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HARMONIC MORPHISMS BETWEEN WEYL SPACES AND TWISTORIAL MAPS II

by Eric LOUBEAU & Radu PANTILIE (*)

This paper is dedicated to the memory of James Eells.

ABSTRACT. — We define, on smooth manifolds, the notions of almost twistorial structure and twistorial map, thus providing a unified framework for all known examples of twistor spaces. The condition of being a harmonic morphism naturally appears among the geometric properties of submersive twistorial maps between low-dimensional Weyl spaces endowed with a nonintegrable almost twistorial structure due to Eells and Salamon. This leads to the twistorial characterisation of harmonic morphisms between Weyl spaces of dimensions four and three. Also, we give a thorough description of the twistorial maps with one-dimensional fibres from four-dimensional Weyl spaces endowed with the almost twistorial structure of Eells and Salamon.

RÉSUMÉ. — Nous définissons, sur les variétés lisses, les notions de structure presque twistorielle et d'application twistorielle, fournissant ainsi un cadre unifié pour tous les exemples d'espace de twisteurs. La condition de morphisme harmonique apparait naturellement dans les propriétés géométriques des applications twistorielles submersives entre espaces de Weyl de faible dimension, équipés d'une structure presque twistorielle non-intégrable due à Eells et Salamon. Ceci mène à la caractérisation twistorielle des morphismes harmoniques entre espaces de Weyl de dimension quatre et trois. De plus, nous donnons une description complète des applications twistorielles à fibres unidimensionelles d'un espace de Weyl de dimension quatre, équipé de la structure presque twistorielle non-intégrable due à Eells et Salamon.

Introduction

A predominant theme in the theory of harmonic maps is their relationship to holomorphicity. Right from their inception, harmonic maps have been recognised by Eells and Sampson to include holomorphic maps between Kähler manifolds, and a few years later, Lichnerowicz distinguished two sub-classes of almost Hermitian manifolds for which this also holds.

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A far harder task is to find conditions forcing harmonic maps to be holomorphic. The clearest situation is between two-spheres since harmonicity then implies conformality, but, in general, additional conditions are required, on the map or on curvature.

From a Riemann surface, or a complex projective space, into a compact irreducible Hermitian symmetric space harmonic stability ensures holomorphicity [4], [12].

In the broader context of compact Kähler manifolds, Siu [19] used a $\partial\overline{\partial}$-Bochner argument to prove, under strong negativity of the target curvature, the holomorphicity of rank four harmonic maps, from which he deduced the biholomorphicity of compact Kähler manifolds of the same homotopy type.

Whilst these results are of great importance in understanding harmonic maps, this scheme bears its own natural limits, if only because it requires the presence of complex structures, with their topological consequences.

To overcome this hurdle, one can replace the codomain with an adequate bundle admitting a natural complex structure, such that harmonicity of any map is given by holomorphicity of its lift to this bundle.

This strategy, which could be traced to Calabi and even Weierstrass, was put into effect by Eells and Salamon [7], who modified a well-known twistor construction of Atiyah, Hitchin and Singer (see Example 3.6, below) to define, in our terminology, an *almost twistorial structure* on any oriented four-dimensional Riemannian manifold (see Example 3.7). This yields a bijective correspondence between conformal harmonic maps and holomorphic curves (see Proposition 4.7). These ideas were extensively pursued by Bryant [3], and Burstall and Rawnsley [5] for (even-dimensional) Riemannian symmetric spaces.

The success of this strategy leads naturally to considering lifts on both the domain and the codomain, hence removing any need of pre-existing almost complex structures. The objective is two-fold: firstly, show that the existence of a holomorphic lift implies harmonicity and, secondly, find natural conditions under which harmonic maps admit holomorphic lifts.

As holomorphic maps are closed under composition, it seems that harmonic morphisms will have an important role in this programme.

Besides, the very conformal nature of their characterization makes Weyl geometry an ideal framework for their study, but it also turns out to provide examples of twistor spaces in dimension two, three and four.

In the complex category, a twistor, that is, a point of the twistor space of a (complex) manifold M, determines an immersed submanifold of M. For example, the twistor space of an anti-self-dual complex-conformal four-dimensional manifold (M^4, c) is the space of self-dual (immersed) surfaces of (M^4, c); also, the twistor space of a three-dimensional Einstein–Weyl space (M^3, c, D) is the space of coisotropic surfaces of (M^3, c) which are totally-geodesic with respect to D (see [16]).

A complex analytic map $\varphi : M \to N$ between manifolds endowed with twistorial structures is twistorial if it maps (some of the) twistors on M to twistors on N (see [16]).

In this paper we extend the notions of almost twistorial structure and twistorial map, to the smooth category. We show that, *in the smooth category, a twistor on a manifold M (endowed with a twistorial structure) is a pair (R, J) where R is an immersed submanifold of M and J is a linear CR-structure on the normal bundle of R in M* (Remark 3.2). These submanifolds may well just be points. For example, the twistor space of a four-dimensional anti-self-dual conformal manifold (M^4, c) is formed of pairs (x, J_x) where $x \in M$ and J_x is a positive orthogonal complex structure on (T_xM, c_x).

On the other hand, the twistor space of a three-dimensional Einstein–Weyl space (see Example 3.3) is formed of pairs (γ, J) where γ is a geodesic and J is an orthogonal complex structure on the normal bundle of γ.

As in the complex category, a (smooth) map $\varphi : M \to N$ between manifolds endowed with twistorial structures is twistorial if it maps twistors on M to twistors on N (Definition 4.1, Remark 4.2). It follows that twistorial maps naturally generalize holomorphic maps. Examples of twistorial maps have been previously used to obtain constructions of Einstein and anti-self-dual manifolds (see [6], [15], [16] and the references therein).

In Section 1, we recall a few basic facts on harmonic morphisms between Weyl spaces. Section 2 is preparatory for Sections 3 and 4, where we give the definitions and some examples of almost twistorial structures and twistorial maps, on smooth manifolds.

In the first part [11] of this work, we introduced the notion of harmonic morphism between (complex-)Weyl spaces and we continued the study (initiated in [16]) of the relations between harmonic morphisms and twistorial maps. As all of the known examples of complex analytic almost twistorial structures have smooth versions, all of the main results of [11] have ‘real’ versions. But not all smooth almost twistorial structures come from complex analytic almost twistorial structures. The first example of such an almost twistorial structure is the almost twistorial structure of Eells and Salamon, mentioned above. We introduce similar almost twistorial structures (Examples 3.5 and 5.2) with respect to which a map (between Weyl spaces of dimensions four and three) is twistorial if and only if it is a harmonic morphism (Theorem 5.4(i)); this implies that *there exists a bijective correspondence between one-dimensional foliations on* (M^4, c, D), *which are locally defined by harmonic morphisms, and certain almost CR-structures on the bundle of positive orthogonal complex structures on* (M^4, c).

In Section 5, we also give the necessary and sufficient conditions for a map with one-dimensional fibres from a four-dimensional Weyl space endowed with the almost twistorial structure of Eells and Salamon to be twistorial (Theorem 5.6). It follows that, all such twistorial maps are harmonic morphisms. Also, a generalized monopole equation (Definition 4.4(1)) is naturally involved.

1. Harmonic morphisms

In this section all manifolds and maps are assumed smooth.

Let M^m be a manifold of dimension m. If m is even then we denote by L the line bundle associated to the frame bundle of M^m through the morphism of Lie groups $\rho_m : \mathrm{GL}(m, \mathbb{R}) \to (0, \infty)$, $\rho_m(a) = |\det a|^{1/m}$, $\big(a \in \mathrm{GL}(m, \mathbb{R})\big)$; obviously, L is oriented. If m is odd then we denote by L the line bundle associated to the frame bundle of M^m through the morphism of Lie groups $\rho_m : \mathrm{GL}(m, \mathbb{R}) \to \mathbb{R}^*$, $\rho_m(a) = (\det a)^{1/m}$, $\big(a \in \mathrm{GL}(m, \mathbb{R})\big)$; obviously, $L^* \otimes TM$ is an oriented vector bundle. We say that L is *the line bundle of* M^m (cf. [6]).

