since $\delta(D) \leqslant 2$. If $\delta(D') \geqslant 1$ then $\delta(D) \leqslant 1$ since $\delta(rg_2^1) = 2$, and we get $\dim |D| = r \leqslant \frac{1}{2}(2r+1-\delta(D)) \leqslant \frac{1}{2}(d-\delta(D))$. Consequently, if $\deg(D') \geqslant 2$ or if $\delta(D') \geqslant 1$, then the inequality (Clif 1) works. If not, $|D| = rg_2^1$ and

$$\dim |D| = \frac{1}{2}(d - \beta(D)).$$

If $|D| = rg_2^1$, with $0 < r < g-1$ and r odd, then $|K - D| = (g-1-r)g_2^1$ since $|K| = (g-1)g_2^1$. Since g is odd (see the remark after the proof), then $g - 1 - r$ is odd. □

Remark 3.11. — A real hyperelliptic curve such that $\delta(g_2^1) = 2$ is given by the real polynomial equation $y^2 = f(x)$, where f is a monic polynomial of degree $2g+2$, with g odd, and where f has no real roots. The converse also holds (see [11, Prop. 4.3]).

Let D be a special and effective divisor of degree d on a real curve X such that $\dim |D| = \frac{1}{2}(d - \beta(D)) - k$ with $k \in \mathbb{N}$. According to Theorem 3.8, we have $\dim |D| \leqslant \frac{1}{2}(d - \delta(D)) + k + 1$. Similarly, if $\dim |D| = \frac{1}{2}(d - \delta(D)) + k + 1$ with $k \in \mathbb{N}$ then, using Theorem 3.8, $\dim |D| \leqslant \frac{1}{2}(d - \beta(D)) - k$. Consequently, we will say that D is extremal (for the real Clifford inequalities) if $\dim |D| = \frac{1}{2}(d - \beta(D)) - k = \frac{1}{2}(d - \delta(D)) + k + 1$ for some $k \in \mathbb{N}$.

Before making some remarks concerning extremal divisors, we state a consequence of Theorem 3.8.

Proposition 3.12. — *Let D be a special and effective divisor of degree d on a real curve X such that D does not satisfy the inequality (Clif 1). Then*

$$\dim |D| \leqslant \frac{1}{2}\Big(d - \frac{1}{2}(s-2)\Big).$$

Proof. — Let $r = \dim|D|$. Since D does not satisfy the inequality (Clif 1) we have $r = \frac{1}{2}(d-\delta(D)) + k + 1$ for a $k \in \mathbb{N}$. By Theorem 3.8 we get $r \leqslant \frac{1}{2}(d-\beta(D)) - k$. Hence $2k+1 \leqslant \frac{1}{2}(\delta(D)-\beta(D)) = \delta(D) - \frac{1}{2}s$. Consequently, $r = \frac{1}{2}(d-\delta(D)) + k + 1 \leqslant \frac{1}{2}(d - \frac{1}{2}(s-2))$. □

Remark 3.13. — If X is an hyperelliptic curve with $\delta(g_2^1) = 2$ and $|D| = rg_2^1$ with r odd, then $\dim|D| = \frac{1}{2}(d-\beta(D)) = \frac{1}{2}(d-\delta(D)) + 1$ *i.e.*, D is extremal for $k = 0$.

We claim that the following two statements are equivalent:

- The divisor D is extremal.
- The inequality of Proposition 3.12 is an equality for D *i.e.*, $\dim|D| = \frac{1}{2}\big(d - \frac{1}{2}(s-2)\big)$ and D does not satisfy the inequality (Clif 1).

Indeed, if D is extremal then s is even, $2k = \delta(D) - \frac{s}{2} - 1$ and $\dim|D| = \frac{1}{2}\big(d - \frac{1}{2}(s-2)\big)$. Conversely, if $\dim|D| = \frac{1}{2}\big(d - \frac{1}{2}(s-2)\big)$ and if D does not satisfy the inequality (Clif 1), we get that s is even. The integers d and $-\frac{s}{2}+1$ have the same parity and since D does not satisfy the inequality (Clif 1), $\frac{1}{2}(d-\beta(D))-1 = \frac{1}{2}(d-s+\delta(D))-1 \geqslant \frac{1}{2}(d-\delta(D))$ by Theorem A. Hence $\delta(D) - \frac{s}{2} - 1 \geqslant 0$ and $\delta(D) - \frac{s}{2} - 1$ is even since d and $\delta(D)$ have the same parity. We set $k = \frac{1}{2}(\delta(D) - \frac{s}{2} - 1)$ (we have $k \in \mathbb{N}$) and we obtain $\dim|D| = \frac{1}{2}(d-\beta(D)) - k = \frac{1}{2}(d-\delta(D)) + k + 1$ *i.e.*, D is extremal.