Similar considerations apply to any vector bundle.

Let $\varphi : M \to N$ be a submersion and let $\mathscr{H}$ be a distribution on M, complementary to the fibres of φ. Let $L_{\mathscr{H}}$ and L_N be the line bundles of $\mathscr{H}$ and N, respectively. As $L_{\mathscr{H}}^{2n} = \big(\Lambda^n(\mathscr{H})\big)^2$ and $L_N^{2n} = \big(\Lambda^n(TN)\big)^2$, where $n = \dim N$, the differential of φ induces a bundle map Λ from $L_{\mathscr{H}}^2$ to L_N^2. If n is odd then we also have a bundle map λ from $L_{\mathscr{H}}$ to L_N; obviously, $\Lambda = \lambda^2$. Furthermore, if n is odd, $\mathrm{d}\varphi$ and λ induce a bundle map from $L_{\mathscr{H}}^* \otimes \mathscr{H}$ to $L_N^* \otimes TN$, which will also be denoted by $\mathrm{d}\varphi$; note that, $\mathrm{d}\varphi : L_{\mathscr{H}}^* \otimes \mathscr{H} \to L_N^* \otimes TN$ is orientation preserving, on each fibre.

Let c be a conformal structure on M; that is, c is a section of $L^2 \otimes \big(\odot^2 T^*M\big)$ which is 'positive-definite'; that is, for any positive section s^2 of L^2 we have $c = s^2 \otimes g_s$, where g_s is a Riemannian metric on M; then g_s is a *representative* of c. Therefore, c corresponds to a Riemannian metric on the vector bundle $L^* \otimes TM$ (see [6]). Obviously, c also corresponds to a reduction of the frame bundle of M to $\mathrm{CO}(m, \mathbb{R})$, where $m = \dim M$; the total space of the reduction corresponding to c is formed of the *conformal frames* on (M, c).

If $\dim M$ is odd then local sections of L correspond to *oriented local representatives* of c; that is, (local) representatives of c, on some oriented open set of M. Let $\mathscr{H}$ be a distribution on M. Then c induces a conformal structure $c|_{\mathscr{H}}$ on $\mathscr{H}$ and, it follows that, we have an isomorphism, which depends of c, between L^2 and $L^2_{\mathscr{H}}$.

Definition 1.1 (cf. [2]). — *Let (M, c_M) and (N, c_N) be conformal manifolds. A map $\varphi : (M, c_M) \to (N, c_N)$ is* horizontally weakly conformal *if at each point $x \in M$, either $\mathrm{d}\varphi_x = 0$ or $\mathrm{d}\varphi_x|_{(\ker \mathrm{d}\varphi_x)^\perp}$ is a conformal linear isomorphism from $\big((\ker \mathrm{d}\varphi_x)^\perp, (c_M)_x|_{(\ker \mathrm{d}\varphi_x)^\perp}\big)$ onto $\big(T_{\varphi(x)}N, (c_N)_{\varphi(x)}\big)$.*

Let (M, c) be a conformal manifold. A connection D on M is *conformal* if $Dc = 0$; equivalently, D is the covariant derivation of a principal connection on the bundle of conformal frames on (M, c). If D is torsion-free then it is called a *Weyl connection* and (M, c, D) is a *Weyl space*.

Definition 1.2 (cf. [2]). — *(i) Let (M, c, D) be a Weyl space. A* harmonic function*, on (M, c, D), is a function f, (locally) defined on M, such that* $\mathrm{trace}_c(D\mathrm{d}f) = 0$.

(ii) A map $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ between Weyl spaces is a harmonic map *if* $\mathrm{trace}_c(D\mathrm{d}\varphi) = 0$, *where D is the connection on $\varphi^*(TN) \otimes T^*M$ induced by D^M, D^N and φ.*

(iii) A map $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ between Weyl spaces is a harmonic morphism *if for any harmonic function $f : V \to \mathbb{R}$, on (N, c_N, D^N), with V an open set of N such that $\varphi^{-1}(V)$ is nonempty, $f \circ \varphi : \varphi^{-1}(V) \to \mathbb{R}$ is a harmonic function, on (M, c_M, D^M).*

Obviously, any harmonic function is a harmonic map and a harmonic morphism, if $\mathbb{R}$ is endowed with its conformal structure and canonical connection.

The following result is basic for the theory of harmonic morphisms (see [11], [2]).

Theorem 1.3. — *A map between Weyl spaces is a harmonic morphism if and only if it is a harmonic map which is horizontally weakly conformal.*

2. Complex distributions

Unless otherwise stated, all manifolds and maps are assumed smooth.

Definition 2.1. — *A* complex distribution *on a manifold M is a complex subbundle $\mathscr{F}$ of $T^{\mathbb{C}}M$ such that $\dim(\mathscr{F}_x \cap \overline{\mathscr{F}}_x)$, $(x \in M)$, is constant. If $\mathscr{F} \cap \overline{\mathscr{F}}$ is the zero bundle then $\mathscr{F}$ is an* almost CR-structure *on M, and $(M, \mathscr{F})$ an* almost CR-manifold.

Example 2.2. — 1) Let F be an almost f-structure on a manifold M; that is, F is a section of $\mathrm{End}(TM)$ such that $F^3 + F = 0$ [21]. We denote by T^0M, $T^{1,0}M$, $T^{0,1}M$ the eigendistributions of $F^{\mathbb{C}} \in \Gamma\big(\mathrm{End}(T^{\mathbb{C}}M)\big)$ corresponding to the eigenvalues 0, i, $-$i, respectively. Then $T^0M \oplus T^{0,1}M$ is a complex distribution on M and $T^{0,1}M$ is an almost CR-structure on M. We say that $T^0M \oplus T^{0,1}M$ is *the complex distribution associated to* F; similarly, $T^{0,1}M$ is *the almost CR-structure associated to* F.

2) Let M be endowed with a complex distribution $\mathscr{F}$ and let $N \subseteq M$ be a submanifold. Suppose that $\dim(T^{\mathbb{C}}_xN \cap \mathscr{F}_x)$ and $\dim(T^{\mathbb{C}}_xN \cap \mathscr{F}_x \cap \overline{\mathscr{F}}_x)$, $(x \in N)$, are constant. Then $T^{\mathbb{C}}N \cap \mathscr{F}$ is a complex distribution on N which we call *the complex distribution induced by* $\mathscr{F}$ *on* N.

In particular, if N is a real hypersurface in the complex manifold M then $T^{\mathbb{C}}N \cap T^{0,1}M$ is an almost CR-structure on N.

Next, we define the notion of holomorphic map between manifolds endowed with complex distributions.

Definition 2.3. — *Let $\mathscr{F}^M$ and $\mathscr{F}^N$ be complex distributions on M and N, respectively. A map $\varphi : (M, \mathscr{F}^M) \to (N, \mathscr{F}^N)$ is* holomorphic *if $\mathrm{d}\varphi(\mathscr{F}^M) \subseteq \mathscr{F}^N$.*

A map $\varphi : (M, F^M) \to (N, F^N)$ between manifolds endowed with f-structures is holomorphic *if $\varphi : (M, \mathscr{F}^M) \to (N, \mathscr{F}^N)$ is holomorphic where $\mathscr{F}^M$ and $\mathscr{F}^N$ are the complex distributions, on M and N, associated to F^M and F^N, respectively.*

Let F be an almost f-structure on M and J an almost complex structure on N. Then, a map $\varphi : (M, F) \to (N, J)$ is holomorphic if and only if $\mathrm{d}\varphi \circ F = J \circ \mathrm{d}\varphi$.

Definition 2.4. — *Let M be a manifold. A subbundle E of $T^{\mathbb{C}}M$ is* integrable *if for any sections X, Y of E we have that $[X, Y]$ is a section of E.*

A CR-structure *is an integrable almost CR-structure; a* CR-manifold *is a manifold endowed with a CR-structure.*

An almost f-structure is integrable *if its associated complex distribution is integrable; an* f-structure *is an integrable almost f-structure.*

Note that a complex distribution $\mathscr{F}^M$ on M is integrable if and only if for any point $x \in M$, there exists a holomorphic submersion $\varphi : (U, \mathscr{F}^M|_U) \to (N, \mathscr{F}^N)$, from some open neighbourhood U of x, onto some CR-manifold $(N, \mathscr{F}^N)$, such that $\ker \mathrm{d}\varphi = \big(\mathscr{F}^M \cap \overline{\mathscr{F}^M}\big)|_U$. If $U = M$ and φ has connected fibres then $\mathscr{F}^M$ is called *simple*; then, obviously, up to a CR-diffeomorphism, φ is unique; we call it the *holomorphic submersion corresponding to $\mathscr{F}^M$*.

Let E be a complex vector bundle over M endowed with a complex linear connection ∇ (that is, ∇ is a complex-linear map $\Gamma(E) \to \Gamma\big(\mathrm{Hom}_{\mathbb{C}}\,(T^{\mathbb{C}}M, E)\big)$ such that $\nabla(sf) = (\nabla s)f + s \underset{\mathbb{C}}{\otimes} \mathrm{d}f$ for any section s of E and any complex-valued function f on M; equivalently, we have $\nabla^{\mathbb{C}} = \nabla \oplus \overline{\nabla}$, with respect to the decomposition $E^{\mathbb{C}} = E \oplus \overline{E}$).

Let $\pi : \mathrm{Gr}_k(E) \to M$ be the bundle of complex vector subspaces of complex dimension k of E; we shall denote by $\mathscr{H}\big(\subseteq T(\mathrm{Gr}_k(E))\big)$ the connection induced by ∇ on $\mathrm{Gr}_k(E)$. Note that, as $\mathrm{Gr}_k(E)$ is a bundle whose typical fibre is a complex manifold and its structural group a complex Lie group acting holomorphically on the fibre, we have $(\ker \mathrm{d}\pi)^{\mathbb{C}} = (\ker \mathrm{d}\pi)^{1,0} \oplus (\ker \mathrm{d}\pi)^{0,1}$.

The following result, which we do not imagine to be new, will be useful later on. We omit the proof.

Proposition 2.5. — *Let p be a section of* $\mathrm{Gr}_k(E)$ *(equivalently, p is a complex vector subbundle of complex rank k of E). Let $X \in T^{\mathbb{C}}_{x_0}M$ for some $x_0 \in M$; denote by $p_0 = p(x_0)$.*

(a) *The following assertions are equivalent:*

(i) $(\mathrm{d}p)^{\mathbb{C}}\,(X) \in \mathscr{H}^{\mathbb{C}}_{p_0}$.

(ii) $\nabla^{\mathbb{C}}_X s \in p^{\mathbb{C}}_0$ *for any section s of $p^{\mathbb{C}}$.*

(b) *The following assertions are equivalent:*

(iii) $(\mathrm{d}p)^{\mathbb{C}}\,(X) \in \mathscr{H}^{\mathbb{C}}_{p_0} \oplus (\ker \mathrm{d}\pi)^{0,1}_{p_0}$.

(iv) $\nabla_X s \in p_0$ *for any section s of p.*

Note that, if X is real (that is, $X \in TM$) then the assertions (i),…,(iv), of Proposition 2.5, are equivalent.

3. Almost twistorial structures

In this section, we define, in the smooth category, the notion of (almost) twistorial structure.

Definition 3.1 (cf. [17], [16]). — *An* almost twistorial structure, *on a manifold M, is a quadruple $\tau = (P, M, \pi, \mathscr{F})$ where $\pi : P \to M$ is a locally trivial fibre space and $\mathscr{F}$ is a complex distribution on P which induces almost complex structures on each fibre of π; if $\mathscr{F}$ is induced by an almost f-structure F then, also, (P, M, π, F) is called an almost twistorial structure.*

The almost twistorial structure $\tau = (P, M, \pi, \mathscr{F})$ *is* integrable *if* $\mathscr{F}$ *is integrable. A* twistorial structure *is an integrable almost twistorial structure; the leaf space of* $\mathscr{F} \cap \overline{\mathscr{F}}$ *is called the* twistor space *of* τ*.*

A twistorial structure $\tau = (P, M, \pi, \mathscr{F})$ *is called* simple *if* $\mathscr{F}$ *is simple; if* τ *is simple with* $\varphi : (P, \mathscr{F}) \to Z$ *the corresponding holomorphic submersion then* $\mathrm{d}\varphi(\mathscr{F})$ *is a CR-structure on* Z*.*

Remark 3.2. — Let $\tau = (P, M, \pi, \mathscr{F})$ be a twistorial structure and let Z be its twistor space. Each $z \in Z$ determines a pair (R_z, J_z) where R_z is an immersed submanifold of M and J_z is a linear CR-structure on the normal bundle of R_z.

Indeed, let $R_z = \varphi^{-1}(z)$, where $\varphi : P \to Z$ is the (continuous) projection whose fibres are the leaves of $\mathscr{F} \cap \overline{\mathscr{F}}$. Then $\pi|_{R_z} : R_z \to M$ is an immersion. Also, let $(\ker \mathrm{d}\pi)^{0,1} = \mathscr{F} \cap (\ker \mathrm{d}\pi)^{\mathbb{C}}$. Then the restriction to R_z of the quotient of $\mathscr{F}$ through $TR_z \oplus (\ker \mathrm{d}\pi)^{0,1}$ defines a linear CR-structure J_z on the normal bundle $(\pi|_{R_z})^*(TM)/TR_z$ of R_z in M.

If (M, J) is a complex manifold then, obviously, (M, M, Id_M, J) is a twistorial structure whose twistor space is (M, J).

Also, the CR manifolds constructed in [10] and [18] can be easily seen to be natural examples of twistor spaces.

Next, we formulate the examples of almost twistorial structures with which we shall work.

Example 3.3 ([9], [17]). — Let (M^3, c, D) be a three-dimensional Weyl space. Let $\pi : P \to M$ be the bundle of nonzero skew-adjoint f-structures on (M^3, c). Obviously, P is also the bundle of nonzero skew-adjoint f-structures on the oriented Riemannian bundle $(L^* \otimes TM, c)$. Therefore, P is isomorphic to the sphere bundle of $(L^* \otimes TM, c)$. In particular, the typical fibre and the structural group of P are $\mathbb{C}P^1$ and $\mathrm{PGL}(2, \mathbb{C})$, respectively.

We could also define P as follows: firstly, note that, there exists a unique oriented Riemannian structure, on the vector bundle E of skew-adjoint endomorphisms on (M, c), with respect to which $[A, B] = A \times B$, for any $A, B \in E$; then P is the sphere bundle of E.

The bundle P is also isomorphic with the bundle of oriented two-dimensional subspaces on M^3. Therefore, there exists a bijective correspondence between one-dimensional foliations on M^3, with oriented orthogonal complement, and almost f-structures on (M^3, c). Furthermore, under this bijection, conformal one-dimensional foliations correspond to (integrable) f-structures.

Let k be a section of L^*. We define a conformal connection ∇ on (M,c) by $\nabla_X Y = D_X Y + \frac{1}{2}\, kX \times Y$ for any vector fields X and Y on M. We say that ∇ is *the connection associated to D and k.*

Let $\mathscr{H} \subseteq TP$ be the connection induced by ∇ on P. We denote by $\mathscr{H}^0$, $\mathscr{H}^{1,0}$ the subbundles of $\mathscr{H}^{\mathbb{C}}$ such that, at each $p \in P$, the subspaces $\mathscr{H}^0_p$, $\mathscr{H}^{1,0}_p \subseteq \mathscr{H}^{\mathbb{C}}_p$ are the horizontal lifts of the eigenspaces of $p^{\mathbb{C}} \in \mathrm{End}(T^{\mathbb{C}}_{\pi(p)}M)$ corresponding to the eigenvalues 0, i, respectively.