Looking at the example of hyperelliptic curves, we may ask the following questions:

Do there exist extremal divisors and what geometric properties does this imply for X?

In case $k = 0$ and D is extremal, does it follow that X is an hyperelliptic curve with $\delta(g_2^1) = 2$?

Before giving an answer to these questions in Theorem 3.18, we state some classical results concerning extremal complex curves and special divisors on complex curves that easily extend to real curves. Recall that a non-degenerate curve X in $\mathbb{P}^r_{\mathbb{R}}$ is called extremal if the genus is maximal with respect to the degree of X (*cf.* [3, p. 117]).

Lemma 3.14 ([7], Lem. 3.1). — *Let D and E be divisors of degree d and e on a curve X of genus g and suppose that $|E|$ is base point free. Then*

$$\ell(D) - \ell(D-E) \leqslant \frac{e}{2}$$

if $2D - E$ is special.

The previous lemma applies in case D is semi-canonical *i.e.*, $2D = K$.

Lemma 3.15 ([5], Lem. 1.2.3). — *Let g_k^1 be a base point free pencil on a curve X of genus g such that $k \neq 0, 1$. Let $n \in \mathbb{N} \setminus \{0\}$ such that $n \leqslant \frac{2g-2}{k(k-1)}$. Then* $\dim ng_k^1 = n$ *and ng_k^1 is base point free.*

Lemma 3.16 ([1], [6] p. 200 and [3] p. 122). — *Let X be an extremal curve of degree $d > 2r$ in $\mathbb{P}_{\mathbb{R}}^r$ $(r \geqslant 3)$. Then one of the followings holds:*

(i) *X lies on a rational normal scroll Y in $\mathbb{P}_{\mathbb{R}}^r$ (Y is real, see* [3] *p. 120). Write $d = m(r-1)+1+\varepsilon$ where $m = [\frac{d-1}{r-1}]$ and $\varepsilon \in \{0, 1, 2, \dots, r-2\}$. The curve $X_{\mathbb{C}}$ has only finitely many base point free pencils of degree $m+1$ (in fact, only 1 for $r > 3$, and 1 or 2 if $r = 3$). These pencils are swept out by the rulings of $Y_{\mathbb{C}}$. Moreover $X_{\mathbb{C}}$ has no g_m^1.*

(ii) *X is the image of a smooth plane curve X' of degree $\frac{d}{2}$ under the Veronese map $\mathbb{P}_{\mathbb{R}}^2 \to \mathbb{P}_{\mathbb{R}}^5$.*

Lemma 3.17. — *A special pencil (i.e., a complete special g_d^1) which does not satisfy* (Clif 1) *is composed by divisors of the form $P_1 + \ldots + P_s$ with $P_1, \dots, P_s$ real points of X such that no two of them belong to the same connected component of $X(\mathbb{R})$.*

Proof. — Let D be an effective special divisor such that $\dim |D| = r = 1 > \frac{1}{2}(d - \delta(D))$. We get $2 = d - \delta(D) + 2k + 2$ for a certain $k \in \mathbb{N}$. Since $d \geqslant \delta(D)$ (D is effective), we have $k = 0$ and $d = \delta(D)$. Now the proof follows from Lemma 3.2. □

We give an answer to the previously asked questions in the following theorem.

Theorem 3.18. — *Let D is an effective and special divisor of degree d. Then D is extremal in the sense that*

$$r = \dim |D| = \frac{1}{2}(d - \delta(D)) + k + 1 = \frac{1}{2}(d - \beta(D)) - k$$

if and only if $k = 0$, X is an hyperelliptic curve with $\delta(g_2^1) = 2$ and $|D| = rg_2^1$ with r odd.

Proof. — If X is an hyperelliptic curve with $\delta(g_2^1) = 2$ and $|D| = rg_2^1$ with r odd, we have already seen that D is extremal.

For the rest of the proof, we assume D is extremal in the sense that

$$r = \dim |D| = \frac{1}{2}(d - \delta(D)) + k + 1 = \frac{1}{2}(d - \beta(D)) - k$$

for a $k \in \mathbb{N}$. Using the remark 3.13, we have $r = \frac{1}{2}\big(d - \frac{1}{2}(s-2)\big)$.