We define the almost f-structure $\mathcal{F}$ on P with respect to which $T^0P = \mathscr{H}^0$ and $T^{1,0}P = (\ker \mathrm{d}\pi)^{1,0} \oplus \mathscr{H}^{1,0}$. Then $\tau = (P, M, \pi, \mathcal{F})$ is an almost twistorial structure on M which is integrable if and only if (M^3, c, D) is Einstein–Weyl (see [13])

Remark 3.4. — Let (M^3, c) be a three-dimensional conformal manifold. Let $\pi : P \to M$ be the bundle of nonzero skew-adjoint f-structures on (M^3, c).

Let D be a Weyl connection on (M^3, c) and let ∇ be a conformal connection on (M^3, c). The following assertions are equivalent:

(i) ∇ and D induce, by applying the construction of Example 3.3 (with $\mathscr{H}$ the connection on P induced by ∇ and D, respectively), the same almost f-structure on P.

(ii) ∇ and D are projectively equivalent.

(iii) There exists a section k of L^* such that ∇ is the connection associated to D and k.

Example 3.5 (cf. [7]). — Under the same hypotheses as in Example 3.3, we define the almost f-structure $\mathcal{F}'$ on P with respect to which we have $T^0P = \mathscr{H}^0$ and $T^{1,0}P = (\ker \mathrm{d}\pi)^{0,1} \oplus \mathscr{H}^{1,0}$. Then $\tau' = (P, M, \pi, \mathcal{F}')$ is a nonintegrable (that is, always not integrable) almost twistorial structure on M.

Example 3.6 ([1]). — Let (M^4, c, D) be a four-dimensional oriented Weyl space. Let $\pi : P \to M$ be the bundle of positive orthogonal complex structures on (M^4, c). Obviously, P is also the bundle of positive orthogonal complex structures on the oriented Riemannian bundle $(L^* \otimes TM, c)$. Let E be the adjoint bundle of $(L^* \otimes TM, c)$ and let $*_c$ be the involution of E induced by the Hodge star-operator of $(L^* \otimes TM, c)$, under the isomorphism $E = \Lambda^2(L \otimes T^*M)$. Then $E = E_+ \oplus E_-$ where $E_\pm$ is the vector bundle, of rank three, formed of the eigenvectors of $*_c$ corresponding to the eigenvalue ± 1. There exists a unique oriented Riemannian structure $< \cdot, \cdot >$ on $E_\pm$ with respect to which $AB = - < A, B > \mathrm{Id}_{TM} \pm A \times B$ for any $A, B \in E_\pm$. It follows that P is the sphere bundle of E_+.

Similarly to Example 3.3, there exists a bijective correspondence between two-dimensional distributions $\mathscr{F}$ on M^4, with oriented orthogonal complements, and pairs (J,K) of almost Hermitian structures on (M^4,c), with J positive and K negative, such that $J|_{\mathscr{F}^\perp} = K|_{\mathscr{F}^\perp}$.

Let $\mathscr{H} \subseteq TP$ be the connection induced by D on P. We denote by $\mathscr{H}^{1,0}$ the subbundle of $\mathscr{H}^{\mathbb{C}}$ such that, at each $p \in P$, the subspace $\mathscr{H}^{1,0}_p \subseteq \mathscr{H}^{\mathbb{C}}_p$ is the horizontal lift of the eigenspace of $p^{\mathbb{C}} \in \mathrm{End}(T^{\mathbb{C}}_{\pi(p)}M)$ corresponding to the eigenvalue i. We define the almost complex structure $\mathcal{J}$ on P with respect to which $T^{1,0}P = (\ker \mathrm{d}\pi)^{1,0} \oplus \mathscr{H}^{1,0}$. Then $(P,M,\pi,\mathcal{J})$ is an almost twistorial structure on M which is integrable if and only if (M^4,c) is anti-self-dual.

Example 3.7 ([7]). — Let (M^4,c,D) be a four-dimensional oriented Weyl space. With the same notations as in Example 3.6, let $\mathcal{J}'$ be the almost complex structure on P with respect to which $T^{1,0}P = (\ker \mathrm{d}\pi)^{0,1} \oplus \mathscr{H}^{1,0}$. Then $\tau' = (P,M,\pi,\mathcal{J}')$ is a nonintegrable almost twistorial structure on M.

4. Twistorial maps

We start this section with the definition of twistorial maps (cf. [17], [16]).

Definition 4.1. — *Let $\tau_M = (P_M, M, \pi_M, \mathscr{F}^M)$ and $\tau_N = (P_N, N, \pi_N, \mathscr{F}^N)$ be almost twistorial structures and let $\varphi : M \to N$ be a map. Suppose that there exist a locally trivial fibre subspace $\pi_{M,\varphi} : P_{M,\varphi} \to M$ of $\pi_M : P_M \to M$ and a map $\Phi : P_{M,\varphi} \to P_N$ with the properties:*

1) $\mathscr{F}^M$ induces a complex distribution $\mathscr{F}^{M,\varphi}$ on $P_{M,\varphi}$ and almost complex structures on each fibre of $\pi_{M,\varphi}$ such that $\mathrm{d}\pi_M(\mathscr{F}^M_p) = \mathrm{d}\pi_{M,\varphi}(\mathscr{F}^{M,\varphi}_p)$, for any $p \in P_{M,\varphi}$.

2) $\varphi \circ \pi_{M,\varphi} = \pi_N \circ \Phi$.

Then $\varphi : (M,\tau_M) \to (N,\tau_N)$ is a twistorial map *(with respect to Φ) if the map $\Phi : (P_{M,\varphi}, \mathscr{F}^{M,\varphi}) \to (P_N, \mathscr{F}^N)$ is holomorphic. If $\mathscr{F}^{M,\varphi}$ and $\mathscr{F}^N$ are simple complex distributions, with $(P_{M,\varphi}, \mathscr{F}^{M,\varphi}) \to Z_{M,\varphi}$ and $(P_N, \mathscr{F}^N) \to Z_N$, respectively, the corresponding holomorphic submersions onto CR-manifolds, then Φ induces a holomorphic map $Z_\varphi : Z_{M,\varphi} \to Z_N$ which is called the* twistorial representation *of φ.*

Remark 4.2. — With the same notations as in Definition 4.1, we have that $\tau_{M,\varphi} = (P_{M,\varphi}, M, \pi_{M,\varphi}, \mathscr{F}^{M,\varphi})$ is an almost twistorial structure on M. Obviously, $\tau_{M,\varphi}$ is integrable if τ_M is integrable. Then from (1) it follows that the twistor space $Z_{M,\varphi}$ of $\tau_{M,\varphi}$ is a topological subspace of the twistor space Z_M of τ_M. Moreover, for any $z \in Z_{M,\varphi}$ the pair (R_z, J_z) of Remark 3.2 applied to $\tau_{M,\varphi}$ is equal to the pair determined by z as a point of Z_M. Furthermore, as Φ is holomorphic, *we have that* $\varphi(R_z) \subseteq R_{Z_\varphi(z)}$ *and the map between the normal bundles of* R_z *and* $R_{Z_\varphi(z)}$*, induced by the differential of* φ*, intertwines* J_z *and* $J_{Z_\varphi(z)}$ (here, we have identified R_z with $\pi_{M,\varphi}(R_z)$ and, similarly, for $R_{Z_\varphi(z)}$).

If τ_M is simple then $\tau_{M,\varphi}$ is also simple and $Z_{M,\varphi}$ is a submanifold of Z_M. Moreover, if we denote by $\mathcal{C}_M$ and $\mathcal{C}_{M,\varphi}$ the CR-structures of Z_M and $Z_{M,\varphi}$, respectively, then $\mathcal{C}_{M,\varphi} = \mathcal{C}_M \cap T^{\mathbb{C}} Z_{M,\varphi}$.