We set $\delta = \delta(D)$ and $\beta = \beta(D)$.

By Lemma 3.5, we may assume $|D|$ is base point free. Remark that if we prove the theorem for the moving part of $|D|$ then D is in fact base point free by Proposition 3.10 (ii) since $\dim |D| > \frac{1}{2}(d-\delta(D))$. It follows from Riemann-Roch and the equality $\dim |D| = \frac{1}{2}(d-\frac{1}{2}(s-2))$ that one may assume, replacing D by $K-D$ if necessary, that $0 \leqslant d \leqslant g-1$. We now copy the proof of Theorem 3.8.

The inequality (3.3) remains valid. The inequalities (3.1) and (3.2) are now equalities

$$r = \frac{1}{2}(d-\delta) + k + 1. \tag{3.7}$$

and

$$r = \frac{1}{2}(d-\beta) - k. \tag{3.8}$$

By (3.8), s is even. The inequality (3.4) becomes $r \geqslant \frac{1}{2}s$. Using (3.7) and (3.8), we obtain $\delta = d - 2r + 2k + 2$, and $\beta = s - \delta = d - 2r - 2k$. Hence $s = 2d - 4r + 2$. Using (3.4) again, we obtain

$$r \geqslant \frac{1}{3}(d+1). \tag{3.9}$$

We have one of the two following possibilities:

(i) D is simple.

The linear system $|D|$ embeds birationally X in $\mathbb{P}^r_{\mathbb{R}}$ as a curve of degree d. Using the facts that $d \leqslant g-1$ and $2r \leqslant d \leqslant 3r-1$ (by (3.9) and Clifford's theorem), we get

$$2 \leqslant \left[2 + \frac{1}{r-1}\right] \leqslant m = \left[\frac{d-1}{r-1}\right] \leqslant \left[3 + \frac{1}{r-1}\right] \leqslant 4.$$

If $m = 2$, replacing in (3.6) and using (3.9), we get $d \geqslant g$, contradicting the fact that D was supposed of degree $\leqslant g-1$. If $m = 3$ or $m = 4$ (if $m = 4$ then $r = 2$), a straightforward calculation (replace in (3.6) and use (3.9)) shows that the Castelnuovo's inequality (3.6) is an equality and that $d = 3r - 1$ and $g = 3r$ is the only possibility. By (3.8) and (3.7), we have $\delta = r + 1 + 2k$ and $s - \delta = r - 1 - 2k$. Hence $s = 2r$. By [2, Lem. 2.9], D is semi-canonical *i.e.*, $2D = K$.

At this moment of the proof there is no contradiction about the existence of such extremal and simple divisor D. The geometric properties of extremal curves (Lemma 3.16) will give this contradiction.

Case 1: $r = 2$.

We identify X via $|D|$ with a smooth plane quintic curve. The contradiction is given by $\delta \geqslant 3$ and the fact that X has a unique pseudo-line (the definition of a pseudo-line is in the next section).

Case 2: Either $r > 5$ or $r = 5$ and X is not a smooth plane curve.

Then $X_{\mathbb{C}}$ has a unique g^1_4 (Lemma 3.16). Hence this g^1_4 is real (*cf.* [11])

i.e., there is an effective divisor E of degree 4 such that $|E| = g^1_4$. By Lemma 3.15, $2E$ is base point free and $\dim |2E| = 2$. Let $r' = \dim |D-2E|$. Since D is semi-canonical, we have $r' \geqslant 5-4 = 1$ by Lemma 3.14. Hence $D-2E$ is special, moreover $\delta(D-2E) = \delta$. We claim that $D-2E$ is also extremal for the same integer k from (3.7) and (3.8): By Lemma 3.14, we have $r' \geqslant r-4$. Hence there is an integer k' such that $r' = \frac{1}{2}(d - \delta(D)) + k + 1 + k' - 4 = \frac{1}{2}(\deg(D-2E) - \delta(D-2E)) + k + k' + 1$ and $r' = \frac{1}{2}(d-\beta(D)) - k + k' - 4 = \frac{1}{2}(\deg(D-2E) - \beta(D-2E)) - k + k'$. If $k' > 0$ then $r' > \frac{1}{2}(\deg(D-2E) - \delta(D-2E)) + k + k'$ and $r' > \frac{1}{2}(\deg(D-2E) - \beta(D-2E)) - k - k'$ and this contradicts Theorem 3.8. Consequently $k' = 0$ and the claim is proved.