Let (M, J_M) and (N, J_N) be complex manifolds and let $\tau_M = (M, M, \mathrm{Id_M}, J_M)$ and $\tau_N = (N, N, \mathrm{Id}_N, J_N)$ be the corresponding twistorial structures. Then it is obvious that a map $\varphi : (M, \tau_M) \to (N, \tau_N)$ is twistorial (with respect to φ) if and only if $\varphi : (M, J_M) \to (N, J_N)$ is holomorphic.

Example 4.3 ([9], [6]). — Let (M^4, c_M) be a four-dimensional oriented conformal manifold and let (N^3, c_N, D^N) be a three-dimensional Weyl space. Let $\tau_M = (P_M, M, \pi_M, \mathcal{J})$ be the almost twistorial structure of Example 3.6, associated to (M^4, c_M), and let $\tau_N = (P_N, N, \pi_N, \mathcal{F})$ be the almost twistorial structure of Example 3.3, associated to (N^3, c_N, D^N).

Let $\varphi : M^4 \to N^3$ be a submersion. Let $\mathscr{V} = \ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^\perp$. Then the orientation of M^4 corresponds to an isomorphism, which depends of c_M, between $\mathscr{V}$ and the line bundle of $\mathscr{H}$. Therefore $(\mathscr{V}^* \otimes \mathscr{H}, c|_{\mathscr{H}})$ is an oriented Riemannian vector bundle. We define $\Phi : P_M \to P_N$ by

$$\Phi(p) = \tfrac{1}{\|\,\mathrm{d}\varphi(V^* \otimes p(V))\|}\, \mathrm{d}\varphi(V^* \otimes p(V)),$$

where $\{V\}$ is any basis of $\mathscr{V}_{\pi_M(p)}$ and $\{V^*\}$ its dual basis, $(p \in P_M)$.

Let $I^{\mathscr{H}}$ be the $\mathscr{V}$-valued two-form on $\mathscr{H}$ defined by $I^{\mathscr{H}}(X, Y) = -\mathscr{V}[X, Y]$, for any sections X and Y of $\mathscr{H}$. Then $*_{\mathscr{H}} I^{\mathscr{H}}$ is a horizontal one-form on M^4, where $*_{\mathscr{H}}$ is the Hodge star-operator of $(\mathscr{V}^* \otimes \mathscr{H}, c|_{\mathscr{H}})$. Denote by D_+ the Weyl connection on (M^4, c_M) defined by $D_+ = D + *_{\mathscr{H}} I^{\mathscr{H}}$ where D is the Weyl connection of $(M^4, c_M, \mathscr{V})$ (see [11]).

The following assertions are equivalent:

(i) $\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ is twistorial (with respect to Φ).

(ii) $\varphi : (M^4, c_M) \to (N^3, c_N)$ is horizontally-conformal and $\varphi^*(D^N) = \mathscr{H} D_+$ as partial connections on $\mathscr{H}$, over $\mathscr{H}$.

Let $\varphi : (M^4, c_M) \to N^3$ be a submersion from a four-dimensional oriented conformal manifold to a three-dimensional manifold. Let L be the line bundle of N^3. As the orientation of M^4 corresponds to an isomorphism between $\mathscr{V}$ and $\varphi^*(L)$, the pull-back by φ of any section of L^* is a (vertical) one-form on M^4. Similarly, if E is a vector bundle over N^3, the pull-back by φ of any section of $L^*\otimes E$ is a $\varphi^*(E)$-valued one-form on M^4.

Next, we recall the following definitions (see [16] and the references therein).

Definition 4.4. — *Let P be a principal bundle on N^3 endowed with a (principal) connection Γ and let A be a section of $L^*\otimes\mathrm{Ad}P$.*

1) Let N^3 be endowed with a conformal structure c_N and a Weyl connection D^N. The pair (A,Γ) is called a monopole on (N^3,c_N,D^N) *if*

$$R=*_N(D^N\otimes\nabla)(A)$$

where R is the curvature form of Γ and ∇ is (the covariant derivation of) the connection induced by Γ on $\mathrm{Ad}P$.

2) Let (M^4,c_M) be a four-dimensional oriented conformal manifold and let $\varphi:M^4\to N^3$ be a submersion. The connection $\widetilde{\Gamma}$ on $\varphi^*(P)$ defined by

$$\widetilde{\Gamma}=\varphi^*(\Gamma)+\varphi^*(A)$$

is called the pull-back by φ of (A,Γ).

Next, we prove the following (cf. [16] and the references therein):

Proposition 4.5. — *Let (M^4,c_M) be a four-dimensional oriented conformal manifold and let (N^3,c_N,D^N) be a three-dimensional Weyl space; denote by τ_M and τ_N the almost twistorial structures of Examples 3.6 and 3.3, associated to (M^4,c_M) and (N^3,c_N,D^N), respectively.*

Let P be a principal bundle over N^3 endowed with a connection Γ and let A be a nowhere zero section of $L^*\otimes\mathrm{Ad}P$. Also, let $\varphi:(M^4,c_M)\to(N^3,c_N)$ be a surjective horizontally conformal submersion with connected fibres.

Then any two of the following assertions imply the third:

(i) $\varphi:(M^4,\tau_M)\to(N^3,\tau_N)$ *is twistorial.*

(ii) (A,Γ) *is a monopole on* (N^3,c_N,D^N).

(iii) $\widetilde{\Gamma}$ *is anti-self-dual.*

Proof. — Let $\widetilde{R}$ be the curvature form of $\widetilde{\Gamma}$. A straightforward calculation shows that, up to an anti-self-dual term, the following equality holds

$$\widetilde{R}=\varphi^*(R)+\varphi^*\big((D^N\otimes\nabla)(A)\big)+\big(\varphi^*(D^N)-\mathscr{H}D_+\big)\wedge\varphi^*(A)\tag{4.1}$$

where we have used the isomorphism $\Lambda^2(T^*M)=\Lambda^2\big(\varphi^*(T^*N)\big)\oplus\big(\varphi^*(TN)\otimes\mathscr{V}^*\big)$, induced by c_M. The proof follows. $\square$

Remark 4.6. — The fact that (4.1) holds, up to an anti-self-dual term, does not require φ be horizontally conformal; moreover, this relation characterizes $\mathscr{H}D_+$ among the partial connections on $\mathscr{V}$, over $\mathscr{H}$.

It follows that *a submersion from a four-dimensional oriented conformal manifold to a three-dimensional Weyl space is twistorial, as in Example 4.3, if and only if it pulls-back any (local) monopole to an anti-self-dual connection*; the 'only if' part is essentially due to [16] whilst the 'if' part is an immediate consequence of Proposition 4.5 (see also [16] and the references therein).

We end this section with the following result which follows quickly from Proposition 2.5.

Proposition 4.7 (cf. [7]). — *Let (M^m, c, D) be a Weyl space, $m = 3, 4$, and let k be a section of the dual of the line bundle of M^m; if $m = 4$ assume M^4 oriented and $k = 0$. Let τ'_M be the almost twistorial structure on M^m given by Examples 3.5 or 3.7 according to $m = 3$ or $m = 4$, respectively.*

Let N^2 be an oriented surface in M^m. Let $\tau_N = (N, N, Id_N, J)$ where J is the positive Hermitian structure of $(N^2, c|_N)$.

The following assertions are equivalent:

(i) $(N^2, \tau_N) \to (M^m, \tau'_M)$ *is twistorial (with respect to p_N).*

(ii) N^2 *is a minimal surface in (M^m, c, D) and $k|_N = 0$.*

5. Harmonic morphisms and twistorial maps between Weyl spaces of dimensions four and three

We start this section by introducing a natural generalization of the almost twistorial structure of Example 3.5.

Example 5.1. — Let (M^3, c) be a three-dimensional conformal manifold endowed with two Weyl connections D' and D''. Let k be a section of L^* and let ∇ be the connection associated to D'' and k.

Let $\pi : P \to M$ be the bundle of nonzero skew-adjoint f-structures on (M^3, c). We denote by $\mathscr{H}^0$ the subbundle of $T^{\mathbb{C}}P$ such that, at each $p \in P$, the subspace $\mathscr{H}^0_p \subseteq T^{\mathbb{C}}_p P$ is the horizontal lift, with respect to D', of the eigenspace of $p^{\mathbb{C}} \in \mathrm{End}(T^{\mathbb{C}}_{\pi(p)}M)$ corresponding to the eigenvalue 0. Also, we denote by $\mathscr{H}^{1,0}$ the subbundle of $T^{\mathbb{C}}P$ such that, at each $p \in P$, the subspace $\mathscr{H}^{1,0}_p \subseteq T^{\mathbb{C}}_p P$ is the horizontal lift, with respect to ∇, of the eigenspace of $p^{\mathbb{C}} \in \mathrm{End}(T^{\mathbb{C}}_{\pi(p)}M)$ corresponding to the eigenvalue i.

We define the almost f-structure $\mathcal{F}''$ on P with respect to which $T^0P = \mathscr{H}^0$ and $T^{1,0}P = (\ker \mathrm{d}\pi)^{0,1} \oplus \mathscr{H}^{1,0}$. Then $\tau'' = (P, M, \pi, \mathcal{F}'')$ is a nonintegrable almost twistorial structure on M (the nonintegrability of τ'' follows easily from the proof of the integrability result presented in [14]).

We shall also need the following.

Example 5.2. — Let (M^4, c, D) be a four-dimensional oriented Weyl space. Let $\pi : P \to M$ be the bundle of positive orthogonal complex structures on (M^4, c).

Let $\varphi : M^4 \to N^3$ be a submersion; denote by $\mathscr{V} = \ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^\perp$. For any $p \in P$ let l'_p and l''_p be the lines in $T^{\mathbb{C}}_{\pi(p)}M$ spanned by $V - \mathrm{i}\, p(V)$ and $X - \mathrm{i}\, p(X)$, respectively, for any $V \in \mathscr{V}_{\pi(p)}$ and $X \in p(\mathscr{V}_{\pi(p)})^\perp \cap \mathscr{H}_{\pi(p)}$.

We define the almost CR-structure $\mathscr{C}'_\varphi$ on P which, at each $p \in P$, is the direct sum of $(\ker \mathrm{d}\pi_p)^{0,1}$ and the horizontal lift, with respect to D, of l'_p.

Similarly, we define the almost CR-structure $\mathscr{C}''_\varphi$ on P which, at each $p \in P$, is the direct sum of $(\ker \mathrm{d}\pi_p)^{0,1}$ and the horizontal lift, with respect to D, of l''_p.

Then $\tau'_\varphi = (P, M, \pi, \mathscr{C}'_\varphi)$ and $\tau''_\varphi = (P, M, \pi, \mathscr{C}''_\varphi)$ are nonintegrable almost twistorial structures on M^4 (the nonintegrability of τ'_φ and τ''_φ follows easily from the proof of the integrability result presented in [14]).

Remark 5.3. — Let (N^3, c_N) be a three-dimensional conformal manifold. At least locally, we may assume that $(L^* \otimes TN, c)$ is the adjoint bundle of a rank two complex vector bundle E with group $\mathrm{SU}(2)$. Furthermore, under this identification, P_N is isomorphic to PE. As $\mathrm{SO}(3,\mathbb{R}) = \mathrm{SU}(2)/\mathbb{Z}_2 = \mathrm{PSU}(2)$, any connection on $(L^* \otimes TN, c)$ corresponds to a connection on the $\mathrm{PSU}(2)$-bundle P_N. Also, $(L^* \otimes TN, c)$ is the adjoint bundle of the $\mathrm{PSU}(2)$-bundle P_N.

Similarly, if (M^4, c_M) is a four-dimensional oriented conformal manifold then the bundle P_M of positive orthogonal complex structures on (M^4, c_M) is a $\mathrm{PSU}(2)$-bundle.

Furthermore, a submersion $\varphi : (M^4, c_M) \to (N^3, c_N)$ is horizontally conformal if and only if $\Phi : P_M \to P_N$ is a morphism of $\mathrm{PSU}(2)$-bundles (in general, Φ is a morphism of $\mathrm{SL}(3,\mathbb{R})$-bundles). Also, note that, if φ is horizontally conformal then $P_M = \varphi^*(P_N)$ as $\mathrm{PSU}(2)$-bundles.

The following result shows the importance of the almost twistorial structures of Example 5.2.

Theorem 5.4. — *Let (M^4, c_M, D^M) be a four-dimensional oriented Weyl space and let P_M be the bundle of positive orthogonal complex structures on (M^4, c_M).*

Let (N^3, c_N, D^N) be a three-dimensional Weyl space and let k be a section of the dual of the line bundle of N^3; denote by ∇ the connection associated to D^N and k. Let τ'_N be the almost twistorial structure of Example 3.5, associated to (N^3, c_N, D^N, k).

Let $\varphi : M^4 \to N^3$ be a submersion; denote by $\mathscr{V} = \ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^\perp$. Let τ'_φ and τ''_φ be the almost twistorial structures of Example 5.2, associated to (M^4, c_M, D^M) and φ.

(i) *The following assertions are equivalent:*

(i1) $\varphi : (M^4, \tau'_\varphi) \to (N^3, \tau'_N)$ *is twistorial.*

(i2) $\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)$ *is a harmonic morphism.*

(ii) *The following assertions are equivalent:*

(ii1) $\varphi : (M^4, \tau''_\varphi) \to (N^3, \tau'_N)$ *is twistorial.*

(ii2) $\varphi : (M^4, c_M) \to (N^3, c_N)$ *is horizontally conformal, $\mathscr{V}(D^M - D) = \frac{1}{2}\,k$ and $\varphi^*(D^N) = \mathscr{H} D^M + \frac{1}{2} *_{\mathscr{H}} I^{\mathscr{H}}$ as partial connections, over $\mathscr{H}$, where D is the Weyl connection of $(M^4, c_M, \mathscr{V})$.*

(ii3) $\varphi : (M^4, c_M) \to (N^3, c_N)$ *is horizontally conformal and the partial connections on P_M, over $\mathscr{H}$, induced by D^M and $\varphi^*(\nabla)$ are equal.*

Proof. — (i) By Remark 5.3, if (i1) holds then $\varphi : (M^4, c_M) \to (N^3, c_N)$ is horizontally conformal.

Let $\tau'_\varphi = (P_M, M, \pi_M, \mathscr{C}'_\varphi)$ and let $\tau'_N = (P_N, N, \pi_N, \mathcal{F}')$. By associating to each orthogonal complex structure on (M^4, c_M) its eigenspace corresponding to $-\mathrm{i}$, we identify $\pi_M : P_M \to M$ with the bundle of self-dual spaces on (M^4, c_M).

Similarly, by associating to each skew-adjoint f-structure on (N^3, c_N) the sum of its eigenspaces corresponding to 0 and $-\mathrm{i}$, we identify $\pi_N : P_N \to N$ with the bundle of (complex) two-dimensional degenerate spaces on (N^3, c_N).

Under these identifications the natural lift $\Phi : P_M \to P_N$ of φ is given by $\Phi(p) = \mathrm{d}\varphi(p)$, for any $p \in P_M$.

Let q be a local section of P_N, over some open set $V \subseteq N$, and let p be the local section of P_M, over $\varphi^{-1}(V)$, such that $\Phi \circ p = q \circ \varphi$. At least locally, we may assume that p is generated by $Y = U + \mathrm{i}X$ and Z, with U vertical, and X and Z basic. Thus, q is generated by $\mathrm{d}\varphi(X)$ and $\mathrm{d}\varphi(Z)$.