Since X is not hyperelliptic, $D-2E$ is simple (see the part of the proof concerning non-simple extremal divisors). Hence $g = 3r'$ and we get a contradiction, since $r' < r$.

Case 3: $r = 5$ and X is a smooth plane curve.
By Lemma 3.16, X is the image of a smooth plane curve of degree 7 under the Veronese embedding $\mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^5_{\mathbb{R}}$. Hence X has a unique very ample $g^2_7 = |E|$. Using Lemma 3.14 (D is semi-canonical), the linear system $|D-E|$ is a g^i_7 with $i \geqslant 2$. Since E calculates the Clifford index of $X_{\mathbb{C}}$ [7, p. 174] (see also [7] for the definition of the Clifford index), we have $|D-E| = |E|$ *i.e.*, $|D| = |2E|$. So $\delta = 0$, which is impossible.

Case 4: $r = 4$.
Similarly to *Case 2*, X has a $g^1_4 = |E|$. Let $D' = D - E$. Applying Lemma 3.14, we get $\dim |D'| \geqslant 2$. By Riemann-Roch $\ell(K - (2D' - E)) = \ell(2D' - E) - 10 + 12 - 1 > 0$. Consequently, according to Lemma 3.14, $\ell(D' - E) = \ell(D - 2E) > 0$. More precisely either $\ell(D-2E) = 1$ or X would have a g^1_3 contradicting Lemma 3.16. So $|D| = |2E + D''|$, with D'' an effective divisor of degree 3. Hence $\delta \leqslant 3$, which is again impossible.

Case 5: $r = 3$ and $X_{\mathbb{C}}$ has a unique $g^1_4 = |E|$.
By Lemma 3.14 and since X has no g^1_3, we get $|D| = |2E|$ and a contradiction on δ.

Case 6: $r = 3$ and $X_{\mathbb{C}}$ has two g^1_4, $|E|$ and $|F|$, which are real.
We know that X lies on a unique quadric S, and $|E|$ and $|F|$ correspond to the rulings of S. More precisely X is of bi-degree $(4,4)$ on $S \cong \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$. By Lemma 3.14 for D and E, and since X has no g^1_3, we get $|D| = |2E|$ or $|D| = |E + F|$. If $|D| = |2E|$ then $\delta = 0$, which gives the contradiction. So we assume $|D| = |E+F|$. Since $s = 6$ and $\delta = 4 + 2k$, we have two possibilities for k: either $k = 0$ or $k = 1$. Since X is of bi-degree $(4,4)$ on S, we have $\delta(E) = 0$ or 2 or 4. If $\delta(E) = 4$ then E does not

satisfy (Clif 1) and Lemma 3.17 gives a contradiction since $\delta(E) < s$. We conclude that $\delta(E)$ is equal either 0 or 2, and it is the same for F. Since $\delta = \delta(E+F) \leqslant \delta(E) + \delta(F)$, we obtain $k = 0$. Since $\delta = \delta(E+F) = 4$, we have $\delta(E) = \delta(F) = 2$. Moreover, there exist two distinct real connected components C and C' of $X(\mathbb{R})$ such that the degree of the restriction of E (resp. F) to C is odd (resp. even) and the degree of the restriction of F (resp. E) to C' is odd (resp. even). The divisors E and F are induced by a basis $\{E', F'\}$ of the first homology group $H_1(S(\mathbb{R}), \mathbb{Z}/2)$ with coefficient in $\mathbb{Z}/2$ of the torus $S(\mathbb{R})$. The conditions on the intersection of C and C' with the rulings imply that $[C] = E'$ and $[C'] = F'$ in $H_1(S(\mathbb{R}), \mathbb{Z}/2)$. Therefore $[C].[C'] = E'.F' = 1$ and $C \cap C' \neq \emptyset$, giving a contradiction since C and C' are distinct real connected components.

Case 7: $r = 3$ and $X_{\mathbb{C}}$ has two g_4^1, $|E|$ and $|\bar{E}|$ which are complex and switched by the complex conjugation.
We argue similarly as in the previous case, but on the complex curve $X_{\mathbb{C}}$. We obtain that $|D| = |E + \bar{E}|$ *i.e.,* $\delta = 0$, which is impossible.