We shall assume that, for some $x_0 \in \varphi^{-1}(V)$, we have $\mathrm{d}q(\mathrm{d}\varphi(X_{x_0}))$ horizontal, with respect to the connection induced by ∇ on P_N. Then Proposition 2.5, the fundamental equation (see [11]) and a straightforward calculation show that $\mathrm{d}p(\overline{Y}_{x_0}) \in (\mathscr{C}'_\varphi)_{p(x_0)}$ if and only if $c_M\big(\mathrm{trace}_{c_M}(\widetilde{D}\mathrm{d}\varphi), Z\big) = 0$, where $\widetilde{D}$ is the connection induced by D^M and D^N on $\varphi^*(TN) \otimes T^*M$.

This proves that (i2)$\Longrightarrow$(i1).

To prove the converse, assume that (i1) holds. Then there exists a unique $A \in (\mathscr{C}'_\varphi)_{p(x_0)}$ such that $\mathrm{d}\Phi(A) = \mathrm{d}q(\mathrm{d}\varphi(X_{x_0}))$. From the fact that $\mathrm{d}\pi_M\big((\mathcal{C}'_\varphi)_{p(x_0)}\big)$ is the line spanned by $\overline{Y}_{x_0}$, it follows quickly that $\mathrm{d}\pi_M(A) = -\overline{Y}_{x_0}$. Together with $\mathrm{d}\Phi(A) = \mathrm{d}\Phi(\mathrm{d}p(-\overline{Y}_{x_0}))$ this gives $A = \mathrm{d}p(-\overline{Y}_{x_0})$. Hence, $\mathrm{d}p(\overline{Y}_{x_0}) \in (\mathscr{C}'_\varphi)_{p(x_0)}$ and, therefore, $c_M\big(\mathrm{trace}_{c_M}(\widetilde{D}\mathrm{d}\varphi), Z\big) = 0$.

(ii) By Remark 5.3, if (ii1) holds then $\varphi : (M^4, c_M) \to (N^3, c_N)$ is horizontally conformal.

Similarly to above, let p be a basic local section of P_M; that is, there exists a local section q of P_N such that $\Phi \circ p = q \circ \varphi$. At least locally, we may assume that p is generated by $Y = U + \mathrm{i}X$ and Z, with U vertical, and X and Z basic. Thus, q is generated by $\mathrm{d}\varphi(X)$ and $\mathrm{d}\varphi(Z)$.

To prove (ii1) $\iff$ (ii2), we shall assume that p is horizontal, at some point $x_0 \in M$, with respect to the connection induced by D^M on P_M (this could be done as follows: firstly, define p over some hypersurface, containing x_0 and transversal to the fibres of φ, such that p is horizontal at x_0; then, extend p, to an open neighbourhood of x_0, so that to be basic); in particular, $\mathrm{d}p(\overline{Z}_{x_0}) \in (\mathscr{C}''_\varphi)_{p(x_0)}$.

Assertion (ii1) is equivalent to the fact that, for any such p, we have $\mathrm{d}q(\mathrm{d}\varphi(X_{x_0}))$ contained in the eigenbundle of $\mathcal{F}'$ corresponding to the eigenvalue i. By using Proposition 2.5, it quickly follows that (ii1) $\iff$ (ii2).

Let h be an oriented representative of c_N and let g be a representative of c_M such that $\varphi : (M^4, g) \to (N^3, h)$ is a Riemannian submersion. Let α^M and α^N be the Lee forms of D^M and D^N with respect to g and h, respectively.

The equivalence (ii2) $\iff$ (ii3) follows from the fact that any two of the following assertions imply the third:

(a) $\mathrm{d}p(\mathscr{H}_{x_0})$ is horizontal, with respect to the connection induced by D^M on P_M.

(b) q is horizontal at $\varphi(x_0)$, with respect to the connection induced by ∇ on P_N.

(c) At x_0 we have $\alpha^M|_{\mathscr{V}} = \frac{1}{2}\,k$ and $\alpha^N = \alpha^M|_{\mathscr{H}} + \frac{1}{2} *_{\mathscr{H}} I^{\mathscr{H}}$, where we have identified k and α^N with their pull-backs by φ and, in the first equality, we have used the isomorphism between $\mathscr{V}$ and the pull-back by φ of the line bundle of N^3 induced by φ and the orientation of M^4.

The proof is complete. □

Remark 5.5. — Let (M^4, c, D) be a four-dimensional oriented Weyl space and P the bundle of positive orthogonal complex structures on (M^4, c).

With the same notations as in Example 5.2, by Theorem 5.4(i), *the relation* $\varphi \longleftrightarrow \mathscr{C}'_\varphi$ *induces a bijective correspondence between one-dimensional foliations on* (M^4, c, D)*, which are locally defined by harmonic morphisms, and certain almost CR-structures on* P*.*

Next, we prove the following.

THEOREM 5.6. — *Let* (M^4, c_M, D^M) *be a four-dimensional oriented Weyl space and let* τ'_M *be the almost twistorial structure of Example 3.7, associated to* (M^4, c_M, D^M)*.*

Let (N^3, c_N) *be a three-dimensional conformal manifold endowed with two Weyl connections* D' *and* D''*. Let* k *be a section of the dual of the line bundle of* N^3 *and let* τ''_N *be the almost twistorial structure of Example 5.1, associated to* (N^3, c_N, D', D'', k)*.*

Let $\varphi : M^4 \to N^3$ *be a submersion. The following assertions are equivalent:*

(i) $\varphi : (M^4, \tau'_M) \to (N^3, \tau''_N)$ *is twistorial.*

(ii) $\varphi : (M^4, c_M) \to (N^3, c_N)$ *is horizontally conformal and the connection induced by* D^M *on the bundle of positive orthogonal complex structures on* (M^4, c_M) *is the pull-back by* φ *of* (A, ∇)*, where* $A = (D'' - D')^{\sharp_{c_N}}$ *and* ∇ *is the connection associated to* D'' *and* k*.*

(iii) *The following assertions hold:*

(iii1) $\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D')$ *is a harmonic morphism;*

(iii2) $\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ *is twistorial, where* τ_M *and* τ_N *are the almost twistorial structures of Examples 3.6 and 3.3, associated to* (M^4, c_M) *and* $(N^3, c_N, 2D'' - D')$*, respectively.*

(iii3) $\mathscr{V}(D^M - D) = \frac{1}{2}\, k$ *where* $\mathscr{V} = \ker \mathrm{d}\varphi$ *and* D *is the Weyl connection of* $(M^4, c_M, \mathscr{V})$*.*

Proof. — From Theorem 5.4, it follows that assertion (i) holds if and only if $\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D')$ is a harmonic morphism, $\mathscr{V}(D^M - D) = \frac{1}{2}\, k$ and $\varphi^*(D^N) = \mathscr{H} D^M + \frac{1}{2} *_{\mathscr{H}} I^{\mathscr{H}}$ as partial connections, over $\mathscr{H}$. Together with the fundamental equation, this quickly gives (i) $\Longleftrightarrow$ (iii).

Let $\tau'_M = (P_M, M, \pi_M, \mathscr{J}')$. From Theorem 5.4, it follows that assertion (i) holds if and only if $\varphi : (M^4, c_M) \to (N^3, c_N)$ is horizontally conformal, the partial connections on P_M, over $\mathscr{H}$, induced by D^M and $\varphi^*(\nabla)$ are equal, and we have $\mathscr{H}(D^M - D) = D' - D'' + \frac{1}{2} *_{\mathscr{H}} I^{\mathscr{H}}$, as partial connections, over $\mathscr{H}$.