(ii) D is not simple.
Here $|D|$ induces a non-trivial covering map $\varphi : X \to X'$ of degree t on a curve X' of genus g'. There is an effective divisor $D' \in \mathrm{Div}(X')$ such that $|D'|$ is a $g^r_{\frac{d}{2}}$ and such that $D = \varphi^*(D')$. Moreover, following the proof of Theorem 3.8, we see that D' is non-special, X' is an M-curve, $t = 2$, $r = \frac{1}{2}(d - 2g')$, $\delta = 2\delta(D') = 2g' + 2 = s$. The identities (3.7) and (3.8) say that $k = 0$ and that $g' = 0$ *i.e.,* that X is an hyperelliptic curve. By Proposition 3.10 we get that $\delta(g_2^1) = s = 2$ and $|D| = rg_2^1$ with r odd (if r is even, it contradicts (3.7)). $\square$

The Clifford type inequalities from Theorem 3.8 seem to be the best possible since in the previous proof, the extremal cases for these inequalities correspond to extremal Castelnuovo curves. As in the complex situation, these inequalities become equalities in non-trivial cases, only if the curves are hyperelliptic.

From Theorem 3.8, Theorem 3.18, Proposition 3.12 and Remark 3.13, we may derive Theorem B stated in the introduction.

We show now that the inequalities of Theorem B may become equalities.

Example 3.19. — Let X be an hyperelliptic curve X such that $\delta(g_2^1) = 2$. If D is an element of the g_2^1, then D does not satisfy the inequality (Clif 1) and the second inequality of Theorem B (i) is an equality.

Let X be a real trigonal curve, *i.e.,* X has a g_3^1. We assume that $\delta(g_3^1) = 3$ and we take D an element of the g_3^1. By [8, p. 179], such a trigonal curve exists. Then D does not satisfy the inequality (Clif 1), but it gives an example of a divisor for which equality holds in the second inequality of Theorem B (ii).

4. Special real curves in projective spaces

Let $X \subseteq \mathbb{P}^r_{\mathbb{R}}$, $r \geqslant 2$, be a smooth real curve, X is non-degenerate if X is not contained in an hyperplane of $\mathbb{P}^r_{\mathbb{R}}$. We assume, in what follows, that X is non-degenerate. We say that X is special (resp. non-special) if the divisor associated to the sheaf of hyperplane sections $\mathcal{O}_X(1)$ is special (resp. non-special).

Let C be a connected component of $X(\mathbb{R})$. The component C is called a pseudo-line (resp. an oval) if the fundamental class of C is non-trivial (resp. trivial) in $H_1(\mathbb{P}^r_{\mathbb{R}}(\mathbb{R}), \mathbb{Z}/2)$. Equivalently, C is a pseudo-line (resp. an oval) if and only if for each real hyperplane H, $H(\mathbb{R})$ intersects C in an odd (resp. even) number of points, when counted with multiplicities (see [10]).

In this section, we wish to discuss some conditions under which we may bound the genus, the number of pseudo-lines, and the number of ovals of a non-degenerate smooth real curve in $\mathbb{P}^r_{\mathbb{R}}$. For the genus, if X is a smooth plane curve of degree d, we have

$$g = \frac{1}{2}(d-1)(d-2).$$

When $r \geqslant 3$, there is no formula for the genus of X in terms of its degree. The situation is therefore more complicated. However, there is an inequality of Castelnuovo (inequality (3.6)) that we have already seen in the proof of Theorem 3.8.

The following proposition improves the Castelnuovo inequality for non-special real curves of degree d in $\mathbb{P}^r_{\mathbb{R}}$ such that $2r \leqslant d \leqslant 3r$.

Proposition 4.1. — *Let $r \geqslant 2$ be an integer and $X \subseteq \mathbb{P}^r_{\mathbb{R}}$ be a non-degenerate real curve. Let d be the degree of X and δ (resp. β) be the number of pseudo-lines (resp. ovals) of X. Assume $d + 2k < 2r + \delta$ and $d - 2k < 2r + \beta$ for some $k \in \mathbb{N}$. Then X is non-special and*

$$g \leqslant d - r,$$

and equality holds if and only if X is linearly normal i.e., if and only if the restriction map

$$H^0(\mathbb{P}^r_{\mathbb{R}}, \mathcal{O}(1)) \to H^0(X, \mathcal{O}_X(1))$$

is surjective. If $2r \leqslant d$, the inequality $g \leqslant d-r$ improves the Castelnuovo inequality. Under the hypotheses of the proposition, we have $d \leqslant 3r$.

Proof. — Let H be a hyperplane section of X *i.e.,* a divisor obtained by cutting out the curve by a real hyperplane. Then $\dim |H| \geqslant r > \frac{1}{2}(d-\delta(H))+k$ and $\dim |H| \geqslant r > \frac{1}{2}(d-\beta(H))-k$ by the hypotheses. Theorem 3.8 says that H is non-special and by Riemann-Roch,

$$g = d - \dim |H| \leqslant d - r.$$

Clearly, the previous inequality becomes an equality if and only if the map $H^0(\mathbb{P}^r_{\mathbb{R}}, \mathcal{O}(1)) \hookrightarrow H^0(X, \mathcal{O}_X(1))$ is an isomorphism.

If $d \geqslant 2r$, the bound of the Castelnuovo inequality is greater than $d-r$. Under the hypotheses of the proposition and using Harnack inequality, we have $2d < 4r+s \leqslant 4r+g+1$. Since $g \leqslant d-r$, we get $d \leqslant 3r$. □

One may wonder what can be said about the number of pseudo-lines and ovals of X when $X \subseteq \mathbb{P}^r_{\mathbb{R}}$ is a non-special real curve. The following proposition shows that there is no restriction on these numbers except the fact that the number of pseudo-lines should be congruent to $g+r$ modulo 2.

Proposition 4.2. — *Let $r \geqslant 3$ be an integer and X be a real curve. Let δ be an integer $\leqslant s$. There is a smooth embedding $\varphi : X \hookrightarrow \mathbb{P}^r_{\mathbb{R}}$ such that X is non-special curve of degree $g+r$ in $\mathbb{P}^r_{\mathbb{R}}$ and X has δ pseudo-lines provided that $\delta = g+r \mod 2$.*

Proof. — Since $\delta < g+r$ and $\delta = g+r \mod 2$, there is an effective divisor D of degree $g+r$ such that $\delta(D) = \delta$. Choosing D general, D is non-special and D is very ample (see the proof of a theorem of Halphen [9, p. 350]). The morphism associated to $|D|$ gives the result. □

Example 4.3. — Let X be a real curve of genus 6 with 5 real connected components. The previous proposition says that there exist three distinct embeddings of X in $\mathbb{P}^3_{\mathbb{R}}$ such that X is non-special and such that the number of pseudo-lines is successively equal to 1, 3 and 5.

We show now that Theorem A gives an upper bound on the number of ovals or the number of pseudo-lines of special real space curves.

Proposition 4.4. — *Let $X \subseteq \mathbb{P}^r_{\mathbb{R}}$ be a non-degenerate special real curve of degree d. Let δ (resp. β) denote the number of pseudo-lines (resp. ovals) of X. Then, either*

$$\delta \leqslant d - 2r,$$

or

$$\beta \leqslant d - 2r.$$

Moreover, the first inequality is satisfied if s *is equal to* 0, 1, g, $g+1$.

Proof. — By Theorem A, we have two possibilities since the hyperplane section H of X is special.

Firstly, $d-\delta \geqslant 2\dim|H| \geqslant 2r$. Hence $\delta \leqslant d-2r$ and this inequality is satisfied if s is equal to 0, 1, g, $g+1$.

Secondly, $d-\beta \geqslant 2r$ *i.e.*, $\beta \leqslant d-2r$. □

In particular, for semi-canonical curves (curves in a projective space such that twice the hyperplane section is the canonical divisor), the above proposition gives:

Corollary 4.5. — *Let* $X \subseteq \mathbb{P}^r_{\mathbb{R}}$ *be a non-degenerate semi-canonical curve of genus* g. *Let* δ *(resp.* β*) denote the number of pseudo-lines (resp. ovals) of* X. *Then, either* $\delta \leqslant g-1-2r$ *or* $\beta \leqslant g-1-2r$. *Moreover, the first inequality is satisfied if* s *is equal to* 0, 1, g, $g+1$.

Example 4.6. — Let X be a smooth intersection of two cubics in $\mathbb{P}^3_{\mathbb{R}}$. By [7, Thm. 3.6 and p. 192], X is a semi-canonical curve of genus 10 such that $|\mathcal{O}_X(1)|$ is a special g^3_9. Let δ (resp. β) denote the number of pseudo-lines (resp. ovals) of X. By Proposition 4.4, we have either $\beta \leqslant 3$ or $\delta \leqslant 3$. Moreover, the first inequality is satisfied if s is equal to 0, 1, 10, 11. The result is non-trivial only if $s \geqslant 8$, for example it says that the distribution $\delta = 4$, $\beta = 4$ is not allowed.

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