Let $(X_1,\dots,X_4)$ be a positive conformal local frame on (M^4,c_M) such that X_1 is vertical and X_2, X_3, X_4 are basic. Let g be the local representative of c_M induced by $(X_1,\dots,X_4)$. Let α^M and α be the Lee forms, with respect to g, of D^M and D, respectively. Denote by Γ^i_{jk}, $(i,j,k=1,\dots,4)$, the Christoffel symbols of D^M with respect to $(X_1,\dots,X_4)$. Then a straightforward calculation gives the following relations:

$$\begin{aligned}
\Gamma^1_{21}+\Gamma^3_{41} &= \big(\alpha^M-\alpha-\tfrac12 *_{\mathscr{H}} I^{\mathscr{H}}\big)(X_2),\\
\Gamma^1_{31}-\Gamma^2_{41} &= \big(\alpha^M-\alpha-\tfrac12 *_{\mathscr{H}} I^{\mathscr{H}}\big)(X_3),\\
\Gamma^1_{41}+\Gamma^2_{31} &= \big(\alpha^M-\alpha-\tfrac12 *_{\mathscr{H}} I^{\mathscr{H}}\big)(X_4).
\end{aligned}$$

Thus, we have proved that (i) holds if and only if $\varphi:(M^4,c_M)\to(N^3,c_N)$ is horizontally conformal, the partial connections on P_M, over $\mathscr{H}$, induced by D^M and $\varphi^*(\nabla)$ are equal, and the following relations hold:

$$\begin{aligned}
\Gamma^1_{21}+\Gamma^3_{41} &= (D'-D'')(X_2),\\
\Gamma^1_{31}-\Gamma^2_{41} &= (D'-D'')(X_3),\\
\Gamma^1_{41}+\Gamma^2_{31} &= (D'-D'')(X_4).
\end{aligned}$$

It follows that (i) $\iff$ (ii). □

Remark 5.7. — 1) Let (M^4,c_M,D^M) be a four-dimensional oriented Weyl space. Let τ'_M be the almost twistorial structure of Example 3.7, associated to (M^4,c_M,D^M); denote by P_M the bundle of positive orthogonal complex structures on (M^4,c_M).

Let $\varphi:M^4\to N^3$ be a submersion onto a three-dimensional manifold. Let P_N be the bundle of oriented lines on N^3. Then, similarly to Example 4.3, there can be defined a bundle map $\Phi:P_M\to P_N$.

Suppose that there exists an almost twistorial structure τ on N^3 such that $\varphi:(M^4,\tau'_M)\to(N^3,\tau)$ is a twistorial map, with respect to Φ. Then there exist a section k of the dual of the line bundle of N^3, a conformal structure c_N on N^3 and Weyl connections D' and D'' on (N^3,c_N) such that τ is the almost twistorial structure of Example 5.1, associated to (N^3,c_N,D',D'',k).

A similar comment applies to the twistorial maps of Example 4.3.

2) Let (M^4,c_M) be an oriented four-dimensional conformal manifold and let (N^3,c_N,D) be a three-dimensional Weyl space. Denote by τ_M and τ_N the almost twistorial structures, of Examples 3.6 and 3.3, associated to (M^4,c_M) and (N^3,c_N,D), respectively. Let $\varphi:(M^4,\tau_M)\to(N^3,\tau_N)$ be a twistorial map (see [16] and [6] for examples of such maps).

Let D' be a Weyl connection on (N^3, c_N) and k a section of the dual of the line bundle of N^3; denote by $D'' = \frac{1}{2}(D + D')$.

From Theorem 5.6, it follows that there exists a unique Weyl connection D^M on (M^4, c_M) such that $\varphi : (M^4, \tau'_M) \to (N^3, \tau''_N)$ is twistorial, where τ'_M and τ''_N are the almost twistorial structures of Examples 3.7 and 5.1, associated to (M^4, c_M, D^M) and (N^3, c_N, D', D'', k), respectively.

3) Let (M^4, c_M) be a four-dimensional oriented conformal manifold and L the line bundle of M^4. Denote by $P_\pm$ the bundles of positive/negative orthogonal complex structures on (M^4, c_M). As $P_\pm$ are, locally, the projectivisations of the bundles of positive/negative spinors on $(L^* \otimes TM, c_M)$, any conformal connection on (M^4, c_M) corresponds to a pair (Γ_+, Γ_-), where $\Gamma_\pm$ are connections on $P_\pm$.

Let $\varphi : (M^4, c_M) \to (N^3, c_N)$ be a horizontally conformal submersion onto a three-dimensional conformal manifold. Endow (N^3, c_N) with two Weyl connections D' and D'' and let k be a section of the dual of the line bundle of N^3. Let Γ_+ be the connection on P_+ which is the pull-back by φ of (A, ∇), where $A = (D'' - D')^{\sharp_{c_N}}$ and ∇ is the connection associated to D'' and k.

Let Γ_- be a connection on P_- and suppose that the connection D^M corresponding to (Γ_+, Γ_-) is torsion-free. Then, obviously, D^M is a Weyl connection on (M^4, c_M). Moreover, by Theorem 5.6, the map $\varphi : (M^4, \tau'_M) \to (N^3, \tau''_N)$ is twistorial, where τ'_M and τ''_N are the almost twistorial structures of Examples 3.7 and 5.1, associated to (M^4, c_M, D^M) and (N^3, c_N, D', D'', k), respectively.

4) With the same notations as in Theorem 5.6, it can be proved that (A, ∇) is a monopole on $(N^3, c_N, 2D'' - D')$ if and only if $D' = D''$ and ∇ is flat (cf. [8]; see [11] for details about the resulting maps).

The following result is an immediate consequence of Theorem 5.6; note that, the equivalence (ii) $\Longleftrightarrow$ (iii) appears in [11].

Corollary 5.8. — *Let (M^4, c_M, D^M) be a four-dimensional oriented Weyl space and τ'_M the almost twistorial structure of Example 3.7, associated to (M^4, c_M, D^M).*

Let (N^3, c_N, D^N) be a three-dimensional Weyl space and k a section of the dual of the line bundle of N^3. Let τ'_N be the almost twistorial structure of Example 3.5, associated to (N^3, c_N, D^N, k).

Let $\varphi : M^4 \to N^3$ be a submersion. The following assertions are equivalent:

(i) *$\varphi : (M^4, \tau'_M) \to (N^3, \tau'_N)$ is twistorial.*

(ii) $\varphi : (M^4, c_M) \to (N^3, c_N)$ *is horizontally conformal and the connection induced by* D^M *on the bundle of positive orthogonal complex structures on* (M^4, c_M) *is the pull-back by* φ *of* ∇, *where* ∇ *is the connection associated to* D^N *and* k.

(iii) *The following assertions hold:*

(iii1) $\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)$ *is a harmonic morphism;*

(iii2) $\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ *is twistorial, where* τ_M *and* τ_N *are the almost twistorial structures of Examples 3.6 and 3.3, associated to* (M^4, c_M) *and* (N^3, c_N, D^N), *respectively.*

(iii3) $\mathscr{V}(D^M - D) = \frac{1}{2}\,k$ *where* $\mathscr{V} = \ker \mathrm{d}\varphi$ *and* D *is the Weyl connection of* $(M^4, c_M, \mathscr{V})$.

Example 5.9. — 1) Let $\varphi : (M^4, g) \to (N^3, h)$ be a harmonic morphism given by the Gibbons-Hawking or the Beltrami fields construction (see [15]). Then φ satisfies (ii) and (iii) of Corollary 5.8, with D^M and D^N the Levi-Civita connections of g and h, respectively, and a suitable choice of k ([11], [16]). Hence, φ satisfies assertion (i) of Corollary 5.8 (and, also, (i) of Theorem 5.6).

2) Let $\varphi : (M^4, g) \to (N^3, h)$ be a harmonic morphism of Killing type between Riemannian manifolds of dimensions four and three. Then φ satisfies (iii1), (iii2) of Theorem 5.6, with D^M and D' the Levi-Civita connections of g and h, respectively, and a suitable choice of D'' [16]; also, φ satisfies (iii3), with $k = 0$. Hence, φ satisfies assertion (i) of Theorem 5.6.

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